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Algorithms for Computing the Geopotential Using a Simple-Layer Density Model

Rockville, Md.
March 1977

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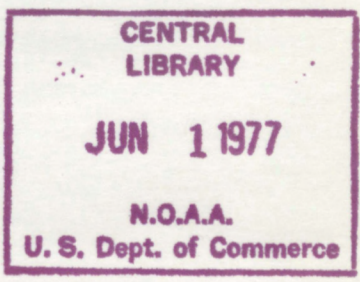
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Algorithms for Computing the Geopotential Using a Simple-Layer Density Model.

Foster Morrison

Geodetic Research and Development Laboratory
National Geodetic Survey

Rockville, Md.
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Algorithms for Computing the Geopotential Using a Simple-Layer Density Model

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1. INTRODUCTION

The geopotential is a scalar function of position in a gravitational field. It is defined as the work done by a unit mass in moving from a point to another point in the field. The geopotential is a useful quantity in many applications, such as in the study of the motion of celestial bodies and in the design of satellite orbits. In this paper, we present algorithms for computing the geopotential using a simple-layer density model. The model is based on the assumption that the density of the Earth is constant in each of the layers. This model is used to compute the geopotential at various points on the Earth's surface. The algorithms are described in detail in the following sections. The first section describes the method of singularity-matching, which is used to compute the geopotential at points near the poles. The second section describes the method of Taylor series, which is used to compute the geopotential at points near the equator. The third section describes the method of numerical cubature, which is used to compute the geopotential at points near the surface. The fourth section describes the annotated computer listing, which is used to compute the geopotential at various points on the Earth's surface. The fifth section describes the program testing, which is used to verify the accuracy of the algorithms. The sixth section describes the applications, which are used to compute the geopotential at various points on the Earth's surface. The seventh section describes the construction of the equal-area blocks, which are used to compute the geopotential at various points on the Earth's surface. The eighth section describes the acknowledgments, which are used to thank the people who helped in the preparation of this paper. The ninth section describes the references, which are used to cite the work of other researchers in the field. The tenth section describes the appendixes, which are used to provide additional information on the algorithms and the computer listing.

II. METHODS IN USE

The geopotential is a scalar function of position in a gravitational field. It is defined as the work done by a unit mass in moving from a point to another point in the field. The geopotential is a useful quantity in many applications, such as in the study of the motion of celestial bodies and in the design of satellite orbits. In this paper, we present algorithms for computing the geopotential using a simple-layer density model. The model is based on the assumption that the density of the Earth is constant in each of the layers. This model is used to compute the geopotential at various points on the Earth's surface. The algorithms are described in detail in the following sections. The first section describes the method of singularity-matching, which is used to compute the geopotential at points near the poles. The second section describes the method of Taylor series, which is used to compute the geopotential at points near the equator. The third section describes the method of numerical cubature, which is used to compute the geopotential at points near the surface. The fourth section describes the annotated computer listing, which is used to compute the geopotential at various points on the Earth's surface. The fifth section describes the program testing, which is used to verify the accuracy of the algorithms. The sixth section describes the applications, which are used to compute the geopotential at various points on the Earth's surface. The seventh section describes the construction of the equal-area blocks, which are used to compute the geopotential at various points on the Earth's surface. The eighth section describes the acknowledgments, which are used to thank the people who helped in the preparation of this paper. The ninth section describes the references, which are used to cite the work of other researchers in the field. The tenth section describes the appendixes, which are used to provide additional information on the algorithms and the computer listing.

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Algorithms for Computing the Geopotential Using a Simple-Layer Density Model

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ABSTRACT.—Several algorithms have been developed for computing the gravitational attraction of a simple-density layer; these are numerical integration, Taylor series, and mixed analytic and numerical integration of a special approximation. A computer program has been written to combine these techniques for computing the higher frequency components of the gravitational acceleration of an artificial Earth satellite. A total of 1640 equal-area, constant surface density ($5^\circ \times 5^\circ$) blocks on an oblate spheroid is used. The special approximation is used in the sub-satellite region, Taylor series in a surrounding zone, and numerical quadrature in the remaining regions. The relative sizes of these zones are readily changed. An auxiliary program can generate all the parameters for different equal-area block configurations. Different orders may be used in the numerical quadrature done in connection with the special approximation. Numerical tests comprising integrations of equations of satellite motion and static gravity simulations indicate the simple-density layer model is not only feasible, but highly practical and very easy to use.

I. INTRODUCTION

Choosing to use a density layer model to represent a gravitational potential field does not specify an algorithm for the computation. Using an analytic solution to a boundary value problem, such as spherical or spheroidal harmonics, specifies the general outline of the algorithm. Quite a bit of discretion remains to the user as to what recursion relations (if any) to apply and what normalization is most efficient: a considerable literature has developed through inquiries into these questions.

Density layer models provide us with many more options. For one thing, we must specify the surface, or boundary, upon which the density is to be defined. The shape of the surface is related to the choice of coordinates, which is another option. Finally, there is a choice of methods of representing the

density on the surface. Intricate surfaces and exotic coordinates both would require more information storage and computation to obtain the gravity vector at a specified point. The choice of analytic or discrete representations of the density upon the surface will affect the efficiency and speed of one's calculations, also.

To a large extent, the application of the gravity model is a decisive factor in its choice. A model suitable for the combination of gravimetry, and satellite derived data, which may be dynamical, geometric, or both, might be too cumbersome for computing the ephemeris of a satellite.

II. METHODS IN USE

Working directly from optical satellite observations, Koch and Morrison (1970) derived a density

layer model for the geopotential. In theory, the surface layer was on the Earth's surface, while in fact the mathematical model was equivalent to point masses fixed onto an approximate surface (Morrison 1971). Improved solutions (Koch and Witte 1971), and (Koch 1974) utilizing additional data have retained this type of algorithm.

Vinti (1971), on the other hand, has used a density layer model with a surface consisting of the smallest sphere enclosing the Earth. Vinti's model avoids the occurrence of impulses on satellite orbits coming close to any of the point masses, but it is computationally no different from using the spherical harmonic model for the geopotential. Moreover, having the surface layer above the Earth's actual surface yields a model unsuitable for computing mean gravity anomalies at the surface or for doing combination solutions.

