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A Note on the Hydrostatic Equation in the NHEML Two-Dimensional Hurricane Model

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A NOTE ON THE HYDROSTATIC EQUATION IN THE NHEML TWO-DIMENSIONAL HURRICANE MODEL

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The finite difference form of the hydrostatic equation derived for the UCLA (1974) general circulation model is examined. It is shown that the equation used to obtain the geopotential height in the lowest model layer has a large inherent truncation error. A modification of the scheme is suggested in which this height is computed by assuming that the lapse rate between the surface and the adjacent model level is dry adiabatic. The UCLA (1974) thickness formulation is then used to compute the heights at the remaining levels. This scheme yields much more accurate heights throughout the depth of the atmosphere.

1. INTRODUCTION

The development of a new two-dimensional numerical hurricane model is an important part of the Modeling Group research effort at the National Hurricane and Experimental Meteorology Laboratory (NHEML). The vertical differencing scheme is the same as that employed in the new UCLA (1974) general circulation model. In the NHEML two-dimensional hurricane model, geopotential heights were found to be in error by approximately 100 m. The error was further confirmed by comparisons with known heights in idealized atmospheres. In this paper, we illustrate these errors and present a technique that remedies the situation.

2. MODEL STRUCTURE AND PERTINENT EQUATIONS

Figure 1 illustrates a generalization of the model structure proposed by UCLA (1974). At the odd k levels, the horizontal velocity, $\mathbb{V}$, temperature, T, specific humidity, q, and the geopotential height, $\phi = gz$ (g is the gravitational acceleration), are defined. At even k levels, the vertical motion, $\pi\dot{\sigma}$, where $\dot{\sigma} = d\sigma/dt$ (t is time), is defined. The vertical coordinate is

$$\sigma = \frac{p_k - p_{k_I}}{\pi}, \qquad (1)$$

where

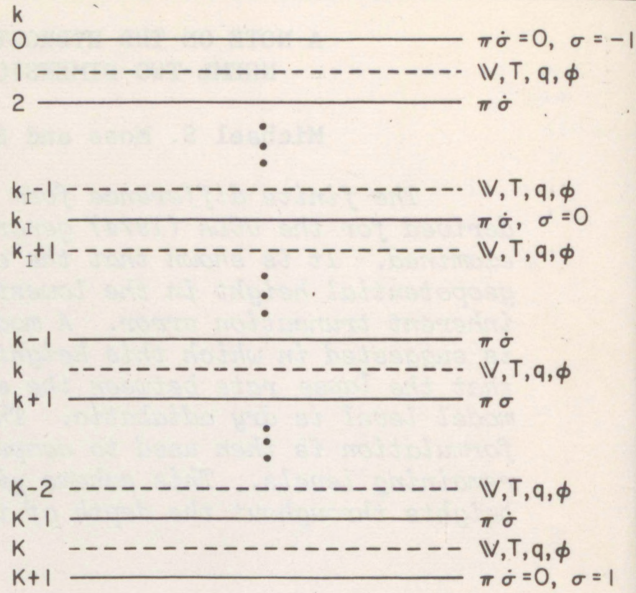

Figure 1. UCLA (1974) generalized model structure.

$$\pi = \begin{cases} \pi_u = p_{k_I} - p_o & \text{for} \quad p_o \le p_k \le p_{k_I} \\ \pi_L = p_{K+1} - p_{k_I} & \text{for} \quad p_{k_I} < p_k \le p_o . \end{cases} \qquad (2)$$

p_k is the pressure at the kth level, π_u is the pressure difference between the level k_I and the top of the model atmosphere (level 0), and π_L is the pressure difference between the surface (level K+1) and k_I (fig. 1). It should be noted that p_{k_I} is a constant pressure and for $p \le p_{k_I}$ the isobaric surfaces are also sigma surfaces. Figure 2 shows the vertical structure of the UCLA (1974) model for $p \ge p_{k_I} = 100$ mb. Figure 3 illustrates a vertical structure similar to that used in the NHEML two-dimensional hurricane model. Here $p_{k_I} = p_o = 0$ ($\pi = \pi_L = p_{K+1} - p_o$) and we are left with the original σ-coordinate system of Phillips (1957).

