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Extended Free Net Adjustment Constraints

Haim B. Papo

Rockville, MD

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CONTENTS

Abstract 1

1. Introduction 1

2. Theoretical model 2

3. Two applications 7

4. Numerical example in 2-D 9

5. Conclusions 11

Acknowledgments 11

Appendix A. Skew reference systems 12

Appendix B. Iterations of the extended free net adjustment solution 14

References 16

EXTENDED FREE NET ADJUSTMENT CONSTRAINTS[1]

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ABSTRACT. The contribution of geodetic measurements to the establishment of a control network can be partitioned into global and local (individual) components. The global component epitomized in a number of geometrically meaningful parameters can be estimated together with the individual point coordinates. The additional rank defect created by the extension of the parameter list is corrected by free net adjustment constraints that are extended beyond those needed for a solution of the network datum problem. Two applications of extended free net adjustment are outlined and illustrated by elementary numerical examples. A non-Cartesian (skew) reference system discussed in appendix A provides an exotic interpretation of the estimated global and individual parameters. Iterations of the extended free net adjustment are treated in appendix B, featuring two distinctly different sets of preliminary values of the parameters.

1. INTRODUCTION

Free net adjustment techniques play a major role in the analysis of geodetic networks. Optimal error propagation properties combined with a meaningful and unique datum that is established without interfering with the inner geometry of the network (minimal constraints) have made this method of analysis extremely popular. Free net adjustment has been employed extensively in deformation analysis as well as in 4-D (time dependent) analysis of geodetic networks. Thus far, however, in all its forms and variations it has been used exclusively as a means for solving the inherent datum problem of the geodetic network. A line has been drawn (Wolf 1977, 1978) beyond which free net adjustment constraints have been considered inapplicable. Such a line, real or imaginary, is highly challenging. Recent studies in 4-D analysis of geodetic networks by this author have renewed interest in extending the application of free net adjustment constraints beyond the datum-defect-solution barrier.

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[2]Prepared during a grant period (September 1984 through February 1985) while serving as a Visiting Senior Scientist in Geodesy, National Research Council, National Academy of Sciences, Washington, D.C.

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Attention has been drawn by Wolf (1978) to the so-called smearing effect in adjustment computations where due to inadequate modeling a deterministic part of the measurements is "smeared" onto the residuals. A generalization of the same idea can be used as a starting point for analyzing the contribution of geodetic measurements to the establishment of control networks. It is well known that measured distances, for example, contribute implicitly to datum definition (scale) of the network. We may be interested in having an explicit quantitative estimate of that contribution where conventionally the datum content of the measurements is "smeared" onto the control point coordinates. Another example where the conventional approach is inadequate is encountered in 4-D analysis of geodetic networks. It is often necessary to define datum in 4-D by minimizing only the irregular part of the velocities of the reference points while obtaining parameter estimates of the systematic part. The above and other examples can be treated effectively by extending the conventional free net adjustment technique. The solution as proposed in this paper is one of minimal constraints with complete freedom in selecting the function to be minimized for a specific datum definition.

2. THEORETICAL MODEL

Consider a set of n measurements made with the objective of obtaining the estimate of u parameters. The measurements are expressed as a function of the parameters through a mathematical model, which when linearized around preliminary values of the parameters results in a system of observation equations:

$$L + V = C \cdot W \tag{1}$$

where C is the design matrix, W represents corrections to the approximate values of the parameters W°, L is the vector of differences between observed and computed measurements, and V is the vector of measurement corrections.

If the measurements consist of distances, elevation differences, angles, or azimuths (but not coordinates), and the parameters are coordinates of points in a network, it is well known that the u parameters are nonestimable due to datum defect of the system of observation equations (Meissl 1982; Pope 1971).

We make the following simplifying assumptions:

The geometry of the measurements is such that there are no configuration defects in addition to the datum defect (Welsch 1979).

The number of measurements is redundant with respect to the number of estimable parameters, i.e., the inequality $n>(u-d)$ is satisfied, where d is the size of the datum defect.

