NOAA Technical Memorandum ERL WPL-40

GRAVITY WAVE PROPAGATION IN THE PRESENCE OF A CURRENT WITH AN ARBITRARY VERTICAL PROFILE

Bob Weber

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ABSTRACT

This paper determines analytically the effect of current shears upon the phase velocity of gravity waves on the oceans surface and the resulting consequences for an HF sea echo radar which measures surface currents. Past treatments of this subject have restricted the current profile in order to avoid singularities in the Rayleigh equation or they have required the use of numerical methods which do not readily allow generalizations to be drawn. It is found that the phase velocities of these waves depend upon the current at all depths but with more weight given to the current near the surface. For weak shears and deep water, the weighting approaches the exponential weighting derived by Stewart and Joy (1974). Therefore, an HF sea echo radar does not measure the current just at the surface with the result that its estimate of the surface current will be in some error due to a shear. However, large errors (amounting to more than 10% of the phase velocity of the gravity wave causing the sea echo) should be detectable by measuring the excess separation of the advancing and receding Bragg lines in the sea echo Doppler spectrum.

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Introduction

While the study of gravity waves on the ocean's surface is one of the oldest in hydrodynamics, it is also a study that has been revitalized in recent years by fresh theoretical inquiry and by new experimental techniques. In particular, the sea echoes obtained with high frequency (HF) radars can be used to measure both surface gravity waves [Hasselman (1971) and Barrick et al. (1974)] and currents near the ocean's surface [Barrick et al. (1974) and Stewart and Joy (1974)] over hundreds of square kilometers in real time. A portable radar system [Barrick et al. (1977)] is currently under development at WPL* and has actually been employed over the past year to map currents in the Gulf Stream off the east coast of Florida and in the Lower Cook Inlet of Alaska. In the interest of brevity, we shall henceforth refer to this radar unit as CSR (for Current-Sensing Radar).

The radar does not measure the current directly, but rather it measures the phase velocity of certain ocean waves which propagate radially with respect to the radar and which have wavelengths equal to half the HF wavelength; this is referred to as the Bragg mechanism. Since the phase velocity of these ocean waves is known with confidence for deep water in the absence of currents, any deviation of the measured phase velocity from this accepted velocity is normally attributed to currents. Of course, Barrick and Weber [1977], as well as others have shown that nonlinear wave-wave interactions can also contribute a significant shift to this phase velocity, especially in high sea conditions. Thus, the connection between wave phase velocity and current velocity is not always trivial.

Another potential complication in the interpretation of the phase velocity of the Bragg waves is caused by vertical shears in the current near the surface. For currents that are constant with depth, there is no problem because the phase velocity is simply shifted by an amount proportional to this current velocity. However, when the current varies with depth below the surface, the shift in the phase velocity is an integrated result of the current over some depth [Stewart and Joy (1974), Benjamin (1962), and Fenton (1973)]. Past treatments of this problem have restricted the current to special profiles in order to avoid (1) singularities in the governing differential equations of (2) the use of numerical solutions for these equations. This then is the purpose of the present paper: (1) to determine analytically the effect of arbitrary vertical current shears upon the phase velocities of gravity waves and (2) to calculate the resulting impact upon the current measuring capability of the CSR system.

We are not concerned here with how the near-surface currents are generated or why they exhibit certain spatial and temporal patterns, although these will no doubt be elucidated with the help of radar techniques. In general, these currents affect not only the phase velocity but also the heights [Longuet-Higgins and Stewart (1960) and Huang et al. (1972)] and the directions [Evans (1974)] of gravity waves. In addition, if the present study is carried to higher perturbation orders, then it is found that sea echo Doppler spectra are not uniformly shifted by the current when a shear is present but, rather, are possibly distorted. This subject will not be considered further here but will be deferred to a later study.

Equations of Motion

The propagation of gravity waves on the ocean's surface in the presence of currents is controlled by the usual equations [Phillips (1969)] which express the conservation of mass, the conservation of momentum, and the continuity of the air-water boundary. The first conservation equation is normally simplified by neglecting density variations on time and distance scales that are meaningful to a discussion of gravity waves. Hence, the water velocity $\vec{u}$ is required to have zero divergence.

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{1}$$

Of course, some ocean circulation is induced by changes in the water density due to temperature, pressure, or salinity gradients. But, these contributions to (1) are small and are not considered here.

The velocity $\vec{u}$ is separated into two distinct components, one of which is the background current $\vec{u}_o$ that exists in the absence of waves. The other current component $\vec{v}$ contains all of the contributions due to the waves, including the effects of wave-current interactions. Now, the divergence condition (1) must be obeyed in the absence of waves. As a result, the current $\vec{u}_o$ separately has zero divergence and, hence, so does the wave-induced velocity $\vec{v}$. We make an additional restriction upon the current $\vec{u}_o$ for the present problem. That is, it is assumed that the horizontal variations in the current are negligible and that $\vec{u}_o$ is a function only of the depth below the mean sea level at $z = 0$. This restriction combined with the divergence condition (1) implies that the vertical current $\vec{u}_o \cdot \hat{z}$ equals a constant which we shall take to be zero.

The Navier-Stokes equation expresses the conservation of momentum. If we ignore the Coriolis term and the viscosity of water, then this equation is

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = \vec{g} - \frac{\vec{\nabla} p}{\rho} \qquad (2)$$

where $\vec{g} = -g\hat{z}$ is the acceleration due to gravity, p is the pressure, and ρ is the water density. Just as the velocity $\vec{u}$ was separated into two parts, we decompose the pressure p into the zero-wave pressure p_o and the wave-induced partial pressure q. We require that Equation (2) must be obeyed in the absence of waves when $\vec{u} = \vec{u}_o$ and $p = p_o$. If the current is constant with time and homogeneous with horizontal position, then the pressure $p_o = -\rho g z$ is simply the hydrostatic pressure.