In the derivation of the finite difference form of the hydrostatic equation for the UCLA model, it is demanded that total energy be conserved under an adiabatic, frictionless process. The resultant equations become (fig. 1)

$$\phi_K = \phi_{K+1} + \sum_{k=k_I+1}^{K}{}' \pi_L \sigma_k \frac{RT_k}{p_k} \Delta\sigma_k \qquad (3)$$

$$-\sum_{k=k_I+1}^{K-2}{}' \sigma_{k+1}\, c_p \hat{\Theta}_{k+1} (p_{k+1}^{\kappa} - p_k^{\kappa})$$

and

$$\phi_k = \phi_{k+2} + c_p (p_{k+2}^{\kappa} - p_k^{\kappa}) \hat{\Theta}_{k+1} \qquad (4)$$

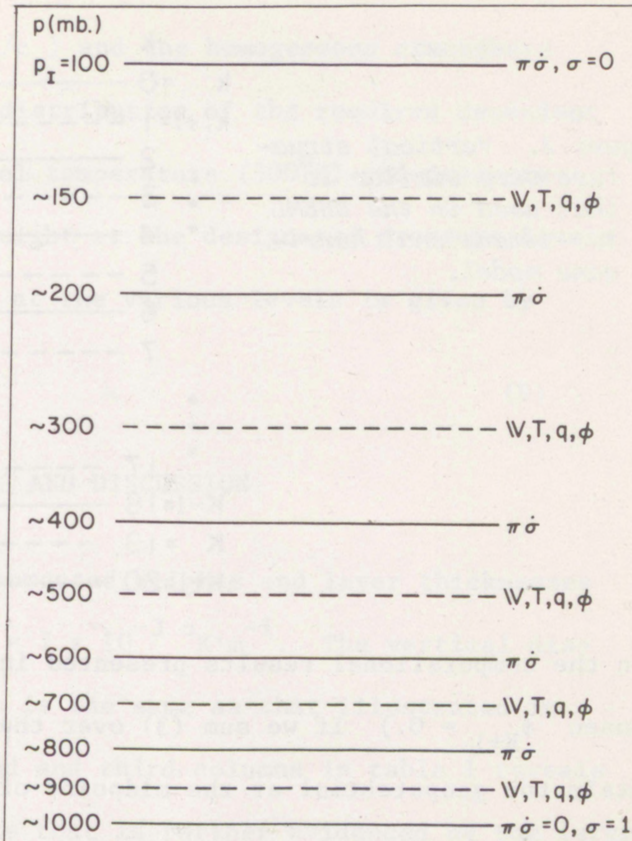

Figure 2. Vertical structure of the lower layers of the UCLA (1974) general circulation model.

where $\sum'$ is summation over odd k, R is the gas constant for dry air, c_p is the specific heat capacity at constant pressure for dry air, $\kappa = R/c_p$,

$$\Delta\sigma_k = \sigma_{k+1} - \sigma_{k-1} \,, \qquad (5)$$

$$\hat{\Theta}_{k+1} = \frac{\ln\Theta_k - \ln\Theta_{k+2}}{\dfrac{1}{\Theta_{k+2}} - \dfrac{1}{\Theta_k}} \,, \qquad (6)$$

