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BENDING OF A RAY IN A RANDOM INHOMOGENEOUS MEDIUM[1]

R.M. Jones

Abstract

When a wave propagates through a random, inhomogeneous medium, the random component of the refractive index affects not only the random component of the ray direction, but also the mean component. Likewise, the mean component of the refractive index affects not only the mean component of the ray direction, but also the random component. This report derives equations based on a perturbation expansion that can be used to calculate the mean and random component of the ray direction in a medium in which the mean and random component of the refractive index vary arbitrarily in three dimensions. The equations are suitable for numerical integration in a ray tracing program.

1. Summary

This report develops a formula for the average ray direction and the mean square deviation from the average for propagation of a ray in a random, inhomogeneous medium. Evaluation of the formula involves integrating three simultaneous equations along the average ray path and is thus suited for calculation by a ray tracing program. It is valid under the following restrictions:

1. Ray theory applies.
2. The medium is isotropic, so that the ray direction is the same as the wave normal direction.
3. The change in the average refractive index within a correlation length is negligible.
4. The random component of the refractive index is much smaller than the average refractive index.
5. The mean square deviation of the ray direction from the average remains small.
6. Three-point and higher correlations are neglected.
7. Moments of higher order than two are neglected.
8. The ray direction is a Gaussian-distributed random variable.

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[1] Presented at the U.R.S.I. International Symposium on Electromagnetic Wave Theory, USSR, Tbilisi, September 9-15, 1971.

The formula actually gives the ensemble average $<\vec{\sigma}>$ of the unit vector $\vec{\sigma}$ in the ray direction. This ensemble average $<\vec{\sigma}>$ determines the average ray direction. In addition $<\vec{\sigma}>$ determines the mean square angular deviation $<\varphi^2>$ from the average ray direction because $<\sigma> = <\cos\varphi> = \exp(-\frac{1}{2}<\varphi^2>)$. That is,

$$<\varphi^2> = -\mathrm{Log}_e <\sigma>^2 = -\mathrm{Log}_e \left(<\sigma_j><\sigma_j> \right) .$$

As in all the equations in this paper, summation from 1 to 3 is implied over repeated indexes.

The three components $<\sigma_i>$ of $<\vec{\sigma}>$ are found by integrating the following three (for i = 1, 2, 3) differential equations along the average ray path:

$$\frac{d<\sigma_i>}{ds} = \frac{1}{<n><\sigma>} \left(1 + \frac{<\mu^2>}{<n>^2} \right) \left(\nabla_i <n> - \gamma <\sigma_i> \right) +$$

$$+ \frac{1}{<n>^2 <\sigma>^2} \left(\left(\gamma^2 <\sigma_i> - \frac{\nabla_j <n> \nabla_j <n> \nabla_i <n>}{\gamma} \right) I + \right.$$

$$+ \left(2\gamma - \frac{\nabla_j <n> \nabla_j <n>}{\gamma} \right) J_i +$$

$$+ \left(-2 <\sigma_j> \nabla_i <n> + \frac{\nabla_j <n> \nabla_i <n>}{\gamma} - <\sigma_i> \nabla_j <n> + \right.$$

$$\left. + \frac{\nabla_\ell <n> \nabla_\ell <n>}{\gamma} <\sigma_i><\sigma_j> \right) J_j +$$

$$\left. + <\sigma_i> K_{\ell\ell} + \frac{K_{i\ell} \nabla_\ell <n>}{\gamma} \right)$$

where the total refractive index is

$$n = \langle n \rangle + \mu \quad ,$$

the random part has a mean of zero with the following two-point correlation function

$$\langle \mu(\vec{r}_1)\mu(\vec{r}_2) \rangle = \langle \mu^2 \rangle C_n(\vec{r}_1 - \vec{r}_2) \quad ,$$

the magnitude of the average of the unit vector in the ray direction is

$$\langle \sigma \rangle = | \langle \vec{\sigma} \rangle | = \sqrt{\langle \sigma_j \rangle \langle \sigma_j \rangle} \quad ,$$

$$\gamma = \langle \sigma_j \rangle \nabla_j \langle n \rangle \quad ,$$

$\nabla_i \langle n \rangle$ is the i th component of $\vec{\nabla} \langle n \rangle$,

