NOAA Technical Memorandum ERL WPL-56

LOGARITHMIC NORMALIZATION TO REMOVE UNKNOWN SIGNAL-GAIN FACTORS BEFORE AVERAGING

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noaa NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION / Environmental Research Laboratories

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ABSTRACT

To increase the effective number of samples when averaging radar-echo spectra, unknown multiplicative factors (such as path losses) that vary from spectrum to spectrum must first be removed. A method of logarithmically normalizing the separate spectra is presented here, and the errors incurred in its use are determined. In terms of accuracy and ease of implementation, logarithmic normalization is better than alternative methods. Examples illustrating its application to HF sea echo are provided, and overall errors in extracting sea-state parameters are obtained and compared with the errors resulting from other normalization procedures.

1. INTRODUCTION

Radar-echo Doppler spectra are used in many remote-sensing applications, including radio oceanography and radio meteorology. Because the radar echo originates from a statistical ensemble of random scatterers (such as the waves on the ocean surface), the echo signal is a random variable. Hence, a single Doppler power spectrum output from the receiver/processor FFT (Fast Fourier Transform) generally has a random-like appearance about a mean that can vary over a wide dynamic range. The power at each spectral point is a chi-squared (χ^2) random variable with two degrees of freedom, because the complex echo signal voltage is a Gaussian random variable (Barrick and Snider, 1977).

In order to reduce errors incurred in extracting geophysical parameters from such random echo spectra, it is desirable to average as many independent spectral samples as possible. In many situations, however, each spectrum may be multiplied by a different, unknown factor, originating for example from (i) a varying path loss between samples (e.g., samples from different ranges or over different ionospheric paths); (ii) varying system gain. Adding such spectra together — with unequal multiplicative factors — has the undesirable effect of reducing the number of degrees of freedom in the result, causing the averaging process to be less effective. For example, suppose six power spectra are to be averaged together, each having ten independent samples (20 degrees of freedom). If the system constants and path-loss factors were equal for each of the six, the sum would consist of 60 independent samples (120 degrees of freedom), giving a fluctuation of the power about the mean of $100\% \times (1/\sqrt{60})$. On the other hand, if one of the six were larger than the others by a constant factor of, say, 15, then the remaining five would have little effect when added to the dominant one because they are so small; the result would have not much more than 10 independent samples, and the percent fluctuation of the sum would be $\sim 100\% \times (1/\sqrt{10})$. Therefore it would be desirable to devise some way of estimating the unknown multiplicative factors and dividing them out before averaging.

If the mean of all the points in the echo spectrum were fairly constant (i.e., the shape were flat), one could merely add all of the echo spectral power points together and divide the entire spectrum by that constant. This is called total-power or energy normalization, and is treated in Barrick (1980). In most geophysical situations, however, the spectral mean is not constant as, for example, in the case of HF sea echo (see Fig. 1). There, the first-order spectral peaks are much higher than the second-order spectral region, and these first-order peaks are usually defined by two or three spectral points. Any attempted power or energy normalization will then be a noisy process, giving rise to an F-distribution with only a few degrees of freedom in the denominator (Barrick, 1980). Hence that type of normalization is not ideally suited to sea echo (and many other geophysical echo spectra). Other types of normalization have been examined, including a

variance minimization scheme (Georges et al., 1980). Compared with the logarithmic normalization discussed here, however, the variance minimization is more costly in terms of computer time, and produces no measurably greater accuracy. Furthermore, logarithmic normalization has the desirable feature that finite-sampling errors can be readily estimated in closed form, without the need for lengthy Monte-Carlo simulations.

The purpose of this technical memorandum is: (i) to explain and demonstrate (by examples) the use of logarithmic normalization; and (ii) to present expressions and tables for errors incurred in its use resulting from finite sample sizes.

2. DEFINITION AND USE OF LOGARITHMIC NORMALIZATION

Figure 1 shows two examples of HF sea-echo Doppler spectra measured by the NOAA CODAR system on January 19, 1978, off Pescadero, California. The upper spectrum was measured from a 1.2 km annular range cell 5.4 km from the radar, whereas the lower one came from 17.4 km. Because the signal path loss for the farther range cell is greater, its additive noise level (at the edges of the spectrum) is higher with respect to the sea-echo. We would like to average the usable sea-echo signal regions for the two spectra (along with many others at ranges in between). Because the path loss is unknown, and the system AGC (Automatic Gain Control) is variable, there is no way of knowing the necessary multiplicative factor *a priori*.

