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ABSTRACT

This report is a companion to an earlier report in which the relationship between the variances of temperature and vertical velocity and their dependence on the height gradient of temperature was examined by solving the equations of motion, continuity and energy for a special linearized model. That report led to the conclusion that $\theta_o g^{-1} d\theta/dz \propto \overline{\theta^2}/\overline{w^2}$ where θ and w are the potential temperature and vertical velocity perturbations, θ_o is the unperturbed potential temperature. This report continues the analysis of the above model and examines the morphology of the perturbations and their dynamic stability. It is shown that the value of Richardson Number for "just unstable" conditions tends to 1/8 instead of the value of 1/4 found for height-phase-coherent disturbances.

INTRODUCTION

Remote sensing systems such as acoustic sounders and radars usually depend on what is commonly called Bragg Scatter from turbulent inhomogeneities in the refractive index of the medium. The power backscattered is proportional to the power at the Bragg scale in the spectrum of the refractive index. For backscatter the Bragg scale is one-half the wavelength of the radar. Furthermore, the velocity variance in the medium is a most important factor in the width, or 2nd moment, of the Doppler spectrum. These quantities, potentially measurable by the radars, are of great interest if they can be related to the mean gradients with height of properties, such as temperature, that are of interest to the meteorologist. Efforts in this area have in the past been directed toward the use of various turbulence relationships derived from similarity theory and the energy and variance budget equations (e.g., Ottersten, 1969; Gage et al., 1980; Gossard et al., 1982).

In a recent report by Gossard and Frisch (1986), an alternative approach was used in which the linearized equations of motion, continuity and energy were solved for a relationship between mean temperature gradient and the variances of temperatue and vertical velocity for a model in which it was assumed that the perturbations were not coherent with height, but in fact moved on the average with the wind at the height where they were embedded. Such a model is in contrast with coherent models of deep gravity wave systems (such as lee waves locked to a topographic feature) in which the wave crests and troughs vary in a phase coherent manner with height. The results, relating the variances of temperature and vertical velocity to the mean temperature gradient, were shown to be consistent with what would be inferred from the turbulent flux equation if the contribution of the pressure covariance terms was neglected.

MORPHOLOGY AND KINEMATICS

From Eq. (13) of TM ERL WPL 134, we note that

$$\frac{s}{u_o'} - \kappa\frac{\sigma^2}{u_o'} = -\frac{1}{2}\left[\frac{\kappa\sigma^2}{u_o'}(1 - Pr) + \frac{nk}{\sigma^2}\right] \pm \sqrt{\frac{1}{4}\left[\frac{\kappa\sigma^2}{u_o'}(1 - Pr) + \frac{nk}{\sigma^2}\right]^2 - \frac{m^2}{\sigma^2} Ri} \qquad (1a)$$

where ν and κ are the eddy coefficients of viscosity and thermal conductivity respectively, $\sigma^2 = k^2 + \ell^2 + n^2 \equiv m^2 + n^2$ where k, ℓ, n are the wavenumbers in the x,y,z directions, x,y,z are the two horizontal and the vertical axes respectively, $u_o' = \partial u_o/\partial z$ taken to lie in the x,z plane, u_o is the mean horizontal wind, Ri is the Richardson Number equal to $N^2/u_o'^2$ where N is the Vaisala-Brunt frequency $(g/\theta)(d\theta/dz)$ and θ is the potential temperature. The turbulent

Prandtl Number $Pr = \nu/\kappa$ and the Stokes Operator $D/Dt = \partial/\partial t + u_o\, \partial/\partial x + v_o\, \partial/\partial y = -s$.