$$I = \frac{\langle \mu^2 \rangle}{\langle n \rangle^2} \int_0^\infty C_n\left(\rho \frac{\langle \vec{\sigma} \rangle}{\langle \sigma \rangle}\right) d\rho \quad ,$$

$$J_k = \frac{\langle \mu^2 \rangle}{\langle n \rangle} \int_0^\infty \nabla_k C_n\left(\rho \frac{\langle \vec{\sigma} \rangle}{\langle \sigma \rangle}\right) d\rho \quad ,$$

$$K_{k\ell} = \langle \mu^2 \rangle \int_0^\infty \nabla_k \nabla_\ell C_n\left(\rho \frac{\langle \vec{\sigma} \rangle}{\langle \sigma \rangle}\right) d\rho \quad .$$

The derivation begins with a standard differential equation for the bending of a ray in an isotropic medium. As might be suspected from some of the limitations required for a valid result, the derivation involves expanding the ray equation in powers of $\sigma_j - \langle \sigma_j \rangle$ and keeping only linear terms. The resulting equation is solved by successive approximations to give multiple integrals. Finally, an ensemble average is taken, neglecting three-point (and higher) correlations and moments of higher order than two.

2. Introduction

In a homogeneous, random medium, the average ray direction remains constant, while the mean square deviation from the average increases with propagation distance through the medium. Formulas that give the mean square deviation of the ray direction from the average for small deviations are well known (e.g., Chernov, 1960).

In an inhomogeneous medium the average ray direction is not constant and in fact depends on the randomness of the medium. The purpose of this report is to calculate the average ray direction and the mean square deviation from the average for a ray traveling through an inhomogeneous medium. The calculations for an inhomogeneous medium are much more complicated than for a homogeneous medium. They would be even more complicated without limiting the applicability of the calculations to situations that satisfy the following restrictions.

1. Ray theory applies.

2. The medium is isotropic, so that the ray direction is the same as the wave-normal direction.

3. The change in the average refractive index within a correlation length is negligible. This is necessary because the derivation depends on dividing the medium into segments small enough that the gradient of the average refractive index is nearly constant but still much larger than the correlation length for the random component of the refractive index.

4. The random component of the refractive index is much smaller than the average refractive index. This is necessary because the derivation involves an expansion in powers of the ratio of the random component of the refractive index to the average refractive index. Most media probably satisfy this restriction.

5. The mean square deviation of the ray direction from the average ray direction remains small. This is necessary because the derivation involves an expansion in powers of the ratio of difference between the instantaneous unit vector in the ray direction and the average of a unit vector in the ray direction to the average of a unit vector in the ray direction. This approximation will eventually break down, even in a homogeneous medium, for long path lengths. The final solution is therefore valid only for relatively short path lengths.

6. Three-point and higher correlations are neglected. This limitation simplifies the calculations. Higher correlations could be added although their effect would probably be small in most cases.

7. Moments of higher order than two are neglected. This simplifies the calculations. Higher moments could be added.

8. The ray direction is a Gaussian-distributed random variable.

3. Derivation Method

The average ray direction and the mean square deviation from the average are determined by the average of a unit vector in the ray direction. It is clear that the average of a unit vector in the ray direction gives the average ray direction. It is not immediately clear that it also gives the mean square deviation from the average. Section 4 shows this to be true, if the ray direction is a Gaussian-distributed random variable, and then derives the required formula.

The calculation of the average of a unit vector in the ray direction for an inhomogeneous medium can be broken into two parts. First, we consider a medium in which the average refractive index has a constant gradient and the correlation functions for the random part of the refractive index are constant. Second, we simply consider the inhomogeneous medium to be made up of segments that satisfy the above criteria, and we then write the final solution as an integral over these segments.

We begin by writing the general ray equation for the bending of a ray in an isotropic medium in terms of a unit vector $\vec{\sigma}$ in the ray direction. We then choose a coordinate system in which the gradient of the average refractive index is in the z direction, and we rewrite the ray equation so that z is the independent variable instead of ray path length. We let our medium be a slab bounded by $z = 0$ and $z = \Delta z$.

Next we convert to tensor notation for ease in factoring vectors out of dot products. We then write the unit vector $\vec{\sigma}$ as the sum of the average unit vector $< \vec{\sigma} >$ plus a small perturbation $\Delta \vec{\sigma}$ and expand the ray equation in powers of $\vec{\Delta\sigma}$. We then neglect all powers of $\vec{\Delta\sigma}$ but linear terms. Keeping higher powers would give terms in the final result containing three-point and higher correlations or terms proportional to the square of the slab thickness Δz.