Therefore, the propagation of gravity waves in the presence of a current field $\vec{u}_o$ is governed by

$$\vec{\nabla} \cdot \vec{v} = 0 \tag{3}$$

and

$$\left(\frac{\partial}{\partial t} + \vec{u}_o \cdot \vec{\nabla}\right)\vec{v} + \frac{\partial \vec{u}_o}{\partial z} (\vec{v} \cdot \hat{z}) + (\vec{v} \cdot \vec{\nabla})\vec{v} = - \frac{\vec{\nabla} q}{\rho} \tag{4}$$

where the current $\vec{u}_o$ is horizontal and a function of z only. In addition to these equations, we have the boundary conditions which state that the sea surface $z = \eta$ remain continuous

$$\vec{v} \cdot \hat{z}\Big|_{z=\eta} = \left[\frac{\partial}{\partial t} + (\vec{u}_o + \vec{v})\Big|_{z=\eta} \cdot \vec{\nabla}\right]\eta \tag{5}$$

and that the surface pressure vanish

$$q\Big|_{z=\eta} = \rho g \eta \tag{6}$$

where the atmospheric pressure is neglected. The displacement η of the surface from the mean level $z = 0$ is caused by gravity waves which may have an arbitrary spectral distribution. The changes in the sea level due to tides and storm surges and the corresponding vertical velocities of the surface are not included in these expressions. In addition, there is the boundary condition at the bottom $z = -d$ which states that the velocity $\vec{v} = 0$. In the interest of consistency, the background current $\vec{u}_o$ should also be forced to zero at the bottom.

Method of Solution

The sea surface can quite generally be represented by a Fourier integral or series given by

$$\eta(\vec{r},t) = \sum_{\vec{k},\omega} \eta(\vec{k},\omega) \exp[i(\vec{k} \cdot \vec{r} - \omega t)] \tag{7}$$

where $\vec{r} = x\hat{x} + y\hat{y}$ gives the horizontal position in the plane ($z = 0$) of the undisturbed surface and where $\vec{k}$ and ω are the wavevector and frequency, respectively. The velocity can be similarly expanded to give

$$\vec{v}(\vec{r},z,t) = \sum_{\vec{k},\omega} \vec{v}(\vec{k},z,\omega) \exp[i(\vec{k}\cdot\vec{r} - \omega t)] \tag{8}$$

and the partial pressure is expressed by

$$q(\vec{r},z,t) = \sum_{\vec{k},\omega} q(\vec{k},z,\omega) \exp[i(\vec{k}\cdot\vec{r} - \omega t)] \ . \tag{9}$$

The z-dependence of the velocity and pressure is a function of the particular vertical profile for the current and will be determined from the equations of motion and the boundary conditions.

Following the procedure outlined by Barrick and Weber (1977), the Fourier coefficients are expanded in perturbation series as follows:

$$\eta(\vec{k},\omega) = \sum_{n=1}^{\infty} \eta_n(\vec{k},\omega) \tag{10}$$

$$\vec{v}(\vec{k},z,\omega) = \sum_{n=1}^{\infty} \vec{v}_n(\vec{k},z,\omega) \tag{11}$$

and

$$q(\vec{k},z,\omega) = \sum_{n=1}^{\infty} q_n(\vec{k},z,\omega) \tag{12}$$

where the subscript indicates the perturbation order. In addition to the perturbation expansions of the waveheight, velocity, and pressure, we also expand the wave frequency

$$\omega = \sum_{n=0}^{\infty} \omega_n \tag{13}$$

so that we allow for the most general dispersion relation. Therefore, the phase velocities will depend upon the waveheights through the higher-order terms, which arise from the nonlinearities in the equations of motion. The wavevectors $\vec{k}$ and the first-order waveheights $\eta_1(\vec{k},\omega)$ are the independent parameters in terms of which all the higher-order perturbation variables are to be derived. In order for the perturbation expansions to converge, the slopes on the surface must be small. That is,

$$\sum_{\vec{k},\omega} |\eta(\vec{k},\omega)\vec{k}| < 1 .$$

By using the above Fourier transformations in conjunction with the perturbation expansions, it is possible to derive a self-consistent and general solution for gravity waves.

The first-order divergence condition is obtained from (3) and is given by

$$i\vec{k} \cdot \vec{v}_1(\vec{k},z,\omega) + \frac{\partial}{\partial z} (\hat{z}\cdot\vec{v}_1(\vec{k},z,\omega)) = 0 \ . \qquad (14)$$

Meanwhile, the first-order part of Equation (4) is

$$-i(\omega_o - \vec{k} \cdot \vec{u}_o(z)) \vec{v}_1(\vec{k},z,\omega) +$$

$$+ \left[\frac{\partial}{\partial z} \vec{u}_o(z)\right](\hat{z} \cdot \vec{v}_1(\vec{k},z,\omega)) = \frac{-i\vec{k}q_1(\vec{k},z,\omega)}{\rho} - \frac{\hat{z}}{\rho}\frac{\partial}{\partial z} q_1(\vec{k},z,\omega) \ . \qquad (15)$$

Of course, there are an indefinite number of additional equations which express the higher-order perturbation variables in terms of the lower-order ones. As a result, once we obtain the solutions to these first-order equations, then we can derive all of the higher-order results (at least, in principle).

The solutions to (14) and (15) are restricted by the boundary conditions (5) and (6) at the surface which, to first order, become

$$\hat{z} \cdot \vec{v}_1(\vec{k},0,\omega) = -i(\omega_o - \vec{k} \cdot \vec{u}_o(0))\eta_1(\vec{k},\omega) \qquad (16)$$

and

$$q_1(\vec{k},0,\omega) = \rho g \eta_1(\vec{k},\omega) \qquad (17)$$

respectively. We note that these boundary conditions place restrictions upon the dynamic variables at the mean surface (z = 0). This form for the boundary conditions was obtained by expanding the dynamic variables in Taylor series in terms of the z-coordinate about the mean surface before the Fourier transformation was performed upon (5) and (6).