III. APPROXIMATION BY TAYLOR SERIES

The fundamental formula for density layer models is

$$U = G \int_{\sigma} \int \frac{\chi d\sigma}{r^*} \quad (1)$$

where $\mathbf{r}^* = \mathbf{r} - \mathbf{r}_s$ (See appendix I for explanation of undefined terms). If the surface, σ , is a sphere, we may use

$$d\sigma = r^2 \cos\phi d\lambda d\phi$$

and compute (1) as an iterated integral

$$U = G \int_0^{2\pi} \int_{-\pi}^{+\pi} \frac{\chi(\phi, \lambda) r^2 \cos\phi d\lambda d\phi}{r^*} \quad (2)$$

Even if the surface is not exactly a sphere, (2) may be used provided the surface is starlike with respect to the origin. (For a starlike surface a straight line starting at the origin intersects the surface only once, i.e., the radial coordinate of the surface is a single valued function of ϕ and λ .) Even if the surface has high frequency ripples and a resulting very large surface area, the total mass of the surface will be very little affected because the surface density is divided by the slope of the topography. An analogous expression could be derived for spheroidal coordinates. Another alternative is to remove r^2 from the numerator of (2) and use density per unit of solid angle: the position of the layer would enter only through r^* . For the numerical tests made, only (2) was used.

Given (2), there are various options for expressing the integrand. The simplest, perhaps, is to write

$$h(\phi, \lambda) = \frac{G\chi(\phi, \lambda)r^2 \cos\phi}{r^*} \quad (3)$$

and expand h into a Taylor series. (See appendix II.) Since the domain of validity of such a series would be restricted, expansions would have to be done within different regions on the Earth, usually tesseræ bounded by lines of latitude and longitude. The result is of the form

$$U = \sum_i (\Delta T)_i + U_0 \quad (4)$$

$$(\Delta T)_i \equiv \iint [h_0 + h_{\phi\phi} + h_{\lambda\lambda} + \frac{1}{2}(h_{\phi\phi}\phi^2 + 2h_{\phi\lambda}\phi\lambda + h_{\lambda\lambda}\lambda^2)] d\phi d\lambda \quad (5)$$

if we truncate the series at the second-order terms. The function U_0 is a spherical harmonic expansion used for the central force term, oblateness and long-wavelength parts of the potential. Eq. (5), an integral of a polynomial, is evaluated quite easily. The derivatives are somewhat complicated, but straightforward.

To obtain the gravity force, the gradient operator is applied

$$\mathbf{g} = \nabla U, \quad (6)$$

$$\nabla = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x_s} + \hat{\mathbf{j}} \frac{\partial}{\partial y_s} + \hat{\mathbf{k}} \frac{\partial}{\partial z_s} \right)^T$$

Numerical tests have been made at altitudes of 300 and 1000 km, using $15^\circ \times 15^\circ$ and $5^\circ \times 5^\circ$ patches. Results are satisfactory for the 300-km altitude only when the smaller patches are used. This is due to the fact that the Taylor series has many properties of the point mass. Specifically, these properties are

$$h_0 = O(1/r^*)$$

$$h_{\phi\phi} = O(1/r^{*2}),$$

$$h_{\phi\lambda} = O(1/r^{*3}), \text{ etc.}$$

Nevertheless, the method is fast and accurate for altitudes above 1000 km and softens the impulses of a purely point mass approximation.

Spherical coordinates are used for the sake of rapid computation, and for the area differential the spherical approximation

$$d\sigma = r^2 \cos\phi d\lambda d\phi + O(e^2)$$

suffices. Simplifications occur if h is expanded about a point whose latitude is the average of the north and south edges of the block and similarly

for the longitude. (See fig. 1.) When h is expanded to the second order and integrated, the odd order terms drop out

$$(\Delta T)_i = \phi_B \lambda_B \left[h_0 + \frac{1}{24} (h_{\phi\phi} \phi_B^2 + h_{\lambda\lambda} \lambda_B^2) \right] \quad (7)$$

where ϕ_B and λ_B are the extent of the block in latitude and longitude.

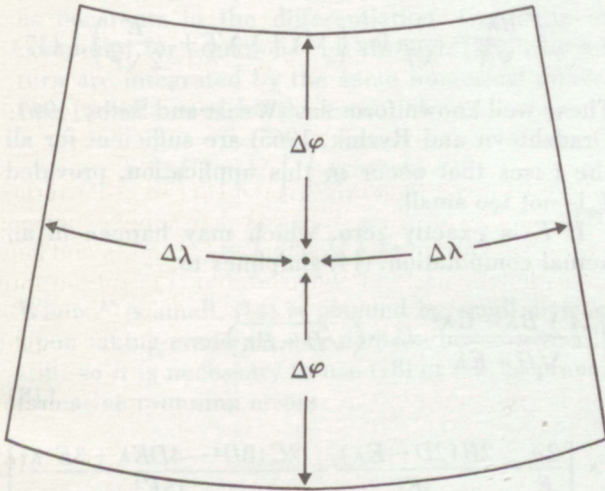


FIGURE 1.—The block over which ΔT is evaluated. The origin is in the "center."

IV. THE METHOD OF SINGULARITY-MATCHING

The limitations of the Taylor series approximations to U are caused by the presence of the square root function in the denominator of (2)

$$r^* = \sqrt{(\mathbf{r} - \mathbf{r}_s)^2}.$$

here $(\mathbf{r} - \mathbf{r}_s)^2$ is an analytic function of the coordinates everywhere, but r^* is not at $r^* = 0$, and $1/r^*$ is not well approximated by Taylor series in that region.

The method of singularity-matching provides approximation techniques for computing such things as convergent improper integrals where exact analytic solutions cannot be found. The troublesome integrand $h(x)$ is factored by an elementary function $p(x)$ with the same type of singularities so that one may write

$$h(x) = p(x) f(x). \quad (8)$$

Now $f(x)$ may be expanded as a Taylor series. The desired results may be obtained provided the expressions $x^n p(x)$, $n=0, \dots, N$ can be integrated analytically. Often this can be done by repeated application of integration by parts.

Computing the potential of a surface layer is complicated by the facts that (1) is a double integral and, in most cases, r^* is close to but not exactly zero. We begin by converting (1) to an iterated integral in the spherical coordinates ϕ and λ , as was done for a Taylor series expansion. For the factor with a singularity we choose

$$p(\lambda) = (D + E\lambda + F\lambda^2)^{-1/2},$$

where D , E and F are functions of ϕ . For practical applications at satellite altitudes, it is simplest to use the Taylor series expansion for r^{*2} to compute D , E and F ; then we have

$$f(\lambda) = G \chi r^2(\phi) \cos \phi = \text{"constant"}$$

which greatly simplifies the results. The integration with respect to ϕ now may be done by a conventional numerical quadrature. Newton-Cotes formulas of order 3 through 9 were programmed (Abramowitz and Stegun 1964).