and

$$\Theta_k = \frac{1}{p_k^{\kappa}} T_k \,. \qquad (7)$$

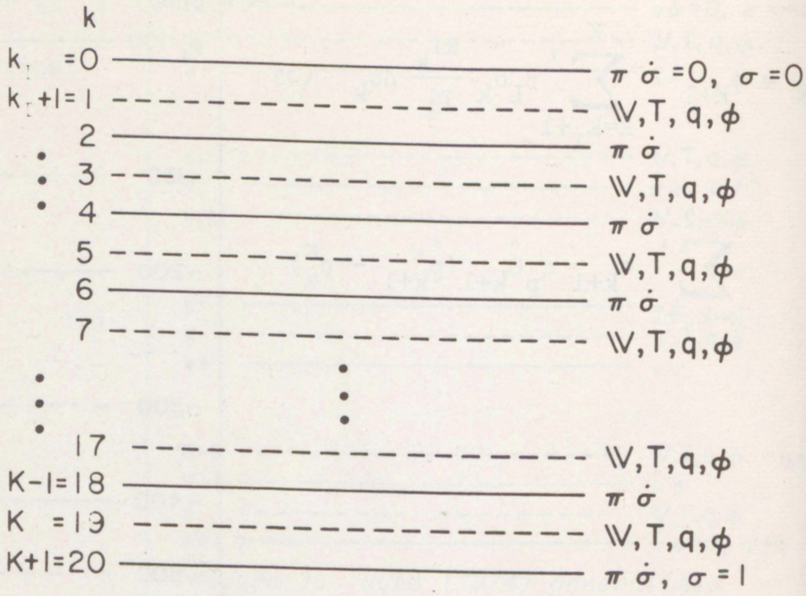

Figure 3. Vertical structure very similar to that used in the NHEML two-dimensional hurricane model.

(In the computational results presented in the following section, we have chosen $\phi_{K+1} = 0.$) If we sum (3) over the depth of the atmosphere, we obtain the geopotential at the midpoint of the lowest model layer. Equation (4) is then used to obtain the thickness of the remaining layers and, with (3), enables us to compute the geopotentials at the remaining levels.

An obvious test of this system is to make comparisons with known heights for an idealized atmosphere. The generality of the constant lapse rate atmosphere makes it practical for this purpose. It can be shown (Hess, 1958) that the height for this atmosphere is given by

$$Z_k = \frac{T_{K+1}}{\gamma} \left\{ 1 - \left(\frac{p_k}{p_{K+1}} \right)^{\frac{R\gamma}{g}} \right\} \tag{8}$$

where $\gamma (= -\partial T/\partial Z)$ is the (constant) lapse rate, T_{K+1} is the sea level temperature and p_{K+1} is the sea level pressure. Some special subsets of the constant lapse rate atmosphere are the isothermal atmosphere ($\gamma = 0$), the dry adiabatic atmosphere ($\gamma = g/c_p$) and the homogeneous atmosphere ($\gamma = g/R$). We obtain the vertical distribution of the required dependent variables by specifying the sea level temperature (300°K) and pressure (1000 mb), and then computing the height at the designated pressure levels from (8). Finally, the temperature at the various levels is given by

$$T_k = T_{K+1} - \gamma Z_k \quad . \tag{9}$$

3. RESULTS AND DISCUSSION

Table 1 lists the actual and computed heights and layer thicknesses for an idealized atmosphere with $\gamma = 7 \times 10^{-3}\ {}^{\circ}K \cdot m^{-1}$. The vertical distribution of the dependent variables is the same as that illustrated in figure 3. A comparison of the second and third columns in table 1 reveals a large error in the computed heights that is further evidenced by the large root-mean-square (RMS) error. On the other hand, a comparison of the fourth and fifth columns along with the corresponding RMS deviation shows that the thickness calculations (4) are quite accurate. Therefore, the error clearly arises from the use of (3) to obtain the geopotential height of the lowest level. Hydrostatic calculations for the UCLA (1974) vertical structure (fig. 2) yield essentially the same results, albeit the RMS height errors are less (table 2).