The Rayleigh Equation

The scalar equation (14) and the vector equation (15) can be combined to produce a single second-order differential equation for the partial pressure $q_1(\vec{k},z,\omega)$ or for one of the vector components of the velocity $\vec{v}_1(\vec{k},z,\omega)$. We shall choose the differential equation for the z-component of velocity

$$\frac{\partial^2}{\partial z^2}(\hat{z}\cdot\vec{v}_1(\vec{k},z,\omega)) = \left[k^2 - \frac{\frac{\partial^2}{\partial z^2}\vec{k}\cdot\vec{u}_o(z)}{(\omega_o - \vec{k}\cdot\vec{u}_o(z))}\right]\hat{z}\cdot\vec{v}_1(\vec{k},z,\omega) \qquad (18)$$

which is known as the inviscid Orr-Sommerfeld equation or the Rayleigh equation. Peregrine (1976) reviews some solutions to this equation for special current profiles. Meanwhile, Lin (1961) and Yih (1972) approach this equation through the theory of hydrodynamic stability. Certainly, (18) indicates that there might be a problem if the phase velocity equals the projected current (i.e., $\omega_o = \vec{k}\cdot\vec{u}_o(z)$) at some critical depth below the surface. It is well known that one of the solutions to this equation could contain singularities at these critical depths, which are physically unrealistic. A more complete description is needed in these cases, although the solutions to Rayleigh's equation may still be of interest for locations away from these critical depths. We shall avoid questions of stability and refer the reader to the voluminous literature on this subject. Therefore, the present study does not apply to those situations where either the current profile or wave is unstable.

The Rayleigh equation (18) contains the independent variable z, the independent parameter k, and the arbitrary stable background current

$\vec{u}_o(z)$. The wave frequency ω_o however can not assume just any value because the surface boundary conditions will determine the dispersion relation. The dependent variable $\vec{z} \cdot \vec{v}_1(\vec{k},z,\omega)$ is now expressed as

$$\hat{z} \cdot \vec{v}_1(\vec{k},z,\omega) = \hat{z} \cdot \vec{v}_1(k,0,\omega) \sinh[k(z+d)] \exp \int_o^z dz' h(z')/\sinh(kd) \tag{19}$$

where h(z) contains the influence of the background current and carries the dimensions of length^{-1}. It is obvious from (18) that h(z) must vanish for linear current shears because, in that case, the second-derivative of $\vec{u}_o(z)$ vanishes. The form of (19) was chosen so that the velocity will vanish at the bottom z = -d. When we substitute the expression (19) in the linear and homogeneous second-order differential equation (18), we obtain the following non-linear and non-homogeneous first-order differential equation or Riccati equation (Davis [1960]).

$$\frac{\partial}{\partial z} h(z) + h^2(z) + \alpha_o(z)\, h(z) = \beta_o(z) \tag{20}$$

where

$$\alpha_o(z) = 2k \coth[k(z+d)] \tag{21}$$

and

$$\beta_o(z) = -\frac{\frac{\partial^2}{\partial z^2} \vec{k}\cdot\vec{u}_o(z)}{(\omega_o - \vec{k}\cdot\vec{u}_o(z))} \, . \tag{22}$$

The function h(z) can be redefined to be equal to

$$h(z) = \frac{\beta_o(z)}{\alpha_o(z)} + h_1(z) \, . \tag{23}$$

In this case, (20) produces an equation for $h_1(z)$ formally identical to (20).

$$\frac{\partial h_1(z)}{\partial z} + h_1^2(z) + \alpha_1(z)\, h_1(z) = \beta_1(z) \tag{24}$$

where

$$\alpha_1(z) = \alpha_o(z) + 2\,\frac{\beta_o(z)}{\alpha_o(z)} \tag{25}$$

and

$$\beta_1(z) = -\frac{\partial}{\partial z}\left(\frac{\beta_o(z)}{\alpha_o(z)}\right) - \left(\frac{\beta_o(z)}{\alpha_o(z)}\right)^2 . \tag{26}$$

Because (24) is formally identical to (20), we can repeat the above process ad infinitum so that

$$h(z) = \sum_{n=0}^{\infty} \frac{\beta_n(z)}{\alpha_n(z)} \tag{27}$$

where each consecutive term in this infinite series is generated by recurrence relations like (25) and (26). In other words,

$$\alpha_{n+1}(z) = \alpha_n(z) + \frac{2\beta_n(z)}{\alpha_n(z)} \tag{28}$$

and

$$\beta_{n+1}(z) = -\frac{\partial}{\partial z}\left(\frac{\beta_n(z)}{\alpha_n(z)}\right) - \left(\frac{\beta_n(z)}{\alpha_n(z)}\right)^2 . \tag{29}$$

A cursory examination of these equations reveals that higher-order terms in (27) contain higher-order derivatives of the current profile and become increasingly more nonlinear in the current velocity itself. It is also apparent that this series can not converge unless there are some restrictions placed upon the size of the various derivatives of the current shear. Later, it will become evident that the proper restrictions are given by

$$\left|\frac{\partial^n}{\partial z^n} \vec{k}\cdot\vec{u}_o(o)\right| << [2k \coth(kd)]^n \sqrt{gk} \ . \qquad (30)$$

However, the convergence of the series (27) is not a prerequisite for the existence of the solution to the Rayleigh equation. In this case, the function h(o) at the surface is approximated by

$$h(o) \simeq (\omega_o - \vec{k}\cdot\vec{u}_o(0))^{-1} \sum_{n=2}^{\infty} [-2k \coth(kd)]^{1-n} \frac{\partial^n}{\partial z^n} \vec{k}\cdot\vec{u}_o(o) \ . \qquad (31)$$

This last expression is the result of (30) and the fact that $|\omega_o - \vec{k}\cdot\vec{u}_o(o)| \simeq \sqrt{gk}$, which will be obvious once we have discussed the dispersion condition. Also, closer examination of h(z) reveals that $(\omega_o - \vec{k} \cdot \vec{u}_o(z))$ is a factor in the denominator of every term.