All of the integrals with respect to λ are of the simple form

$$\int h(\lambda) d\lambda = \int \frac{(A+B\lambda+C\lambda^2)d\lambda}{\sqrt{D+E\lambda+F\lambda^2}} \quad (9)$$

which can be expressed in elementary function for various values of D , E and F , e.g., see Peirce and Foster (1956). It is more efficient to branch to different coding than to compute (9) as a function of a complex variable.

Various improvements and generalizations can be made over what has been programmed. For low altitudes, the matching of the singularity in $p(\lambda)$ and $1/r^*$ should be precise. Gaussian quadrature should be faster and more accurate than Newton-Cotes. Some of the integration with respect to ϕ also could be done analytically to save time and reduce error.

Density variations within a block and the departure of the approximation $p(\lambda)$ from $1/r^*$ can be modeled by expanding

$$f(\lambda) = \frac{G \chi r^2 \cos \phi}{p(\lambda) r^*}$$

in a Taylor series. No matter what order Taylor series is used, the integral may be done quite simply (Gradshteyn and Ryzhik 1965, p. 80), so

any level of approximation can be achieved. Even surfaces of complex shape could be modeled if it were really necessary.

If we define

$$f = G\chi r^2 \cos \phi \quad (10)$$

$$g = (\mathbf{r} - \mathbf{r}_s)^2$$

$$p = g^{-1/2} \quad (11)$$

and use the approximations

$$f = A + B\lambda + C\lambda^2$$

$$A = f_0 + f_{\phi\phi} + f_{\phi\phi\phi^2}/2 \quad (12)$$

$$B = f_{\lambda} + f_{\lambda\phi}\phi$$

$$C = \frac{1}{2}f_{\lambda\lambda}$$

$$g = D + E\lambda + F\lambda^2$$

$$D = g_0 + g_{\phi\phi} + g_{\phi\phi\phi^2}/2 \quad (13)$$

$$E = g_{\lambda} + g_{\lambda\phi}\phi$$

$$F = \frac{1}{2}g_{\lambda\lambda}$$

we can perform the integration with respect to λ in closed form. If a higher approximation to g is required, the integrals will be elliptic and a degree of simplicity will disappear. An alternative to the elliptic integrals is to subdivide the blocks. Doing one of the iterated integrals analytically removes the problem of approximating $1/r^*$ by a Taylor series. The integration with respect to ϕ precedes smoothly by numerical quadratures. Some part of the quadrature with respect to ϕ could be done analytically.

Numerical tests of the potential function have proved very promising at the 300 km altitude even with 15° patches. These results are far more promising than those obtained with the direct numerical integration algorithm (Koch 1971).

If $F \neq 0$, we may use in evaluating (9)

$$\int \frac{A + B\lambda + C\lambda^2}{\sqrt{D + E\lambda + F\lambda^2}} d\lambda \quad (14)$$

$$= \frac{B\sqrt{X}}{F} + C \left[\frac{\lambda}{2F} - \frac{3}{4} \frac{E}{F^2} \right] \sqrt{X} + Q \int \frac{d\lambda}{\sqrt{X}},$$

where $X = D + E\lambda + F\lambda^2$ and

$$Q = A - \frac{BE}{2F} + \frac{(3E^2 - 4DF)C}{8F^2}.$$

If the quantity $q \geq 0$, $q = 4DF - E^2$, then

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{F}} \sinh^{-1} \left(\frac{2F\lambda + E}{\sqrt{q}} \right). \quad (15)$$

To have $q \geq 0$, it is necessary to have $F > 0$, since $D \geq 0$ for this application. If $F < 0$, one can use

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{-F}} \sin^{-1} \left(\frac{-2F\lambda - E}{\sqrt{-q}} \right) \quad (16)$$

It may happen that $F > 0$, but $q < 0$. Such may be the case if F is relatively small and E large. In this case one has

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{F}} \log \left(\sqrt{X} + \lambda \sqrt{F} + \frac{E}{2\sqrt{F}} \right). \quad (17)$$

These well known formulas (Weast and Selby, 1967; Gradshteyn and Ryzhik, 1965) are sufficient for all the cases that occur in this application, provided F is not too small.

If F is exactly zero, which may happen in an actual computation, (14) simplifies to

$$\int \frac{A + B\lambda + C\lambda^2}{\sqrt{D + E\lambda}} d\lambda = (\sqrt{D + E\lambda}) \quad (18)$$

$$\times \left[\frac{2A}{E} - \frac{2B(2D - E\lambda)}{3E^2} + \frac{2C(8D^2 - 4DE\lambda + 3E^2\lambda^2)}{15E^3} \right]$$

Equation (18) is the application of equations 101, 102, 103 of Peirce and Foster (1956).

When both E and F are very small compared to D , (14) is computed best by expanding the denominator in the binomial series and integrating term by term. This procedure is identical in the formal sense to the Taylor series approximation, but there are differences in the numerical application. For one thing, the quantities D , E , and F , required to perform the tests indicating whether (14), (15), (16), (17), or (18) should be applied, are used at once. Also, the binomial series expansion might not be used for every value of the latitude in the second iterated integral, which is done numerically. The results are

$$\int_{-\delta\lambda}^{+\delta\lambda} (A + B\lambda + C\lambda^2) X^{-1/2} d\lambda = D^{-1/2} \left\{ 2A\delta\lambda \quad (19)\right.$$

$$+ \frac{2}{3} \delta\lambda^3 \left[C - \frac{1}{2} BE^* + \frac{3}{8} AE^{*2} - \frac{1}{2} AF^* \right]$$

$$+ \frac{\delta\lambda^5}{5} \left[\frac{3}{4} CE^{*2} + \frac{3}{4} AF^{*2} + \frac{3}{2} BE^*F^* - CF^* \right]$$

$$\left. + \frac{3}{28} \delta\lambda^7 CF^{*2} + O(\delta\lambda^9) \right\},$$

where $E^* = E/D$, $F^* = F/D$.

Equation (19) is given as a definite integral since all the even powers of $\delta\lambda$ have zero coefficients with the limits taken thus; and these limits are the ones used. All the constants of integration have been omitted from (14)–(18).

There is a definite advantage in interchanging the orders of differentiation and the integration implicit for U in (6). The gradient operator is applied inside the integration (9); the gradients of D , E , and F in (14) are required. The quantities A , B , and C do not depend on the point at which the potential or gravity is being evaluated, so they act as constants in the differentiation. Gradients are computed for equations (14) through (18), which in turn are integrated by the same numerical quadrature formula used for the potential.

$$\begin{aligned} \nabla(\Delta U) &= \nabla \int \int f g^{-1/2} d\phi d\lambda \\ &= \int d\phi \left\{ \nabla \int f g^{-1/2} d\lambda \right\}. \end{aligned} \quad (20)$$

When F is small, (14) is plagued by small divisors. Upon taking gradients, the divisors become smaller still, so it is necessary to use (18) or (19) to prevent excessive rounding errors.