Each spectrum shown has already been averaged. Each consists of the sum of 8 consecutive spectra from that same range cell; the individual spectra were produced from time series 2 m 14 s long (512-point FFTs were employed). The spectra were recorded at 25.4 MHz and their frequency scales have been normalized so that the expected positions of the first-order echoes (shown at ±1) actually occur at 0.5141 Hz; the frequency spacing (resolution) between spectral points is 0.00745 Hz. The figures illustrate that — in terms of absolute power — only a few points of first- and second-order echo dominate each spectrum. Yet we ultimately employ the

echo region between -1.5 and +1.5 normalized frequency for extraction of sea-state information (e.g., see Barrick and Lipa, 1979; Lipa and Barrick, 1980; Lipa, Barrick, and Maresca, 1980). Hence the sum of the spectral power points will produce a normalizing estimator whose variance is large because only a few points in the sum are large. Logarithmic normalization — as defined below — provides an estimator with lower variance.

Define K as the number of independent samples that have already been averaged to form the spectrum for each range cell before logarithmic normalization. (K=8 for the case shown in Fig. 1). Then assume we have L such spectra (in this case, originating from L different ranges) with different, unknown multiplicative factors that we want to estimate and remove before averaging the L spectra together. Within the ℓ-th (of L spectra), let the portion of the spectrum representing the usable signal lie above the dashed line; the remainder is either the additive system/external noise or is too close to this noise to be reliable. Furthermore, the usable echo region varies with range here, being less at the greater distance. We want to be careful to use only that part of the spectrum above the dashed line for normalization purposes, because the additive noise part is not multiplied by the same unknown constant; to use part of the noise would be to contaminate the process.

Because the more distant range cell contains fewer usable signal spectral points, let us (for simplicity in illustrating the method) use the smaller signal band for all spectra. Assume that this "window" of usable signal contains J independent power spectral points. The power sample, therefore designated P_j^{ℓ}, contains K independent samples already, and is the j-th (of J) point within the ℓ-th spectrum; this spectral value here represents either first or second-order sea echo. In logarithmic normalization, both regions will contribute equally effectively to the normalization process, even though many of the second-order spectral points have much lower absolute power than the region near ± 1.

In everything that follows, we assume the following:

a) $J >> 1$ (typically, $J > 10$; this is very reasonable for any HF sea-echo Doppler spectra that are usable at all).

b) K and L need not necessarily be large.

c) P_j^{ℓ} is χ^2 with 2K degrees of freedom; its true mean value is $A_{\ell}p_j$.

d) The unknown constant multiplying the ℓ-th spectrum is defined as A_{ℓ}.

3. THE NORMALZING ESTIMATOR

From the J sea-echo spectral points selected within the allowed window of the ℓ-th power spectrum, we calculate c_{ℓ} as follows:

$$c_{\ell} \equiv \frac{1}{J} \sum_{j=1}^{J} \ell n \, P_j^{\ell} \, .$$

The normalization process then consists of dividing each point in the ℓ-th spectrum by

$$C_{\ell} \equiv \exp[c_{\ell}] = \exp[\frac{1}{J} \sum_{j=1}^{J} \ell n \, P_j^{\ell}] \, .$$

(It is of interest to note that C_{ℓ} is the geometric mean of the P_j^{ℓ}.)

Hence, the renormalized spectral points of the ℓ-th spectrum are

$$P_{Nj}^{\ell} \equiv P_j^{\ell}/C_{\ell} = P_j^{\ell} \times \exp[- \frac{1}{J} \sum_{j=1}^{J} \ell n \, P_j^{\ell}] \, . \qquad (1)$$

4. ERROR INCURRED IN LOGARITHMIC NORMALIZATION

The original spectral points, P_j^{ℓ}, were χ^2 random variables with 2K degrees of freedom. The fact that these χ^2 variables are multiplied by

another random variable in the normalizing process (i.e., $1/C_\ell$) means that: (a) the individual, normalized L spectra is no longer strictly χ^2, and (b) these spectra before averaging will contain more fluctuation than the original, unnormalized spectral points, P_j^ℓ.