We now write the refractive index as a sum of its average value $< n >$ plus its random component μ. The average refractive index varies linearly through the slab because we assume a constant gradient of $< n >$ in the slab. Keeping the linear variation of $< n >$ in the equation gives terms in the final result proportional to the square of the slab thickness. Since we want ultimately to neglect such terms, we can neglect them now by assuming $< n >$ constant within the slab. We then expand the equation in a power series of $\mu / < n >$ and discard powers larger than 2. This corresponds to neglecting moments of higher order than 2 in the ensemble average.

We now solve the differential equation by successive approximations, beginning with $\vec{\Delta\sigma} = 0$. This gives multiple integrals over the slab thickness. We neglect triple integrals and higher order integrals because they lead to terms in the final result that contain three-point and higher order correlations or terms proportional to the square and higher powers of the slab thickness. We also discard other terms in the successive approximation solution proportional to the square of the slab thickness. Taking the ensemble average of the remaining single and double integrals gives the change in the average of the unit vector after having gone through the slab.

The change in the average value of the unit vector is proportional to the approximate straight-line distance traveled in the slab. Taking the limit of small slab thickness allows rewriting the equation as a differential equation for the average of the unit vector in the ray direction. We assume that this differential equation applies at each point in the inhomogeneous medium.

Finally, the special case of a homogeneous medium is compared with the result of Chernov (1960).

4. The Derivation

We assume that the refractive index n is composed of an average value plus a random component

$$n = \langle n \rangle + \mu \quad (1)$$

where the brackets < and > indicate an ensemble average. The ray equation in an isotropic medium can be written

$$\frac{d(n\vec{\sigma})}{ds} = \vec{\nabla} n \quad (2)$$

where $\vec{\sigma}$ is a unit vector in the ray direction (same as the wave normal direction since the medium is isotropic), and s is geometrical path length. We are considering a medium in which $\vec{\nabla}\langle n \rangle$ and all the statistical properties of μ are constant. We choose the z axis parallel with $\vec{\nabla} < n >$. Our medium is a slab with thickness Δz and with boundaries perpendicular to the z axis. We assume that a representative ray enters the slab at $z = 0$ and leaves at $z = \Delta z$. Thus no reflection occurs within the slab, only a small amount of bending. We now change the independent variable in the ray equation from s to z by means of the following equation:

$$\frac{ds}{dz} = \frac{|\nabla < n >|}{\vec{\sigma} \cdot \vec{\nabla} < n >} . \tag{3}$$

Using

$$\frac{dn}{ds} = \vec{\sigma} \cdot \vec{\nabla} n \tag{4}$$

along with (3) in (2) gives

$$\frac{d\vec{\sigma}}{d z} = \frac{|\nabla < n >|}{\vec{\sigma} \cdot \vec{\nabla} < n > n} \left(\vec{\nabla} n - \vec{\sigma} \, (\vec{\sigma} \cdot \vec{\nabla} n) \right) . \tag{5}$$

Equation (5) can be rewritten in tensor notation

$$\frac{d\sigma_i}{dz} = \frac{|\nabla < n >|}{n} \frac{\nabla_i n - \sigma_i (\sigma_j \nabla_j n)}{\sigma_k \nabla_k < n >} = \frac{|\nabla < n >|}{n} \frac{(\delta_{ij} - \sigma_i \sigma_j) \nabla_j n}{\sigma_k \nabla_k < n >} \tag{6}$$

where summation from 1 to 3 is implied over repeated indices, and δ_{ij} is the Kronecker δ. Equation (6) explicitly gives each component of (5) for i equal 1, 2, or 3.

$$\sigma_k = < \sigma_k > + \Delta \sigma_k . \tag{7}$$

Then

$$\sigma_k \nabla_k < n > = < \sigma_k > \nabla_k < n > + \nabla_k < n > \Delta \sigma_k$$

$$= < \vec{\sigma} > \cdot \vec{\nabla} < n > + \nabla_k < n > \Delta \sigma_k \tag{8}$$

$$= < \vec{\sigma} > \cdot \vec{\nabla} < n > \left(1 + \frac{\nabla_k < n >}{< \vec{\sigma} > \cdot \vec{\nabla} < n >} \Delta \sigma_k \right) ,$$

and

$$\sigma_i \sigma_j = \left(< \sigma_i > + \Delta \sigma_i \right) \left(< \sigma_j > + \Delta \sigma_j \right)$$

$$= < \sigma_i > < \sigma_j > + < \sigma_j > \Delta \sigma_i + < \sigma_i > \Delta \sigma_j + \Delta \sigma_i \Delta \sigma_j \ . \tag{9}$$