To optimize the numerical integration method, Koch (1971) and Fröhlich and Koch (1974) have studied different configurations of the evaluation points. Recently, Fröhlich (1975) published some algorithms of very high accuracy.

V. PROGRAMMING OF THE ALGORITHMS

To construct a program suitable for application to anticipated high precision satellite altimetry data, the computer programs for these algorithms were changed to use a scheme of 1640 equal-area blocks on an oblate spheroid; these blocks are about $5^\circ \times 5^\circ$ at the equator and fairly uniform in shape everywhere (see figs. 2 and 3). Blocks near the poles are somewhat extended in longitude, for example, those at the poles cover 90° in longitude. For this reason these blocks are subdivided into roughly 5° longitude bands and the algorithms are applied within each subblock generated. This is illustrated in figure 4. To save computer time, different algorithms are used depending on the distance from the satellite to the center of the block.

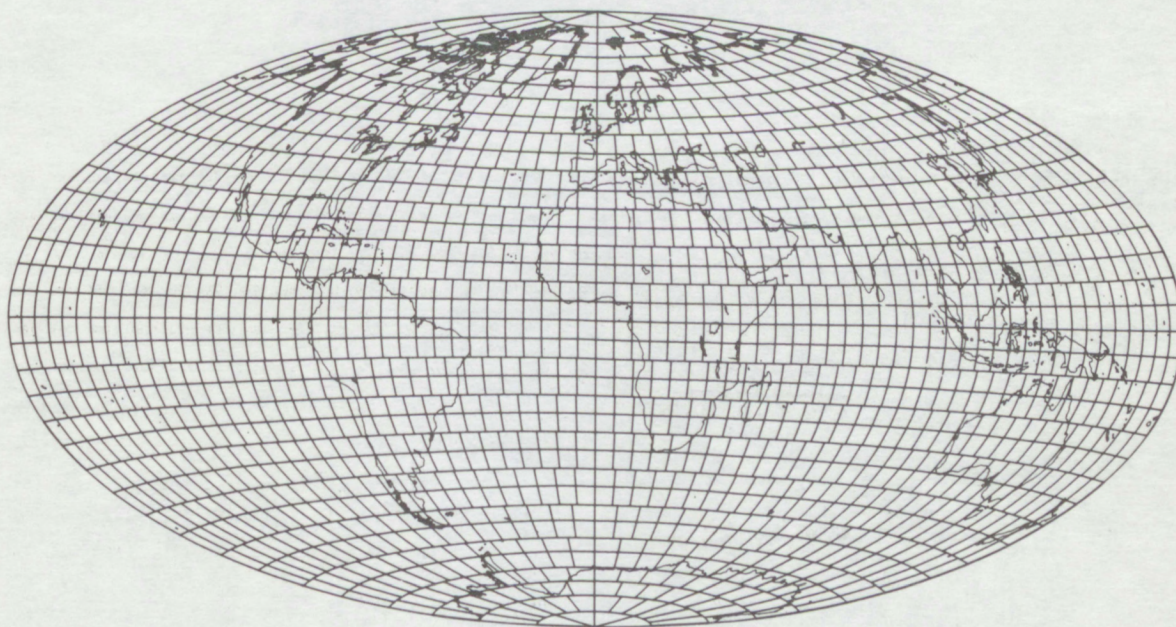


FIGURE 2.—1640 five-degree equal-area blocks. Aitoff equal-area projection.

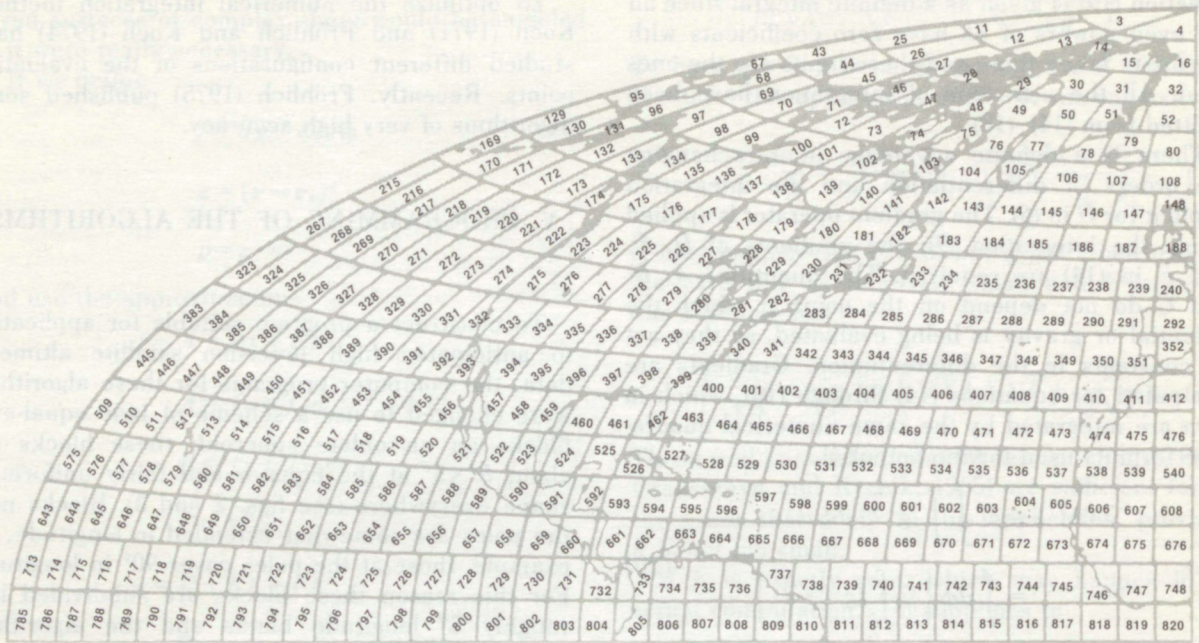


FIGURE 3a.—Equal-area blocks with index numbers. NW quadrant.

