NOAA Technical Report NOS 81

Formulas for Positioning At Sea by Circular, Hyperbolic, and Astronomic Methods

Rockville, Md.
February 1980
Revised July 1980

U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
National Ocean Survey

NOAA TECHNICAL REPORTS

National Ocean Survey Series

Survey (NOS) provides charts and related information for the safe navigation of marine and air commerce. s other Earth science data—from geodetic, hydrographic, oceanographic, gravimetric, and astronomic stigations, and measurements—to protect life and property and to meet the needs of engineering, scientific, ndustrial interests.

e reports deal with new practices and techniques, the views expressed are those of the authors and do not survey policy. NOS series NOAA Technical Reports is a continuation of, and retains the consecutive he former series, Environmental Science Services Administration (ESSA) Technical Reports Coast and and the earlier series, C&GS Technical Bulletins.

llowing reports is available from the National Technical Information Service (NTIS), U.S. Department of Commerce, Sills Bldg., 5285 Port Royal Road, Springfield, VA 22161. (NTIS accession numbers given in parentheses.) Write to source for price.

NOAA TECHNICAL REPORTS

NOS 45 Pacific SEAMAP 1961–70 Data for Area 15524-10: Longitude 155°W to 165°W, Latitude 24°N to 30°N, bathymetry, magnetics, and gravity. J. J. Dowling, E. E. Chiburis, P. Dehlinger, and M. J. Yellin, January 1972. (COM-72-51029)

NOS 46 Pacific SEAMAP 1961-70 Data for Area 15530-10: Longitude 155°W to 165°W, Latitude 30°N to 36°N, bathymetry, magnetics, and gravity. J. J. Dowling, E. F. Chiburis, P. Dehlinger, and M. J. Yellin, January 1972. (COM-73-50145)

NOS 47 Pacific SEAMAP 1961–70 Data for Area 15248-14: Longitude 152°W to 166°W, Latitude 48°N to 54°N, bathymetry, magnetics, and gravity. J. J. Dowling, E. F. Chiburis, P. Dehlinger, and M. J. Yellin, April 1972. (COM-72-51030)

NOS 48 Pacific SEAMAP 1961–70 Data for Area 16648-14: Longitude 166°W to 180°W, Latitude 48°N to 54°N, bathymetry, magnteics, and gravity. J. J. Dowling, E. F. Chiburis, P. Dehlinger, and M. J. Yellin, April 1972. (COM-72-51028)

NOS 49 Pacific SEAMAP 1961-70 Data for Areas 16530-10 and 17530-10: Longitude 165° W to 180°W, Latitude 30°N to 36°N, bathymetry, magnetics, and gravity. E. F. Chiburis, J. J. Dowling, P. Dehlinger, and M. J. Yellin, July 1972. (COM-73-50173)

NOS 50 Pacific SEAMAP 1961–70 Data for Areas 16524-10 and 17524-10: Longitude 165°W to 180°W, Latitude 24°N to 30°N, bathymetry, magnetics, and gravity. E. F. Chiburis, J. J. Dowling, P. Dehlinger, and M. J. Yellin, July 1972. (COM-73-50172)

NOS 51 Pacific SEAMAP 1961–70 Data for Areas 15636-12, 15642-12, 16836-12, and 16842-12: Longitude 156°W to 180°W, Latitude 36°N to 48°N, bathymetry, magnetics, and gravity. E. F. Chiburis, J. J. Dowling, P. Dehlinger, and M. J. Yellin, July 1972. (COM-73-50280)

NOS 52 Pacific SEAMAP 1961–70 data evaluation summary. P. Dehlinger, E. F. Chiburis, and J. J. Dowling, July 1972. (COM-73-50110)

NOS 53 Grid calibration by coordinate transfer. Lawrence Fritz, December 1972. (COM-73-50240)

NOS 54 A cross-coupling computer for the Oceanographer's Askania Gravity Meter. Carl A. Peterson and Thomas E. Brown, February 1973. (COM-73-50317)

NOS 55 A mathematical model for the simulation of a photogrammetric camera using stellar control. Chester C. Slama, December 1972. (COM-73-50171)

NOS 56 Cholesky factorization and matrix inversion. Erwin Schmid, March 1973. (COM-73-50486)

NOS 57 Complete comparator calibration. Lawrence W. Fritz, July 1973. (COM-74-50229)

NOS 58 Telemetering hydrographic tide gauge. Charles W. Iseley, July 1973. (COM-74-50001)

NOS 59 Gravity gradients at satellite altitudes. B. Chovitz, J. Lucas, and F. Morrison, November 1973. (COM-74-50231)

NOS 60 The reduction of photographic plate measurements for satellite triangulation. Anna-Mary Bush, June 1973. (COM-73-50749)

NOS 61 Radiation pressure on a spheroidal satellite. James R. Lucas, July 1974. (COM-74-51195/AS)

NOS 62 Earth's gravity field and station coordinates from Doppler Data, Satellite Triangulation, and Gravity Anomalies. Karl-Rudolf Koch, February 1974. (COM-74-50490/AS)

NOS 63 World maps on the August epicycloidal conformal projection. Edwin Schmid, May 1974. (COM-74-11746/AS)

NOS 64 Variability of tidal datums and accuracy in determining datums from short series of observations. Robert Lawrence Swanson, October 1974. (COM-75-10275)

NOS 65 NGS-1. The statistics of residuals and the detection of outliers. Allen J. Pope, May 1976, 133 pp. (PB258428)

NOS 66 NGS-2. Effect of geoceiver observations upon the classical triangulation network. Robert E. Moose, and Soren W. Henriksen, June 1976, 65 pp. (PB260921)

NOS 67 NGS-3. Algorithms for computing the geopotential using a simple-layer density model. Foster Morrison, March 1977, 41 pp. (PB266967)

NOS 68 NGS-4. Test results of first-order class III leveling. Charles T. Whalen and Emery Balaze, November 1976, 30 pp. (PB265421)

NOS 69 Tidal hydrodynamics in the Strait Juan de Fuca—Strait of Georgia. Bruce B. Parker, January 1977, 56 pp. (PB270191)

NOS 70 NGS-5. Selenocentric geodetic reference system. Frederick J. Doyle, A. A. Elassal, and J. R. Lucas, February 1977, 53 pp. (PB266046)

NOS 71 NGS-6. Application of digital filtering to satellite geodesy. C. C. Goad, May 1977, 73 pp. (PB270192)

NOS 72 NGS-7. Systems for the determination of polar motion. Soren W. Henriksen, May 1977, 55 pp. (PB274698/AS)

NOS 73 NGS-8. Control leveling. Charles T. Whalen, May 1978, 23 pp. (PB286838)

NOS 74 NGS-9. Survey of the McDonald Observatory radial line scheme by relative lateration techniques. William E. Carter and T. Vincenty, June 1978, 33 pp. (PB287427)

NOS 75 NGS-10. An algorithm to compute the eigenvectors of a symmetric matrix. E. Schmid, August 1978, 5 pp. (PB287923)

NOS 76 NGS-11. The application of multiquadric equations and point mass anomaly models to crustal movement studies. Rolland L. Hardy, November 1978, 63 pp. (PB293544)

NOS 77 Numerical simulation of sedimentation and circulation in rectangular marina basins. David R. Askren, January 1979, 137 pp. (PB294107)

(Continued on inside back cover)

NOAA Technical Report NOS 81

Formulas for Positioning At Sea by Circular, Hyperbolic, and Astronomic Methods

James Collins

Coastal Mapping Division
Rockville, Md.
Revised July 1980

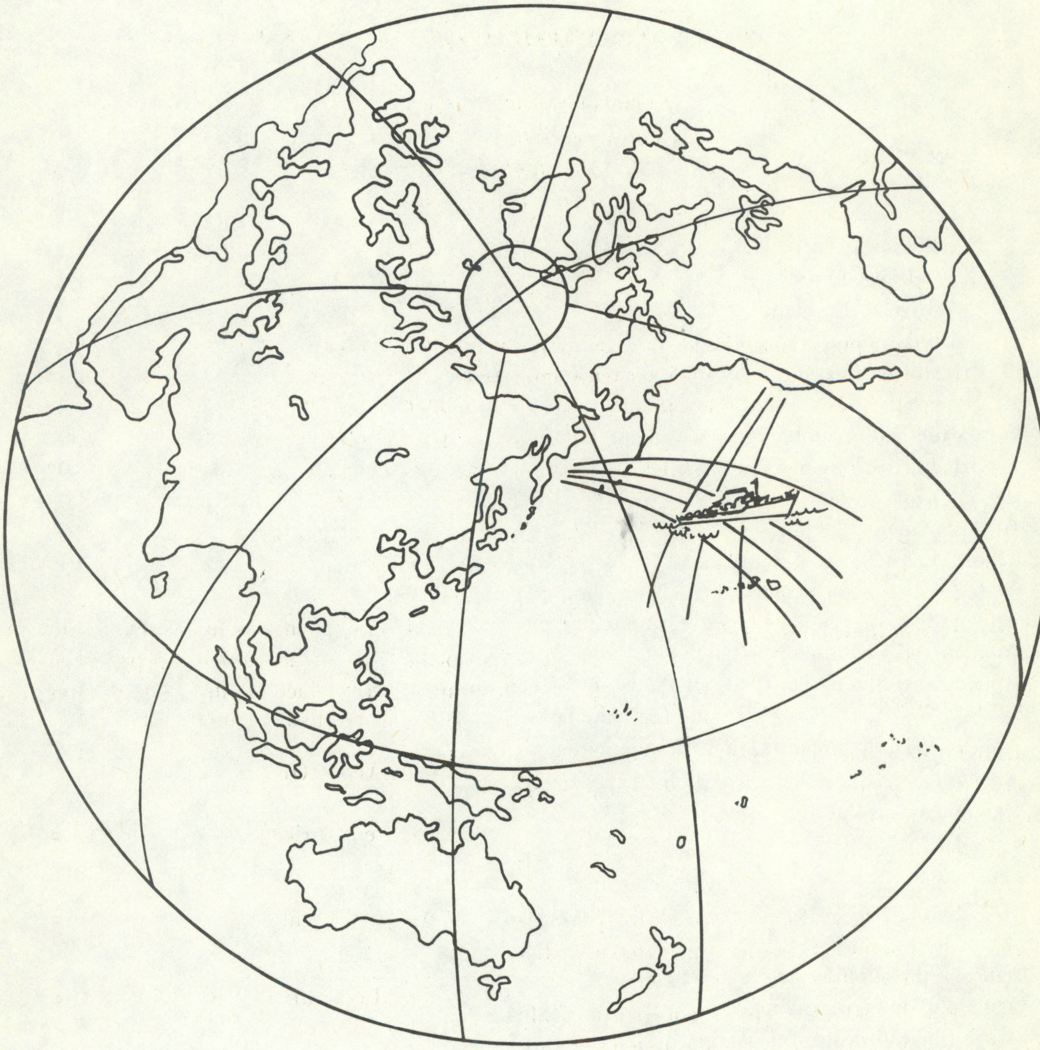
U.S. DEPARTMENT OF COMMERCE
Philip M. Klutznick, Secretary

National Oceanic and Atmospheric Administration
Richard A. Frank, Administrator

National Ocean Survey
Herbert R. Lippold, Director

CONTENTS

FIGURES



POSITIONING AT SEA

FIGURE 1.

Formulas for Positioning at Sea by Circular, Hyperbolic, and Astronomic Methods

Commander James Collins, NOAA
Chief, Coastal Mapping Division
NOAA, National Ocean Survey
Rockville, Maryland

ABSTRACT

Today's positioning of a vessel is accomplished by using either hyperbolic (LORAN), circular (Raydist), celestial, or satellite navigation systems. The use of these systems with onboard computers often has created problems for the navigator and computer programer who is faced with a variety of formulas for computing ships' positions. This report is an attempt to clarify and simplify the formulas and procedures required to correctly compute the position of a vessel on the high seas.

1. INTRODUCTION

The sequence in which material will be presented is:

- Computation of a long line inverse distance.
- Computation of positions from circular or range-range positioning systems.
- Computation of positions from hyperbolic positioning systems.
- Sight reduction and computation of celestial positions;
- Solution of positioning problem using any combination of the above three systems.

The last item, where the navigator wishes to use a celestial fix in combination with electronic navigation systems, is presented so that the navigator will have maximum flexibility in using all his positioning information.

Before giving the formulas, a brief discussion of the figure of the Earth (ellipsoid) is presented for readers who are unfamiliar with the various terms.