If the logarithmic normalization process is really that good, we would expect a result approaching the one we would have had if normalization had been unnecessary (i.e., there were no unknown A_ℓ's multiplying the spectra). In that case, the sample-average of L spectra each containing K independent samples would have had a standard deviation (normalized) of $1/\sqrt{KL}$.

The analysis in the Appendix shows that the fractional standard deviation of the new spectral points, P_{Nj}^ℓ, after sample-averaging L such renormalized spectra (i.e., $\frac{1}{L}\sum_{\ell=1}^{L} P_{Nj}^\ell$) at the desired j-th frequency point can be expressed as

$$sd(P_{Nj}^\ell) = 1/\sqrt{LK_e} \quad , \tag{2}$$

where

$$K_e \equiv Kf_1/(1 + f_2/J) \tag{3}$$

defines an effective number of independent samples in terms of K. The factors f_1 and f_2 are shown to approach unity for large K, and therefore it is readily seen that $K_e \to K$ for K and J large, as expected.

The derivation of expressions for f_1 and f_2 is performed in the Appendix. These functions, f_1 and f_2, turn out to be functions of K only. Table 1 gives exact values of f_1 and f_2 for small and moderately large K. For values of K larger than 10, adequate accuracy can be obtained from the asymptotic forms:

$$f_1 \to 1 - 1/K \quad ; \tag{4a}$$

$$f_2 \to 1 + 2.5/K \; . \tag{4b}$$

5. FIRST EXAMPLE: SKYWAVE RADAR DETERMINATION OF SIGNIFICANT WAVEHEIGHT

Maresca and Georges (1980) derived a semi-empirical relation between ocean waveheight to the ratio of second-order to first-order Doppler spectral energy. This relationship is

$$k_o h = CR^p,$$

where R is the ratio, C and p are semi-empirically determined constants, h is ocean rms waveheight, and k_o is the radar spatial wavenumber. Barrick (1977) showed that p = 1/2 if one applies a known weighting-function to the second-order echo. Maresca and Georges (1980) show that $p \simeq 0.6$ if one does not weight and if one forms the ratio in a manner that makes the result direction insensitive. For illustration we take p = 0.5 here.

Assume that we have 40 independent spectra that will be used, but these 40 come from sufficiently different ranges and times that we must assume each is multiplied by an unknown ionospheric path loss. We also assume that the half-power width of the first-order Doppler peak includes $\sim$3 spectral points. Hence the <u>effective</u> frequency resolution of the spectra is $3 \times \Delta f$, where Δf is the spacing output from the FFT. Thus we do a 3-point frequency smoothing of the data first; this is equivalent to calculating the area within the first-order region, since that is the sum of three nearly equal values. If we had, say, $\sim$150 points of recognizable sea echo originally output from the FFT (first <u>and</u> second order), this frequency smoothing means there will be only $\sim$50 <u>separately</u> resolved frequency bins remaining. Assume that while the effective width of the first-order region was 3 points, the effective half-power width of the second-order region is 7.

(a) Logarithmic Normalization:

Using the above problem description, we find that the parameters relevant for logarithmic normalization (after the 3-point frequency smoothing) are: L = 40; J = 50; K = 3. Hence, using Eqs. (2) and (3) and Table I, we find that the fractional variance of the renormalized sea-echo spectral points (with 1/3 the frequency resolution) after the 40 spectra have been averaged is

$$\frac{1}{LK_e} = \frac{1 + 1.9747/50}{40 \times 3 \times .7035} = \frac{1}{81.3} \quad .$$

Thus, 81 independent samples is less than 40 × 3 = 120 independent samples that would have been available if normalization had been unnecessary. As mentioned earlier, this greater variance arises because the renormalizing estimator multiplying the spectra before averaging is itself a random variable. Now, waveheight accuracy is given by Eq. (26) of Barrick (1980), where we employ KN = 81.2 and KM = 81.2 × 7/3 in that expression. Hence we obtain a percent standard deviation for waveheight of

$$sd(h) = .066 \times 100\% = 6.6\%.$$

(b) Peak or Quotient Normalization:

For comparison, we look at the waveheight extraction accuracy we would have obtained had we taken the 3-point frequency-smoothed first-order peak and divided each spectrum by this to remove the unknown constant. (Note that in this case, if the frequency smoothing had <u>not</u> been used, the denominator of the quotient would have consisted of only <u>one</u> independent sample, and hence the mean and standard deviation of this quotient would have been <u>infinite</u>, as shown in Barrick (1980).) With the frequency smoothing, however, we have the following parameters relevant to the use of Eq. (28) in Barrick (1980): p = 1/2, K = 40, N = 3, M = 7. Thus we obtain

$$sd(h) = p\langle q\rangle^{p-1}\, SD(q)/\sqrt{K} = .1098 \times 100\% = 11.0\% ,$$

and $h_b/h = \sqrt{3/2} = 1.225$, where the latter is the waveheight bias factor.

Therefore, logarithmic normalization does a better job of removing path losses than the more frequently used peak normalization, in terms of resulting finite-sample errors (i.e., 6.6% compared to 11.0%). Note that in most cases, peak normalization is not much different than total energy normalization, because most of the total echo energy is contained in the narrow first-order peak.

6. SECOND EXAMPLE: SURFACE-WAVE RADAR DETERMINATION OF THE WAVEHEIGHT DIRECTIONAL SPECTRUM

Here we employ as an example narrow-beam surface-wave sea-echo Doppler spectra, as analyzed in Lipa et al. (1980). To do a complete inversion of the relevant integral equation, one must divide the points within the second-order Doppler region by the total first-order energy (or power under the first-order peak). We want to determine here how many effective degrees of freedom we have in the sample-averaged second-order region after this first-order division; for convenience, call the resulting quantity at the j-th spectral point S_j.

Assume that we have 12 range cells we want to average together, but they are multiplied by unknown path losses; therefore, L = 12. The spectra from each range cell, like those shown in Fig. 1, consist of 8 consecutive power FFTs added together (K=8). We want to compare the merits of logarithmic vs. peak (or quotient) normalization. We note that the first-order line contains about 3 samples* between its half-power points. Since this first-order function (according to second-order sea-echo scatter theory, Lipa and

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* The broadening to include three points could arise from the nature of the window used with the FFT or from smearing of the echo due to current variations and wave-wave interactions; we need not be concerned with its origin here.

Barrick, 1980) has been convolved with the rest of the second-order spectrum, the resolution in the spectrum is really no better than $3\Delta f$, where Δf is the frequency spacing between spectral points. Hence some of the remaining randomness can be removed — with no sacrifice in resolution — by doing a running frequency average over the second-order region with a sliding 3-point function. This means that K is actually increased from 8 to 24, but J is divided by three. If there were originally 300 frequency points within the window to be used for analysis, this means J = 100 now. (Note from Eqs. (2) and (3) of this paper that it is always better if J is large to do a sliding average whose width is proportional to that of the first-order line, which will increase K while decreasing J. For peak or quotient normalization this is necessary also in order to produce a more stable, optimal result.)

(a) Logarithmic Normalization:

Since we have K = 24, L = 12, J = 100, the total number of effective samples after logarithmic normalization but before division by the first-order energy is given by

$$\frac{1}{LK_e} = \frac{1 + 1.1062/100}{12 \times 24 \times .9589} = \frac{1}{273.1} \quad ,$$

resulting in 273.1 independent samples rather than the optimal $12 \times 24 = 288$.

Now, the sample error in S_j after division by the first-order energy is obtained from Eq. (26) of Barrick (1980), where p = 1 here. This comes about because the first-order energy region (in the notation there) contains NK = 273.1 independent samples, as does each independent point in the numerator (or the second-order spectral region). Thus we have

$$sd(S_j) = \sqrt{\frac{1}{NK} + \frac{1}{MK}} = \sqrt{\frac{1}{273.1} + \frac{1}{273.1}} = 0.86 \times 100\% = 8.6\%.$$

(b) Peak or Quotient Normalization:

Here, in the notation of Barrick (1980), we have M = N = 24, and for each of the L = 12 spectra, we form an F-distributed quotient with M = N = 24. Then from Eq. (11) of the above reference, we see that the normalized standard deviation is

$$sd(S_j) = \frac{N(2N-1)}{L(N-1)^2(N-2)} = .090 \times 100\% = 9.0\% \ .$$

Therefore, we see here that logarithmic normalization is a bit better, but not all that much (8.6% compared to 9%). The reason for the small difference is that the number of independent samples in the first-order echo peak before dividing it into all of the remaining second-order spectral points is 24, which is already a reasonably stable statistic.