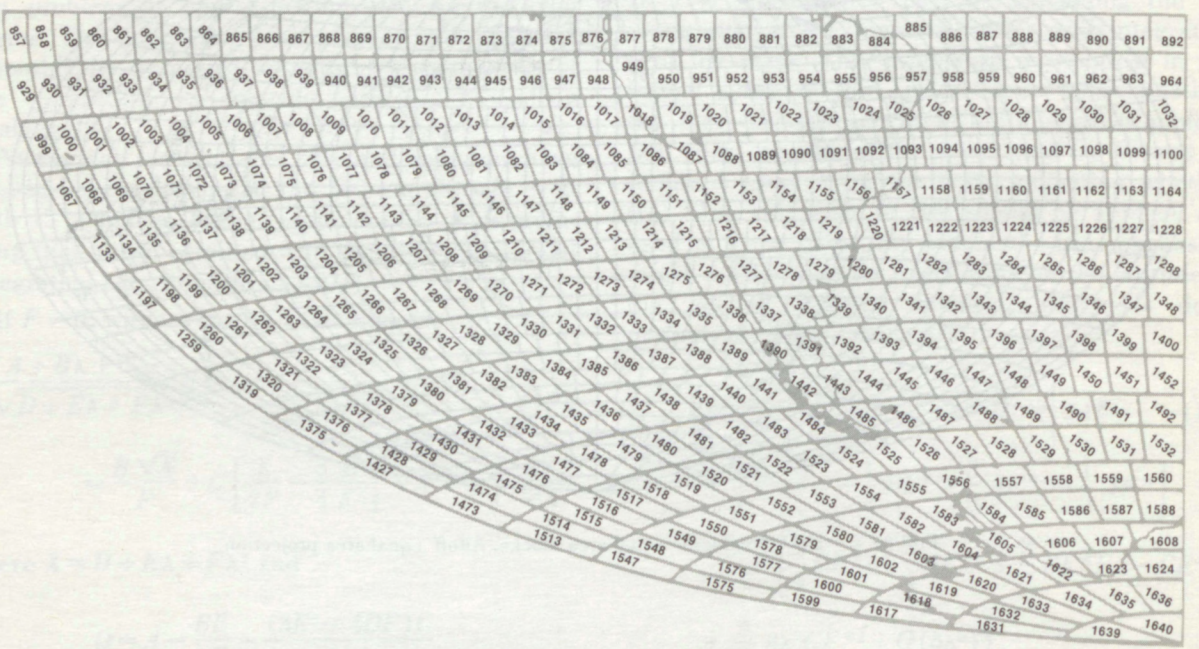


FIGURE 3b.—Equal-area blocks with index numbers. SW quadrant.

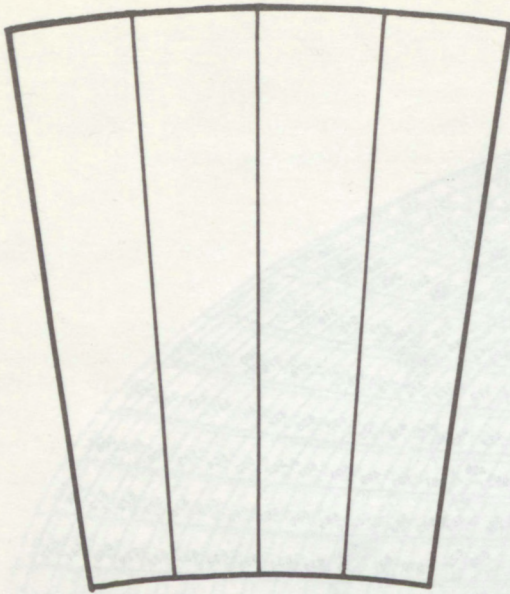


FIGURE 4.—Subdivision of blocks near pole into blocks equal in longitude and latitude.

Density is assumed constant within each block in this production version of the program, so that

$$B=C=0 \quad (21)$$

in expression (14). The method using (14) is employed for the closest blocks, Taylor series for a zone surrounding that, and numerical integration for the farthest blocks. No loss of accuracy is suffered if the distances of $R/4$ and R , (R =Earth radius), are used to define these zones. Accuracy is maintained from the lowest possible altitudes for orbiting satellites out to infinity.

For operational purposes and to facilitate incorporation into large-scale orbital analysis programs, some optimization has been done. The equal-area block system has an eight-fold symmetry which has been used to reduce storage requirements. Certain sines and cosines are handled as complex exponential functions, which can save time and memory. Some loops were written out explicitly to sacrifice memory for central processor time. It was found that the Taylor series and numerical quadrature algorithms can be computed using the short single-precision word length of the IBM 360 with double-precision accumulation of the geopotential and gravity. The more complicated formulas which arise from applying (2) require double precision at almost every step, but just for certain troublesome satellite locations. More detailed study is required on this question, though the use of double precision is a satisfactory if not optimal solution.

Some time is saved by direct application of the Newton-Raphson algorithm instead of the compiler subroutine to compute the square root needed to evaluate the distance from the satellite to a given block. Since the blocks whose contributions are evaluated successively are always adjacent, very accurate first approximations to this distance are available, except for the case of the first block, of course. The compiler function is used to initialize the process.

VI. PROGRAM TESTING

Timing and debugging tests were made using 90 satellite positions—half at 300 km, most of the rest at 1000 km and one each at 10R, 100R, 1000R, 1,0000R, and 1,000,000R. If the density on an oblate spheroid is chosen so that

$$\chi = \chi(\phi) = 1/r^2(\phi) \quad (22)$$

the potential will be nearly that due to a point mass equal to the mass of the spheroid and located at its center. This was determined through comparisons with a spherical shell of constant density. If we set $U = \mu/r_s$ in (1) and attempt to solve the integral equation for χ , we can observe that $d\sigma = r^2(\phi) \cos \phi d\phi d\lambda$, which in turn implies that (22) is an approximate solution good to an order of about e^2 . A higher order analytic solution to the problem would be useful for testing, but none has as yet been found either through attempts at solution or in the literature.

The program has been inserted as a force model in two orbit computation programs. The first is a small program on the CDC 6600 that generates only orbital elements. The geopotential modeling program was then converted for use on the IBM 360, optimized, and inserted into the well-known GEODYN program, which is now operational on the NOAA IBM 360-195 at Suitland, Md.

Program linkage and compatibility have been tested by using (22) as the density distribution and integrating a two-body problem perturbed only by the computational errors in the surface density model program. The results in Keplerian elements a , e , and I are given in figures 5, 6, and 7. Both the semi-major axis, a , and the eccentricity, e , exhibit periodic changes only, at least for the 9 hour period for which the integration was done. Since the mean value of a obviously is not the same as the initial value, drift occurs in the mean anomaly. In practice, the process of orbital adjustment would rectify this problem. A very large secular or long-period perturbation is present in the inclination, I . Runs made (on the CDC 6600) with an inclination of 59.4° did not exhibit this effect, so it is most

likely the result of resonance caused by the truncation errors in the force model. A force model with zero total density and with no errors equivalent to low order harmonics would be much less prone to cause such problems (Morrison 1972). This is a very severe test in many ways. The model will be used with a spherical harmonic model of, perhaps, degree and order eight, so it will be used to model the last few parts per million of the gravity field. A thorough analysis of the dynamical effects of these computational errors on an actual satellite orbit remains to be done. A qualitative analytic analysis of the problem has been done by Morrison (1972). More computer simulations and a numerical verification of that study would be desirable.