2. THE ELLIPSOID

2.1 General

All computations are performed on an earth ellipsoid. Figure 2A shows this ellipsoid and its associated parameters. The difference in longitude is the angular difference (in the plane of the Equator) between planes containing the meridians of the surface points. Some common parameters of the ellipsoid that will be used are:

SQUARES OF:

First eccentricity $$e^2 = \frac{a^2 - b^2}{a^2}$$

Second eccentricity $$e'^2 = \frac{a^2 - b^2}{b^2}$$

Third eccentricity $$m = \frac{a^2 - b^2}{a^2 + b^2}$$

Fourth eccentricity $$e_0{}^2 = \frac{a^2 - b^2}{ab}$$

2.2 Reduced Latitude

Geodetic or geographic latitude (ϕ, phi) is the latitude generally given for points and represents the angle between the equatorial plane and the normal (or perpendicular) at the point. The reduced and geocentric latitudes (β, beta, and ψ, psi) are mathematical quantities given by the relations:

$$\tan \beta = \frac{b}{a} \tan \phi \quad \text{and} \quad \tan \psi = \frac{b^2}{a^2} \tan \phi$$

The geodetic and geocentric latitude is shown in figure 2B.

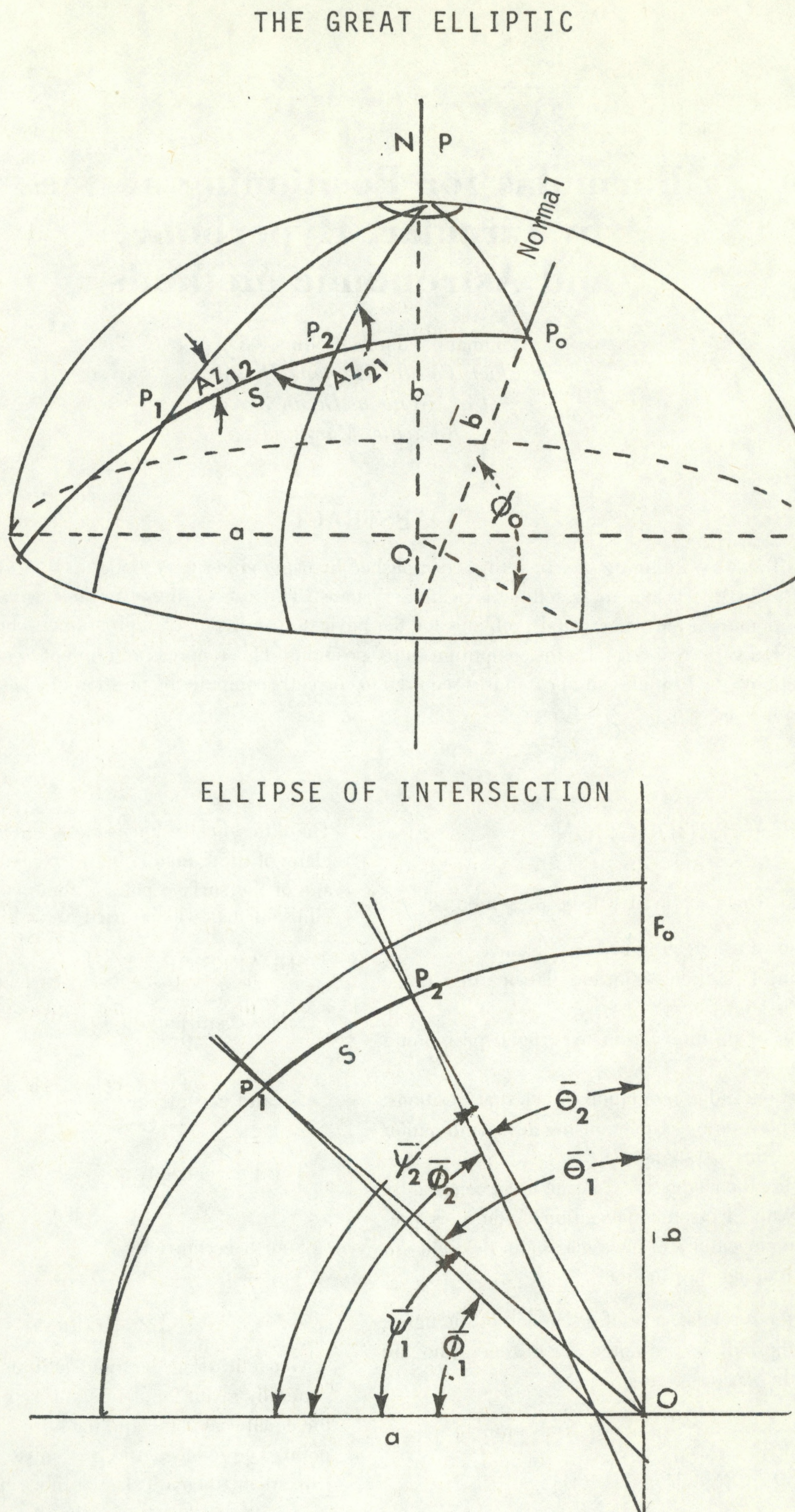

THE GREAT ELLIPTIC
N P
Normal
P2
Po
P1
AZ12
AZ21
S
b
b̄
Øo
a
O
ELLIPSE OF INTERSECTION
Fo
P2
P1
S
Θ̄2
Θ̄1
Ψ̄2
Φ̄2
Ψ̄1
Φ̄1
b̄
a
O

FIGURE 2.

The reduced and geocentric latitudes will be used in inverse computations since they simplify most expressions. The result of any position computation, however, will be geodetic or geographic latitude.

2.3 Surface Lines

Since the Earth is represented by an ellipsoid of rotation, any plane that cuts this ellipsoid will produce an ellipse. For example, a plane cutting the ellipsoid through the Equator will produce an ellipse of intersection with zero eccentricity (or circle). A plane cutting the ellipsoid through the poles will produce an ellipse with the maximum eccentricity (e). Planes cutting the ellipsoid at angles other than zero or 90 degrees with the Equator will produce an ellipse of intersection with an eccentricity between zero and e.

Plane curvature (curved in one plane only) surface lines on the ellipsoid result from cutting the ellipsoid with a plane. To calculate the distance between two points on an ellipsoid, these plane curve surface lines may be used to approximate the geodetic line or "geodesic" which is the shortest line between the points. Two plane curves that are often used to represent surface distances between points on the ellipsoid are the normal section and the great elliptic.

2.3.1 The Normal Section

The normal section is formed by cutting the ellipsoid with a plane that contains the normal (see figure 2) of one point, and goes through the second point. This plane curve is the most frequently used, and a full discussion of its characteristics can be found in Bomford (1962) and many other texts.

2.3.2 The Great Elliptic

The great elliptic is formed by cutting the ellipsoid with a plane that contains the center of the ellipsoid and two surface points. Since it is the easiest geometric curve on the ellipsoid to compute and since difference between the geodesic, normal section, and great elliptic are rather small, this curve will be used to compute geodetic distances on the ellipsoid in this report. Computing great elliptic distance between points is analogous to computing great circle distances on a sphere. Great elliptic azimuths (Az_{12}, fig. 2) are also similar to azimuths computed from great circle computations. For a more detailed explanation the reader is referred to Bomford (1962) and Collins (1971).

2.4 Inverse Problem

The computation of the distance and azimuth of a line between two points is called the inverse solution. The formulas for solution of the inverse distance by great elliptic formulas are given below:

Given: Point one (P_1) with Latitude ϕ_1, and Longitude λ_1

Point two (P_2) with Latitude ϕ_2, and Longitude λ_2

Find: The surface distance from P_1 to P_2 (S), the forward azimuth from P_1 to P_2 (Az_{12}), and the back azimuth from P_2 to P_1 (Az_{21}).

The forward azimuth is given by:

$$\cot Az_{12} = \frac{[\tan \psi_2 - \tan \psi_1 \cos \Delta\lambda] \cos \psi_1}{\sin \Delta\lambda}$$

and the reverse or back azimuth by:

$$\cot Az_{21} = \frac{[\tan \psi_1 - \tan \psi_2 \cos (-\Delta\lambda)] \cos \psi_2}{\sin (-\Delta\lambda)}$$

where

$$\tan \psi_1 = \frac{b^2}{a^2} \tan \phi_1 \quad \text{and} \quad \tan \psi_2 = \frac{b^2}{a^2} \tan \phi_2$$

and, $\Delta\lambda = \lambda_2 - \lambda_1$.

The distance from P_1 to P_2 (S) is a portion of the ellipse of intersection and can be found by first determining the geocentric latitudes of points 1 and 2 on this ellipse of intersection ($\bar{\psi}_i$). (See figure 2.)

The geocentric latitudes on the ellipse of intersection are given by:

$$\tan \bar{\psi}_1 = \frac{\tan \psi_1}{\cos Az_{12}}$$

and

$$\tan \bar{\psi}_2 = \frac{\tan \psi_2}{\cos Az_{21}}$$

The distance (S) can now be found by the simple calculation of the elliptic integral over the portion of the ellipse between P_1 and P_2.

The ellipse of intersection is of dimensions a and $\bar{b}$ where:

$$\bar{b} = b\sqrt{1 + e'^2 \cos^2 \beta_0} \quad \text{Note: } e'^2 = \frac{a^2 - b^2}{b^2}$$

and

$$\tan^2 \beta_0 = \frac{a^2}{b^2}\left[\frac{\tan^2 \psi_1 + \tan^2 \psi_2 - 2 \tan \psi_1 \tan \psi_2 \cos \Delta\lambda}{\sin^2 \Delta\lambda}\right]$$

The best procedure for finding the distance (S) is to compute distance S_1 (point P_1 to P_0) and S_2 (point P_2 to P_0) and subtract S_2 from S_1:

thus $$S = S_1 - S_2$$

The reason for our using this procedure is that when long distances are computed, the quantity S_2 becomes positive and S_1 and S_2 are automatically added to give the correct length of S.

There are many ways to evaluate the elliptic integral, such as repetitive incremental summing, the spiral approximation, and the classical series expansion and step-by-step integration. Since this last method is the most widely accepted, it will be used here.

The formulas for this computation according to Gan'Shin (1969) are

$$S = R_0[\psi - A \sin 2\psi - B \sin 4\psi \cdots]$$

where

$$R_0 = \sqrt{\frac{1}{2}(a^2 + b^2)}\left(1 - \frac{m^2}{16}\right)$$

and

$$m = \frac{a^2 - b^2}{a^2 + b^2}.$$

ψ is expressed in radians.

To evaluate the distances S_1 and S_2, the polar angles instead of geocentric latitudes are used.

let

$$\bar{\theta}_1 = 90° - \bar{\psi}_1 \quad \text{and} \quad \bar{\theta}_2 = 90° - \bar{\psi}_2$$

The distance formula now becomes:

$$S_1 = R_0[\bar{\theta}_1 + A \sin 2\bar{\theta}_1 - B \sin 4\bar{\theta}_1 \cdots]$$

$$S_2 = R_0[\bar{\theta}_2 + A \sin 2\bar{\theta}_2 - B \sin 4\bar{\theta}_2 \cdots]$$

and

$$S = [S_1 - S_2].$$

Since we are evaluating distances of a small ellipse, the third eccentricity will be:

$$\bar{m} = \frac{a^2 - \bar{b}^2}{a^2 + \bar{b}^2}$$

where

$$\bar{b} = b\sqrt{1 + e'^2 \cos^2 \beta_0}$$

and the average radius of the intersection ellipse is:

$$R_0 = \sqrt{\frac{1}{2}(a^2 + \bar{b}^2)}\left(1 - \frac{\bar{m}^2}{16}\right)$$

The constants A and B now become:

$$A = \frac{\bar{m}}{4} + \frac{21}{128}\bar{m}^3 + \cdots$$

$$B = \frac{\bar{m}^2}{64} + \cdots$$

In summary, the steps necessary to solve the inverse problem are:

(where $i = 1, 2$ and $j = 2, 1$)

Step	*Operation*	*Comment*
1	$\Delta\lambda = \lambda_j - \lambda_i$	Find difference in longitude.
2	$\tan \psi_i = \frac{b^2}{a^2} \tan \phi_i$	Find geocentric latitudes for both points.
3	$\tan Az_{ij} = \frac{\sin \Delta\lambda}{(\tan \psi_j - \tan \psi_i \cos \Delta\lambda) \cos \psi_i}$ NOTE: $Az_{21};\Delta\lambda = \lambda_1 - \lambda_2$	Compute forward and back azimuths. (Use arctan function that returns azimuth in all four quadrants.)
4	$\theta_i = \arctan \frac{\cos Az_{ij}}{\tan \psi_i}$	Compute polar angles in plane of intersecting ellipse.
5	$\beta_0 = \arctan \frac{a}{b} \frac{\sqrt{\tan^2 \psi_i + \tan^2 \psi_j - 2 \tan \psi_i \tan \psi_j \text{ cso } \Delta\lambda}}{\sin \Delta\lambda}$	Compute maximum reduced latitude of great elliptic
6	$\bar{b} = b\sqrt{1 + e'^2 \cos^2 \beta_0}$ where: $e'^2 = \frac{a^2 - b^2}{b^2}$	Compute semiminor axis of ellipse of intersection.
7	$\bar{m} = \frac{a^2 - \bar{b}^2}{a^2 + \bar{b}^2}$	Compute third eccentricity of ellipse of intersection by second expression.
8	$R_0 = \sqrt{\frac{1}{2}(a^2 + \bar{b}^2)}\left[1 - \frac{\bar{m}^2}{16}\right]$	Compute average radius of ellipse of intersection.
9	$A = \frac{\bar{m}}{4}$	Compute constants for evaluating elliptic integral.
10	$S_i = R_0[\bar{\theta}_i - A \sin 2\bar{\theta}_i]$	Compute S_1 and S_2 ($i = 1,2$) using truncated formula.
11	$S = [S_1 + S_2]$	Add S_2 to S_1 to get S (absolute value).