7. SUMMARY

This memorandum presents and demonstrates a technique for correcting radar echo Doppler spectra that are contaminated with unknown multiplicative path-loss/system-gain factors before these spectra are averaged together. This logarithmic normalization technique has been found to be as good as or better than any other method. It is especially advantageous when each separate Doppler spectrum (or a small number) is contaminated with a different, unknown factor. When the number of spectra contaminated with the same factor is large, all normalization methods give essentially the same accuracy. The entire logarithmic normalization scheme can be implemented digitally with no more than three lines of FORTRAN code.

Simple, explicit expressions are derived that give the errors involved in logarithmic normalization as a function of the number of samples and spectral points. The use of this technique for two different examples involving HF sea echo is demonstrated and the accuracy of logarithmic normalization is compared with another popular normalization method.

ACKNOWLEDGMENTS

The author expresses appreciation for initial suggestions of the technique by Dr. T. M. Georges and helpful discussions and criticisms from Jack P. Riley and Dr. Belinda J. Lipa.

APPENDIX: DERIVATION OF ERRORS IN LOGARITHMIC NORMALIZATION

Errors incurred due to finite sample sizes have been analyzed in Barrick (1980) for "peak" (or quotient) normalization, as well as for other mathematical techniques developed for extracting sea-state information from radar echoes. This Appendix derives similar expressions for finite-sample-size errors that arise from logarithmic normalization.

In the equations defining c_ℓ and C_ℓ preceding (1), we express P_j^ℓ as the product of the unknown path-loss factor A_ℓ, of the power-mean p_j at the j-th frequency point, and of x_j^ℓ, a <u>normalized</u> χ^2 random variable having 2K degrees of freedom. (Hence the mean and variance of x_j^ℓ are $\langle x_j^\ell \rangle = 1$ and $\langle (x_j^\ell)^2 \rangle - \langle x_j^\ell \rangle^2 = 1/K$). Define c_ℓ:

$$c_\ell = \frac{1}{J} \sum_{j=1}^{J} \ell n P_j^\ell = \frac{1}{J} \sum_{j=1}^{J} \ell n (A_\ell p_j x_j^\ell) \ .$$

We now use the product law of logarithmic arguments to express the logarithm as sums:

$$c_\ell = \frac{1}{J} \sum_{j=1}^{J} \ell n A_\ell + \frac{1}{J} \sum_{j=1}^{J} \ell n p_j + \frac{1}{J} \sum_{j=1}^{J} \ell n x_j^\ell \ .$$

But we define

$$\ell n p \equiv \frac{1}{J} \sum_{j=1}^{J} \ell n p_j ,$$

and note that $\frac{1}{J} \sum_{j=1}^{J} \ell n A_\ell = \ell n A_\ell$, to obtain

$$C_\ell = \exp(c_\ell) = \exp\{\ell n A_\ell + \ell n p + \frac{1}{J} \sum_{j=1}^{J} \ell n x_j^\ell\} \ ,$$

or

$$C_\ell = pA_\ell \exp\{\frac{1}{J} \sum_{j=1}^{J} \ell n x_j^\ell\} .$$

Define a new random variable as

$$y_\ell \equiv \frac{1}{J} \sum_{j=1}^{J} \ell n x_j^\ell . \qquad \text{(A-1)}$$

Then the renormalized random spectral points are:

$$P_{Nj}^\ell = P_j^\ell / C_\ell = P_j^\ell \exp\{-y^\ell\}/(A^\ell p) .$$

But:

$$P_j^\ell = A_\ell p_j x_j^\ell .$$

Hence we have

$$P_{Nj}^\ell = \left(\frac{p_j}{p}\right) x_j^\ell, \; e^{-y_\ell} \qquad \text{(A-2)}$$

In this equation, the factor (p_j/p) is a constant independent of ℓ, and x_j^ℓ is a random variable as well as y_ℓ. As seen from (A-2), the unknown factor A_ℓ has been effectively divided out, but at the expense of the additional fluctuating factor e^{-y_ℓ}.