As a means of analyzing the contribution of the measurements the vector W is partitioned into global and local (individual) components through the introduction of a vector of parameters Y that represents contribution of the measurements to the global component of the point coordinates. The elements of Y are conceived as the parameters of a transformation ω (a mapping of X onto W):

$$\omega(Y^{a}) : (X^{a}) \rightarrow (W^{a}) \tag{2}$$

where

$$X^a = X^o + X \; ; \quad W^a = W^o + W \quad \text{and} \quad Y^a = Y^o + Y.$$

The mapping ω is general (Koch and Papo 1985), restricted only by the condition that $D=\partial W/\partial X$ has to be a square, full rank matrix. In the special case of a linear mapping, called also isomorphism, the vectors X and W are similar in size and nomenclature (Wolf 1978; Shilov 1980). Strang (1977) defines the two isomorphic spaces (W and X) as different and yet identical for "all algebraic purposes." X represents point-coordinate corrections where the global content of the measurements has been withheld. An important characteristic of the parameters Y is their complete irrelevance to the actual datum defect of the system of observation equations. (See discussion following equation (5').) Their number (f) is limited by the inequality: $f<(u-d)$. Additional properties of Y are discussed below.

The observation eqs. (1) are rewritten now in terms of the corrections (X,Y) to the respective approximate values (X^o,Y^o). Those values (X^o,Y^o) serve for the initial (zeroth) iteration of the solution (Pope 1972). Subsequently the adjusted parameters X^a and Y^a are substituted for X^o and Y^o, respectively, and the solution is iterated until convergence.

$$L + V = C(D,F) \begin{vmatrix} X \\ Y \end{vmatrix} = (A,B) \begin{vmatrix} X \\ Y \end{vmatrix} \tag{1'}$$

where

$D(Y^o) = \partial W/\partial X$ is a u by u full rank matrix.

$F(X^o) = \partial W/\partial Y$ is a u by f matrix of full column rank. This implies that the elements of Y are independent.

The rank of (D,F) is u. However, the rank of (A,B) in eq. (1') is only u-d (same as the column rank of C). This means that the number of estimable parameters in the extended system is still u-d. The size of the null space of (A,B) is thus d+f, which means that d+f linear conditions between the parameters are necessary to obtain a minimally constrained solution.

The (1') system is partitioned while paying attention to the size and nature of its defects

$$\underset{d}{A_0 \; X_0} + \underset{f}{A_{10} \; X_{10}} + \underset{u-d-f}{A_{11} \; X_{11}} + \underset{f}{B \; Y} = L + V \tag{1''}$$

The A_0 matrix, which pertains to X_0, is selected so that the remaining part of A is of full (u-d) rank (Perelmuter 1979). The partitioning process is continued by selecting now A_{10} so that the remaining (A_{11},B) is of full rank. The four matrices A, (A,B), (A_{10},A_{11}), and (A_{11},B) span the same u-d linear space where only the last two are of full rank. The matrix A_{11} which is of full column rank spans a (u-d-f) dimensional subspace of (A,B). It follows then that A_{10} and B span the same f-dimensional linear subspace of (A,B). We shall see in the following sections that it is fairly easy to perform the above partitionings by following simple geometric considerations.

A trivial minimally constrained solution is obtained by setting to zero $\bar{X}_0$ and $\bar{X}_{10}$ and then solving for $\bar{X}_{11}$ and $\bar{Y}$. The partitioning of the (1') system into (1"), however, is not unique. Each set of A_0, A_{10}, A_{11} would produce slightly different parameter estimates $(\bar{X},\bar{Y})$ due to change of base (Pope 1971) and due to errors in the measurements. Transformation from one minimally constrained solution into another is obtained as follows (Wolf 1977):

$$\begin{aligned} \bar{\bar{X}} &= \bar{X} + D^{-1} E p + R q \\ \bar{\bar{Y}} &= \bar{Y} \qquad\qquad\quad + q \end{aligned} \tag{3}$$

where
- p is a vector of d datum transformation parameters,
- q is a vector of f variations in the Y parameters,
- E is a u by d matrix known as Helmert's matrix. Its columns span the null space of C.
- R is a u by f matrix of full column rank which represents an apparent functional relationship between X and Y. However R is not a conventional matrix of partial derivatives as X and Y are independent by definition. The linear dependency is enforced only as means of guaranteeing the invariance of V in spite of the introduction of the additional parameters (Y).