In examining the form of the perturbations and their stability, it is useful to consider special cases that are simple enough for analytical study. Consider the case of $Pr = 1$. Then

$$\left[\frac{s}{u_o'} - \frac{\kappa\sigma^2}{u_o'} + \frac{nk}{2\sigma^2}\right]^2 = \frac{1}{4}\frac{n^2k^2}{\sigma^4} - \frac{m^2}{\sigma^2} Ri \; . \tag{1b}$$

Furthermore, if $\kappa\sigma^2/u_o'$ is negligibly small compared with s/u_o',

$$n^2 + \frac{u_o'}{s} kn + m^2 \left[Ri\left(\frac{u_o'}{s}\right)^2 + 1\right] = 0 \tag{1c}$$

where we note that the center term changes sign with u_o' or n. Then, for n pure real, the real part of s is

$$s_r = -\frac{1}{2}\frac{u_o' kn}{n^2 + m^2} \tag{2a}$$

and the imaginary part of s, say $s_i = \omega$, is given by

$$\omega^2 = \frac{m^2}{n^2 + m^2} N^2 - \frac{1}{4} u_o'^2 k^2 \frac{n^2}{(n^2 + m^2)^2} \tag{2b}$$

in agreement with (13d) of TM ERL WPL 134. Therefore, for this special case, n is given as a function of the intrinsic frequency ω by

$$n^2 = -\frac{m^2}{2}\left(2 - \frac{N^2}{\omega^2} + \frac{1}{4}\frac{k^2}{m^2}\frac{u_o'^2}{\omega^2}\right) \pm \sqrt{\frac{m^4}{4}\left(2 - \frac{N^2}{\omega^2} + \frac{1}{4}\frac{k^2}{m^2}\frac{u_o'^2}{\omega^2}\right)^2 - m^4\left(1 - \frac{N^2}{\omega^2}\right)} \; . \tag{3}$$

Thus the condition for n to be pure real is that the + sign be chosen and that

$$1 < N^2/\omega^2 < 2 + \frac{1}{4}(k^2/m^2)(u_o'^2/\omega^2) \; . \tag{4a}$$

When $u_o' = 0$, the condition for $n^2 > 0$ becomes the gravity wave condition

$$N^2/\omega^2 - 1 > 0 \; . \tag{4b}$$

Written in terms of Richardson Number, Eq. (3) becomes

$$\frac{n^2}{m^2} = \frac{1}{2}\left(2 - \frac{u_o'^2}{\omega^2} Ri + \frac{1}{4}\frac{u_o'^2}{\omega^2}\frac{k^2}{m^2}\right)\left[-1 \pm \sqrt{1 - \frac{4\left(1 - \frac{u_o'^2}{\omega^2} Ri\right)}{\left[2 - \frac{u_o'^2}{\omega^2} Ri + \frac{1}{4}\frac{u_o'^2}{\omega^2}\frac{k^2}{m^2}\right]^2}}\,\right] \cdot (5)$$

The plots of $n^2/m^2 = L_h^{\,2}/L_v^{\,2}$, from Eqs. (3) and (5), are shown in Fig. 1 for various shear conditions ($\omega^2/u_o'^2$) where n^2/m^2 vs. N^2/ω^2 are the dashed curves and n^2/m^2 vs. Ri are the solid curves. For given $\omega^2/u_o'^2$ the ratio of $L_v^{\,2}/L_h^{\,2}$ clearly decreases as N^2/ω^2 and Ri increase.