As the measure of error, we determine the variance of the sample average

$$P_{Nj} = \frac{1}{L} \sum_{\ell=1}^{L} P_{Nj}^{\ell} , \qquad \text{i.e.,} \tag{A-3}$$

$$\langle P_{Nj}^2 \rangle - \langle P_{Nj} \rangle^2 = \left(\frac{p_j}{p}\right)^2 \left\{ \frac{1}{L^2} \sum_{k=1}^{L} \sum_{\ell=1}^{L} \left[\langle x_j^k \, x_j^{\ell} \, e^{-y_k - y_{\ell}} \rangle - \langle x_j^k \, e^{-y_k} \rangle \langle x_j^{\ell} \, e^{-y_{\ell}} \rangle \right] \right\} .$$

The expression in brackets {...} is the "normalized" or fractional variance; it is ultimately to be compared to the same fractional variance if the exponentials e^{-y_k} and $e^{-y_{\ell}}$ were not present, which can be seen to be 1/KL. The above equation is easily seen to reduce to

$$\langle P_{Nj}^2 \rangle - \langle P_{Nj} \rangle^2 = \frac{1}{L} \left(\frac{p_j}{p}\right) [\langle x_j^2 \, e^{-2y} \rangle - \langle x_j \, e^{-y} \rangle^2] , \tag{A-4}$$

where we have dropped the ℓ, k superscripts on x, y.

Equation (A-4) is now simplified by using the following facts and/or assumptions:

(a) x_j^{ℓ} and y_{ℓ} are statistically independent; this assumption is very reasonable since y_{ℓ} is the sum of a large number of terms, each of which (i.e., $\ell n x_i^{\ell}$) is independent of x_j^{ℓ}, except the one for i=j. For large J, the effect of the one correlated term in y_{ℓ}, compared with the large number of uncorrelated terms, becomes vanishingly small.

(b) It has been already stated that x_j^{ℓ} is independent of x_j^k for $k \neq \ell$, and y_k is independent of y_{ℓ} for $k \neq \ell$. This is <u>not</u> an approximation, but a fact resulting from the statistical independence of the L separate power spectra.

(c) We assume that J is large enough (J > 10) that y (or y_{ℓ}) is essentially Gaussian-distributed by virtue of the Central Limit Theorem of statistics; i.e., $y = \frac{1}{J} \sum_{j=1}^{J} \ell n x_j$ is the sum of enough random variables that it is reasonably close to Gaussian.

Assumption (a) permits us to write:

$$\langle P_{Nj}^2 \rangle - \langle P_{Nj} \rangle^2 = \left(\frac{p_j}{p}\right)^2 \frac{1}{L} \left[\langle x_j^2 \rangle \langle e^{-2y} \rangle - \langle x_j \rangle^2 \langle e^{-y} \rangle^2\right] ,$$

or

$$\langle P_{Nj}^2 \rangle - \langle P_{Nj} \rangle^2 = \left(\frac{p_j}{p}\right)^2 \frac{1}{L} \left[\left(1 + \frac{1}{K}\right)\langle e^{-2y} \rangle - \langle e^{-y} \rangle^2\right] , \qquad \text{(A-5)}$$

since $\langle x_j \rangle = 1$ and $\langle x_j^2 \rangle = 1 + 1/K$.

Assumption (c) means that we can express the statistics of y for a Gaussian distribution in terms of two parameters: the mean of y (call it "a") and the variance of y (call it "σ^2"). We now proceed to derive these quantities exactly, using integral identities from Gradshteyn and Ryzhik (1965), Eq. 4.352, No. 1; and 4.358, No. 2. (An error in the latter formula has been noted and corrected here.)

$$a = \langle y \rangle = \frac{1}{J} \sum_{j=1}^{J} \langle \ell n x_j \rangle = \langle \ell n x \rangle = \frac{K^K}{\Gamma(K)} \int_0^\infty (\ell n x) x^{K-1} e^{-Kx} \, dx$$

or

$$a = \psi(K) - \ell n K, \qquad \text{(A-6)}$$

where $\psi(K)$ is Euler's Psi Function, which has the following specific representations for integer K

$$\underline{K = 1}: \quad \psi(K) = \psi(1) = -\gamma = -0.57721566490, \quad (\gamma = \text{Euler's const.})$$

$$\underline{K \geq 2}: \quad \psi(K) = -\gamma + \sum_{n}^{K-1} \frac{1}{n} . \qquad \text{(A-7)}$$