This noniterative algorithm will give great elliptic distances to millimeter accuracies for all lengths of line. If the B and m^3 terms are disregarded, the precision of 1 meter can be obtained in the computations, which should be sufficient for most positioning problems.

2.5 Programing Considerations

The formulas for computing great elliptic distances given here are general and can be easily programed. If sufficient regard is given to signs of the angles, the distance and azimuths can be obtained directly without resorting to computer logic. It is especially helpful to use an arctangent function that returns values in all four quadrants. (Evaluate numerator and denominator separately.) An example of this type of function is the rectangular to polar coordinate conversion used by Hewlett-Packard handheld calculators.

For example, if the azimuths are evaluated from a function that returns azimuth in all four quadrants, the correct azimuths from 0 to 360 degrees will be obtained. The sign ($\pm$) of the cosines of these azimuths will then determine whether the polar angle (θ) (fig. 2) is positive or negative. A negative value of θ will result in a negative distance for S_1 or S_2. If one of these distances is negative and the other positive, the distances will sum to give the difference between the values which for the particular case is the correct value of S. (See examples, Appendix A.) If line end points are in different hemispheres, the distance is found by subtracting S from (ΠR_0) (the meridianal length).

3. CIRCULAR OR RANGE-RANGE METHODS

The circular positioning method, where the vessels range to two shore stations, will be the first method considered; however, the approach will work equally well with hyperbolic or circular hyperbolic system. This method of computing positions is iterative in that an assumed position is used to begin the computation. A deduced reconing or (D.R.) position should, in most cases, be adequate so that only one computational cycle is necessary.

Referring to figure 3: let the deduced reconing position be the origin of a coordinate system. The inversed distances to circular electronic control station 1 and 2 will be S_1 and S_2, and the azimuths will be Az_1 and Az_2, respectively. The differences between the inversed distances and the measured distances (dial reading converted to meters) will be:

$$\Delta S_1 = S'_1 - S_1$$

$$\Delta S_2 = S'_2 - S_2$$

where the primed values are the observed values.

The intersection of the dashed lines of figure 3 thus represents the true position of the vessel. The equations of these dashed lines in slope intercept form will be:

$$Y = p_i X + d_i$$

where $d_i = \Delta S_i / \cos Az_i \qquad i = 1, 2, \ldots$ station 1, station 2

$$p_i = -\tan Az_i$$

Since there are two equations in two unknowns, a solution is possible. The solution will be presented later.

4. HYPERBOLIC SYSTEMS

Hyperbolic position systems such as LORAN and Hi-Fix essentially provide difference in distances to two slave stations and one or two master stations.

Figure 4 shows typical hyperbolic geometry. Where N is an integer indicating a lane reading, v the propagation velocity, and f the comparison frequency.

$$N_1 = \frac{f_1}{v}[S_{01} + S_0 - S_1]$$

and

$$N_2 = \frac{f_2}{v}[S_{02} + S_0 - S_2]$$

where ℓ' is the lane width at the base line

$$\ell' = \frac{v}{2f}$$

By definition, "the direction of a hyperbola at any point coincides with the bisector of the angle formed by lines joining the point to a pair of stations", Laurila (1976). Thus the azimuth of the L hyperbola at P is:

$$Az'_2 = Az_2 + (1/2)(Az_0 - Az_2)$$

and the azimuth of the R hyperbola at P is:

$$Az'_1 = Az_0 + (1/2)(Az_1 - Az_0)$$

Let P be a D.R. position with coordinates θ and λ. By computing inverses from this assumed point to all three stations (master and two slave), the azimuths of the hyperbolas and the theoretical lane readings at the assumed point can be computed (N_2 & N_1).

Figure 5 shows the geometry at assumed point P. The dashed lines represent the observed lane counts (N'_2 & N'_1). The differences between the observed and computed values can now be formed.

$$\Delta N_1 = N'_1 - N_1$$

$$\Delta N_2 = N'_2 - N_2$$

The differences can be converted into meters by the lane width at point $P(\ell)$:

CIRCULAR

COMPUTED POSITION

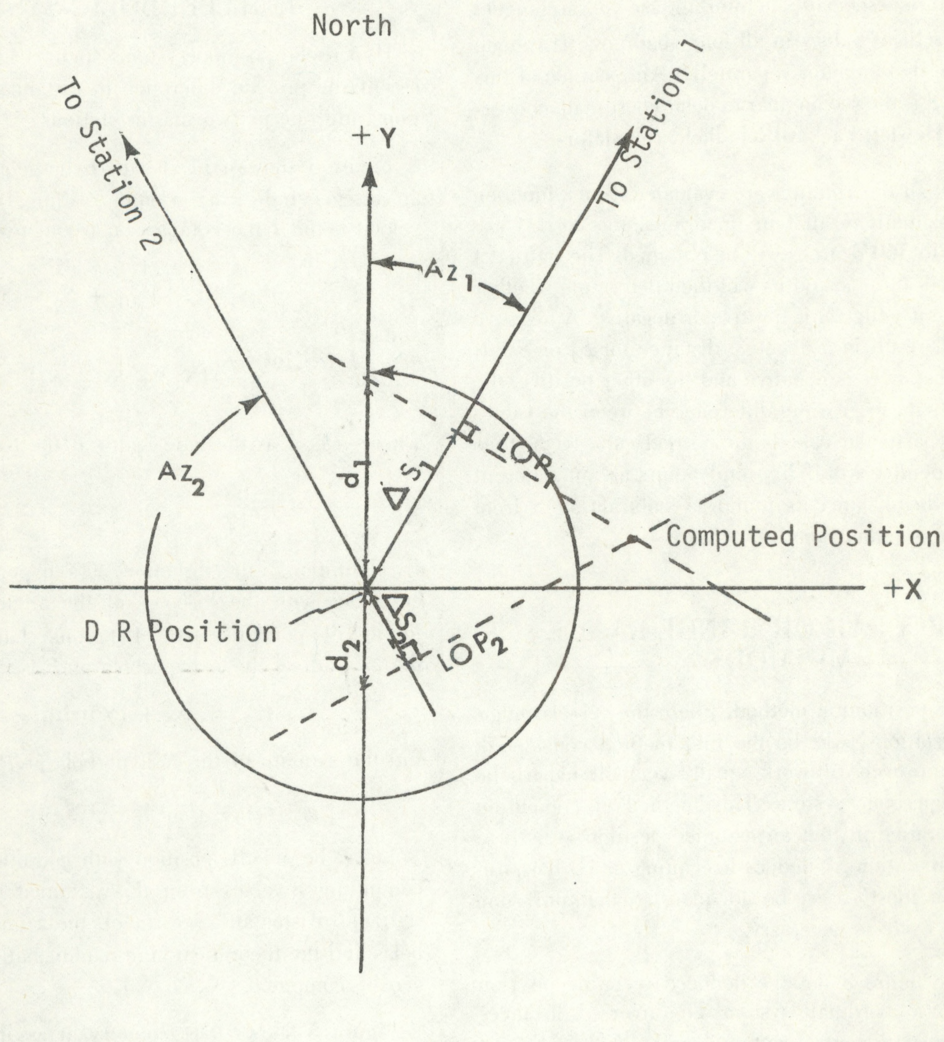


FIGURE 3.

HYPERBOLIC SYSTEMS

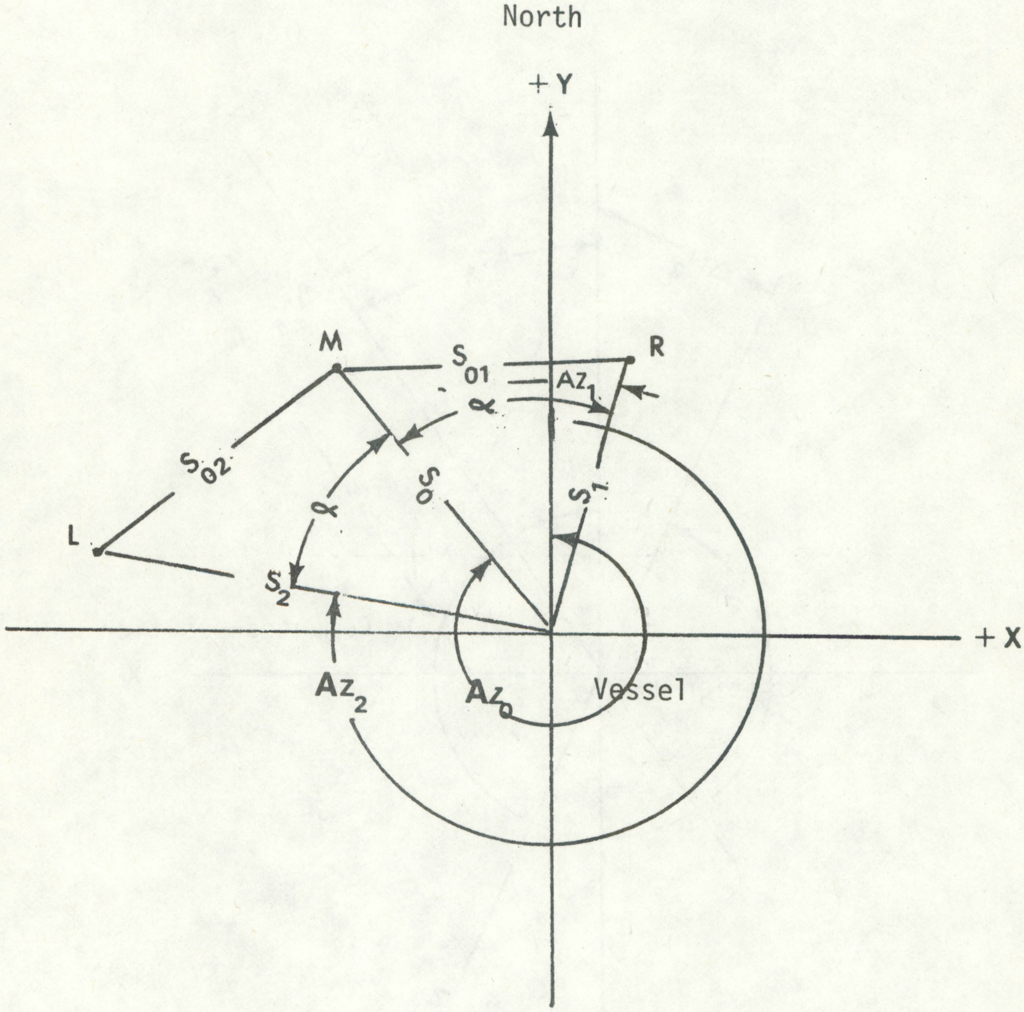


FIGURE 4.