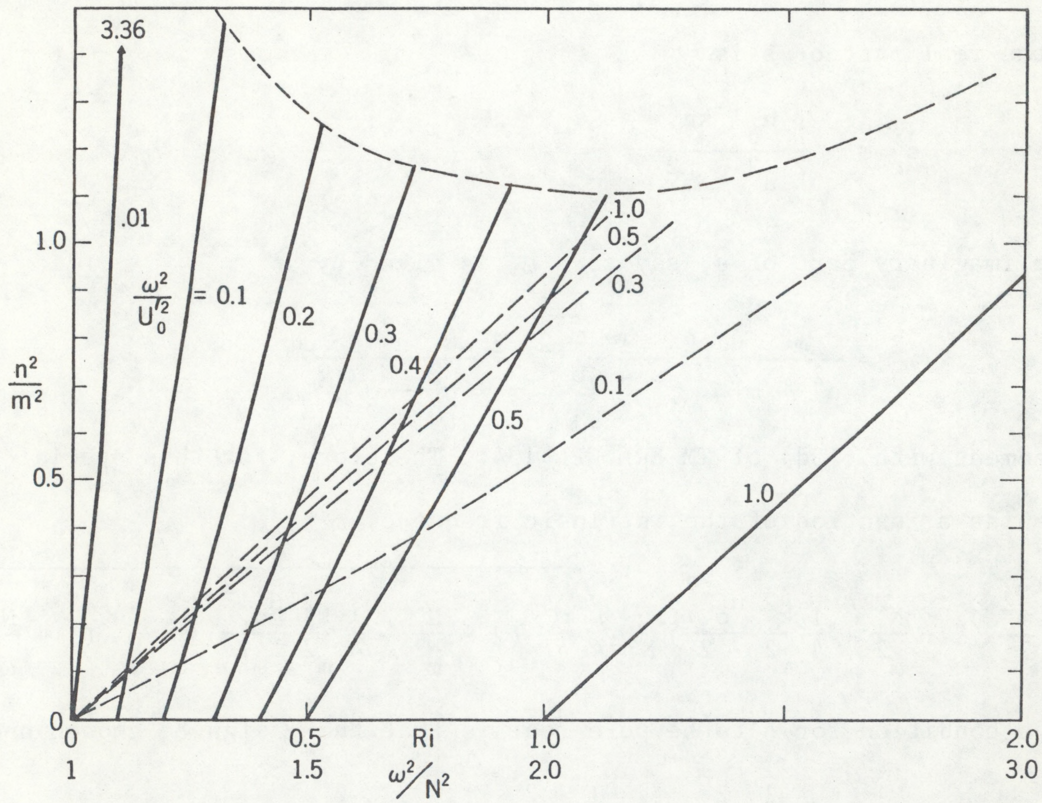

Figure 1. Plot of $n^2/m^2 = L_h^{\,2}/L_v^{\,2}$ vs. N^2/ω^2 (dashed curves) and Richardson Number (solid curves) for various shear conditions ($\omega^2/u_o'^2$) for a model in which the eddy viscosity and eddy conductivity are negligibly small and in which the perturbed features move with the local mean wind. Note that the larger Ri and N^2/ω^2, the larger is $L_h^{\,2}/L_v^{\,2}$.

When s is real, there is no realistic condition for which n is real for this simplified model unless N^2 is negative, in fact, unless $N^2/s^2 < -1$.

Note that the functional relation (1c) depends on the sign of n (and the sign of u_o'); that is,

$$\frac{n}{k} = \frac{1}{2} \frac{u_o'}{s} \left[-1 \pm \sqrt{1 - 4 \frac{m^2}{k^2} \frac{s^2}{u_o'^2} \left(Ri \frac{u_o'^2}{s^2} + 1\right)}\right] . \qquad (6)$$

Thus, choosing u_o' positive, n/k is positive when the plus sign on the radical is chosen and negative (say n/k = p/k) when the negative sign is chosen. We are interested in standing "wave" interference patterns in the vertical plane containing the mean wind and its shear vector, so we add the two solutions. As a simple example we choose the two dimensional case with $\ell = 0$.

We have formulated the problem for an observer drifting with the local mean wind, and have chosen $D/Dt \equiv \partial/\partial t + u_o(z)\, \partial/\partial x \equiv -s$ to be invariant with height. Therefore, for the drifting observer,

$$W = W_A\, e^{-st}\, e^{i(kx + nz)} .$$

For example, choosing the case when s is pure imaginary, (say $s = i\omega$)

$$W = W_A\, e^{i(kx + nz - \omega t)}$$

and ω is therefore the intrinsic frequency noted by the drifting observer. Its formal relationship to the frequency relative to a fixed point, $\sigma(z) \equiv i\partial/\partial t$, is

$$\sigma = \omega + k\, u_o'\, z$$

so that, relative to the fixed reference system,

$$W = W_A\, e^{i(kx + nz - \sigma t)} = W_A\, e^{i[kx + nz - (\omega + k\, u_o'\, z)t]}$$

where we have chosen to analyze the case in which ω is independent of height.