Note that p and q are small quantities of the order of the measurement errors. They are linearly independent which means that $[D^{-1}E, R]$ is a full rank (d+f) matrix.

For both $\bar{X},\bar{Y}$ and $\bar{\bar{X}},\bar{\bar{Y}}$ to be minimally constrained solutions, V in (1') has to remain invariant under the above transformation (3). Substitute first $\bar{X},\bar{Y}$ and then $\bar{\bar{X}},\bar{\bar{Y}}$ (eq. (3)) into the right-hand side of (1') and equate:

$$(A,B) \begin{vmatrix} \bar{X} \\ \bar{Y} \end{vmatrix} = (A,B) \begin{vmatrix} \bar{X} \\ \bar{Y} \end{vmatrix} + (A\ B) \begin{vmatrix} D^{-1}E, & R \\ 0, & I \end{vmatrix} \begin{vmatrix} p \\ q \end{vmatrix} \tag{4}$$

Equation (4) holds for an arbitrary nonzero $\begin{vmatrix} p \\ q \end{vmatrix}$ only if

$$(A,B)\begin{vmatrix} D^{-1}E, & R \\ 0, & I \end{vmatrix} = (0,0) \qquad (5)$$

which results in

$$A \quad D^{-1} \; E = C \; D \; D^{-1} \; E = C \; E = 0 \qquad (6)$$

and

$$C \quad D \quad R + C \; F = C \; (D \; R + F) = 0. \qquad (6')$$

Equation (6) is well known and identifies E as a basis of the null space of the C matrix. A solution of equation (6'), which also complies with the stated properties of p and q, leads to: $R=-D^{-1}F$. Equation (5) is written again:

$$(A,B)\begin{vmatrix} D^{-1} & (E,-F) \\ & (0,\ I) \end{vmatrix} = (0,0) \qquad (5')$$

where $D^{-1}(E,-F)$ is a matrix of full (d+f) rank. The linear independence of E and F means that Y cannot substitute for the datum parameters.

Equation (5') signifies the fact that $\begin{vmatrix} D^{-1} & (E,-F) \\ & (0,\ I) \end{vmatrix}$ is a basis of the null space of the (A,B) matrix (Koch and Papo 1985).

Of all the minimally constrained solutions defined through eq. (3) there is only one which satisfies the following minimum condition:

$$\hat{X}^T \quad \hat{X} = \min \qquad (7)$$

where

$$\hat{X} = X^a - X^r = X + \Delta X \; ; \; \Delta X = X^\circ - X^r.$$

Here X^r is a set of preliminary values of X^a which unlike X° is kept fixed throughout the iterations of the solution. (See appendix B.) It provides a stable (fixed) basis for the datum of the free-net solution.

Equation (7) is differentiated with respect to $\begin{vmatrix} p \\ q \end{vmatrix}$ resulting in:

$$\begin{aligned} D^{-1} \; E^T \hat{X} &= 0 \\ D^{-1}(-F^T)\hat{X} &= 0 \end{aligned} \qquad (8)$$

Equations (8) represent d+f independent linear conditions between the $\hat{X}$ parameters which can correct the defects of the system. The first in (8) is the well known free-net adjustment constraints equation which corrects the datum defect. The second equation in (8) constitutes an extension of the free-net constraints. It corrects the additional defect caused by the introduction of Y. Equations (8) are written again with $H = (E,-F)(D^{-1})^T$, also substituting $X+\Delta X$ for $\hat{X}$:

$$H^T \hat{X} = 0 = H^T X + H^T \Delta X \tag{8'}$$

and is used as a basis for defining the linear relationship between (X_0^T, X_{10}^T) and X_{11}^T

$$\begin{vmatrix} X_0 \\ X_{10} \end{vmatrix} = G_{11}^T X_{11} + (-I, G_{11}^T) \Delta X \tag{9}$$

where $\quad G_{11}^T = -(H_0^T, H_{10}^T)^{-1} H_{11}^T.$

It can be shown that due to the particular pattern of partitioning the square nonsymmetric matrix (H_0^T, H_{10}^T) is of full rank and has a regular inverse (Pope 1971). Equation 9 is substituted in (1") resulting in the following full rank system. See also Papo and Perelmuter (1983).