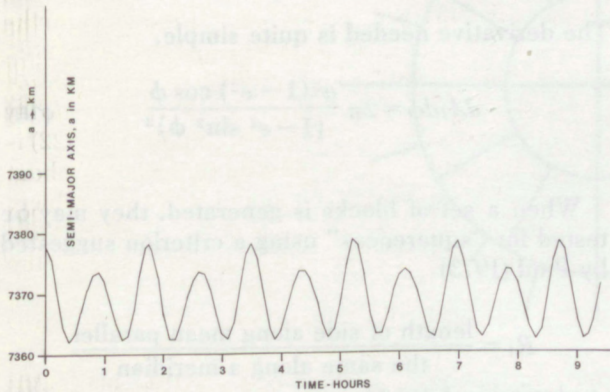


FIGURE 5.—The semi-major axis of the orbit of a particle moving in a central force field modeled by the surface density algorithm and equation (22).

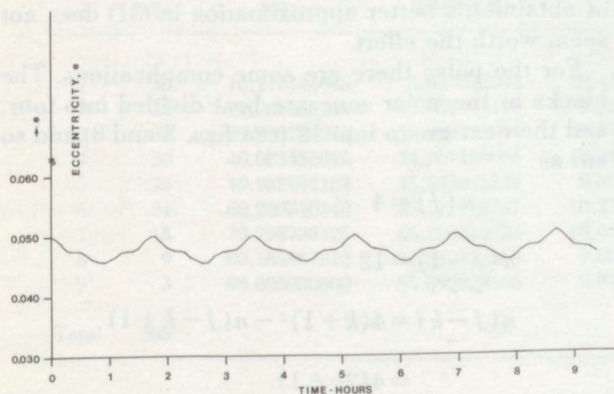


FIGURE 6.—The eccentricity of the orbit of the particle.

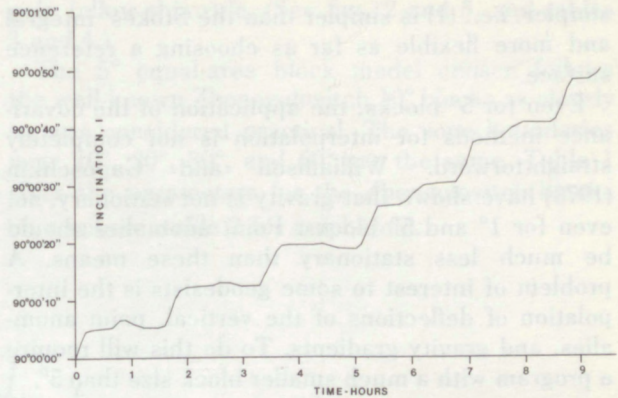


FIGURE 7.—The inclination of the orbit of the particle.

VII. APPLICATIONS

Most of the work in developing the density layer method of geopotential modeling has been devoted to finding approximation techniques for computing (1). Inasmuch as this is part of the art of computing, as opposed to a purely mathematical problem, it is "solved" in a satisfactory way, but time and experience will bring improvements of some sort. More analysis and programming are needed for adjusting the density values from combinations of various kinds of data.

The present program, using 5° blocks, is suitable for using satellite tracking, satellite altimetry, and gravity anomalies which have been averaged over areas of about 5° "squares." The model corresponds, more or less, to a spherical harmonic expansion to degree and order 36. A study of the frequency response of the model is included in Morrison (1976). Since the data quantities and density values can be related through linear integral transforms, either exactly or to a high degree of approximation, the data can be combined in a consistent and optimum way by covariance methods. The density layer method has been used by advocates of discrete variable representations, whereas the techniques known as least-squares collocation and spherical harmonic sampling functions have been used, basically, orthogonal function interpolation. The motivation to develop the discrete approach has been mostly intuitive, based on the observation that when you have to add a new term to your expansion in eigenfunctions for every new data point, you are making unnecessary work for yourself and your computer, and a discrete variable method is better. It seems best to represent the long-wave portions of the geopotential in spherical harmonics and the short-wave parts with the surface coating. The advantage of the surface coating over a discrete variable representation of the gravity is that the upward continuation transformation is much

simpler, i.e., (1) is simpler than the Stokes' integral and more flexible as far as choosing a reference surface.

Even for 5° blocks, the application of the covariance methods for interpolation is not completely straightforward. Williamson and Gaposchkin (1973) have shown that gravity is not stationary, not even for 1° and 5° blocks. Point anomalies should be much less stationary than these means. A problem of interest to some geodesists is the interpolation of deflections of the vertical, point anomalies, and gravity gradients. To do this will require a program with a much smaller block size than 5°.

VIII. CONSTRUCTING THE EQUAL-AREA BLOCKS

Equal-area blocks have been a popular means for determining a uniform sampling of data distributed over spherelike domains (Rapp, 1971). Making the blocks exactly equal in area is fairly simple for spheres or oblate spheroids, and eliminates the need to store and reference a lot of weight factors in operations such as numerical integration.

The area of the zone between the equator and latitude ϕ on an oblate spheroid is given by Unguendoli (1972):

$$A(\phi) = 2\pi a^2(1-e^2) \int_0^\phi \frac{\cos \xi d\xi}{(1-e^2 \sin^2 \xi)^2} \quad (23)$$

where a = semi-major axis, and e = meridian eccentricity. There is no need to resort to series to evaluate (23), since the integrand is a rational function if we use the substitution

$$x = e \sin \xi. \quad (24)$$

The result is

$$A(\phi) = \pi a^2(1-e^2) \left[\frac{\sin \phi}{1-e^2 \sin^2 \phi} + \frac{1}{2e} \log \frac{1+e \sin \phi}{1-e \sin \phi} \right] \quad (25)$$

Strictly speaking, (25) is not defined for $e=0$, but

$$\lim_{e \rightarrow 0} A(\phi) = 2\pi a^2 \sin \phi, \quad (26)$$

which is the correct result for a sphere. For small values of e it is necessary to evaluate the logarithm portion of the expression by series and cancel the divisor e to avoid loss of numerical significance.