Table 3 lists the actual and computed individual sums on the right side of (3) for various constant lapse rate atmospheres. The actual values of the sums are given by the following expressions (UCLA Staff Members, 1974):

$$\sum_{k=k_I+1}^{K}{}' \pi_L \sigma_k \frac{RT_k}{p_k} \Delta\sigma_k \equiv \sum_{k=k_I+1}^{K}{}' \phi_k \Delta\sigma_k \tag{10}$$

Table 1. Actual and computed {via equations (3) and (4)} heights at the given atmospheric levels for a constant lapse rate atmosphere where $\gamma = 7 \times 10^{-3}$ $^{o}K \cdot m^{-1}$; also actual and computed thicknesses of the layers

(Vertical structure essentially the same as that of the NHEML two-dimensional hurricane model (fig. 3))

p (mb)	Z_{a_i}: actual height (m)	Z_{c_i}: computed height (m)	Layer (mb)	ΔZ_{a_i}: actual thickness (m)	ΔZ_{c_i}: computed thickness (m)
950	448.3	590.2			
			950 - 850	956.0	956.1
850	1404.3	1546.3			
			850 - 750	1050.1	1050.1
750	2454.4	2596.4			
			750 - 650	1168.1	1168.0
650	3622.5	3764.4			
			650 - 550	1320.8	1321.0
550	4943.3	5085.4			
			550 - 450	1528.1	1528.1
450	6471.4	6613.5			
			450 - 350	1827.1	1827.3
350	8298.5	8440.8			
			350 - 250	2303.4	2303.7
250	10601.9	10744.5			
			250 - 150	3206.9	3208.1
150	13808.8	13952.6			
			150 - 50	5857.7	5867.5
50	19666.5	19820.1			
	RMS = 143.3 m			RMS = 1.19 m	

Table 2. Actual and computed {via equations (3) and (4)} heights at the given atmospheric levels for a constant lapse rate atmosphere where $\gamma = 7 \times 10^{-3}\ ^{o}K \cdot m^{-1}$; also, actual and computed thicknesses of the layers

(Vertical structure same as that of the Arakawa-Mintz (1974) UCLA general circulation model (fig. 2))

p (mb)	Z_{a_i}: actual height (m)	Z_{c_i}: computed height (m)	Layer (mb)	ΔZ_{a_i}: actual thickness (m)	ΔZ_{c_i}: computed thickness (m)
900	915.9	1007.1			
			900 - 700	2106.0	2106.2
700	3021.8	3113.3			
			700 - 500	2655.1	2655.5
500	5676.9	5768.8			
			500 - 300	3696.6	3697.9
300	9373.5	9466.7			
			300 - 150	4435.3	4438.2
150	13808.8	13904.9			
	RMS = 92.8 m			RMS = 1.604 m	

Table 3. Actual and computed individual sums on the right-hand side of equation (3) for various constant lapse rate atmospheres; also differences between the computed geopotential height and the known height at the midpoint of the lowest atmospheric layer

γ ($^{o}K \cdot m^{-1}$)	Abs. differences between known and computed heights at 950 mb (m)	$\sum_{k=k_I+1}^{K} (a)_{actual}$ ($m^2 \cdot sec^{-2}$)	$\sum_{k=k_I+1}^{K} (a)_{computed}$ ($m^2 \cdot sec^{-2}$)	$\frac{\left\lvert \sum_{k=k_I+1}^{K} (a)_{actual} - \sum_{k=k_I+1}^{K} (a)_{computed} \right\rvert}{g}$ (m)	$\sum_{k=k_I+1}^{K} (b)_{actual}$ ($m^2 \cdot sec^{-2}$)	$\sum_{k=k_I+1}^{K} (b)_{computed}$ ($m^2 \cdot sec^{-2}$)	$\frac{\left\lvert \sum_{k=k_I+1}^{K-2} (b)_{actual} - \sum_{k=k_I+1}^{K-2} (b)_{computed} \right\rvert}{g}$ (m)
$4 \cdot 10^{-3}$	195.7	75336.4	77274.9	197.8	70933.3	70954.1	2.1
$7 \cdot 10^{-3}$	141.9	70285.5	71691.5	143.4	65892.2	65906.9	1.5
g/C_p	106.0	66151.2	67190.3	106.0	61766.7	61766.9	0.0
g/R	15.7	43050.0	43050.0	0.0	38746.2	38590.8	15.7