$$\left| A_{11} + (A_0, A_{10}) G_{11}^T \right| X_{11} + B \ Y = L + V \tag{1'''}$$

According to appendix B, L is defined as: $L = L^b - L^\circ - (A_0, A_{10}) (-I, G_{11}^T) \Delta X$. An estimate of X_{11} and Y can be obtained now to be followed by evaluation of X_0 and X_{10} from eq. (9).

The condition to be satisfied by $\hat{X}$ can be defined also differently as follows (Wolf 1977):

$$\hat{X}^T P_X \hat{X} = \min \tag{7'}$$

where P_X is a general, symmetric, positive-semidefinite matrix. As an example we may set: $P_X = D^T D$. Matrix $D(Y^\circ)$ can be evaluated (eq. (1')) from estimates of Y^a

obtained at the preceding iteration of the solution of normal equations. Such a choice of P_X implies the minimization of $\hat{W}^T \hat{W}$ instead of $\hat{X}^T \hat{X}$. Depending on the particular properties of X, P_X may assume the characteristics of an autocovariance matrix (Pope, personal communication, 1985) and bring us to the realm of collocation. See also Hein and Kisterman (1981).

3. TWO APPLICATIONS

The datum defect (d) of an observational system depends on the dimension of the space and on the measurement types. Certain types of measurements have the potential for a dual contribution to the computation of a network as shown by Pope (personal communication, 1985) and also in Papo (1985). They contribute in determining the relative positions of the points and also in defining the datum of the network. Examples of such measurements are distances, azimuths, and elevation differences (in a 3-D net).

We denote by e the maximum datum defect of a system in i-dimensional (i-D) space where $e=2,4,7$ for $i=1,2,3$. In general we would have: $d<e$ due to one or more of the above datum defining measurements.

In the first application we seek to control the contribution of those measurements to the adjustment of a network by holding back their datum definition content. As an example we consider distances measured in 2-D space. Their datum content is scale. The linear mapping ($\omega(Y)$) is simple:

$$W = Y X \quad \text{where} \quad Y = s \tag{10}$$

The distinction between W and X is that in X the datum content of the distances has been withheld while W contain the complete contribution of the measured distances. In the above 2-D case we have:

$$d = 3 \quad ; \quad f = 1 \quad ; \quad d + f = e = 4$$

$$D = I \text{ as well as } A = C \text{ due to } Y^o = 1$$

$$H^T = \left| \begin{array}{ccc|c|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ -y_1 & x_1 & -y_2 & x_2 & -y_3 & x_3 & -y_4 & \cdots \\ \hline -x_1 & -y_1 & -x_2 & -y_2 & -x_3 & -y_3 & -x_4 & \cdots \end{array} \right| = \left| \begin{array}{c} E^T \\ \\ -F^T \end{array} \right| \tag{11}$$

$$\qquad H_0^T \qquad\qquad H_{10}^T \qquad\qquad H_{11}^T$$

Note that H is equivalent to E of a 2-D system with a maximum ($e=4$) datum defect. Overconstraining ($e>d$) is avoided by the introduction of $Y=s$ as an unknown parameter. The estimated Y constitutes the contribution of the measured distances to defining the datum of the network.

The geometric considerations which assisted us in partitioning the X, A and H matrices are the same as one would use when selecting a basis for a trivial minimally constrained solution:

The $X_0 = \begin{vmatrix} x_1 \\ y_1 \\ x_2 \end{vmatrix}$ coordinate corrections set to zero define a datum, i.e., origin and orientation. The additional defect for scale (when s is defined as unknown), is corrected by defining $X_{10} = y_2$ and then setting it to zero.