Using (8) and (9) in (6) gives

$$\frac{d\sigma_i}{dz} = \frac{|\nabla < n >|}{< \vec{\sigma} > \cdot \vec{\nabla} < n >} \ \frac{\nabla_j n}{n} \ \frac{\left(\delta_{ij} - < \sigma_i > < \sigma_j > - < \sigma_j > \Delta \sigma_i - < \sigma_i > \Delta \sigma_j - \Delta \sigma_i \Delta \sigma_j \right)}{\left(1 + \frac{\nabla_k < n >}{< \vec{\sigma} > \cdot \vec{\nabla} < n >} \Delta \sigma_k \right)} \ . \tag{10}$$

We can now expand (10) in powers of $\Delta\sigma$, discard powers of 2 and higher, and collect terms, changing dummy indices where necessary to give

$$\frac{d\sigma_i}{dz} = \frac{|\nabla < n >|}{< \vec{\sigma} > \cdot \vec{\nabla} < n >} \ \frac{\nabla_j n}{n} \left(\delta_{ij} - < \sigma_i > < \sigma_j > + \right.$$

$$\left. + \left(- < \sigma_j > \delta_{ik} - < \sigma_i > \delta_{jk} - \left(\delta_{ij} - < \sigma_i > < \sigma_j > \right) \frac{\nabla_k < n >}{< \vec{\sigma} > \cdot \vec{\nabla} < n >} \right) \Delta \sigma_k \right) . \tag{11}$$

Substituting (1) into (11), we obtain

$$\frac{d\sigma_i}{dz} = \frac{| < \sigma > | \, | \nabla < n > |}{< \vec{\sigma} > \cdot \vec{\nabla} < n >} \ \frac{\nabla_j < n > + \nabla_j \mu}{1 + \frac{\mu}{< n >}} \left(A_{ij} + B_{ijk} \Delta \sigma_k \right) \tag{12}$$

where

$$A_{ij} = \frac{1}{<n>|<\sigma>|}\left(\delta_{ij} - <\sigma_i><\sigma_j>\right) \tag{13}$$

and

$$B_{ijk} = \frac{-1}{<n>|<\sigma>|}\left(<\sigma_j>\delta_{ik} + <\sigma_i>\delta_{jk} + \left(\delta_{ij} - <\sigma_i><\sigma_j>\right)\frac{\nabla_k <n>}{<\vec{\sigma}>\cdot\vec{\nabla}<n>}\right). \tag{14}$$

Expanding (12) in powers of μ and discarding powers higher than 2 gives

$$\frac{d\sigma_i}{dz} = \frac{|<\sigma>||\nabla<n>|}{<\vec{\sigma}>\cdot\vec{\nabla}<n>}\left(\nabla_j<n> + \nabla_j\mu\right)\left(1 - \frac{\mu}{<n>} + \frac{\mu^2}{<n>^2}\right)\left(A_{ij} + B_{ijk}\Delta\sigma_k\right), \tag{15}$$

which can be rewritten

$$\frac{d\sigma_i}{dz} = \alpha_i + \beta_{ik}\Delta\sigma_k \tag{16}$$

where

$$\alpha_i = \frac{|<\sigma>||\nabla<n>|}{<\vec{\sigma}>\cdot\vec{\nabla}<n>}\left(1 - \frac{\mu}{<n>} + \frac{\mu^2}{<n>^2}\right)\left(\nabla_j<n> + \nabla_j\mu\right)A_{ij} \tag{17}$$

and

$$\beta_{ik} = \frac{|<\sigma>||\nabla<n>|}{<\vec{\sigma}>\cdot\vec{\nabla}<n>}\left(1 - \frac{\mu}{<n>} + \frac{\mu^2}{<n>^2}\right)\left(\nabla_j<n> + \nabla_j\mu\right)B_{ijk}. \tag{18}$$