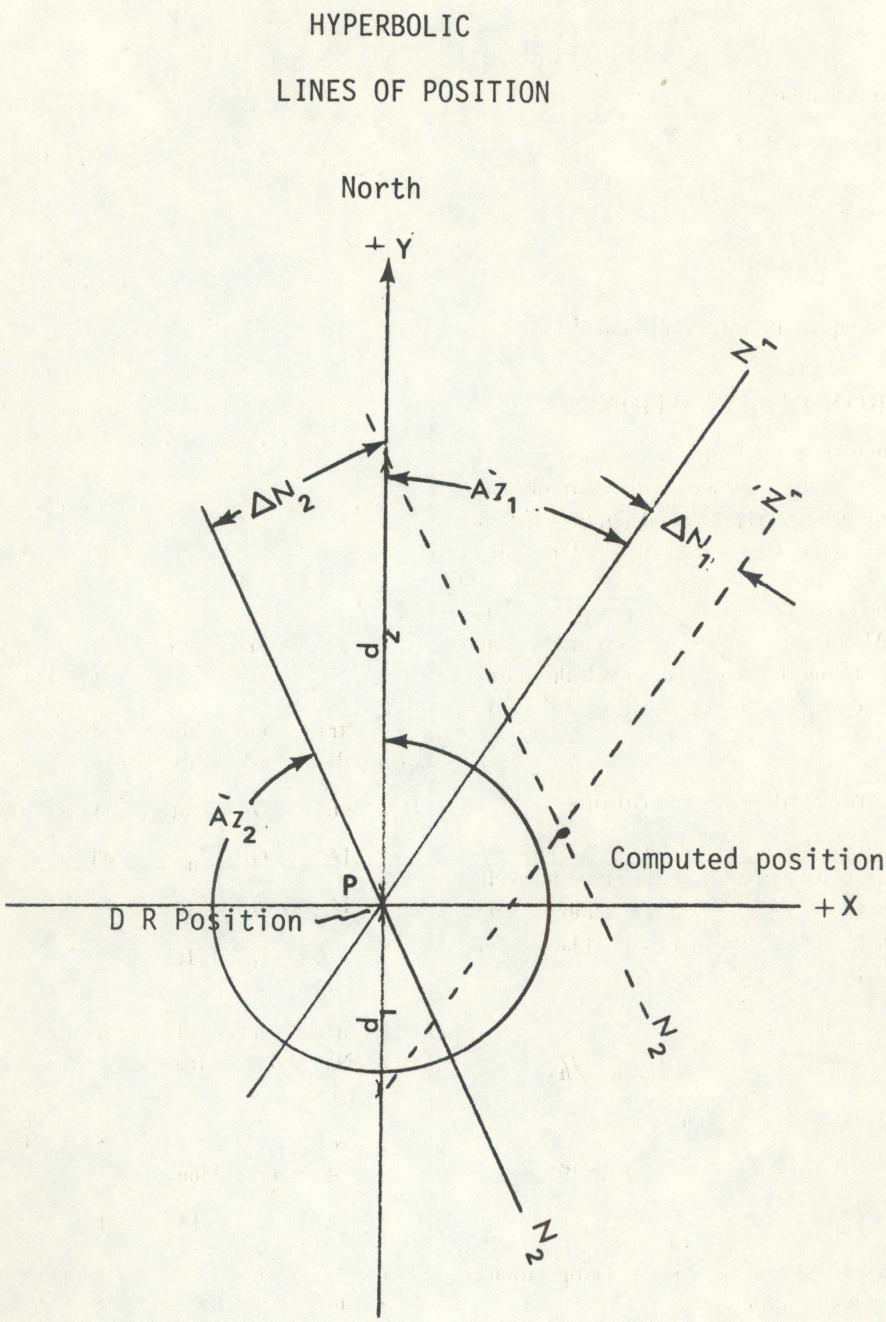

HYPERBOLIC
LINES OF POSITION
North
+Y
+X
ΔN_2
Az_1
ΔN_1
N_1
N_1'
Az_2
P
D R Position
Computed position
N_2
N_2

FIGURE 5.

$$\ell_i = \ell_i' \frac{1}{\sin(1/2)\,\alpha_i}$$

$$\Delta N_2 \text{ (in meters)} = (\Delta N)_2(\ell_2)$$

$$\Delta N_1 = (N_1)(\ell_1)$$

The intersection of the dashed lines (fig. 5) represents the true location of the vessel. As before, these lines can be expressed by linear equations of the form:

$$Y = p_i X + d_i$$

where $$d = \frac{\Delta N_i}{\sin Az_i} \quad i = 1, 2$$

$$p_i = \cot Az_i$$

Solution of these equations will be presented later.

5. ASTRONOMIC POSITIONING

The entire astronomic positioning computation could be fully automated if so desired. However, the partially automated method is used here. This method requires that the daily, sidereal hour angle (SHA) and declination (δ) be entered for each star. It is felt that, in most cases, this information must be looked up each day in order to identify the star observed. Also it would require an inordinate amount of computer storage to file the coordinates of all the stars in an ephemeris and compute daily coordinates from a given epoch.

5.1 Corrections to Observations

Separate provision must be made for index correction and corrections for the Sun, Moon, and planets. These nonstellar bodies were not considered, since they are seldom used in conjunction with other stars and perhaps should be weighted differently in any case.

The refraction and dip corrections are:

$$\text{Corn (in min.)} = -0.97\,[\cot H_0 + \sqrt{h}]$$

where H_0 = obs altitude

h = height of the eye in feet

5.2 Computations

Following is a list of symbols used in the computation of position from astronomic sightings:

Symbol	*Data*
V	Speed of advance (vessel)
C	Course made good (vessel)
h	Height of eye in feet
CCD	Consecutive calendar day of first observation (1 . . . 365)
ϕ & λ	Assumed position of observer (D.R. at time of observation)
For each star	
GMT	Greenwich Mean Time of observation
SHA_*	Sidereal Hour Angle of star
δ_*	Declination of star $+N$, $-S$
H_0	Observed altitude corrected for index error

The first calculation to perform is the computation of the Greenwich Hour Angle of Aries (GHA♈) for zero hours GMT on the day of observation. The Greenwich Hour Angle of Aries at zero hours GMT on January 1 is extracted from the Nautical Almanac for the current year. This hour angle is then updated to the zero hour of the date of observation by the formula:

$$GHA♈_0 = GHA♈_0 + 36000^0.7689(T) + 0^0.000389(T^2)$$

where:

$$T = \frac{\text{Consecutive calendar day of observation} - 1}{365.25}$$

Subtract even multiples of 360° from calculated $GHA♈_0$. The $GHA♈_0$ need only be computed once per day.

Compute the GHA♈ at the time of observation.

$$GHA♈ = GHA♈_0 + (\text{GMT of observation})\,K$$

$$GHA♈ = GHA♈_0 + (h + min/60 + s/3600)\,K$$

$$K = 15°.04107$$

Using the sidereal hour angle (SHA) of each star given in the Nautical Almanac, we can now compute the star's GHA by:

Greenwich Hour Angle Star = Greenwich Hour Angle Aries + Sidereal Hour Angle Star

$$GHA_* = GHA♈ + SHA_*$$

The hour angle of the star is found by subtracting the D.R. longitude (in degrees) from the Greenwich Hour Angle (in degrees) of the star. West longitudes are positive and east longitudes are negative (for celestial computations only).

$$HA = GHA_* - \lambda$$

It is now possible to compute the altitude (H) that the given star would have if it were observed from the D.R. position.

$$\sin H = \sin\phi \sin\delta_* + \cos\phi \cos\delta_* \cos(\text{HA})$$

The difference (in minutes) between the computed and observed altitudes is formed by:

$$\Delta H = (H_0 - H)(60);$$

positive is toward the star from assumed position.

The azimuth of the star (clockwise from north) is:

$$\tan Az = \frac{\sin(-\text{HA})}{[\tan\delta_* - \tan\phi\cos(-\text{HA})]\cos\phi}$$

The signs of the numerator and denominator of this expression will place the azimuth angle in the proper quadrant.

6. ADJUSTMENT FOR SHIP'S MOTION

The preceding calculation will give the information necessary to plot one line of position (LOP) for each star observed. When plotted, these lines of position would be displaced along the vessel's course line due to the motion of the vessel between observations. A graphic adjustment is generally made for this displacement of LOP's by advancing or retarding them along the vessel's track.

The advancement of retardation of lines of position to reduce all lines to the same instant or epoch can be accomplished mathematically as well as graphically. In figure 6, the vessel's position at time (t_i) is represented by the origin of a plane coordinate system; the azimuth of the vessel's course (C) and the vessel's speed (V) are also indicated. In figure 6, LOP_2 at t_2, is shown by the solid line. When this line of position is retarded to the epoch of t_1, it is shown by a dashed line labeled LOP_2 at t_1.

The line of position one (LOP_1) can be represented mathematically by an equation of the form:

$$Y = p_1 X + d_1$$

where the constants

$$p_1 = (-)\tan(Az_1)$$

and,

$$d_1 = \frac{\Delta H_1}{\cos(Az_1)} \qquad \text{(See figure 6.)}$$

If the vessel were stationary during the entire observation period, similar equations could be written for each observation and the set of equations could be solved to determine the position. In most cases, however, the vessel will move between observations.

Where V is the ship's speed in knots and $t_2 - t_1$ is the difference in time (in hours) between the instant of observation for stars two and one, the distance the vessel travels between observations is:

$$SV = V(t_2 - t_1)$$

Figure 6 shows the second LOP taken at t_2 as a solid line marked LOP_2 at t_2. This second LOP must be retarded by distance SV_2 on course C, if it is to be solved simultaneously with LOP_1. This is usually done graphically with a parallel rule and dividers. Another way to achieve this retardation of LOP_2 is to replot it using Az_2 and ΔH_2 where:

$$\Delta H_2' = \Delta H_2 + SV_2 \cos(C - Az_2)$$

Note: SV will be negative if LOP is retarded.

Plotting LOP_2, using Az_2 and $\Delta H_2'$ will give LOP_2 at t_1 (shown by a dashed line). This LOP can be represented mathematically by:

$$Y = p_2 X + d_2$$

where

$$p_2 = -\tan(Az_2)$$

$$d_2 = \frac{\Delta H_2'}{\cos Az_2}$$

Since there are now two equations in two unknowns, both referenced to the same coordinate system, a solution is possible. In general, however, there will be more than two LOP's.

6.1 Least Squares Solution for Three or More Lines of Position

The least squares solution can be performed for all celestial fixes. If two stars are observed, there is no redundancy; however, the following least squares algorithm will give an identical solution to the simultaneous solution of the two equations. In general when three or more stars are observed, the least square solution will provide the best fix.*

As demonstrated previously, the lines of position associated with star observations, hyperbolic, and circular systems can be written in the form:

$$Y = pX + d$$

where

let

$p = \cot Az$; or $p = -\tan Az$

$d = Y$ axis intercept

$X, Y =$ the coordinates of the "intersection" of all the LOP's

$n =$ number of observation

$i = 1 \cdots n,$

*This holds only if there is no bias in the observations. If corrections are made to the lines of position, either away from or toward the substellar points of the stars, the correction must be made before equations are solved.