In the second application we define ω as a 2-D linear transformation of X (affine-symmetric):

$$\begin{vmatrix} u \\ v \end{vmatrix}_i = \begin{vmatrix} g_1 & g_3 \\ g_3 & g_2 \end{vmatrix} \quad \begin{vmatrix} x \\ y \end{vmatrix}_i = \begin{vmatrix} x & 0 & y \\ 0 & y & x \end{vmatrix}_i \begin{vmatrix} g_1 \\ g_2 \\ g_3 \end{vmatrix} \; ; \; Y = \begin{vmatrix} g_1 \\ g_2 \\ g_3 \end{vmatrix} \; ; \; Y^0 = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \qquad (12)$$

In this case we are interested in holding back the homogeneous deformation signal contained in the measured distances. As before, W are coordinates based on the complete contribution of the measurements while X are based on the same measurements whose global deformation content, however, has been withheld. In the present case we have:

$$d = 3 \quad ; \quad f = 3 \quad ; \quad d + f = 6 > e = 4$$

$$H^T = \left|\begin{array}{ccc|ccc|c} 1 & 0 & 1 & 0 & 1 & 0 & 1 \;\cdots\cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \;\cdots\cdots \\ -y_1 & x_1 & -y_2 & x_2 & -y_3 & x_3 & -y_4 \;\cdots\cdots \\ \hline -x_1 & 0 & -x_2 & 0 & -x_3 & 0 & -x_4 \;\cdots\cdots \\ 0 & -y_1 & 0 & -y_2 & 0 & -y_3 & 0 \;\cdots\cdots \\ -y_1 & -x_1 & -y_2 & -x_2 & -y_3 & -x_3 & -y_4 \;\cdots\cdots \end{array}\right| = \begin{vmatrix} E^T \\ \\ -F^T \end{vmatrix} \qquad (13)$$

$$H_0^T \qquad\qquad H_{10}^T \qquad\qquad H_{11}^T$$

The estimated Y parameters represent the homogeneous deformation content of the measured distances. An exotic interpretation of X and Y is discussed in appendix A.

The geometric rationale in planning the partitioning is as follows:

$$X_0 = \begin{vmatrix} x_1 \\ y_1 \\ x_2 \end{vmatrix} \quad ; \quad X_{10} = \begin{vmatrix} y_2 \\ x_3 \\ y_3 \end{vmatrix}$$

As in the first case, setting X_0 to zero can define the datum. The deformation pattern of the network is defined (fixed) by setting to zero the X_{I0} corrections. In fact any three non-collinear points in the 2-D network could provide a basis for minimally constraining the solution.

4. NUMERICAL EXAMPLE IN 2-D

Let us have an elementary 2-D plane network composed of four points. Their preliminary and simulated (true) Cartesian coordinates are shown in figure 1 and are also given in table 1.

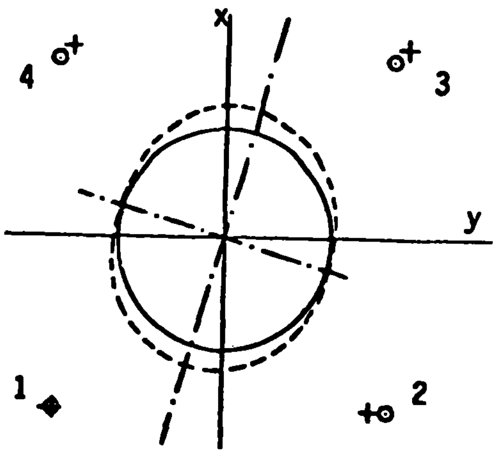

Figure 1. A 2-D network.

Table 1.--Coordinates in a 2-D network

	(0) Preliminary (Fixed) x	y	(+) True x	y
1	-10	-10	-10	-10
2	-10	10	-10	9
3	10	10	11	11
4	10	-10	11	-9

All six distances have been measured as shown in the first two rows of table 2. ε are the simulated (true) errors in the sense "measured minus ε equal true." For simplicity the weight matrix was assumed to be a (6 by 6) unit matrix.