Equation (16) can be rewritten in integral form

$$\Delta\sigma_i(z) = \int_0^z \alpha_i(z_1)\,dz_1 + \int_0^z \beta_{ik}(z_1)\,\Delta\sigma_k(z_1)\,dz_1 . \tag{19}$$

Equation (19) can be solved by successive approximations. For the zeroth order approximation, we choose

$$\Delta \sigma_i (z) = 0 \quad . \tag{20}$$

Substituting (20) into (19) gives for the first-order approximation

$$\Delta \sigma_i (z) = \int_0^z \alpha_i (z_1)\, dz_1 \quad . \tag{21}$$

Substituting (21) into (19) gives the second-order approximation

$$\Delta \sigma_i (\Delta z) = \int_0^{\Delta z} \alpha_i (z_1)\, dz_1 + \int_0^{\Delta z} \beta_{ik} (z_1) \int_0^{z_1} \alpha_k (z_2)\, dz_2 \, dz_1 \quad . \tag{22}$$

It is unnecessary to go to higher order approximations, because that would add only terms with triple and higher order integrals and terms proportional to the square of the slab thickness.

Now we take the ensemble average of (22). If we substitute (17) into the single integral in (22) and expand the product, we have 6 terms in the single integral. Appendix A evaluates the ensemble averages of these single integrals. If we substitute (17) and (18) into the double integral in (22) and expand the products, we have 36 terms in the double integral. Some of these terms can be separated into the product of two single integrals and evaluated using Appendix A. Many of the ensemble averages of the remaining double integrals are zero because of the assumptions we made. Appendix B evaluates the ensemble averages of the non-zero double integrals.

Substituting the formulas for the integrals from Appendices A and B into (22) and discarding terms proportional to the square of the slab thickness gives

$$< \Delta\sigma_i > = A_{ij} \nabla_j < n > \left(1 + \frac{<\mu^2>}{<n>^2} \right) \Delta s +$$

$$+ B_{ijk} A_{k\ell} \left(I \nabla_j < n > \nabla_\ell < n > - J_\ell \nabla_j < n > + J_j \nabla_\ell < n > - K_{j\ell} \right) \Delta s . \tag{23}$$

The integrals I, J_j , and $K_{j\ell}$ are given by (B-16), (B-20), and (B-21). Notice that we have changed from slab thickness to average path length in the slab by using (3). We can now let the slab thickness approach zero, and rewrite (23) as a differential equation for the average of the unit vector in the ray direction.

$$\frac{d < \sigma_i >}{ds} = A_{ij} \nabla_j < n > \left(1 + \frac{<\mu^2>}{<n>^2} \right) +$$

$$+ B_{ijk} A_{k\ell} \left(I \nabla_j < n > \nabla_\ell < n > - J_\ell \nabla_j < n > + J_j \nabla_\ell < n > - K_{j\ell} \right). \tag{24}$$

Substituting (13) and (14) into (24) gives (after much algebra)

$$\frac{d < \sigma_i >}{ds} = \frac{1}{< n > < \sigma >} \left(1 + \frac{<\mu^2>}{<n>^2} \right) \left(\nabla_i < n > - \gamma < \sigma_i > \right) +$$

$$+ \frac{1}{< n >^2 < \sigma >^2} \left(\left(\gamma^2 < \sigma_i > - \frac{\nabla_j < n > \nabla_j < n > \nabla_i < n >}{\gamma} \right) I + \right.$$

$$+ \left(2\gamma - \frac{\nabla_j < n > \nabla_j < n >}{\gamma} \right) J_i +$$

$$+\Big(-2<\sigma_j>\nabla_i<n> + \frac{\nabla_j<n>\nabla_i<n>}{\gamma} - <\sigma_i>\nabla_j<n> +$$

$$+ \frac{\nabla_\ell<n>\nabla_\ell<n>}{\gamma}<\sigma_i><\sigma_j>\Big)J_j +$$

$$+ <\sigma_i>K_{\ell\ell} + \frac{K_{i\ell}\nabla_\ell<n>}{\gamma}\Big) \qquad (25)$$

where

$$<\sigma> = |<\vec{\sigma}>| = \sqrt{<\sigma_j><\sigma_j>} \quad , \qquad (26)$$

$$\gamma = <\vec{\sigma}>\cdot\vec{\nabla}<n> = <\sigma_j>\nabla_j<n> \quad , \qquad (27)$$

the integral I is given by (B-16), J_k is given by (B-20), $K_{k\ell}$ is given by (B-21), and C_n that appears in (B-16), (B-20), and (B-21) is the two-point correlation function defined by (B-4).