CORRECTION
for
SHIPS'MOTION

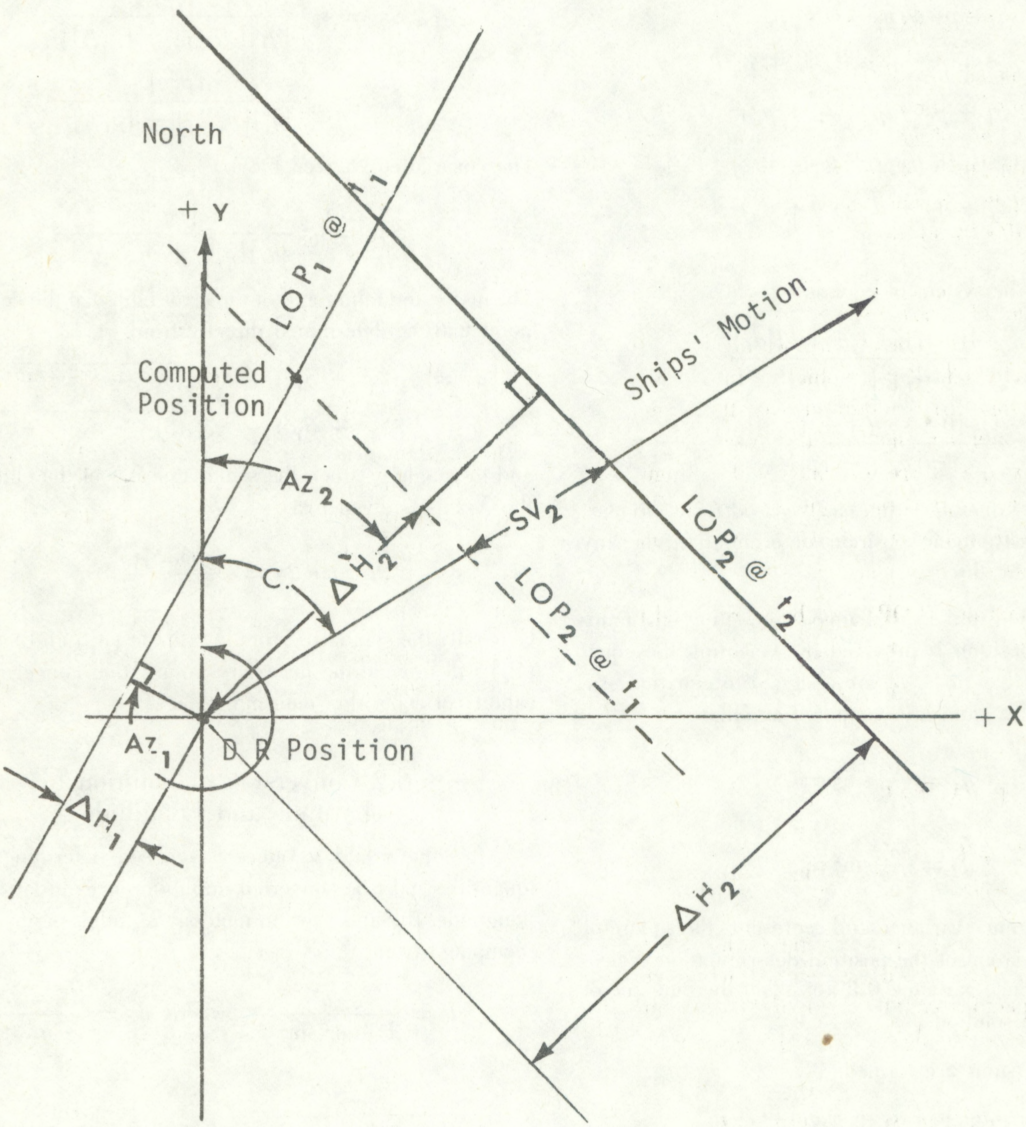


FIGURE 6.

Thus, for n lines of position, or observations, (n) number of equations can be written:

$$p_1X - Y = d_1$$
$$p_2X - Y = d_2$$
$$p_3X - Y = d_3$$
$$\vdots$$
$$p_nX - Y = d_n$$

let the following symbols represent quantities indicated:

$$[p_i] = p_1 + p_2 + p_2 + \cdots p_n$$
$$[p_i^2] = p_1^2 + p_2^2 + p_3^2 \cdots + p_n^2$$
$$[p_i]^2 = (p_1 + p_2 + p_2 \cdots + p_n)^2$$
$$[p_id_i] = (p_1d_1 + p_2d_2 + p_3d_3 + \cdots p_nd_n)$$
$$[d_i] = (d_1 + d_2 + d_3 \cdots + d_n)$$

The solution to the system of equations is:

$$X = \frac{n(-)[p_id_i] + [p_i][d_i]}{n[p_i^2] - [p_i]^2}$$

$$Y = \frac{[p_i][(-)p_id_i] + [p_i^2][d_i]}{n[p_i^2] - [p_i]^2}$$

Therefore the same algorithm can be used for a circular, hyperbolic, or astronomic position determination, or any combination of the three systems.

A better solution to the simultaneous reduction of a number of lines of position is provided by weighting the equations. Weights $q_1, q_2, q_3 \ldots$ are assigned to equation systems 1, 2, 3, . . . , and are computed as follows:

$$q_1 = \frac{s_0^2}{e_1^2} \sin^2 \alpha_1$$

$$q_2 = \frac{s_0^2}{e_2^2} \sin^2 \alpha_2$$

where s_0^2 is reference variance and represents the square of the standard deviation of the position determination. Selection of the reference variance will not affect the outcome of the least squares solution.

The following sums are formed:

$$[p_iq_i] = p_iq_i + p_2q_2 + p_3q_3 + \cdots$$
$$[p_i^2q_i] = p_1^2q_1 + p_2^2q + p_3^2q_3 + \cdots$$
$$[p_iq_i]^2 = [p_iq_i]^2$$
$$[p_iq_id_i] = p_1q_1d_1 + p_2q_2d_2 + p_3q_3d_3 + \cdots$$
$$[d_iq_i] = d_1q_1 + d_2q_2 + d_3q_3 + \cdots$$
$$[q_i] = q_1 + q_2 + q_3 + \cdots$$

The solution of X and Y is given by:

$$X = \frac{[q_i][-p_iq_id_i] + [p_iq_i][d_iq_i]}{[p_i^2q_i][q_i] - [p_iq_i]^2}$$

$$Y = \frac{[p_iq_i][-p_iq_id_i] + [p_i^2q_i][d_iq_i]}{[p_i^2q_i][q_i] - [p_iq_i]^2}$$

The variance and standard deviations of the coordinates (X,Y) can be computed from:

$$S_X^2 = s_0^2 \frac{[q_i]}{[q_i][p_i^2q_i] - [p_iq_i]^2}$$

$$S_Y^2 = s_0^2 \frac{[p^2q]}{[q_i][p_i^2q_i] - [p_iq_i]^2}$$

The covariance is given by:

$$S_{XY}^2 = s_0^2 \frac{[pq]}{[q_i][p_i^2q_i] - [p_iq_i]^2}$$

The major and minor axis of the error ellipse at the computed point may be determined directly from:

$$a^2 = \tfrac{1}{2}(S_X^2 + S_Y^2) + \sqrt{\tfrac{1}{4}(S_X^2 - S_Y^2)^2 + S_{XY}^4}$$
$$b^2 = \tfrac{1}{2}(S_X^2 + S_Y^2) - \sqrt{\tfrac{1}{4}(S_X^2 - S_Y^2)^2 + S_{XY}^4}$$

and the angle between the semimajor axis of the ellipse and the X axis is given by:

$$\tan 2\gamma = \frac{(-)2S_{XY}^2}{S_X^2 - S_Y^2}$$

Generally the standard errors in latitude (S_Y) and longitude (S_X) will provide the necessary information concerning the validity of a position determination.

6.2 Conversion of Solution to Latitude and Longitude

After the x and y values have been determined, these quantities must be converted into differences in latitude and longitude. If x and y are in meters, the radius of curvature is computed from:

$$R = \frac{b}{1 - e^2 \sin^2 \phi} \qquad \text{where: } e^2 = \frac{a^2 - b^2}{a^2}$$

and

$$\Delta\phi = \frac{Y}{R}\rho \qquad \text{where: } \rho = \frac{180^\circ}{\pi}$$

$$\Delta\lambda = \frac{X}{R}\rho \cos \Phi\text{: for E long } (+).$$

If X and Y are in minutes of arc as in the astronomic solution

$$\Delta\phi = Y$$

$$\Delta\lambda = X \cos \phi,$$

the computed position of the vessel will be:

$$\phi = \phi_0 + \Delta\phi$$

$$\lambda = \lambda_0 + \Delta\lambda$$

Care should be used if astronomic LOP's are used in conjunction with either circular or hyperbolic systems. The y axis intercept is in meters for the electronic system and in minutes of arc for the celestial method. Conversion from one to the other can be accomplished using the average radius of curvature (R).

Appendix A

Example: Long Line Computation

Two examples of long line computations are given to illustrate the formulas presented. The step number refers to the numbers given on pages in Section 2.4.

Example 1:

Given: $\phi_1 = 40°00'00.00''$N
$\lambda_1 = 18°00'00.000''$W
$\phi_2 = 57°06'00.851''$N
$\lambda_2 = 45°08'40.841''$E
$a = 6378206.4$ m $\quad b = 6356583.8$ m

Step	*Value*
1.	$\Delta\lambda = 63°.14467806$
2.	$\psi_1 = 39°.80850328$
	$\psi_2 = 56°.92252243$
3.	$Az_{12} = 45°.06275676$
	$Az_{21} = 274°.8902013$
4.	$\bar{\theta}_1 = 40°.28161054$
	$\bar{\theta}_2 = 03°.178014800$
5.	$\beta_0 = 57°.14699203$
6.	$\bar{b} = 6{,}362{,}954.762$
7.	$\bar{m} = 0.002394070$
8.	$\bar{R}_0 = 6{,}370{,}582.863$
9.	$A = 0.000598518$
10.	$S_1 = 4{,}475{,}056.154$
	$S_2 = 352{,}933.872$
11.	$S = 4{,}827{,}990.026$
True distance	$= 4{,}827{,}989.697$ m

Example 2:

Given: $\phi_1 = 40°00'00''$N
$\lambda_1 = 18°00'00''$W
$\phi_2 = 49°16'35.187''$N
$\lambda_2 = 02°19'56.359''$W
$a = 6378206.4$ m $\quad b = 6356583.8$ m

Step	*Value*
1.	$\Delta\lambda = 15°.66767806$
2.	$\psi_1 = 39°.80850328$
	$\psi_2 = 49°.08394186$
3.	$Az_{12} = 45°.01846721$
	$Az_{21} = 236°.0604368$
4.	$\bar{\theta}_1 = 40°.30348800$
	$\bar{\theta}_2 = (-)25°.82253819$
5.	$\beta_0 = 57°.17559379$
6.	$\bar{b} = 6{,}362{,}944.916$
7.	$\bar{m} = 0.002395617$
8.	$\bar{R}_0 = 6{,}370{,}577.943$
9.	$A = 0.000598904$
10.	$S_1 = 4{,}477{,}482.297$
	$S_2 = 2{,}868{,}153.084$
11.	$S = 1{,}609{,}329.213$
True distance	$= 1{,}609{,}329.582$ m

The differences between the true and computed distances can be attributed to the truncation of terms in the series expansion of the elliptic integral. Additionally, calculations were performed using an HP-21 which gives errors beyond the ninth decimal place.

The full series expansion for evaluating the elliptic integral using the geocentric latitude is given below for those desiring greater computational precision.

$$S_i = R_0(\theta - A \sin 2\theta_i + B \sin 4\theta_i - C \sin 6\theta_i \cdots)$$

$$A = \frac{m}{4} + \frac{21}{128} m^3$$

$$-B = \frac{1}{64} m^2$$

$$-C = \frac{5}{128} m^3$$

where S_i is the distance along the great elliptic from the pole of the ellipse to the point, and θ_i is the geocentric polar angle of the point (fig. 2B).

Appendix B

Example: Stellar Sight Reduction

An example of the computation of the Greenwich Hour Angle, altitude, and azimuth of a star is given in this appendix to demonstrate the formulas contained in this report.

Example: The problem given on page 510 of Bowditch is used.

Example 4.—During evening twilight on June 1, 1958, the 1730 DR position of a ship is lat. 40°39′.2 S, long. 75°01′.2 E. At GMT $12^h31^m17^s$ the navigator observes Arcturus with a marine sextant having no IC, from a height of eye of 38 feet. The hs is 7°55′.2.

Required—The *a*, *Zn*, and *AP*.

Solution—

	June 1		
GMT	12^h31^m17 June 1		
12^h30^m	76°59′		
1^m17^s	19′		
SHA	146°33′		
GHA	223°51′		
a	75°09′E		
LHA	299°00′		
t	61°00′E		
d	19°24′N		
aL	41°00′S		
ht	7°14′.0	*d* diff. 6′	
corr.	(+)4′.5		
Hc	7°18′.5	$\Delta d(+)10.75$ *Z*	*S* 123°.8E
Ho	7°42′		
a	24*T*	*al* 41°00′S	
Zn	056°.2	*a* 75°09′E	

	Arcturus	
+	☆	−
IC		
D		6′
R		7′
sum		13′
corr.	(−)	13′
hs		7°55′
Ho		7°42′

Counting January 1 as the first day, June 1 is the 152d day of the year.

$$\text{GHA}\Upsilon_0 \text{ on January 1, 1958} = 100°08'.4$$
$$= 100°.140000000$$

$$T \text{ (for June 1)} = \frac{152 - 1}{36525} = 0.004134155$$

The Greenwich Hour Angle of Aries for zero hours, June 1 is:

$$\text{GHA}\Upsilon_0 = 100°.140000000 + 36{,}000°.7689(T) + 0°.000389(T^2)$$

$$\text{GHA}\Upsilon_0 = 248°.9727476 = 248°58'.36$$

The tabulated value for June 1, 1958, is: 258°58′.3
The value needs to be computed only once per day.