The First-Order Dispersion Relation

Up until now we have not really used the boundary conditions at the surface. In all of the equations, the wave frequency ω_o enters as some unspecified parameter. By using the boundary equations (16) and (17) with the equations of motion (14) and (15), the following dispersion relation is easily derived.

$$(k + h(o))(\omega_o - \vec{k}\cdot\vec{u}_o(o))^2 + \frac{\partial}{\partial z} \vec{k}\cdot\vec{u}_o(o)(\omega_o - \vec{k}\cdot\vec{u}_o(o)) = gk^2 \ . \qquad (32)$$

When the function h(o) vanishes at the surface, this result reduces to that of Plate and Trawle (1970) who used Biesel's (1950) model to calculate the change in wave speed due to a linear current shear. It also goes without saying that this expression reduces to the correct deep water dispersion relation in the absence of any currents (namely, $\omega_o^2 = gk$).

Apart from changing the phase velocity of a gravity wave, the current shear has one other surprising effect upon radar sea echoes. Whereas a constant current simply Doppler shifts or displaces in frequency the entire sea echo spectrum, a current shear introduces a slight non-uniform frequency shift in different regions of the spectrum. For example, the advancing and receding first-order sea-echo spectral peaks may be spread apart by a current shear in excess of the expected separation $2\sqrt{gk}$. This non-uniform Doppler shift arises from those terms in (32) which contain odd powers of the wave frequency. It should be noted that the non-uniform Doppler shift occurs in the presence of any current shear, even a linear one, and that the present generalization of vertical current profiles predicts nothing fundamentally new in this respect.

We recall that h(o) in (31) is inversely proportional to the first power of $(\omega_o - \vec{k}\cdot\vec{u}_o(o))$. This fact along with the dispersion relation (33) indicates that the phase velocity and the projected current velocity at the surface can never be equal (i.e., $\omega_o \neq \vec{k}\cdot\vec{u}_o(o)$). Therefore, the critical depth at which h(z) has a "singularity" must not be z = 0. This is the reason that the possible singularity in h(z) need not prevent a general treatment of the effects of arbitrary current shears upon phase velocity of surface gravity waves. The dispersion relation then can be formally written as

$$(\omega_o - \vec{k}\cdot\vec{u}_o(o))^2 - 2\,\sigma(\omega_o - \vec{k}\cdot\vec{u}_o(o)) = gk \qquad (33)$$

where σ contains all of the effects of the current shear. That is,

$$\sigma = \sum_{n=1}^{\infty} [-2k \coth(kd)]^{-n} \frac{\partial^n}{\partial z^n} \vec{k}\cdot\vec{u}_o(o) \qquad (34)$$

where we have placed the restriction that $\sigma << \sqrt{gk}$ which implies that the effect of the current shear is merely to perturb the phase velocity. Despite the potential complexity of the vertical current profile (19), the influence on the dispersion relation is simply consolidated in the factor σ.

Now, the two roots to (33) are

$$\omega_o = \pm\,\omega_B + \vec{k}\cdot\vec{u}_o(o) + \sigma \pm \frac{\sigma^2}{2\omega_B} \qquad (35)$$

where we define the Bragg frequency $\omega_B = \sqrt{gk}$ as that which the radar would see scattered from a wave with wavenumber k and where terms with higher powers of σ/ω_B are not explicitly shown. From this expression it is clear that $|\omega_o - \vec{k}\cdot\vec{u}_o(o)| \simeq \omega_B$ and hence (31) follows. The two principal effects of the shear that were mentioned earlier are obvious from this expression. First, the current shifts the phase velocity by an amount which depends not only upon the value of the current velocity at the surface but also upon the details of the vertical profile. Second, the phase velocities of any two waves propagating in opposite directions are spread apart by the current shear.

Implications for Sea Echo Doppler Spectra

In the absence of any currents, the radar sea echo spectra [Barrick et al. (1974)] contain two strong peaks, referred to as Bragg lines. One of these lines is located at a frequency of $+\omega_B$ and is caused by HF scattering from a gravity wave advancing towards the radar. The other spectral line is located at $-\omega_B$ and is caused by a wave receding from the radar. Both of these lines are Doppler shifted in the same direction by an amount (see Figure 1)

$$\omega_D = \sum_{n=0}^{\infty} (-2k)^{-n} \frac{\partial^n}{\partial z^n} \vec{k}\cdot\vec{u}_o(o) \ . \tag{36}$$

This is the result we would get if we averaged the Doppler frequencies (35) for the advancing and receeding Bragg lines. A Doppler velocity $v_D = \omega_D/k$ can then be inferred as the surface current measured by the radar. Therefore, the Doppler shift depends upon the current at the surface, the gradient of the current profile at the surface, the curvature of the profile at the surface, etc.

In a private communication, D. E. Barrick pointed out the series (34) and (36) in terms of the derivatives of the background current can be reformulated in integral form using the following identity.

$$2k \coth(kd) \int_{-\infty}^{o} dz\ \vec{k}\cdot\vec{u}_o(z) \exp[2kz \coth(kd)] =$$

$$\sum_{n=0}^{\infty} [-2k \coth(kd)]^{-n} \frac{\partial^n}{\partial z^n} \vec{k}\cdot\vec{u}_o(o) \ .$$

Thus, the results given here agree with those of Stewart and Joy (1974) in the case of great depth $kd >> 1$ and weak shear $\sigma << \omega_B$. An integral formulation of (35) is then given by

$$\omega_o = \pm\, \omega_B + 2k \coth(kd) \int_{-\infty}^{o} dz\, \vec{k}\cdot\vec{u}_o(z) \exp[2kz \coth(kd)]$$

$$\pm \frac{[\vec{k}\cdot\vec{u}_o(o) - 2k \coth(kd) \int_{-\infty}^{o} dz\, \vec{k}\cdot\vec{u}_o(z) \exp(2kz \coth(kd))]^2}{2\omega_B} .$$

Example: Exponential shear profile $[\vec{u}_o(z) = \vec{u}_o(0)\exp(\alpha z)]$.