5. Significance of the Result

Equation (25) gives a system of 3 (for i = 1, 2, 3) differential equations that can be numerically integrated along the average ray path to give the final value of $<\vec{\sigma}>$ for a given set of initial conditions at the beginning of the ray path. Equation (25) is in a form that can be integrated by a ray tracing program.

The average of the unit vector in the ray direction $<\vec{\sigma}>$ determines the average ray direction. In addition, $<\vec{\sigma}>$ determines the mean square angular deviation $<\varphi^2>$ from the average ray direction in the following way:

Let φ be the angle between the instantaneous ray direction and the average ray direction. Then

$$\cos\varphi = \frac{\sigma_i < \sigma_i >}{< \sigma >} \quad . \tag{28}$$

Taking the ensemble average of (28) gives

$$< \cos\varphi > = \frac{< \sigma_i >< \sigma_i >}{< \sigma >} = \frac{< \sigma >^2}{< \sigma >} = < \sigma > \quad . \tag{29}$$

The angle φ is a random variable with mean zero. If in addition it is Gaussian distributed, then

$$< \cos\varphi > = \exp\left(- < \varphi^2 > / 2 \right) \quad . \tag{30}$$

Combining (29) and (30) gives

$$< \varphi^2 > = -2 \log_e < \sigma > = -\log_e < \sigma >^2 \quad . \tag{31}$$

6. Comparison with Previous Results for a Homogeneous Medium

For the special case that

$$\vec{\nabla} < n > = 0 \quad , \tag{32}$$

(25) shows that $< \vec{\sigma} >$ does not change in direction but only in magnitude, and we have

$$< \sigma >_s^2 = < \sigma >_o^2 + \frac{2}{< n >^2} \int K_{\ell\ell} \, ds \quad . \tag{33}$$

If in addition $K_{\ell\ell}$ is constant, then (33) can be integrated to give

$$< \sigma >_s^2 = < \sigma_o >^2 + 2 \, \frac{K_{\ell\ell}}{< n >^2} \, s \quad . \tag{34}$$

Combining (31) and (34) gives for the mean square angular deviation from the average ray direction after traveling a distance s through the medium

$$< \varphi^2 >_s = - \log_e \left(\exp\left(- < \varphi^2 >_o \right) + 2 \frac{K_{\ell\ell}}{< n >^2} s \right) \tag{35}$$

where $< \varphi^2 >_o$ is the initial mean square angular deviation from the average ray direction. For the special case of a Gaussian correlation function

$$C_n(r) = \exp(-r^2/a^2) \tag{36}$$

(35) becomes

$$< \varphi^2 >_s = - \log_e \left(\exp\left(- < \varphi^2 >_o \right) - 4 \frac{< \mu^2 >}{< n >^2} \sqrt{\pi} \frac{s}{a} \right) \quad . \tag{37}$$

If we now take the case for

$$< \varphi^2 >_o = 0 \tag{38}$$

and

$$< n > = 1 \tag{39}$$

then (37) becomes

$$< \varphi^2 > = - \log_e \left(1 - 4 < \mu^2 > \sqrt{\pi} \frac{s}{a} \right) \quad . \tag{40}$$

For s small enough, we can take the first term in the power series for (40) to give

$$< \varphi^2 > \approx 4 < \mu^2 > \sqrt{\pi} \frac{s}{a} \tag{41}$$

which agrees with the result given by Chernov (1960, page 17).

Reference

Chernov, Lev A., Wave propagation in a random medium, translated from the Russian by R. A. Silverman, McGraw-Hill, New York, 1960.

APPENDIX A

Evaluation of the Ensemble Averages of the Single Integrals

Substituting (17) into the single integral in (22) gives 6 integrals whose ensemble averages must be calculated.