Thus, in general,

$$W = W_A e^{-st}[e^{i(kx - k u_o' zt + nz)} + e^{i(kx - k u_o' zt + pz)}] . \qquad (7)$$

Writing (6) symbolically as

$$\frac{n}{k} = - B + A \quad , \quad \frac{p}{k} = - B - A$$

where $B = 1/2\ u_o'/s$ and $A = 1/2\ u_o'/s[1 - 4(m/k)^2(s/u_o')^2(Ri\ u_o'^2/s^2 + 1)]^{1/2}$ $= B[1 - (m/k)^2\ (1/B^2)\ (Ri\ u_o'^2/s^2 + 1)]$. So (7) becomes

$$W = W_A e^{-st}[e^{ik(x - u_o' zt - Bz + Az)} + e^{ik(x - u_o' zt - Bz - Az)}]$$

and

$$W = 2\ W_A e^{-st} \cos kAz\ \mathcal{R}\ e^{ik[x - u_o' z(t + 1/2s)]} .$$

Plots of $(W/W_A)\ e^{st}$ are shown in Fig. 2. Actually, the two interfering "wave forms" are plotted because we believe this plot is rather more revealing than

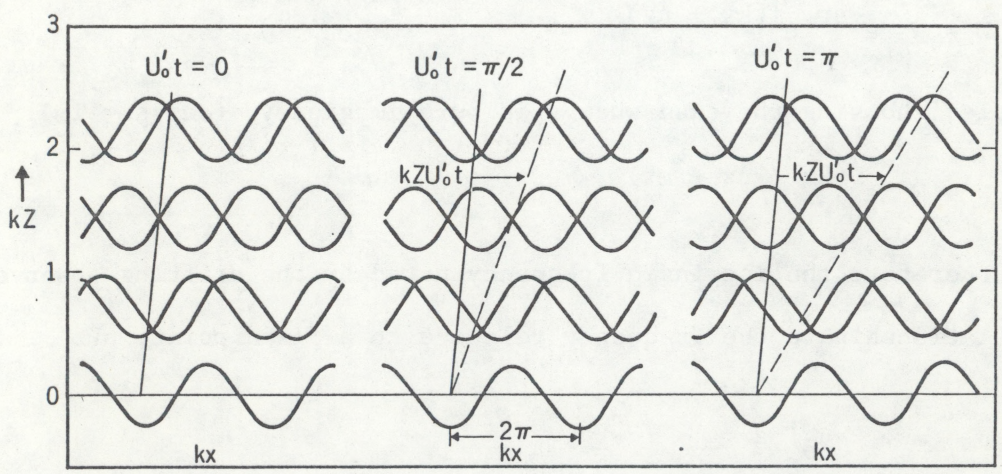

Figure 2. Morphological display of the patterns of interfering waveforms of which the perturbations are composed. Three horizontal frames are successive times in increments of one radian of $u_o't$. Four heights are increasing increments of $\pi/2$ Akz.

the combined value of W. The interpretation in terms of wave patterns is to be avoided, since the solutions apply to Fourier components within the perturbation ensemble. In Figure 2 four different heights are shown separated by intervals of $\pi/2$ radians of AkZ. Three successive times are shown as the three frames on the horizontal axis, separated by u_o't intervals of one radian. The displacement profile u_o't is shown in each frame.