For the variance, we have

$$\sigma^2 = \langle y^2 \rangle - \langle y \rangle^2 = \frac{1}{J} \left[\langle (\ell n x)^2 \rangle - \langle \ell n x \rangle^2\right], \text{ with}$$

$$\langle (\ell n x)^2 \rangle = \frac{K^K}{\Gamma(K)} \int_0^\infty (\ell n x)^2 \, x^{K-1} \, e^{-Kx} \, dx = \left[\psi(K) - \ell n K\right]^2 + \zeta(2, K).$$

Hence we have $\sigma^2 = \frac{1}{J} \zeta(2,K)$, where $\zeta(x,y)$ is the Riemann-Zeta Function.

It has the following representations that we shall employ:

$$\underline{K = 1}: \qquad \zeta(2,1) = \frac{\pi^2}{6} \; ;$$

$$\underline{K \geq 2}: \qquad \zeta(2,K) = \frac{\pi^2}{6} - \sum_{n=1}^{K-1} \frac{1}{n^2} \; .$$

We now use the well-known Gaussian probability density function to determine

$$\langle e^{-y} \rangle = e^{\frac{n^2\sigma^2}{2} - na} \quad , \qquad \text{(i.e., n = 1 and 2 in A-5).}$$

From this we obtain

$$\langle P_{Nj}^2 \rangle - \langle P_{Nj} \rangle^2 = \left(\frac{p_j}{p}\right) \frac{1}{L} e^{-2a} \left[\left(1 + \frac{1}{K}\right) e^{2\sigma^2} - e^{\sigma^2} \right] . \qquad \text{(A-10)}$$

But because $\zeta(2,K)$ is bounded, (i.e., the largest it can get is $\zeta(2,1) = \pi^2/6 = 1.6449$), then $\sigma^2 = \frac{1}{J}\zeta(2,K)$ is a very small number since we have assumed J is large. Hence, neglecting terms $O(J^{-2})$, we expand $e^{2\sigma^2}$ and e^{σ^2} and take only the lowest-order terms to obtain the final result:

$$\langle P_{Nj}^2 \rangle - \langle P_{Nj} \rangle^2 = \frac{p_j}{p} \frac{1}{LK} e^{-2[\psi(K) - \ell n\, K]} \left\{1 + \frac{1}{J}(K+2)\zeta(2,K)\right\}, \qquad \text{(A-11)}$$

from which we obtain the form of (2), with f_1 and f_2 in (3) seen from inspection of (A-11) to be

$$f_1 \equiv e^{2[\psi(K) - \ell n\, K]} \qquad \text{(A-12)}$$

and

$$f_2 \equiv (K+2)\zeta(2,K) \; . \qquad \text{(A-13)}$$

We use these expressions along with their exact representations as given by (A-7) and (A-9) to calculate the table of f_1 and f_2 vs. K.

TABLE I. Exact Calculations of f_1 and f_2 vs. K to be used in Eqs. (2) and (3)

K	f_1	f_2	K	f_1	f_2
1	.3152	4.9348	20	.9508	1.1280
2	.5823	2.5798	22	.9552	1.1161
3	.7035	1.9747	24	.9589	1.1062
4	.7708	1.7030	25	.9605	1.1019
5	.8133	1.5493	30	.9670	1.0846
6	.8426	1.4506	35	.9717	1.0724
7	.8639	1.3819	40	.9752	1.0632
8	.8802	1.3314	45	.9779	1.0561
9	.8930	1.2926	50	.9801	1.0505
10	.9033	1.2620	55	.9819	1.0458
11	.9118	1.2372	60	.9834	1.0420
12	.9190	1.2166	65	.9847	1.0387
13	.9250	1.1994	70	.9858	1.0360
14	.9303	1.1846	75	.9867	1.0335
15	.9348	1.1719	80	.9876	1.0314
16	.9388	1.1609	85	.9883	1.0296
17	.9423	1.1512	90	.9889	1.0279
18	.9455	1.1425	95	.9895	1.0264
19	.9483	1.1349	100	.9900	1.0251