Table 2.--Measurements, errors, and corrections

i-j	1-2	1-3	1-4	2-3	2-4	3-4
ε	0.0000	0.2858	-0.0238	-0.0950	0.3257	0.0000
V	.1216	-.1801	.1273	.1281	-.1681	.1155
L	-1.	1.7	1.	1.	-.3	.0

The fundamental "O" solution was made to serve as a reference. This was a standard free net adjustment solution with d = 3: (2) for origin and (1) for orientation. It may be helpful to regard this solution as a special case with $f = 0$, i.e., without any additional parameters Y. The last two rows in table 2 show, respectively, the estimated corrections to the measured distances after three iterations of the solution and the initial (zeroth iteration) differences between observed and computed distances.

Solutions "I" and "II" made use of the extended free net adjustment method following the formulation given in section 3. Table 3 presents the results of all three solutions. The corrections V remained invariant as expected of minimally constrained solutions. Two to three iterations were necessary to converge to five significant digits in V, X, and Y.

Table 3.--Results of "0", "I", and "II" solutions

Parameters	Preliminary value	Estimated corrections "0"	"I"	"II"
x_1	-10	-0.9148	-0.6923	-0.0165
y_1	-10	.0943	.2962	.2604
x_2	-10	-.1953	.0125	.0166
y_2	10	- .7976	- .9852	- .2612
x_3	10	.8986	.6765	- .0157
y_3	10	.4036	.1915	.2483
x_4	10	.2115	.0033	.0157
y_4	-10	.2998	.4975	- .2475
s	1		.0208	
g_1	1			.0555
g_2	1			- .0191
g_3	1			.0351
$X^T X$		2.6251	2.2797	0.2600

From the value of s in solution "I" we learn that the distances define a scale for the network which is larger by about 2 percent as compared to scale defined by the preliminary coordinates.

The interpretation of the results of solution "II" is more involved. The estimated linear transformation matrix (see eq. 12) represents the deformation pattern of the network as reflected in the measurements versus the geometry of the preliminary coordinates. The circle (preliminary) and the ellipse of distortion shown in figure 1 illustrate the results of solution "II". The major axis of the ellipse is inclined with respect to the x axis by 21.°65. The maximum and minimum scale factors are 1.070 and 0.967, respectively. For an exotic interpretation of "II", see appendix A.

5. CONCLUSIONS

Geodesy of this past decade has been dominated by the emergence and establishment of observational systems of a growing variety and redundancy in their potential for determining geodetic networks of global extent. Conflicts and apparent discrepancies between the datum content of different observational systems that are discussed in the geodetic literature can be analyzed and effectively controlled by the application of the extended free net adjustment approach. It should be noted that no information is lost by the proposed method of adjustment. Following the inspection, evaluation, and eventual approval of the Y parameters, the transformation ω can be performed on the X coordinates in order to obtain the conventional W. It appears that extended free net adjustment constraints can provide yet another solution to the "smearing" problem in geodetic networks which is discussed in Wolf (1978).

In 4-D analysis of geodetic networks there are several prospective applications of the proposed method. With the inevitable improvement in measurement accuracy, a stage is reached where no point in the network can be considered as devoid of relative motion. In leveling we have had this situation for some time. If the relative motion of all points in a network is significantly different from zero, it becomes practically impossible to define criteria for selecting a subset of stable reference points. The extended free net adjustment approach coupled with a modified and flexible statistical null-hypothesis procedure as proposed by Pope (private communication, 1985), and also shown in Koch (1984), is bound to tackle the above problem and provide an acceptable solution.

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APPENDIX A.--SKEW REFERENCE SYSTEMS

This appendix will attempt to give an exotic interpretation of the solution of an observation equations system under extended free net adjustment constraints.

Let us have spatial distance measurements made in 3-D space with the objective of estimating positional coordinates of a network of points. The 3-D reference system in which those coordinates are to be defined is not a conventional Cartesian coordinate system. The basic triad of the reference system is not orthonormal: its axes are mutually nonorthogonal and scale is different along each axis. We will denote it as a skew coordinate system.