STABILITY CONDITIONS: THE MOST UNSTABLE SCALES

For the case of $\kappa = 0$, Pr = 1.0, it was found above that for n real and s real a necessary condition for instability was $N^2 < 0$. We now examine the more general case. From Eq. (11) of TM ERL WPL 134 it was found that

$$\kappa\nu\,\sigma^6 - (\nu + \kappa)\,\sigma^4 s + \sigma^2 s^2 - knu_o'\kappa\sigma^2 + knu_o's + m^2 N^2 = 0 \,. \qquad (8)$$

We consider first the case for which a reversible (oscillatory component) exists. Let n, m and k be real, but let s be complex; say $s = s_r + is_i$. Then the imaginary terms give

$$s_i s_r = \frac{1}{2}\,[(\kappa + \nu)(m^2 + n^2) - kn\,u_o'(m^2 + n^2)^{-1}]\,s_i \,. \qquad (9)$$

We seek the most unstable value of m, for given n, so let

$$\frac{\partial s_r}{\partial m} = 0 = \frac{1}{2}\,(\kappa + \nu)\,2m + \frac{1}{2}\,\frac{kn\,u_o'\,2m}{(m^2 + n^2)^2} - \frac{1}{2}\,\frac{n\,u_o'}{m^2 + n^2}\,\frac{\partial k}{\partial m}$$

or, for the 2 D case, for which $m^2 = k^2$ and $\partial k/\partial m = m/k = 1$,

$$\frac{u_o'}{(\kappa + \nu)\sigma^2} = \frac{2k(n^2 + k^2)}{n(n^2 - k^2)}$$

Substituting in (9), we find

$$s_r^2 = \frac{1}{2}(k^2 + n^2)(\kappa + \nu)\left[\frac{n^2 - 3k^2}{n^2 - k^2}\right]$$

Whence we conclude that $k^2/n^2 = 1/3$ for marginal stability ($s_r = 0$) when s is complex.

On the other hand, when s is pure real we choose the $s_i = 0$ solution of (9) and the condition $\partial s/\partial\sigma = 0$ applied to Eq. (8) gives

$$m^2N^2 = -3\,\kappa\nu\sigma^4 m^2 + 2(\nu + \kappa)\,s\sigma^2 m^2 - m^2 s^2 + \frac{1}{2}\frac{n}{m}u_o'\,\kappa m^2\sigma^2 + knu_o'\kappa m^2 - \frac{1}{2}\,n\,m\,u_o' s\,\frac{\partial k}{\partial m} \tag{10}$$

where $\partial k/\partial m = m/k$. Substituting (10) for m^2N^2 in (8) and letting $s = 0$, we find the neutral stability condition.

Two cases are of particular interest: (A) two dimensional disturbances for which $m = k$, and (B) three dimensional disturbances that are symmetrical in the horizontal plane for which $k^2 = \ell^2 = 1/2\, m^2$. For case B, we find

$$\frac{1}{\sqrt{2}}\frac{u_o'}{\kappa\,Pr\,\sigma^2} = 2\frac{m}{n}\left(1 + \frac{n^2}{m^2}\right)\left(\frac{n^2}{m^2} - 2\right)\left(\frac{n^2}{m^2} - 1\right)^{-1} \equiv \frac{1}{a}\frac{M^2}{\sigma^2} \tag{11}$$

where we have defined the turbulent Prandtl Number $Pr = \nu/\kappa$. For case A the expression is the same except k is substituted for m, and $a = 1$ instead of $a = \sqrt{2}$. $M^2 \equiv u_o'/\kappa\,Pr$ is an inverse length squared. Equation (11) gives the most unstable disturbance geometry (i.e., n/m) for any given wind shear condition. When the shear is zero, it gives the Rayleigh condition $n^2 = 2\,m^2$ for the most unstable geometry of "just unstable" convection in a windless fluid. However, there is a great conceptual difference in the models, since convection filaments and cells are coherent with height through the fluid.

$$\frac{N^2}{\kappa^2 \; Pr \; \sigma^4} = (1 + \frac{n^2}{m^2})[2 \, (\frac{n^2}{m^2} - 2)(\frac{n^2}{m^2} - 1)^{-1} - 1] \qquad (12)$$

which, for $n^2 = 2m^2$, gives the Rayleigh convection criterion of -3.