Next, compute the Greenwich Hour Angle of Aries at the instant of the celestial observation:

GHA♈ at $12^h31^m17^s$Z June 1, 1958

$$\text{GHA}\Upsilon = 248°.97275 + 12°.52139(15°.04107) =$$

$$\text{GHA}\Upsilon = 437°.30784$$

Subtract 360°

$$\text{GHA}\Upsilon = 77°.30784$$

Compute the Greenwich Hour Angle:

$$\text{GHA}_* = \text{GHA}\Upsilon + \text{SHA}_*$$

$$\text{GHA}_* = 77°.308 + 146°.55$$

$$\text{GHA}_* = 223°.858$$

Compute the hour angle of the star from the plotting longitude of: 75°09′*E*

$$\text{HA} = \text{GHA}_* - \lambda = 223°.858 - (-75°.15)$$

Hour angles measured + (W)

$$\text{HA} = 299°.008$$

Compute the altitude that the star would have if it were observed from latitude 41°S and longitude 75°09′E.

$$\sin H = \sin\phi \sin\delta + \cos\phi \cos\delta \cos(\text{HA})$$

$$\sin H = \sin(-41°.0)\sin(19°.4) + \cos(-41°.0)\cos(19°.4)\cos(299°.0)$$

$$\sin H = 0.12720$$

$$H = 7°.306$$

The azimuth of the star, when observed from the assumed position is:

$$\tan Az = \frac{\sin(-\text{HA})}{[\tan\delta - \tan\phi\cos(-\text{HA})]\cos\phi}$$

OR:

$$\tan Az = \frac{\sin(-\text{HA})}{\tan\delta\cos\phi - \sin\phi\cos(-\text{HA})}$$

$$\tan Az =$$

$$\frac{\sin(-299^\circ.0)}{[\tan(19^\circ.4) - \tan(-41^\circ)\cos(-299^\circ.0)]\cos(-41^\circ)}$$

$$\tan Az = \frac{+0.87462}{+0.58384}$$

Since the numerator and denominator of the preceding expression are both positive, the angle will be in the first quadrant:

$$Az = 056^\circ.3$$

the computed altitude of $7^\circ.308 = 7^\circ 18'.5$, and the computed azimuth of $056^\circ.3$ checks the values given by Bowditch.

Appendix C

Example: Celestial Lines of Position and Least Squares Positions Computation

The computation of a position from three celestial lines of position is given here to illustrate the formulas in this report. Problem number 1707b, page 462 of Bowditch (1962) is used.

1707b. The 1800 DR position of a ship is lat. 27°02′.2 N., long. 170°17′.0 W. The ship is on course 045°, speed 14 knots. During evening twilight the navigator observes three stars with the following results:

	Dubhe	*Altair*	*Spica*
Time	1815	1821	1830
H_c	34°45′.2	22°11′.8	47°24′.8
H_o	34°51′.3	22°15′.7	47°20′.4
$Az = Z_n$	331°.5	090°.3	219°.9
Assumed ϕ αL	27°00′.0 N	27°00′. N	27°00′.0 N
λ $\alpha\lambda$	170°10′.2 W	170°05′.0 W	169°54′.8W

Required.—The 1830 fix.

Answer.—1830 fix: L27°11′.5 N, 170°00′.5 W.

This example has to be changed slightly to conform to the computational format presented in this report. One assumed position will be used for all three stars. This single position will slightly alter the computed altitude (H_c) and the azimuth to the star.

Let the assumed position be:

$$\Phi\ 27°00' \text{ N}$$
$$\lambda\ 170°05'.0 \text{ W}$$

The plotting data now become:

	Dubhe	*Altair*	*Spica*
Time	1815	1821	1830
H_c	34°42′.72	22°11′.8	47°30′.83
H_o	34°51′.3	22°15′.7	47°20′.4
Az	331°.4	090°.3	220°.0
ϕ	27°00′ N	27°00′ N	27°00′ N
λ	170°05′ W	170°05′ W	170°05′ W

The altitude differences in minutes of arc now become:

$$\Delta H_1 = +8'.5$$
$$\Delta H_2 = +3'.9$$
$$\Delta H_3 = -10'.4$$

The altitude distances of *Dubhe* and *Altair* advanced to 1830 become:

$$\Delta H_1' = \Delta H_1 + SV\cos(C - Az_1)$$
$$\Delta H_1' = +8'.5 + 3'.5\cos(045° - 331°.4)$$
$$\Delta H_1' = +9'.5$$

$$\Delta H_2' = +3'.9 + 1'.48$$
$$\Delta H_2' = +5'.38$$

The equations of the three lines of position advanced to 1830 are:

$$Y = +0.5452X + 10.820$$
$$Y = +190.984X + (-1027.509)$$
$$Y = -0.8391X + 13.576$$

The solution of this set of equations is found by forming the following sums and products:

$$[p] = 0.545 + 190.984 + (-0.839) = 190.690$$

$$[p^2] = 36475.889$$

$$[p]^2 = (190.690)^2 = 36362.676$$

$$[pd] = (5.897 - 196237.779 - 11.392) = -196243.274$$

$$[d] = 10.820 - 1027.509 + 13.576 = -1003.113$$

$$X = \frac{(3)(196243.274) + (190.690)(-1003.113)}{(3)(36475.889) - (36362.676)}$$

$$X = \frac{397446.2040}{73064.991} = 5.44$$

$$Y = \frac{(190.690)(196243.274) + (36475.889)(-1003.113)}{(3)(36475.889) - (36362.676)}$$

$$Y = \frac{832191.480}{73064.991} = +11.39$$

The correct or updated position of the vessel now becomes:

$$\phi = +27°00' + 11'.4 = 27°11'.4 = 27°11'.4 \text{ N}$$

$$\lambda = 170°05'.0 + 5'.4\cos(27°.19) = 170°00'.2 \text{ W}$$

This computed ship's position could be used as the assumed position and the solution reiterated. If this were done, X and Y values would reduce to zero. Generally, if the assumed and computed positions are within 10 miles of one another, one iteration will suffice.

Appendix D

Example: LORAN computation

An example of a LORAN-C position computation is given here to illustrate the hyperbolic formulas contained in this report.

Given:

$Assumed\,Position\,(P): \phi = 20°00'.0\text{ N} \quad \lambda = 40°00'.0\text{ W}$

LORAN-C Station	*Latitude*	*Longitude*
Right Station Nantucket, MA	41°15′11.98″ N	69°58′40.51″ W
Master Station Carolina Bch., NC	34°03′45.61″ N	77°54′47.20″ W
Left Station Jupiter, FL	27°01′57.32″ N	80°06′53.71″ W

Frequency: 1,000,000 Hertz, Baseline extension delay rates:

$$CD_1 = 33000, \quad CD_2 = 12000$$

LORAN Readings: $N_1 = 35340$; $N_2 = 15060$

Compute: Geodetic inverse distances S_2, S_0, S_1, S_{02} and S_{01}. Also compute the azimuths of the three LORAN stations from point P. Using the algorithm given in section 2.4, the following values were computed:

Point	*S(Meters)*	*Az(Degrees)*
L-M = 2-0	807,401.8	—
R-M = 1-0	1,060,720.8	—
P-R = P-1	3,683,268.2	316°.403
P-M = P-0	4,042,672.7	300°.594
P-L = P-2	4,152,665.2	288°.581

Compute baseline lane widths:

$$2\ell' = \frac{\text{Propagation velocity}}{\text{Frequency}} = \frac{299691160.}{1{,}000{,}000}$$

$$2\ell' = 299.69116 \text{ m}$$

$$\ell' = 149.84558 \text{ m}$$

LORAN-C formulas differ slightly from the standard hyperbolic formulas in that they use a reversed sign convention due to the method of numbering lanes. Furthermore, a correction function has been developed to correct for the long over-water ray paths.

The Defense Mapping Agency Hydrographic Center recommends a propagation velocity of 299691.16 km per second and the following correction algorithms be used to compute LORAN-C positions.

1. Distances over 100 statute miles:

($T > 537$ microseconds)

$$\Delta T = \frac{129.04398}{T} - 0.40758 + 0.00064576438(T)$$

2. Distances less than 100 statute miles:

($T < 537$ microseconds)

$$\Delta T = \frac{2.7412979}{T} - 0.011402 + 0.00032774624(T)$$

The formula for computing lane readings is:

$$N = (T_i + \Delta T_i) - (T_m + \Delta T_m) + (T_{mi} + \Delta T_{mi}) + CD$$

Where:

$$T_i = \frac{S_i}{2\ell'}$$

$$T_m = \frac{S_m}{2\ell'}$$

$$T_{mi} = \frac{S_{mi}}{2\ell'}$$

ΔT_i, ΔT_m, ΔT_{mi} are given by the correction equations depending upon the value of T.

CD = the coding delay associated with each pair of stations.

In this example the theoretical or computed lane readings associated with the assumed (D-R) position are given by the following computations:

$$T_1 = \frac{3683268.2}{299.691160} = 12290.21303$$

$$T_2 = \frac{4152665.2}{299.691160} = 13856.48212$$

$$T_0 = \frac{4042672.7}{299.691160} = 13489.46262$$

$$T_{01} = \frac{1060720.8}{299.691160} = 3539.37967$$

$$T_{02} = \frac{807401.8}{299.691160} = 2694.11283$$

$$\Delta T_1 = \frac{129.04398}{12290.21303} - 0.40758 + 0.00064576438\,(12290.21303)$$

$$\Delta T_1 = 7.53950$$

$$\Delta T_2 = 8.54976$$

$$\Delta T_0 = 8.31300$$

$$\Delta T_{01} = 1.91448$$

$$\Delta T_{02} = 1.38008$$

$$N_1 = (T_1 + \Delta T_1) - (T_0 + \Delta T_0) + (T_{01} + \Delta T_{01}) + CD_1$$

$$N_1 = 2341.27107 + 33000 = 35341.27107$$

$$N_2 = (T_2 + \Delta T_2) - (T_0 + \Delta T_0) + (T_{02} + \Delta T_{02}) + CD_2$$

$$N_2 = 3062.74917 + 12000 = 15062.74917$$

Compute the differences between the observed and computed lane readings.

$$\Delta N_1 = 35340 - 35341.27107 = -1.27107$$

$$\Delta N_2 = 15060 - 15062.74917 = -2.74917$$

These differences are converted into meters by multiplying them by the expanded lane values.

Where:

$$\ell_1 = \frac{\ell'}{\sin(\tfrac{1}{2})(Az_1 - Az_0)} = \frac{149.84558\ m}{\sin(7^\circ.905)}$$

$$\ell_1 = 1089.54052$$

$$\ell_2 = \frac{\ell'}{\sin(\tfrac{1}{2})(Az_0 - Az_2)} = \frac{149.84558\ m}{\sin(6^\circ.006)}$$

$$\ell_1 = 1089.54052$$

$$\ell_2 = 1432.11167$$

The differences between observed and computed lane values converted to meters are:

$$\Delta N_1 = (-1.27107)(1089.54052) = -1384.88227$$

$$\Delta N_2 = (-2.74917)(1432.11167) = -3937.11844$$

Compute the azimuths of the hyperbolic lines passing through the assumed (D-R) position

$$A\acute{z}_1 = Az_0 + (\tfrac{1}{2})(Az_1 - Az_0)$$

$$A\acute{z}_2 = Az_2 + (\tfrac{1}{2})(Az_0 - Az_2)$$

$$A\acute{z}_1 = 300^\circ.594 + 7^\circ.905 = 308^\circ.409$$

$$A\acute{z}_2 = 288^\circ.581 + 6^\circ.006 = 294^\circ.587$$

Form equations of lines of position:

$$Y = pX + d$$

$$Y = \cot A\acute{z}_1 + \frac{\Delta N_1}{\sin A\acute{z}_2}$$

$$Y = -0.795X + 1769.548$$

$$Y = -0.458X - 4329.689$$

Note: The minus sign preceding 4329.689 is due to the fact that lane values increase in a counter clockwise direction for the left-hand set of hyperbola. For LORAN-C station configurations the left hyperbola's line of position equation is therefore given by:

$$Y = \cot A\acute{z}_2 - \frac{\Delta N_2}{\sin A\acute{z}_{21}}$$

The solution of these two equations in two unknowns follow the procedure given in Appendix C.