$$(a) = \pi_L \sigma_k \frac{RT_k}{p_k} \Delta\sigma_k$$

$$(b) = \sigma_{k+1} C_p \hat{\theta}_{k+2} \, (p^{\kappa}_{k+2} - p^{\kappa}_{k})$$

and

$$\sum_{k=k_I+1}^{K-2}{}' \sigma_{k+1} c_p \hat{\Theta}_{k+1} (p_{k+2}^{\kappa} - p_k^{\kappa}) \equiv \sum_{k=k_I+1}^{K-2}{}' \sigma_{k+1} (\phi_k - \phi_{k+2}) \tag{11}$$

where the geopotential heights at the given levels are known. Apparent from table 3 is that the sum on the left-hand side of (11) is relatively accurate, whereas the sum on the left-hand side of (10) is less accurate. Even though the percentage error in the latter sum is less than 3 percent for the cases examined here, the sum itself is large; thus it is mainly responsible for the error in the height computed from (3). Figure 4 illustrates the percent error in the height at the lowest level versus the number of levels for an atmosphere with constant $\gamma = 7 \times 10^{-3}\ {}^{o}K \cdot m^{-1}$. Note that the percentage error decreases with an increasing number of levels; hence the error alluded to above is due to truncation. Furthermore, since the sum on the left-hand side of (10) is exact for the homogeneous atmosphere ($\gamma = g/R$), the truncation error arises from the implicit assumption of constant specific volume (RT_k/p_k) over each layer in (10) and (3).

To circumvent the error, we replace (3) with the following expression:

$$\phi_K = \phi_{K+1} + c_p \Theta_K p_{K+1}^{\kappa} \left(1 - \frac{p_K}{p_{K+1}}\right)^{\kappa} \tag{12}$$

in which, now, the lowest geopotential is obtained by extrapolating the temperature dry adiabatically to the surface. Expression (4) is retained to compute the geopotential heights at the remaining levels. The system of equations (12) and (3) is hereafter referred to as the modified UCLA hydrostatics. The assumption of constant Θ between the surface and the

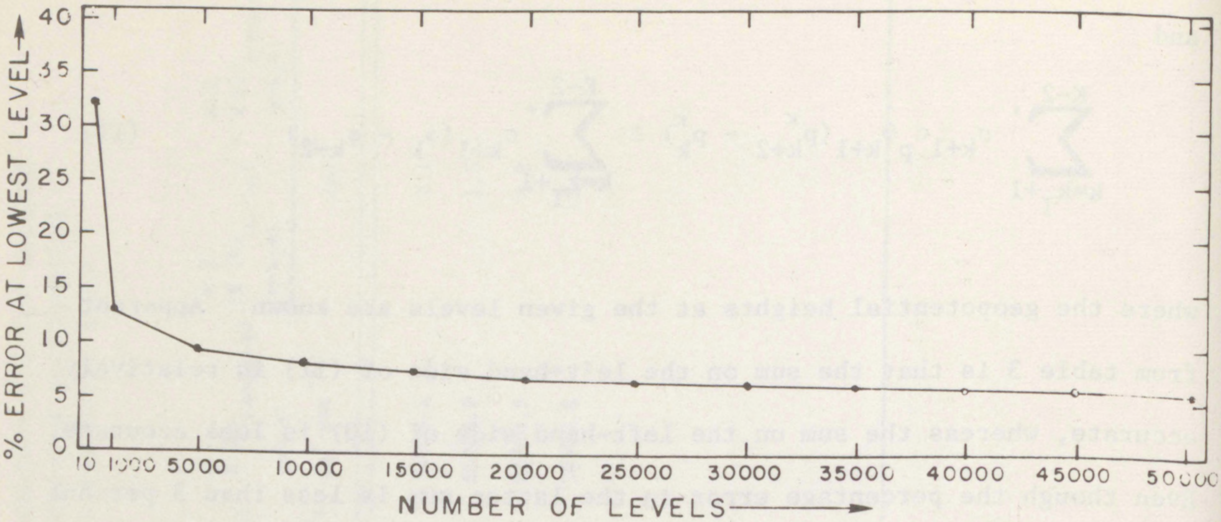

Figure 4. Percent heights error at the lowest applicable level versus the number of levels for an idealized atmosphere with constant $\gamma = 7 \times 10^{-3}$ $^{o}K \cdot m^{-1}$.