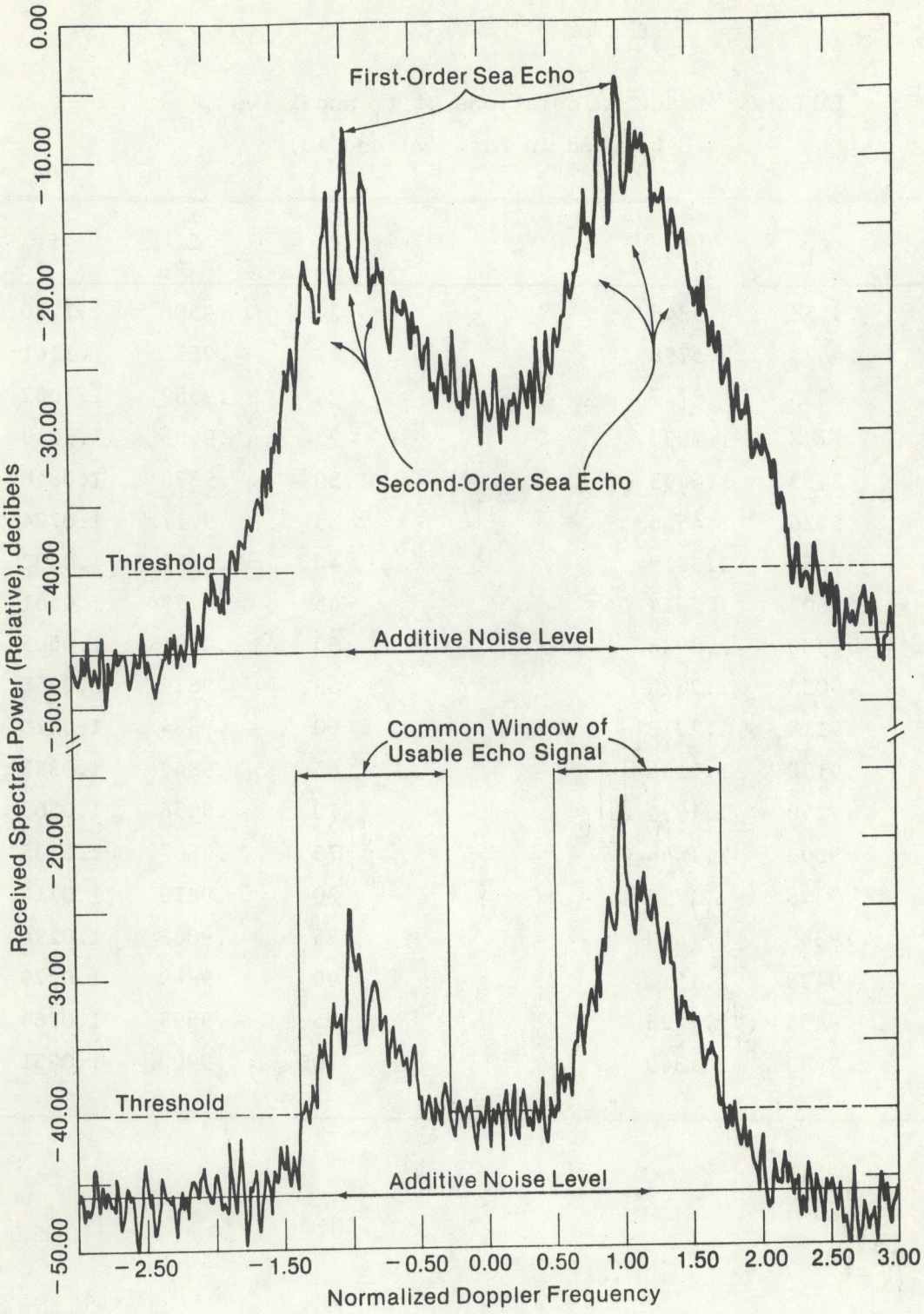

Figure 1. Sea-echo Doppler spectra measured at Pescadero, California 19 January 1978 at 25.4 MHz at two different ranges (upper at 17.4 km, lower at 5.4 km). Each curve is average of 8 independent spectra at same range. Different echo signal levels in each case are due to different unknown path losses and system gains. Threshold for usable sea-echo window is set here at 6 dB above noise level.

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