Form the following sums:

$$[p_i] = -1.253$$

$$[p_i{}^2] = 0.841789$$

$$[p_i]^2 = 1.570$$

$$[p_i d_i] = 576.206902$$

$$[d_i] = -2560.14100$$

The solution of X and Y is given by:

$$X = \frac{(2)(-576.206902) + (-1.253)(-2560.14100)}{2(0.841789) - (1.570009)}$$

$$X = 18098.63$$

$$Y = \frac{(-1.253)(-576.206902) + (0.841789)(-2560.14100)}{0.113569}$$

$$Y = -12618.86$$

The average radius of curvature at the assumed point P is:

$$R(\text{ave}) = \frac{6356583.8}{1 - 0.006768658 \sin^2(20^\circ)}$$

$$R(\text{ave}) = 6361620.821$$

The x and y values converted to degrees are:

$$\Delta\lambda = \frac{18098.63}{6361620.821}(57.2957795)\cos(20^\circ)$$

$$\Delta\lambda = 0^\circ.153$$

$$\Delta\phi = \frac{-12618.86}{6361620.821}(57.2957795)$$

$$\Delta\phi = -0^\circ.114$$

The computed position of the vessel with LORAN-C readings of $X_1 = 35340$ and $X_2 = 15060$ is:

$$\Phi_c = 20° + (-0°114)$$

$$\Phi_c = 19°.886 = 19°53'.$$

$$\lambda_c = 40° + 0°.153$$

$$\lambda_c = -39°.847 = -39°50.5$$

Note: The minus longitude denotes a west longitude.

A slightly more complex but more correct method of calculating the vessel's position is to assign weights to the respective lines of position. These weights are a function of the lane width at the vessel's position and are easily introduced into the solution.

In this example the respective lane widths at point P are:

$$\ell_1 = \frac{149.846}{\sin(7°.905)} = 1089.5 \text{ m}$$

$$\ell_2 = \frac{149.846}{\sin(6°.006)} = 1432.1 \text{ m}$$

Compute Y the axis intercept for the respective lane patterns.

$$Y_1 = \frac{\ell_1}{\sin Az_1} = \frac{-1089.5}{\sin 308°.5} = 1392.1$$

$$Y_2 = \frac{\ell_2}{\sin Az_1} = \frac{-1432.1}{\sin 294°.6} = 1575.1$$

Select the largest value as a base value and compute the relative weights (q) of the two lines of position by:

$$q_1 = \left[\frac{1575.1}{1392.1}\right]^2 = 1.3$$

$$q_2 = \left[\frac{1575.1}{1575.1}\right]^2 = 1.0$$

Form the following sums:

$$[p_i q_i] = (-0.795)(1.3) + (-0.458)(1) = -1.4915$$

$$[p_i^2 q_i] = (-0.795)^2(1.3) + (-0.458)^2(1) = 1.0313965$$

$$[p_i q_i]^2 = 2.22457225$$

$$[p_i q_i d_i] = (-0.795)(1.3)(1769.548) + (-0.458)(1)(-4329.689) = +154.1697040$$

$$[d_i q_i] = (1769.548)(1.3) + (-4329.689)(1) = -2029.27660$$

$$[q_i] = 1.3 + 1.0 = 2.3$$

The solution of X and Y is given by

$$X = \frac{[q_i][-p_i q_i d_i] + [p_i q_i][d_i q_i]}{[p_i^2 q_i][q_i] - [p_i q_i]^2}$$

$$X = \frac{(2.3)(-154.1697040) + (-1.49150)(-2029.27660)}{(1.0313965)(2.3) - (2.22457)}$$

$$X = \frac{2672.0757}{0.1476397} = 18098.63$$

$$Y = \frac{[p_i q_i][-p_i q_i d_i] + [p_i^2 q_i][d_i q_i]}{[p_i^2 q_i][q_i] - [p_i q_i]^2}$$

$$Y = \frac{(-1.4915)(-154.1697040) + (1.0313965)(-2029.27660)}{0.1476397}$$

$$Y = -12618.86$$

These coordinates are converted to degrees of latitude and longitude difference as before. In this case, the use of weight will not change the solution since there are only two lines of position.

Appendix E

Example: Three Point Fix Computation

This appendix includes an example of a three point fix computation, to complete the set of formulas necessary to position a vessel (Dedrick 1978). Additionally, the determination of the error ellipse at a point is presented for those desiring to compute errors associated with a given position determination.

Given: (Refer to figure 7)

$$\text{Left angle} = \theta_1 = 27°.791$$

$$\text{Right angle} = \theta_2 = 37°.247$$

The coordinates of the three fixed signals A, B, and C

$$X_A = 3000.00 \text{ m} \quad Y_A = -1000.00 \text{ m}$$
$$X_B = 0.00 \text{ m} \quad Y_B = 0.00 \text{ m}$$
$$X_C = -3000.00 \text{ m} \quad Y_C = -\ 500.00 \text{ m}$$

solution:

Compute the base lengths between signals AB and BC.

$$b_1 = ((X_A - X_B)^2 + (Y_A - Y_B)^2)^{1/2}$$

$$b_1 = ((X_C - X_B)^2 + (Y_C - Y_B)^2)^{1/2}$$

$$b_1 = 3162.28 \text{ m}$$

$$b_2 = 3041.38 \text{ m}$$

Compute the coordinates of the centers of the equal angle circles (X_1, Y_1, and X_2, Y_2) for angles θ_1 and θ_2.

Left circle:

$$X_1 = \frac{X_A + X_B}{2} + \frac{b_1}{2 \tan \theta_1} (\cos Az_{BA})$$

$$X_1 = \frac{3000.00}{2} - \frac{3162.28}{1.05408} (0.31623)$$

$$X_1 = 2448.70\ m$$

$$Y_1 = \frac{Y_A + Y_B}{2} + \frac{b_1}{2 \tan \theta_1} (\sin Az_{BA})$$

$$Y_1 = \frac{-1.000.00}{2} + \frac{3162.28}{1.05408} (0.94868)$$

$$Y_1 = 2346.09 \text{ m}$$

$$X_2 = \frac{X_B + X_C}{2} + \frac{b_2}{2 \tan \theta_2} (\cos Az_{BC})$$

$$X_2 = \frac{-3000.00}{2} + \frac{3041.38}{1.52067} (-0.16440)$$

$$X_2 = -1828.80\ m$$

$$Y_2 = \frac{Y_B + Y_C}{2} - \frac{b_2}{2 \tan \theta_2} (\sin Az_{BC})$$

$$Y_2 = \frac{-500.00}{2} + 2000.03\ (0.98639)$$

$$Y_2 = 1722.81$$

Compute the coordinates of the observer's position (X_0, Y_0) directly from the previously computed values.

$$X_0 = \frac{2(Y_2 - Y_1)}{d^2} - (X_1 Y_2 - X_2 Y_1)$$

$$Y_0 = (-)\frac{2(X_2 - X_1)}{d^2} (X_1 Y_2 - X_2 Y_1)$$

Where:

$$d^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$$

$$d^2 = (-1818.80 - 2448.70)^2 + (1722.81 - 2346.09)^2$$

$$d^2 = 18685484.21$$

$$(X_1 Y_2 - X_2 Y_1) = (2448.70)(1722.81) - (-1828.80)(2346.09)$$

$$(X_1 Y_2 - X_2 Y_1) = 8509174.24$$

$$X_0 = \frac{2(1722.81 - 2346.09)(8509174.24)}{18685484.21}$$

$$X_0 \quad -567.67$$

$$Y_0 = \frac{(-)2(-1828.80 -- 2448.70)(8509174.24)}{18685484.21}$$

$$Y_0 = 3895.86$$

The error ellipse at computed point P can now be determined and the errors in latitude and longitude calculated.

Assume for the purpose of illustration that the left and right angles can be measured with a precision of $\pm 5°$.

THREE POINT FIX

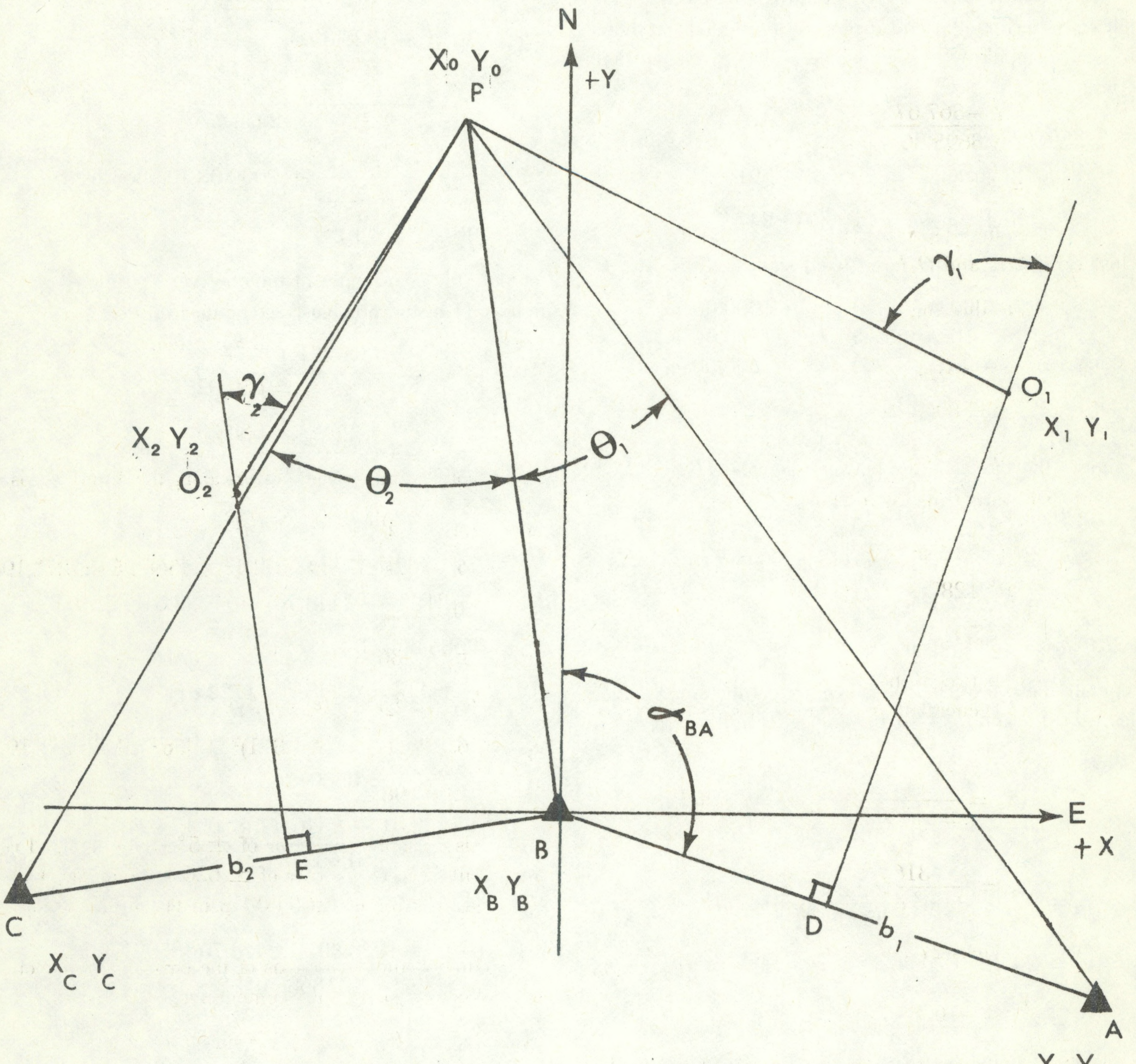


FIGURE 7.

Compute the azimuths of the lines connecting point P and the centers of the equal angle circles (O_1, O_2).

For the left circle the azimuth of the line connecting point O_1 and P is:

$$\alpha_1 = \text{arc tan}\left(\frac{X_0 - X_1}{Y_0 - Y_1}\right)$$

$$\alpha_1 = \text{arc tan}\left(\frac{-567.67 - 2448.70}{3895.86 - 2346.09}\right)$$

$$\alpha_1{}^0 = -62^\circ.8065$$

The azimuth of the line connecting O_2 to P is:

$$\alpha_2 = \text{arc tan}\left(\frac{X_0 - X_2}{Y_0 - Y_2}\right)$$

$$\alpha_2 = \left(\frac{-567.67 + 1828.80}{3895.86 - 1722.81}\right)$$

$$\alpha_2 = 30^\circ.1287$$

Compute the angle between lines $\overline{DO_1}$ and $\overline{O_1P}$ and between lines $\overline{EO_2}$ and $\overline{O_2P}$.