Because shear is a critical factor in our problem, the Richardson Number is an important criterion. Rewriting (12) in terms of $Ri \equiv N^2/u_o'^2$ and applying the geometric constraint imposed by (11) we find

$$Ri \; Pr = \frac{1}{2} \left[\frac{n}{m} \frac{a\sigma^2}{M^2} - (1 + \frac{n^2}{m^2}) \frac{a^2 \sigma^4}{M^4}\right] . \qquad (13)$$

The most unstable geometry (n/m) for the onset of instability is shown in Fig. 3 as a function of wind shear (u_o'), static stability (N^2) and Richardson's Number (Ri). For the two cases A and B, the quantity "a" is 1 or $\sqrt{2}$ respectively.

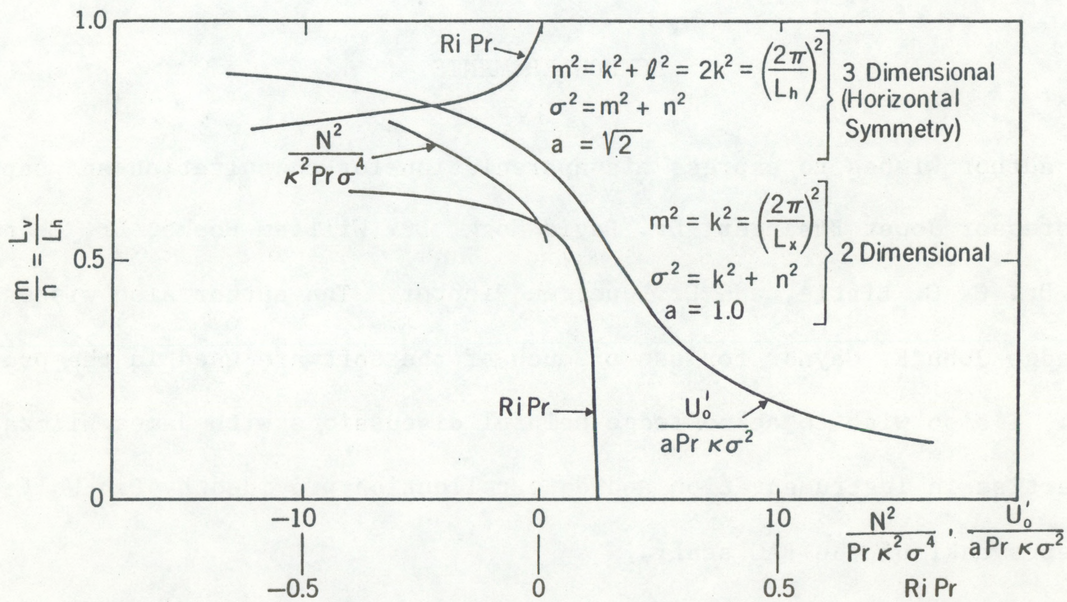

Figure 3. Non-dimensional neutral stability conditions relating geometry (L_v/L_h), to Ri, N^2 and wind shear (u_o'). L_v is the length scale in the vertical direction; L_h is the horizontal length scale.

If (11) is inserted into (13) it is found that

$$Ri\ Pr = \frac{1}{4}\frac{n^2}{m^2}\frac{\frac{n^2}{m^2} - 1}{(1 + \frac{n^2}{m^2})(\frac{n^2}{m^2} - 2)}\left(1 - \frac{1}{2}\frac{\frac{n^2}{m^2} - 1}{\frac{n^2}{m^2} - 2}\right) \tag{14}$$

and, as $m/n \to 0$, we see that

$$Ri\ Pr \to 1/8 \tag{15}$$

in the limit of large positive shear. At the other extreme, as $n/m \to 0$, $Ri\ Pr \to 0$. It is possibly significant that precise measurements of Ri reported by Gossard (1986, Fig. 5) show a minimum substantially less than 1/4--in fact about 1/8 during the time interval 2, zone IV defined in Fig. 2 of that report. Zone IV was a strongly mixed, positively sheared zone shown in Fig. 5 of Gossard et al. (1985).

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