The position of a point P in the skew system is defined by three coordinates (as is usual in 3-D). The position vector of point P ($\vec{OP}$) is evaluated as follows (O is the point of origin):

$$\vec{OP} = x \; \vec{i} + y \; \vec{j} + z \; \vec{k}$$

where the three basis "unit" vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ are of different lengths.

The datum in such a coordinate system is defined by e = 12 parameters:

three	for origin
three	for orientation
three	for scale
three	for nonorthogonality of axes

Spatial distances (measured in 3-D space) are known to reduce the datum defect of scale in a Cartesian system. In a skew reference system, measured distances of a minimum of six mutually nonparallel lines, which are also not parallel to the same plane (no configuration defects), would reduce the three scale and three nonorthogonality defects bringing the remaining defects to six, the same as in a Cartesian system. In a 3-D skew reference system there are no conventional geodetic measurements which are devoid of any datum content. As is well known, however, in a 3-D Cartesian system the measurement of spatial angles does not contribute to datum definition.

Following the examples in section 3 we define Y as the estimable datum parameters of a skew coordinate system. In a 3-D observational system of spatial distances we have:

$$d = 6, \quad e = 12, \quad \text{and} \quad f = e - d = 6$$

The elements of Y are thus the six estimable datum parameters: three for scale and three for nonorthogonality as follows:

$$Y = \begin{vmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{vmatrix} \quad \begin{array}{l} \text{scale in x} \\ \text{scale in y} \\ \text{scale in z} \\ \text{nonorthogonality between y and z} \\ \text{nonorthogonality between x and z} \\ \text{nonorthogonality between x and y} \end{array}$$

The mapping ω is the linear (symmetric) transformation formula:

$$\begin{vmatrix} u \\ v \\ w \end{vmatrix}_i = \begin{vmatrix} g_1 & g_6 & g_5 \\ g_6 & g_2 & g_4 \\ g_5 & g_4 & g_3 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix}_i = \begin{vmatrix} x & 0 & 0 & 0 & z & y \\ 0 & y & 0 & z & 0 & x \\ 0 & 0 & z & y & x & 0 \end{vmatrix}_i \quad Y = F_i\, Y$$

where $(x,y,z)_i$ are the coordinates of point P_i in a skew coordinate system that is defined in scale and nonorthogonality of axes by the elements of Y. The elements of Y are estimable from least squares processing of the measurements. The remaining six datum parameters for origin and orientation are nonestimable. The remaining defect is corrected by six linear constraints (for example free net adjustment constraints: $E^T \hat{X} = 0$). Note that for the initial (zeroth) iteration $Y^{\circ T} = (1,1,1,0,0,0)$ and so $D = I$.

The H matrix in 3-D has the following structure (Papo 1985):

$$H = (E,-F) = \left|\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & z_1 & -y_1 & -x_1 & 0 & 0 & 0 & -z_1 & -y_1 \\ 0 & 1 & 0 & -z_1 & 0 & x_1 & 0 & -y_1 & 0 & -z_1 & 0 & -x_1 \\ 0 & 0 & 1 & y_1 & -x_1 & 0 & 0 & 0 & -z_1 & -y_1 & -x_1 & 0 \\ \hline 1 & 0 & 0 & 0 & z_2 & -y_2 & -x_2 & 0 & 0 & 0 & -z_2 & -y_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right| \begin{array}{c} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \end{array}$$

origin orientation scale nonorthogonality

H in 2-D is obtained from the above H by deleting columns and rows which contain or pertain to z_i.

Appendix A concludes by examining the results of the numerical example in section 4. The X coordinates of solution "II" are defined in a 2-D skew reference system. The parameters of that system are derived from the estimated Y as shown below. Units of scale along the two axes (lengths of the "unit" vectors) in terms of the scale implied by the preliminary coordinates are:

$$s_x = 1 + g_1 = 1.05550$$

$$s_y = 1 + g_2 = 0.98093.$$

The angle θ between the x and y axes is computed from g_3 as follows:

$$\theta = \arccos\,(2\ g_3) = 85.^\circ 97.$$

It can be shown that the network obtained by plotting the adjusted X coordinates in the above skew coordinate system is geometrically equivalent to the one obtained by plotting W of the same points in a Cartesian system.