adjacent information level is close to physical reality, provided that the latter is within the planetary boundary layer (Deardorff, 1972). The results of the modified UCLA hydrostatic computations, for an atmosphere with constant $\gamma = 7 \times 10^{-3}$ $^{o}K \cdot m^{-1}$, are listed in table 4. It is readily apparent that these computations are a substantial improvement over those obtained using the original UCLA hydrostatics (cf. table 1).

A cause for concern, however, are errors that may arise through the violation of the conservation of total energy invoked in the derivation of (3). Therefore, numerical experiments were performed with an adiabatic inviscid version of our two-dimensional hurricane model with both of these hydrostatic formulations. Some of the model parameters are listed in table 5. The vertical distribution of the dependent variables is exactly the same as that illustrated in figure 3. At the initial instant, the radial and tangential winds are 0, the surface pressure is specified so that it monotonically decreases towards the center, and the potential temperature is uniform on the constant sigma surfaces. From the known distribution of

Table 4. Actual and computed {via equation (12)} heights at given atmospheric levels for a constant lapse rate atmosphere where $\gamma = 7 \times 10^{-3}\ {}^{o}K \cdot m^{-1}$; also, actual and computed thicknesses of the layers

(Vertical structure essentially the same as that of the NHEML two-dimensional hurricane model.)

p (mb)	Z_{a_i} : actual height (m)	Z_{c_i} : computed height (m)	Layer (mb)	ΔZ_{a_i} : actual thickness (m)	ΔZ_{c_i} : computed thickness (m)
950	448.3	449.2			
			950 - 850	956.0	956.1
850	1404.3	1405.3			
			850 - 750	1050.1	1050.1
750	2454.4	2455.4			
			750 - 650	1168.1	1168.0
650	3622.5	3623.5			
			650 - 550	1320.8	1321.0
550	4943.3	4944.4			
			550 - 450	1528.1	1528.1
450	6471.4	6472.5			
			450 - 350	1827.1	1827.3
350	8298.5	8299.8			
			350 - 250	2303.4	2303.7
250	10601.9	10603.5			
			250 - 150	3206.9	3208.1
150	13808.8	13811.6			
			150 - 50	5857.7	5867.5
50	19666.5	19679.1			
	RMS = 4.24 m			RMS = 1.19 m	

surface pressure and potential temperature, the temperature, and then the geopotential height, are calculated. The time integration is carried out according to the Matsuno (1966) scheme.

Table 5. Some model parameters used in the NHEML two-dimensional hurricane model

Parameter	Value
Time increment	30 sec
Radial increment	20 km
Radial limit of computational domain	400 km

The results of the experiments, for a 24-hour integration, are illustrated in figure 5 where we have plotted the percentage change in time of the total energy (E_t) from the initial total energy (E_o). Here,

$$E = \int_o^{r_{max}} \int_o^1 \pi(\tfrac{1}{2}|\mathbb{V}|^2 + \phi + c_p T)\,d\sigma\, dr \tag{13}$$

where r_{max} is the radial limit of the computational domain. The total energy loss beginning at about 1 hour in both experiments is caused by the Matsuno (1966) time integration scheme that damps the high speed gravity waves generated by the initial wind-pressure imbalance. Clearly, there is little difference in the conservation of total energy between the two techniques over the 24-hour (2880 time step) period.