Figure 2 shows how the Doppler velocity $v_D = u_o(1 + \frac{\alpha}{2k})^{-1}$ measured from a sea echo frequency spectrum deviates from the surface current u_o for exponential current shears. This same result for the exponential model was obtained by Stewart and Joy (1974) who restricted u_o to small values much less than the wave phase velocity. No such restriction is made here. For this discussion v_D is defined to be the average offset in velocity (from (36)) for the advancing and receding Bragg waves. The results in this figure are valid only if $|\alpha| < 2k$ (so that the terms neglected in (31) will be small), and only if the surface current u_o satisfies the condition $|v_D - u_o| << v_B$, where the Bragg velocity $v_B = \omega_B/k$. The Bragg velocity v_B equals $\sqrt{gc/4\pi\nu_{HF}}$, where ν_{HF} equals the radio frequency and c is the speed of light. We note that this condition does not necessarily limit the size of the current velocity u_o but it does imply that one cannot have both large currents and large shears at the same time. Of course, if these conditions are not satisfied, then the phase velocity can be computed using the more complete (and complicated) expression for h(o) from (27) in the dispersion relation (32). For a radio frequency of 25.4 MHz, the Bragg velocity v_B is about 6 knots, which allows for a very large range of currents and shears. Figure 3 gives the upper limit for the magnitude of the surface current $u_o = v_B \left| 1 + \frac{2k}{\alpha} \right|$ for a range of shears. Actually, the current velocity must be much less than this limit in order for the approximations made earlier to hold. We note that currents which increase with depth are much more restricted than currents which decrease with depth.

In addition, the Bragg lines are separated by an amount equal to (see Figure 1)

$$\Delta = \omega_B(2 + \sigma^2/\omega_B^2) \tag{37}$$

which is somewhat larger when the current varies with depth. This equation can also be used to obtain the relative velocity of two gravity waves with wavenumber k traveling in opposite directions. In general, a vertical shear increases this relative velocity above that value which would exist for a constant current. The fractional spreading of the Bragg lines is given in Figure 4 for different exponential current shears. The fact that the separation of the Bragg lines is greater when the current varies with depth suggests that the radar might possibly afford a means for measuring not only near surface currents but also near surface current shears. Of course, the sensitivity of this method decreases for weaker shears which may be more prevalent in natural situations than the stronger shears. However, there is a positive implication for the radar even in this case. If the shear is not detectable by the radar, then in some cases this may mean that the shear introduces negligible error in the radar measurement of the surface current.

The excess spreading of the Bragg lines is given in Figure 5 for different Doppler velocities v_D inferred from the sea echo Doppler spectrum. The relationship between these two quantities is simply

$$\frac{\Delta - 2\omega_B}{\omega_B} = \left(\frac{v_D - u_o}{v_B}\right)^2 = \frac{\sigma^2}{\omega_B^2} \; . \tag{39}$$

This expression is quite general and applies to any shear profile, not just exponential ones. It is noted that the excess splitting of the receding and advancing Bragg lines is not a sensitive measure of current shears. For example, Figure 5 shows that, if the radar measured Doppler velocity shift and surface current differ by 30 cm/s, then the Doppler velocities from the receding and advancing Bragg lines disagree by about 3 cm/s (or 10%).

Conclusions

One obvious result of this study is the conclusion that HF radars like the CSR system do not simply measure currents just at the surface. Multiple HF techniques have been suggested (Barrick et al. [1974] and Stewart and Joy [1974]) for extracting the current shear and surface current. The splitting of the Bragg lines, discussed here, provides an additional measure for detecting strong vertical shears, although this latter method does not uniquely determine the shear. However, a combination of the multiple-frequency and Bragg-line-splitting techniques may provide both the surface current and the shear profile (i.e., the slope and curvature of the shear) at the surface.

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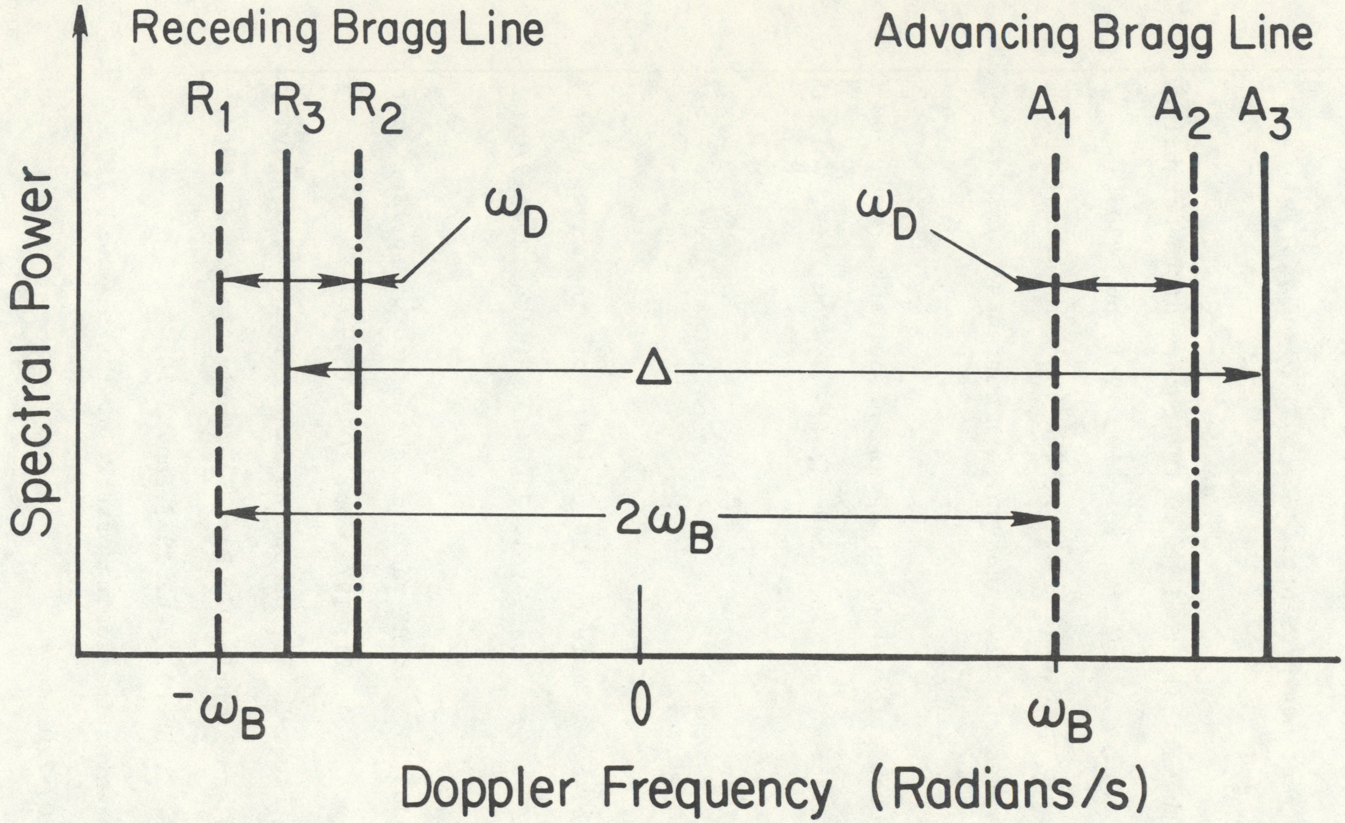