For the left circle this angle is:

$$\gamma_1 = \alpha_1^\circ - (Az_{BA} - 90^\circ)$$

$$\gamma_1 = -62^\circ.8065 - (108^\circ.4349 - 90^\circ)$$

$$\gamma_1 = -81^\circ.2414$$

For the right circle the angle is:

$$\gamma_2 = \alpha_2^\circ - (Az_{BC} + 90^\circ)$$

$$\gamma_2 = 30^\circ.1287 - (-99^\circ.4623 + 90^\circ)$$

$$\gamma_2 = 39^\circ.5910$$

The distance between the equal angle circle, passing through point P generated by angle θ_1, and θ_1 plus 5 degrees is:

$$\Delta_1 = \frac{b_1}{2\sin^2\theta_1}(\cos\theta_1 + \cos\gamma_1)d\theta$$

$$\Delta_1 = \frac{3162.28}{2\sin^2(27^\circ.791)}\left\{(\cos(27.791) + \cos(-81.2414)\right\}\frac{5^\circ}{57^\circ.3}$$

$$\Delta_1 = -658.61 \text{ m}$$

Similarly this change of position for the right equal angle circle resulting from a change of 5° in the angle is:

$$\Delta_2 = \frac{-b_2}{2\sin^2\theta_2}(\cos\theta_2 + \cos\gamma_2)d\theta$$

$$\Delta_2 = \frac{3041.38}{2\sin^2(37^\circ.247)}\left\{\cos(37^\circ.247) + \cos(39^\circ.591)\right\}\frac{5^\circ}{57^\circ.3}$$

$$\Delta_2 = -567.53 \text{ m}$$

Compute the errors in the direction of the equal angle circles. (See figure 8.)

$$e_1 = \frac{\Delta_1}{\sin(\beta_1 - \beta_2)} \quad \text{where: } \beta_1^\circ = \alpha_1 + 90^\circ$$

$$\beta_2^\circ = \alpha_2 - 90^\circ$$

$$e_1 = \frac{-658.16}{\sin(27^\circ.193 - (-59^\circ.871))}$$

$$e_1 = -659.02$$

$$e_2 = \frac{\Delta_2}{\sin(\beta_1 - \beta_2)}$$

$$e_2 = \frac{-567.53}{0.99869}$$

$$e_2 = -568.28$$

Similarly, the displacements along the equal angle circles generated by errors in θ_1 and θ_2 of minus 5° are

$$e_1 = 659.02$$

$$e_2 = 568.28$$

The components of these errors along the X and Y axis are

$$e_X{}^2 = (e_1 \sin\beta_2)^2 + (e_2 \sin\beta_1)^2$$

$$e_X{}^2 = [659.02 \sin(-59^\circ.871)]^2 + [568.28 \sin(27^\circ.193]^2$$

$$e_X{}^2 = 392325.57$$

$$e_X = \pm 626.36$$

$$e_Y{}^2 = (e_1 \cos\beta_2)^2 + (e_2 \cos\beta_1)^2$$

$$e_Y{}^2 = [659.02 \cos(-59^\circ.871)^2 + [568.28 \cos(27^\circ.193)]^2$$

$$e_Y = \pm 604.08$$

In this example, the error of ± 5° in the left and right angles introduced an error of ± 626.36 *m* in the X direction and an error of ±604.08 m in the Y direction.

The major and minor axis of the error ellipse at point P can be computed by first computing:

$$e_{XY}^2 = e_1{}^2 \cos\beta_2 \sin\beta_2 + e_2{}^2 \sin\beta_1 \cos\beta_1$$

$$e_{XY}^2 = (659.02)^2 \cos(-59^\circ.871) \sin(-59^\circ.871) + \cdots$$

$$+ \cdots (568.28)^2 \sin(27.193) \cos(27.193)$$

$$e_{XY}^2 = -57278.33$$

ERROR ELLIPSE

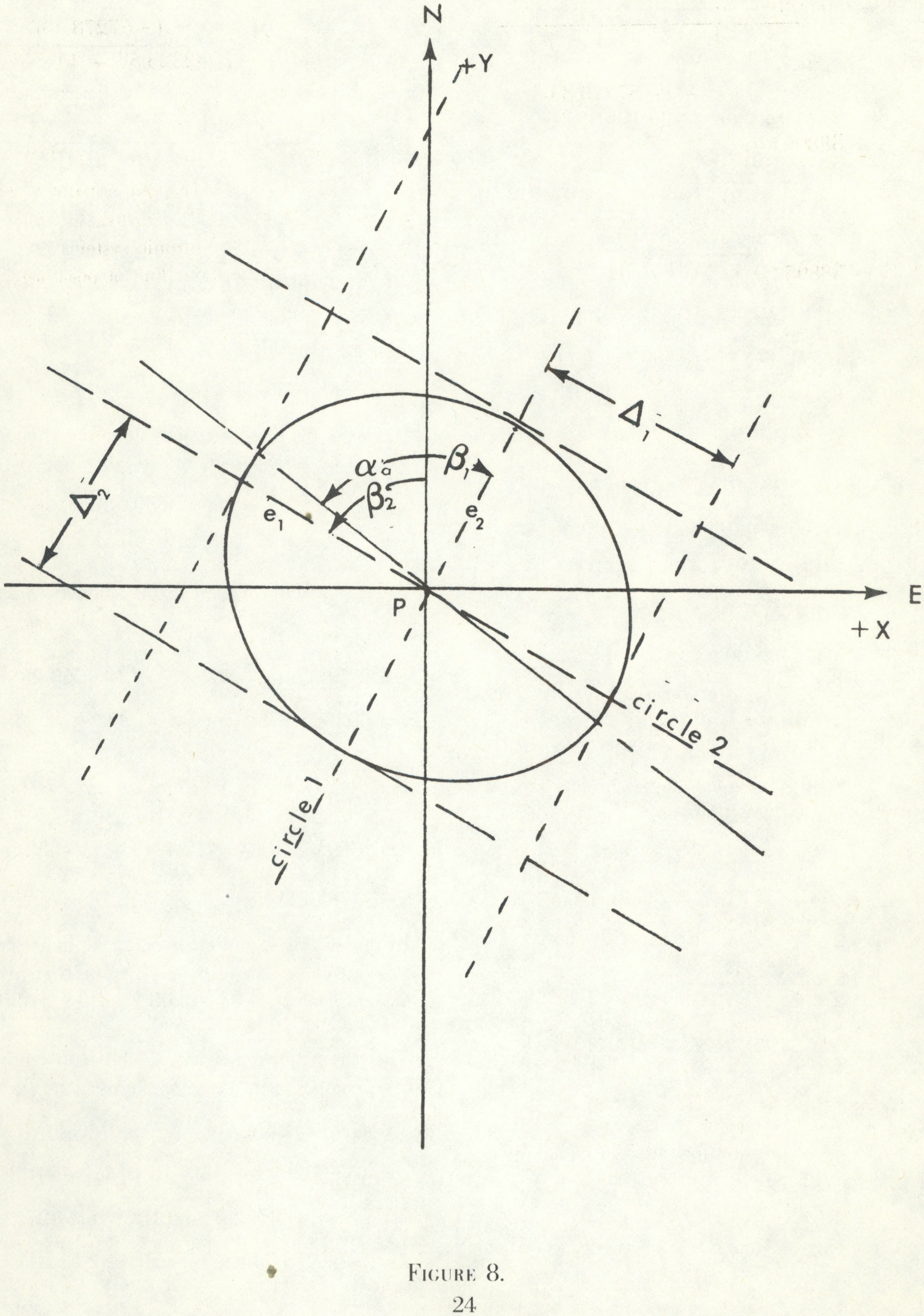


FIGURE 8.

The major (a) and the minor (b) axis of the error ellipse at point P are found by solving the equations:

$$a^2 = \frac{e_X^{\,2} + e_Y^{\,2}}{2} + \{1/4(e_X^{\,2} - e_Y^{\,2})^2 + e_{XY}^4\}^{1/2}$$

$$b^2 = \frac{e_X^{\,2} + e_Y^{\,2}}{2} - \{1/4(e_X^{\,2} - e_Y^{\,2})^2 + e_{XY}^4\}^{1/2}$$

$$a^2 = \frac{(626.36)^2 + (604.08)^2}{2} + \left\{\frac{(626.36^2 - 604.08^2)^2}{4} + (-57278.33)^2\right\}^{1/2}$$

$$a^2 = 378619.75 + 58895.60$$

$$a^2 = 437515.34$$

$$a = 661.45 \text{ m}$$

$$b^2 = 378619.75 - 58895.60$$

$$b^2 = 319724.15$$

$$b = 565.44 \text{ m}$$

The azimuth of the major axis of the error ellipse is given by:

$$\tan \alpha_a = (-)\,\frac{e_{XY}^2}{e_x^{\,2} - a^2}$$

$$\tan \alpha_a = (-)\,\frac{(-57278.33)}{392325.57 - 437515.34}$$

$$\tan \alpha_a = -1.26751$$

$$\alpha_a = -51.7^\circ$$

The computation of the error ellipse parameters given here are general and work equally well for electronic positioning systems. In the case of electronic systems the Δ values are the uncertainties of the lane (line of position) values.

REFERENCES

Bomford, G., 1962: *Geodesy*, Oxford at the Clarendon Press, London, pp. 1–561.

Collins, J., 1971: Logarithmic spirils for computation of distances on the ellipsoid, (Purdue University), Ph. D. Dissertation, pp. 1–161.

Dedrick, K., 1978: Analysis of the three-point fix, Report No. 77-3, Sacramento, Calif., pp. 1–57.

Gan'Shin, V. N., 1969: Geometry of the ellipsoid. Aeronautical Chart and Information Center, St. Louis, Missouri, pp. 1–130.

Laurila, S. H., 1976: *Electronic surveying and navigation*. John Wiley and Sons, Inc., New York, N.Y., pp. 1–545.

Bowditch, N., 1958: *American Practical Navigator*, U.S. Naval Hydrographic Office, Washington, D.C., pp. 1–1524.

☆ U.S. GOVERNMENT PRINTING OFFICE 1980 -311-046/233

NOS 78 Wave sensors survey. Richard L. Ribe, July 1979, 40 pp. (PB80 118581)

NOS 79 NGS-12. Optimization of horizontal control networks by nonlinear programming. Dennis Milbert, August 1979, 42 pp. (PB80 117948)

NOS 80 Circulation and hydrodynamics of the Lower Cape Fear River, North Carolina. Joseph M. Welch and Bruce B. Parker, August 1979, 108 pp. (PB80 117088)

NOAA SCIENTIFIC AND TECHNICAL PUBLICATIONS

The National Oceanic and Atmospheric Administration was established as part of the Department o Commerce on October 3, 1970. The mission responsibilities of NOAA are to assess the socioeconomic impac of natural and technological changes in the environment and to monitor and predict the state of the solid Earth the oceans and their living resources, the atmosphere, and the space environment of the Earth.

The major components of NOAA regularly produce various types of scientific and technical information in the following kinds of publications:

PROFESSIONAL PAPERS — Important definitive research results, major techniques, and special investigations.

CONTRACT AND GRANT REPORTS — Reports prepared by contractors or grantees under NOAA sponsorship.

ATLAS — Presentation of analyzed data generally in the form of maps showing distribution of rainfall, chemical and physical conditions of oceans and atmosphere, distribution of fishes and marine mammals, ionospheric conditions, etc.

TECHNICAL SERVICE PUBLICATIONS — Reports containing data, observations, instructions, etc. A partial listing includes data serials; prediction and outlook periodicals; technical manuals, training papers, planning reports, and information serials; and miscellaneous technical publications.

TECHNICAL REPORTS — Journal quality with extensive details, mathematical developments, or data listings.

TECHNICAL MEMORANDUMS — Reports of preliminary, partial, or negative research or technology results, interim instructions, and the like.

NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NOAA
U.S. DEPARTMENT OF COMMERCE

Information on availability of NOAA publications can be obtained from:

ENVIRONMENTAL SCIENCE INFORMATION CENTER (D822)
ENVIRONMENTAL DATA AND INFORMATION SERVICE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
U.S. DEPARTMENT OF COMMERCE

6009 Executive Boulevard
Rockville, MD 20852

NOAA--S/T 80-273