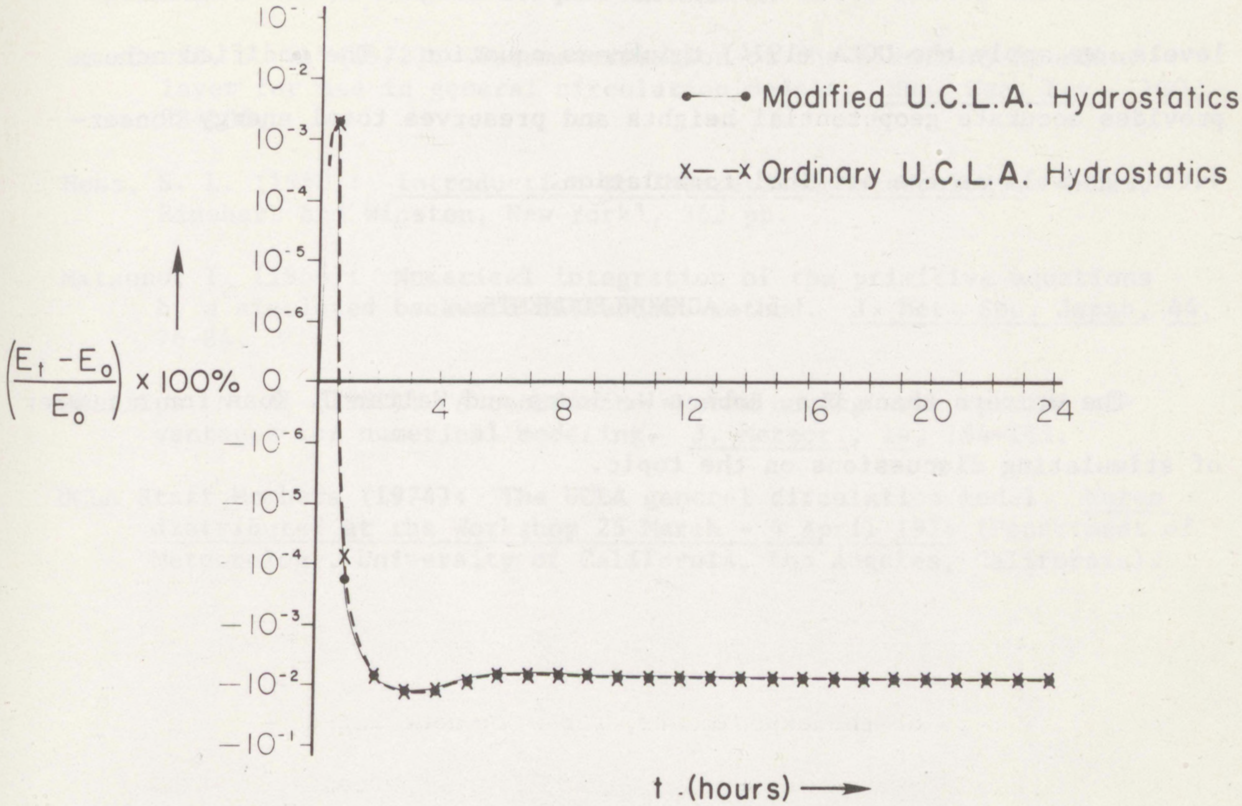

Figure 5. Percent change in the total energy from the initial value as a function of time for the modified and ordinary UCLA (1974) hydrostatics.

4. SUMMARY

The application of the recently proposed (total energy conserving) UCLA (1974) hydrostatic formulation produces erroneous geopotential heights. The error is due to truncation in the equation used to obtain the geopotential height of the lowest model information level. Furthermore, the source of the truncation error itself arises from the implicit assumption of constant density over the individual model layers. To correct the error, we replace the above equation with the assumption that the lapse rate in the lowest half-layer of the model is dry adiabatic. We, therëfore, avoid summing over the entire model depth of the model and integrate only from the earth's surface to the lowest model level. To compute heights at the remaining levels, we apply the UCLA (1974) thickness equation. The modified scheme provides accurate geopotential heights and preserves total energy conservation as well as the original formulation.

5. ACKNOWLEDGMENTS

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