Figure 1. Example of shift in sea-echo spectral Bragg lines due to near-surface currents. R_1 and A_1 are respectively the receding and advancing Bragg lines in the absence of any currents. R_2 and A_2 are these lines uniformly shifted for one current which is constant with depth. Meanwhile, R_3 and A_3 are the Bragg lines for this same surface current but with a depth dependence added.

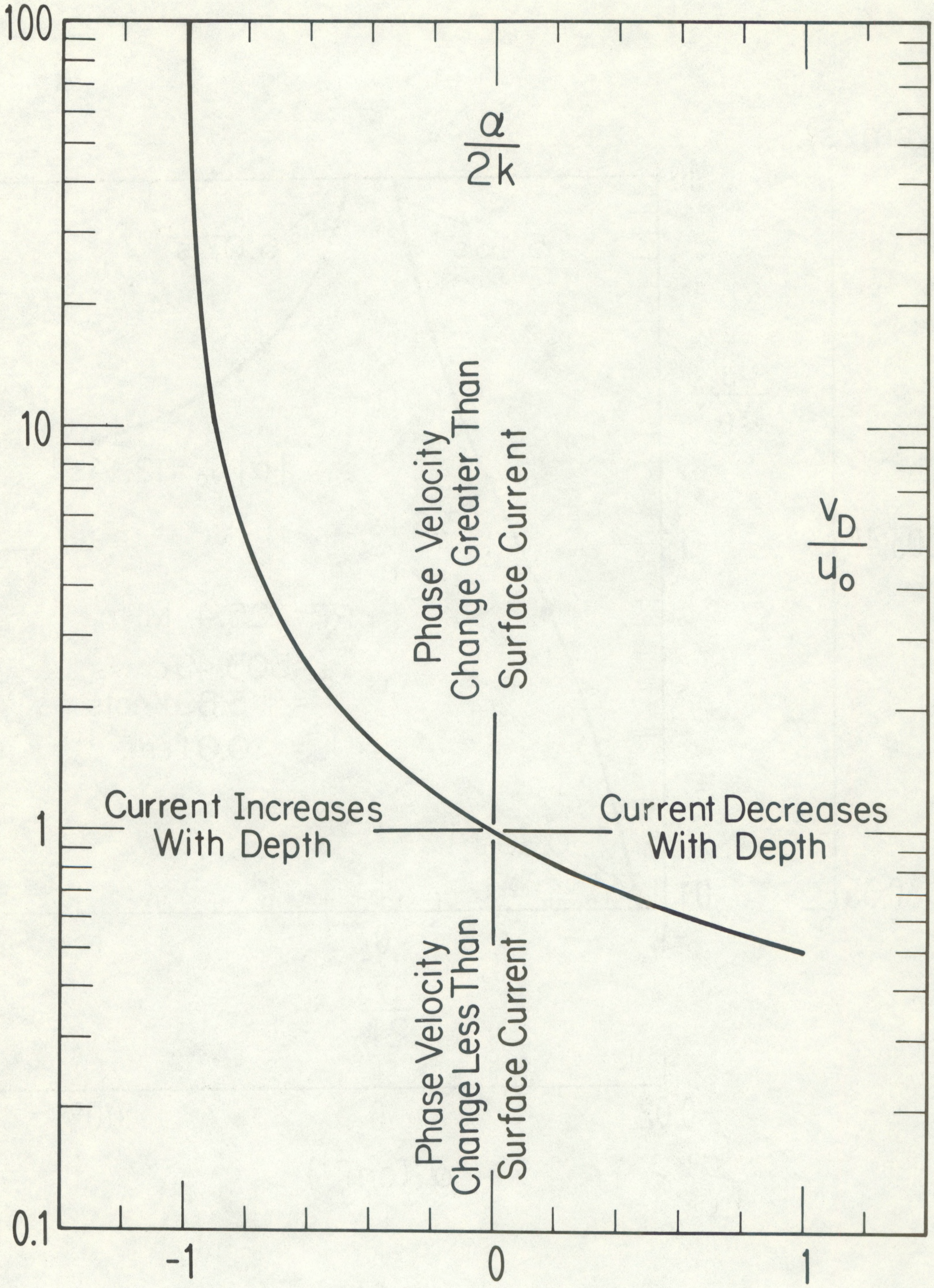

Figure 2. Change v_D in phase velocity for exponential current profile $u(z) = u_o e^{\alpha z}$. The positive z-axis is vertical upward from surface and u_o is the current at the surface. $v_D = u_o(1 + \frac{\alpha}{2k})^{-1}$