The first of these integrals is not a random variable and can easily be evaluated and then rewritten using (3).

$$\int_0^{\Delta z} dz = \Delta z = \frac{<\vec{\sigma}> \cdot \vec{\nabla} <n>}{|<\sigma>|\,|\nabla <n>|}\,\Delta s \qquad \text{(A-1)}$$

The ensemble average of the second integral is zero, since μ has an average of zero.

$$\left\langle \int_0^{\Delta z} \mu\, dz \right\rangle = \int_0^{\Delta z} <\mu>\, dz = \int_0^{\Delta z} 0\, dz = 0 \qquad \text{(A-2)}$$

The ensemble average of the third integral is easily calculated and then rewritten using (3)

$$\left\langle \int_0^{\Delta z} \mu^2\, dz \right\rangle = \int_0^{\Delta z} <\mu^2>\, dz = <\mu^2> \Delta z = <\mu^2> \frac{<\vec{\sigma}> \cdot \vec{\nabla}<n>}{|<\sigma>|\,|\nabla<n>|}\,\Delta s$$

(A-3)

The ensemble average of the fourth integral is zero because μ has an average of zero.

$$\left\langle \int_0^{\Delta z} \vec{\nabla}\mu\, dz \right\rangle = \int_0^{\Delta z} <\vec{\nabla}\mu>\, dz = \int_0^{\Delta z} \vec{\nabla} <\mu>\, dz = 0 \qquad \text{(A-4)}$$

The ensemble average of the fifth integral is zero because we have assumed that $<\mu^2>$ is constant within the slab.

$$\left\langle \int_0^{\Delta z} \mu \vec{\nabla} \mu \, dz \right\rangle = \frac{1}{2} \int_0^{\Delta z} < \vec{\nabla} (\mu^2) > dz = \frac{1}{2} \int_0^{\Delta z} \vec{\nabla} < \mu^2 > dz = 0 \qquad \text{(A-5)}$$

The ensemble average of the sixth integral is zero because we neglect third moments.

$$\left\langle \int_0^{\Delta z} \mu^2 \vec{\nabla} \mu \, dz \right\rangle = \frac{1}{3} \int_0^{\Delta z} \vec{\nabla} < \mu^3 > dz = 0 \qquad \text{(A-6)}$$

APPENDIX B

Evaluation of the Ensemble Averages of the Double Integrals

Because we are neglecting three-point and higher correlations, only four of the 36 double integrals do not factor into the product of two single integrals or have ensemble averages that are not zero.

The first of these integrals is

$$\left\langle \int_0^{\Delta z} \mu(z_1) \int_0^{z_1} \mu(z_2)\, dz_2\, dz_1 \right\rangle , \tag{B-1}$$

which can be rewritten

$$\left\langle \int_0^{\Delta z} \int_0^{z_1} \mu(z_1)\mu(z_2)\, dz_2\, dz_1 \right\rangle . \tag{B-2}$$

Taking the ensemble average inside the integral gives

$$\int_0^{\Delta z} \int_0^{z_1} <\mu^2> C_n(z_2 - z_1)\, dz_2\, dz_1 = <\mu^2> \int_0^{\Delta z} \int_0^{z_1} C_n(z_2 - z_1)\, dz_2\, dz_1 \tag{B-3}$$

where the correlation function C_n is defined by

$$<\mu(\vec{r}_1)\mu(\vec{r}_2)> = <\mu^2> C_n(\vec{r}_1 - \vec{r}_2) . \tag{B-4}$$

We now make a change of integration variables by letting

$$z_0 = \frac{z_2 + z_1}{2} \tag{B-5}$$

and

$$z = z_2 - z_1 . \tag{B-6}$$

Then we have

$$dz_2 \; dz_1 = dz \; dz_o \quad , \tag{B-7}$$

and (B-3) becomes

$$\langle \mu^2 \rangle \int_{o}^{\frac{\Delta z}{2}} \int_{-2z_o}^{o} C_n(z) \, dz \, dz_o + \langle \mu^2 \rangle \int_{\frac{\Delta z}{2}}^{\Delta z} \int_{2(z_o - \Delta z)}^{o} C_n(z) \, dz \, dz_o \quad . \tag{B-8}$$

We assume that the correlation length is much less than the slab thickness. Thus, the correlation function C_n is nearly zero except for z near zero. In that case, we can extend the lower limit of integration of the inner integrals in (B-8) to minus infinity without adding a significant amount to the integrals. Therefore, (B-8) becomes

$$\langle \mu^2 \rangle \int_{o}^{\frac{\Delta z}{2}} \int_{-\infty}^{o} C_n(z) \, dz \, dz_o + \langle \mu^2 \rangle \int_{\frac{\Delta z}{2}}^{\Delta z} \int_{-\infty}^{o} C_n(z) \, dz \, dz_o \quad . \tag{B-9}$$