APPENDIX B.--ITERATIONS OF THE EXTENDED FREE-NET ADJUSTMENT SOLUTION

The theory of free net adjustment is usually presented without considering the need for iterating the solution. In most cases, however, the mathematical model of the measurements is considerably nonlinear and so a single solution (zeroth iteration) is usually inadequate.

In the case of a full rank observation equation system our only concern is to avoid Pope's "pitfalls" (Pope 1972). Assume for simplicity that the nonlinear mathematical model is such that the observables can be expressed as an explicit function of the parameters X^a and Y^a. Linearization is performed about a set of approximate values $X°$ and $Y°$ which are chosen "close" to X^a and Y^a so that the second-and higher-order terms in the Taylor's expansion would come out small and could be neglected. Each solution: $\begin{vmatrix} X^a \\ Y^a \end{vmatrix} = \begin{vmatrix} X° \\ Y° \end{vmatrix} + \begin{vmatrix} X \\ Y \end{vmatrix}$ is used as a basis for another iteration where the new $\begin{vmatrix} X° \\ Y° \end{vmatrix}$ is taken equal to the previous $\begin{vmatrix} X^a \\ Y^a \end{vmatrix}$. "Well-defined" problems converge usually after a few iterations to a stable $\begin{vmatrix} X^a \\ Y^a \end{vmatrix}$ as $\begin{vmatrix} X \\ Y \end{vmatrix}$ approaches zero.

In case of a rank-deficient system (due to the need for datum definition) there is a preliminary set of parameter values X^r which serves as a basis for datum definition (Papo and Perelmuter 1985). The datum of X^r is transferred to the adjusted X^a through the following condition:

$$(X^a - X^r)^T \quad (X^a - X^r) = \min. \tag{B.1}$$

Note that X^r is fixed, unlike $X°$ which changes with each iteration of the solution.

Begin with the observation equations (1'):

$$V + L^b - L° = (A\ ,\ B) \begin{vmatrix} X \\ Y \end{vmatrix} \tag{1'}$$

where

$$X = X^a - X^\circ \quad \text{and} \quad Y = Y^a - Y^\circ .$$

Helmert's condition (B.1) is transformed into d+f extended free net adjustment constraints as follows:

$$H^T \hat{X} = 0 \tag{B.2}$$

where

$$\hat{X} = X + \Delta X \quad \text{and} \quad \Delta X = X^\circ - X^r.$$

Substitute for $\hat{X}$ in (B.2) and partition as in (1") with the result:

$$H^T \hat{X} = 0 = (H_0^T, H_{10}^T) \begin{vmatrix} X_0 \\ X_{10} \end{vmatrix} + H_{11}^T X_{11} + (H_0^T, H_{10}^T, H_{11}^T) \begin{vmatrix} \Delta X_0 \\ \Delta X_{10} \\ \Delta X_{11} \end{vmatrix} \tag{B.2'}$$

from which the following linear relationship is derived:

$$\begin{vmatrix} X_0 \\ X_{10} \end{vmatrix} = G_{11}^T X_{11} + (-I, G_{11}^T) \Delta X \tag{B.3}$$

where $G_{11}^T = -(H_0^T, H_{10}^T)^{-1} H_{11}^T$ as defined in equation (9).

Substitute (B.3) into (1') and regroup with the result:

$$V + L^b - L^\circ + (A_0, A_{10}) \left\{ \begin{vmatrix} X_0 \\ X_{10} \end{vmatrix} - G_{11}^T \Delta X_{11} \right\} = \left\{ \begin{vmatrix} A_{11} + (A_0, A_{10}) G_{11}^T \end{vmatrix}, B \right\} \begin{vmatrix} X_{11} \\ Y \end{vmatrix} \tag{B.4}$$

which is written in compact form as equation (1"')

$$V + L = (\bar{A}_{11}, B) \begin{vmatrix} X_{11} \\ Y \end{vmatrix}. \tag{1"'}$$

As stated in Papo and Perelmuter (1985) the difference $\Delta X = X^a - X^r$ can assume any magnitude without impairing the linearization of the mathematical model of the measurements.

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