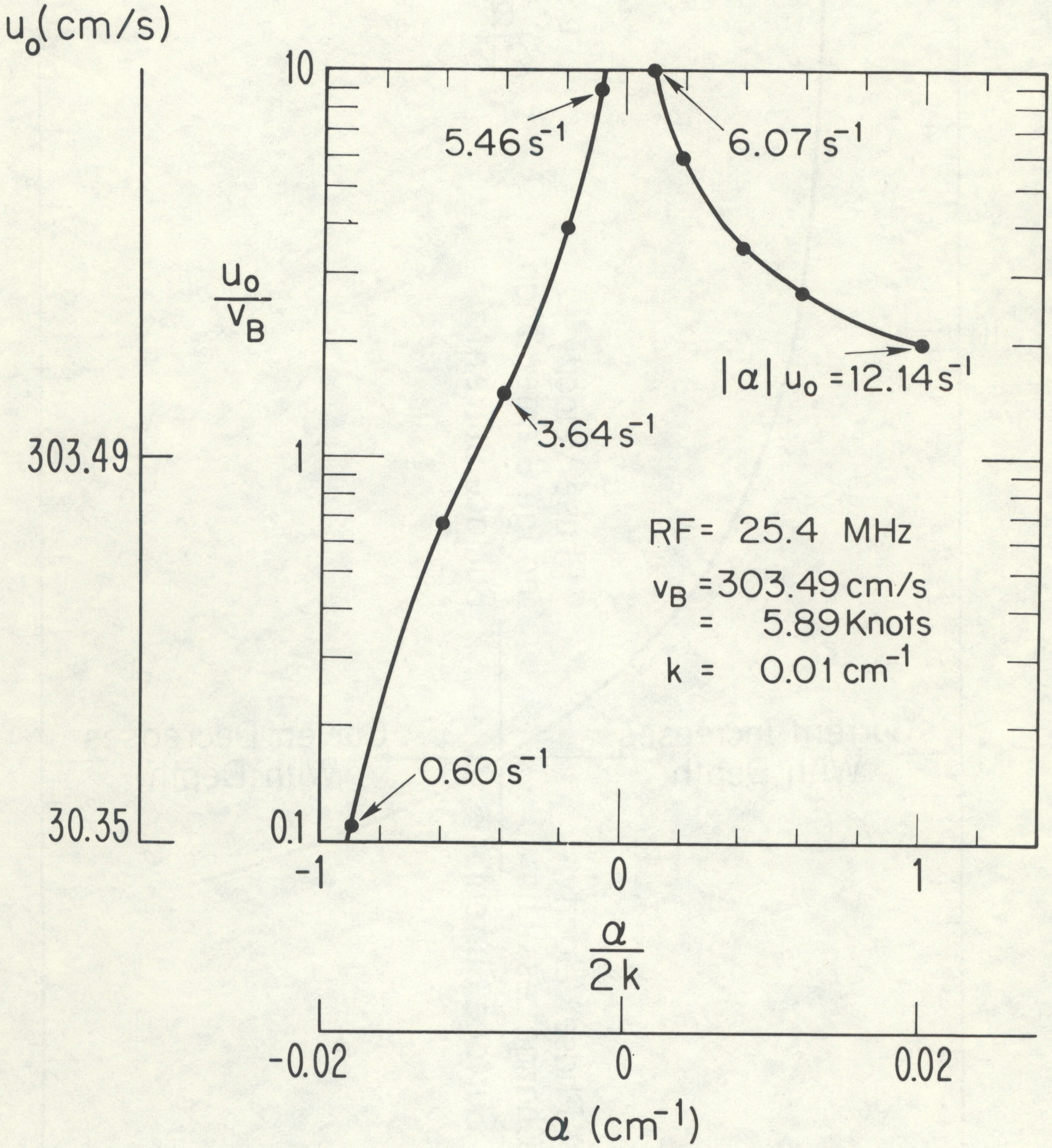

Figure 3. Surface current upper limit for different exponential current shears. The maximum surface current $u_o = v_B \left|1 + \frac{2k}{\alpha}\right|$.