Combining the integrals in (B-9) gives

$$\langle \mu^2 \rangle \int_{o}^{\Delta z} \int_{-\infty}^{o} C_n(z) \, dz \, dz_o \quad . \tag{B-10}$$

Since the inner integral is independent of z_o, the outer integral can easily be evaluated so that (B-10) becomes

$$\langle \mu^2 \rangle \Delta z \int_{-\infty}^{o} C_n(z) \, dz \quad . \tag{B-11}$$

We assume that the correlation function satisfies

$$C_n(-z) = C_n(z) \quad . \tag{B-12}$$

Using (B-12), (B-11) becomes

$$<\mu^2> \Delta z \int_o^\infty C_n(z)\, dz \quad . \qquad (B\text{-}13)$$

We want to express (B-13) in terms of the average path length of the ray in the slab rather than the slab thickness Δz. We can use (3) to make this change. In addition, it is useful to change the integration variable in (B-13) from distance into the slab to distance traveled along the average ray path. Equation (3) can also be used to make this change. Also, the correlation function C_n can be anisotropic so that it depends on the ray direction. This dependence should now be shown explicitly. Thus, (B-13) becomes

$$<\mu^2> \left(\frac{<\vec{\sigma}> \cdot \vec{\nabla} <n>}{|<\sigma>|\,|\nabla <n>|} \right)^2 \Delta s \int_o^\infty C_n\left(\rho \frac{<\vec{\sigma}>}{|<\sigma>|} \right) d\rho \quad . \qquad (B\text{-}14)$$

The final result, giving the equivalence of (B-1) and (B-14), is

$$\left\langle \int_o^{\Delta z} \mu(z_1) \int_o^{z_1} \mu(z_2)\, dz_2\, dz_1 \right\rangle = <n>^2 \left(\frac{<\vec{\sigma}> \cdot \vec{\nabla} <n>}{|<\sigma>|\,|\nabla <n>|} \right)^2 I \Delta s \qquad (B\text{-}15)$$

where

$$I = \frac{<\mu^2>}{<n>^2} \int_o^\infty C_n \left(\rho \frac{<\vec{\sigma}>}{|<\sigma>|} \right) d\rho \quad . \qquad (B\text{-}16)$$

Similarly, the ensemble averages of the other three non-zero integrals can be calculated to give

$$\left\langle \int_0^{\Delta z} \mu(z_1) \int_0^{z_1} \nabla_k \mu(z_2)\, dz_2\, dz_1 \right\rangle = \langle n \rangle \left(\frac{\langle \vec{\sigma} \rangle \cdot \vec{\nabla} \langle n \rangle}{|\langle \sigma \rangle| \, |\nabla \langle n \rangle|} \right)^2 J_k \Delta s$$

(B-17)

$$\left\langle \int_0^{\Delta z} \nabla_k \mu(z_1) \int_0^{z_1} \mu(z_2)\, dz_2\, dz_1 \right\rangle = -\langle n \rangle \left(\frac{\langle \vec{\sigma} \rangle \cdot \vec{\nabla} \langle n \rangle}{|\langle \sigma \rangle| \, |\nabla \langle n \rangle|} \right)^2 J_k \Delta s$$

(B-18)

and

$$\left\langle \int_0^{\Delta z} \nabla_k \mu(z_1) \int_0^{z_1} \nabla_\ell \mu(z_2)\, dz_2\, dz_1 \right\rangle = -\left(\frac{\langle \vec{\sigma} \rangle \cdot \vec{\nabla} \langle n \rangle}{|\langle \sigma \rangle| \, |\nabla \langle n \rangle|} \right)^2 K_{k\ell} \Delta s$$

(B-19)

where

$$J_k = \frac{\langle \mu^2 \rangle}{\langle n \rangle} \int_0^\infty \nabla_k C_n \left(\rho \frac{\langle \vec{\sigma} \rangle}{|\langle \sigma \rangle|} \right) d\rho \qquad \text{(B-20)}$$

and

$$K_{k\ell} = \langle \mu^2 \rangle \int_0^\infty \nabla_k \nabla_\ell C_n \left(\rho \frac{\langle \vec{\sigma} \rangle}{|\langle \sigma \rangle|} \right) d\rho \quad . \qquad \text{(B-21)}$$