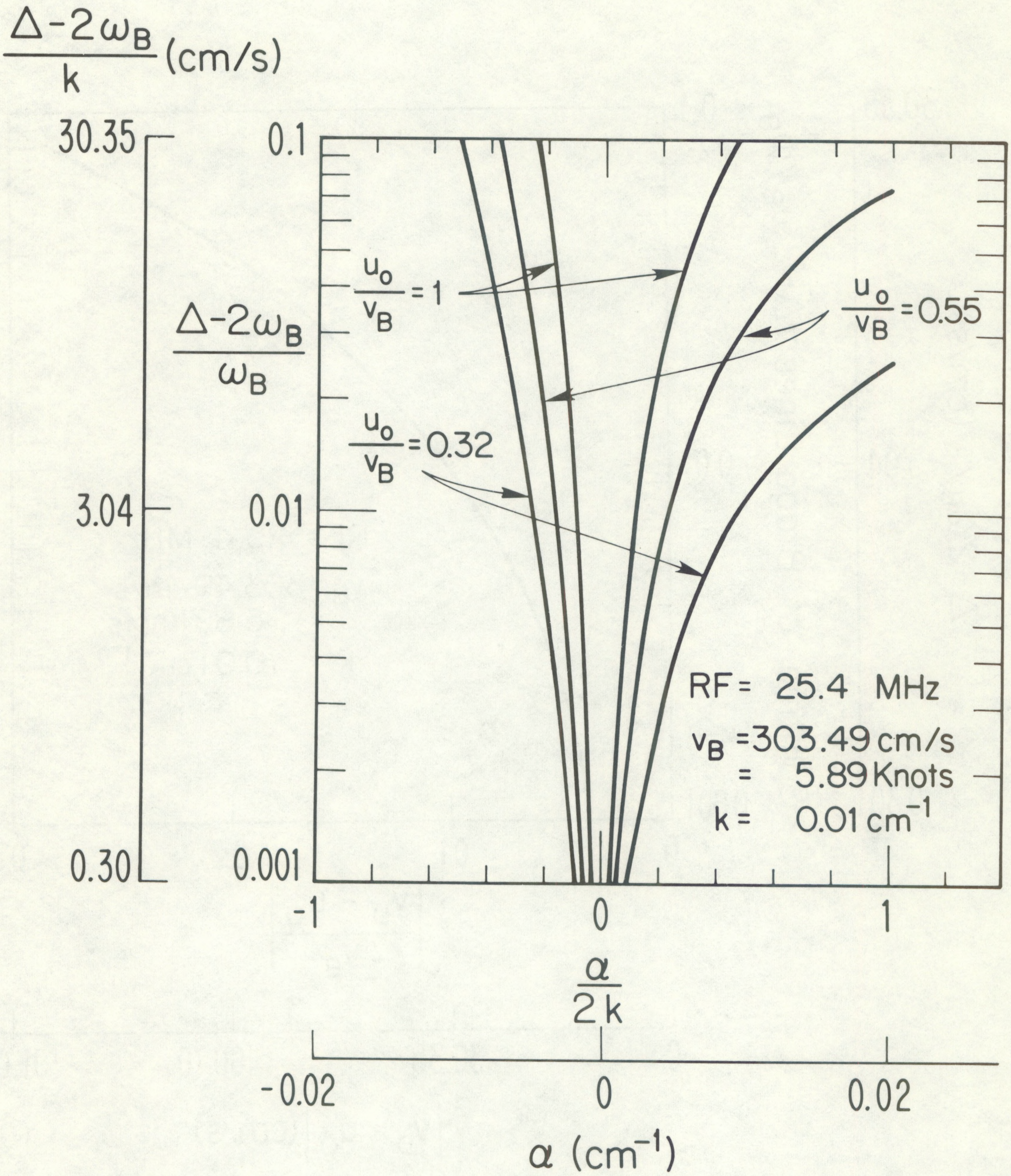

Figure 4. Fractional excess spreading of Bragg lines due to current shear (from Equation (37)).

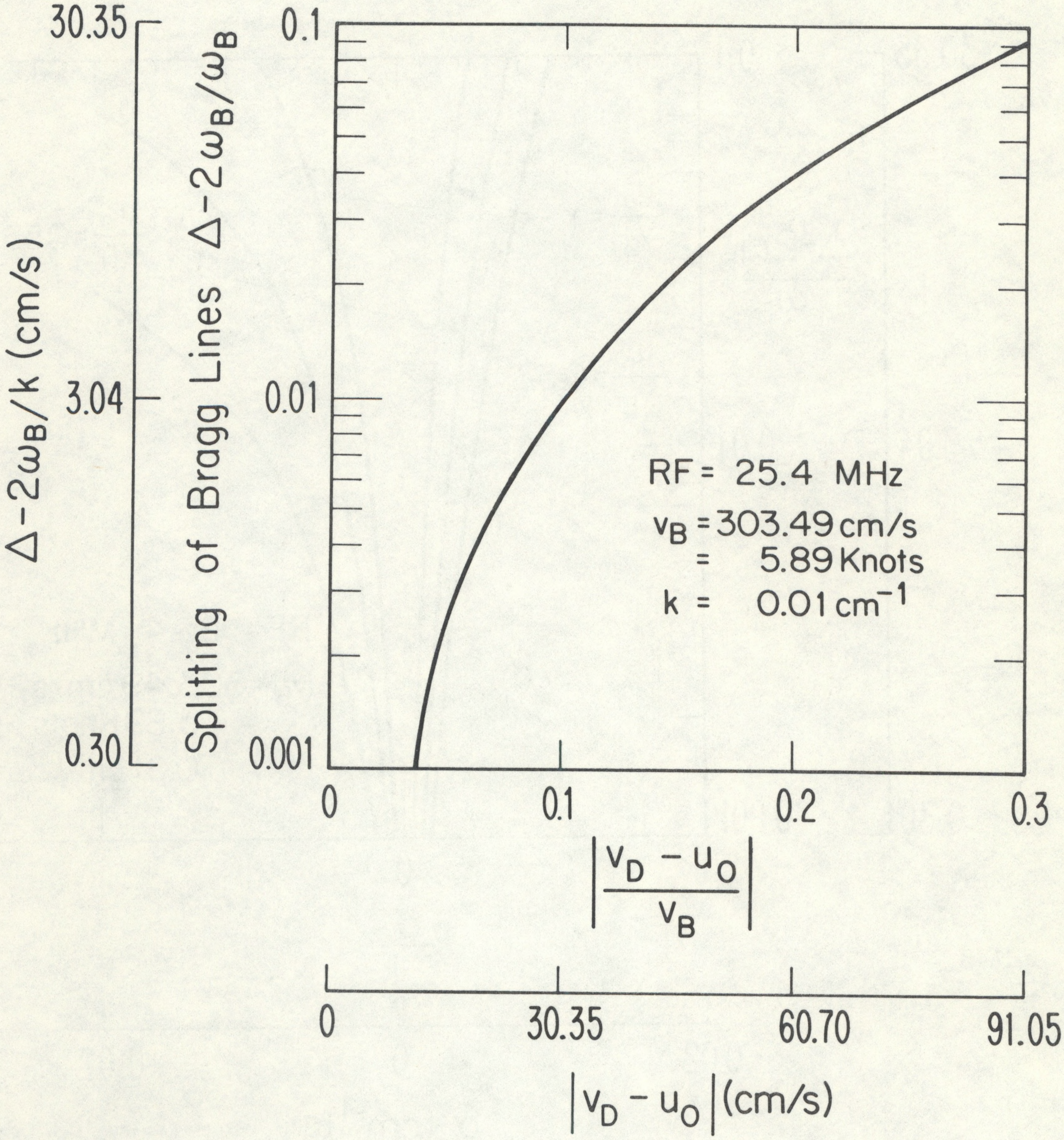

Figure 5. Fractional excess spreading of Bragg lines as a function of the Doppler velocity (from Equation (39)).