To establish a system of equal-area blocks, one needs first to pick a block size, say 5° × 5°. Then a number of zones are chosen, $J=18=90^\circ/5^\circ$ for one hemisphere (the other hemisphere can be obtained by reflection). One next chooses the number of blocks in a zone, starting from

$$n(i) = \left[\frac{360^\circ}{5^\circ} \cos \langle \phi_i \rangle \right], \quad (27)$$

where the zones are numbered starting from the equator: $\langle \phi_i \rangle$ = mean latitude in zone i (and for 5° any other appropriate value may be substituted); the $[\]$ symbolizes truncation to the nearest integer. The area of the zone from 0 to ϕ_j is then

$$A(\phi_j) = \frac{\sum_{i=1}^j n(i)}{\sum_{i=1}^j n(i)} A(90^\circ). \quad (28)$$

To find ϕ_j one substitutes $A(\phi_j)$ from (25) into (28) and uses the Newton-Raphson iteration, since the transcendental equation is not readily solved. For a starting value the spherical approximation may be used.

$$(\phi_j)_0 = \sum_{i=1}^j n(i) / \sum_{i=1}^j n(i). \quad (29)$$

The derivative needed is quite simple.

$$dA/d\phi = 2\pi \frac{a^2(1-e^2) \cos \phi}{(1-e^2 \sin^2 \phi)^2}. \quad (30)$$

When a set of blocks is generated, they may be tested for "squareness" using a criterion suggested by Paul (1973)

$$R_i = \frac{\text{length of side along mean parallel}}{\text{the same along a meridian}} \approx 1. \quad (31)$$

If the values of R_i are not satisfactory one may change the values of $n(i)$ in a trial-and-error way to see if improvement is possible. Setting up a rigorous adjustment of the $n(i)$ by some criterion of obtaining a better approximation in (31) does not seem worth the effort.

For the poles there are some complications. The blocks at the polar zone are best divided into four, and the next group into 12 (see figs. 3 and 8) and so on, as

$$\begin{aligned} n(J) &= 4 \\ n(J-1) &= 12 \\ &\dots \\ n(J-k) &= 4(k+1)^2 - n(J-k+1) \\ &= 4(2k+1) \\ k &\ll J. \end{aligned}$$

This is because the zones are nearly concentric rings at the poles and for those regions

$$R_{j-k} \approx \pi/4$$

gives a better criterion for "squareness." For blocks of 5° only, the two zones nearest either

pole follow this rule. (See figs. 2 and 3, and tables 3 and 4.)

The 5° equal-area block model chosen follows the well-known Zhongolovitch 10° blocks as closely as was considered practical. The zone boundaries near 10°, 20°, 30°, and 60° are the same. Table 1 gives the parameters for the Zhongolovitch blocks on a sphere, table 2 for a spheroid.

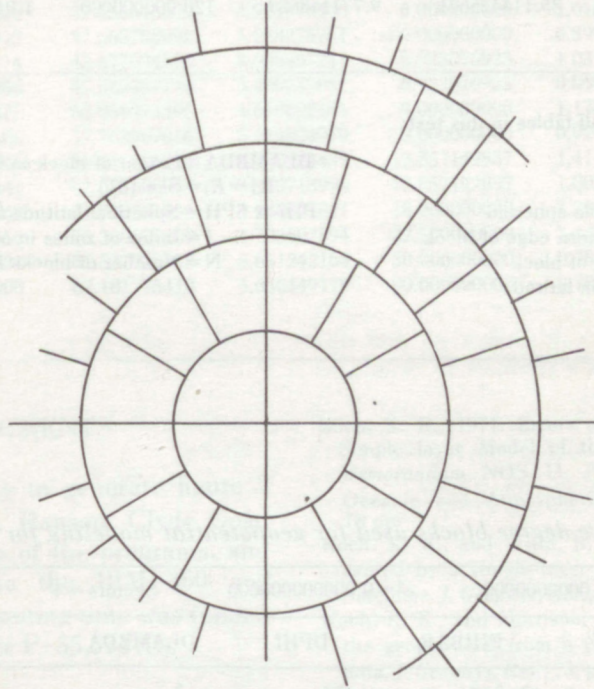


FIGURE 8. — Idealized set of equal-area blocks at pole.

TABLE 1. — Zhongolovitch 10° equal-area blocks for the sphere

$a = 1.0000000000$		$b = 1.0000000000$		$e^2 = 0.000000000000$		Sectors = 1		Area = 0.03064968442527	
J	N	PHI-2	PHIBAR	DPHI	DLAMBDA	RJ	PHI-2 SPH		
1	36	10.114144088	5.057072044	10.114144088	10.000000000	0.984865728	10.114144088		
2	34	19.966058297	15.040101193	9.851914209	10.588235294	1.037923106	19.966058297		
3	32	29.838766211	24.902412254	9.872707913	11.250000000	1.033560986	29.838766211		
4	30	40.083433915	34.961100063	10.244667704	12.000000000	0.959962351	40.083433915		
5	25	49.982997154	45.033215534	9.899563239	14.400000000	1.027967879	49.982997154		
6	21	60.260840340	55.121918747	10.277843186	17.142857143	0.953783344	60.260840340		
7	15	70.298788126	65.279814233	10.037947786	24.000000000	0.999854867	70.298788126		
8	9	80.185857012	75.242322569	9.887068886	40.000000000	1.030564385	80.185857012		
9	3	90.000000000	85.092928506	9.814142988	120.000000000	1.045917823	90.000000000		
Total	205								

TABLE 2.—Zhongolovitch 10° equal-area blocks for the spheroid

$a = 1.0000000000$		$b = 0.9966470765$		$e^2 = 0.006694605000$		Sectors = 1		Area = 0.03058119674137	
J	N	PHI-2	PHIBAR	DPHI	DLAMBDA	RJ	PHI-2 SPH		
1	36	10.158613279	5.079306640	10.158613279	10.000000000	0.980520745	10.092007853		
2	34	20.048575322	15.103594300	9.889962043	10.588235294	1.033621594	19.924951454		
3	32	29.949659806	24.999117564	9.901084484	11.250000000	1.029789802	29.783459904		
4	30	40.209905973	35.079782889	10.260246167	12.000000000	0.957114533	40.020265940		
5	25	50.109306146	45.159606059	9.899400173	14.400000000	1.025712019	49.919815831		
6	21	60.371219506	55.240262826	10.261913360	17.142857143	0.952431222	60.205546609		
7	15	70.380067424	65.375643465	10.008847918	24.000000000	0.999117443	70.258024753		
8	9	80.228851341	75.304459382	9.848783917	40.000000000	1.030310595	80.164278502		
9	3	90.000000000	85.114425671	9.771148659	120.000000000	1.045929007	90.000000000		
Total	205								

The following legend is used for all tables in this text:

- a = Semi-major axis
- b = Semi-minor axis
- e = Eccentricity of the spheroid
- PHI 2 = Latitude of northern edge of block
- PHIBAR = Average latitude of block
- DPHI = Extent of block in latitude
- DLAMBDA = Extent of block in longitude
- RJ = R_i = See (31)
- PHI-2 SPH = Spherical latitude for PHI 2
- J = Index of zones in one hemisphere
- N = Number of blocks in one sector of a zone

TABLE 3.—Five-degree blocks used for geopotential modeling for the sphere

$a = 1.0000000000$		$b = 1.0000000000$		$e^2 = 0.000000000000$		Sectors = 4		Area = 0.07662421106316	
J	N	PHI-2	PHIBAR	DPHI	DLAMBDA	RJ	PHI-2 SPH		
1	18	5.037335850	2.518667925	5.037335850	5.000000000	0.991629292	5.037335850		
2	18	10.114144088	7.575739969	5.076808238	5.000000000	0.976274270	10.114144088		
3	17	14.983246004	12.548695046	4.869101916	5.294117647	1.061314857	14.983246004		
4	17	19.966058297	17.474652151	4.982812293	5.294117647	1.013442458	19.966058297		
5	16	24.803794163	22.384926230	4.837735865	5.625000000	1.075117645	24.803794163		
6	16	29.838766211	27.321280187	5.034972048	5.625000000	0.992560304	29.838766211		
7	15	34.801265705	32.320015958	4.962499494	6.000000000	1.021753384	34.801265705		
8	15	40.083433915	37.442349810	5.282168210	6.000000000	0.901863137	40.083433915		
9	13	45.017042172	42.550238043	4.933608258	6.923076923	1.033751458	45.017042172		
10	13	50.419641072	47.718341622	5.402598900	6.923076923	0.862118040	50.419641072		
11	10	55.035992386	52.727816729	4.616351314	9.000000000	1.180676862	55.035992386		
12	10	60.260840340	57.648416363	5.224847954	9.000000000	0.921752781	60.260840340		
13	7	64.480541509	62.370690925	4.219701169	12.857142857	1.413012628	64.480541509		
14	7	69.485799940	66.983170724	5.005258431	12.857142857	1.004376110	69.485799940		
15	5	73.940639486	71.713219713	4.454839546	18.000000000	1.267816931	73.940639486		
16	4	78.662967149	76.301803318	4.722327663	22.500000000	1.128293217	78.662967149		
17	3	84.338423235	81.500695192	5.675456086	30.000000000	0.781245113	84.338423235		
18	1	90.000000000	87.169211617	5.661576765	90.000000000	0.785078675	90.000000000		
Total	$205 \times 4 = 1640$								

TABLE 4. — Five-degree blocks used for geopotential modeling for the spheroid

$a = 1.000000000$		$b = 0.9966470765$		$e^2 = 0.006694605000$		Sectors = 4	Area = 0.07645299185343
J	N	PHI-2	PHIBAR	DPHI	DLAMBDA	RJ	PHI-2 SPH
1	18	5.059836888	2.529918444	5.059836888	5.000000000	0.987210982	5.026137369
2	18	10.158613279	7.609225084	5.098776391	5.000000000	0.971992256	10.092007853
3	17	15.047475056	12.603044168	4.888861777	5.294117647	1.056801567	14.951263466
4	17	20.048575322	17.548025189	5.001100265	5.294117647	1.009328610	19.924951454
5	16	24.901674946	22.475125134	4.853099625	5.625000000	1.071017871	24.755008330
6	16	29.949659806	27.425667376	5.047984859	5.625000000	0.989068218	29.783459904
7	15	34.921630212	32.435645009	4.971970406	6.000000000	1.018502939	34.741194898
8	15	40.209905973	37.565768092	5.288275761	6.000000000	0.899333629	40.020265940
9	13	45.145343214	42.677624593	4.935437241	6.923076923	1.031256833	44.952912706
10	13	50.545592266	47.845467740	5.400249052	6.923076923	0.860386605	50.356634602
11	10	55.156394527	52.850993396	4.610802261	9.000000000	1.178755745	54.975721014
12	10	60.371219506	57.763807016	5.214824979	9.000000000	0.920586272	60.205546609
13	7	64.580143945	62.475681725	4.208924439	12.857142857	1.411668910	64.430619998
14	7	69.569857941	67.075000943	4.989713996	12.857142857	1.003702695	69.443646351
15	5	74.008705762	71.789281851	4.438847821	18.000000000	1.267271917	73.906491693
16	4	78.712308656	76.360507209	4.703602894	22.500000000	1.128022569	78.638205110
17	3	84.363550821	81.537929738	5.651242164	30.000000000	0.781180387	84.325809778
18	1	90.000000000	87.181775410	5.636449179	90.000000000	0.785081504	90.000000000

Total = 205 × 4 = 1640

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APPENDIX I. NOTATION

Most notations in this paper are standard: symbols undefined in the text are:

G	gravitational constant
\mathbf{r}	position vector on the Earth
\mathbf{r}_s	position of a point where potential is computed
x_s, y_s, z_s	components of \mathbf{r}_s
T	anomalous gravitational potential
U	gravitational potential
x, ξ	dummy variables
λ	longitude
ϕ	latitude (spherical coordinate)
χ	density
σ	surface area

APPENDIX II. DETAILED MATHEMATICAL DEVELOPMENTS

This section is comprised of a detailed presentation of all formulas used. In some cases, options not used in the programs are described so that this material will be suitable for a number of programs tailored for specific purposes.

1. THE TAYLOR SERIES METHOD

A. Development of the Potential

$$T = \sum \Delta T$$

$$\Delta T = \int_{-\Delta\phi}^{\Delta\phi} \int_{-\Delta\lambda}^{\Delta\lambda} \frac{\chi(\phi, \lambda) r^2 \cos \phi}{r^*} d\lambda d\phi \quad (1.1)$$

where

$$\mathbf{r}^* = \mathbf{r} - \mathbf{r}_s$$

and $r^* = \|\mathbf{r}^*\|$

The integrand is abbreviated to

$$f = \frac{\chi r^2 \cos \phi}{r^*} \quad (1.2)$$

and

$$\frac{\partial f}{\partial \phi} = -\frac{\chi r^2 \sin \phi}{r^*} - \frac{\partial r^*}{\partial \phi} \frac{\chi r^2 \cos \phi}{r^{*2}} + \frac{\cos \phi}{r^*} \left[r^2 \frac{\partial \chi}{\partial \phi} + 2r\chi \frac{\partial r}{\partial \phi} \right] \quad (1.3)$$

$$\frac{\partial f}{\partial \lambda} = -\frac{\partial r^*}{\partial \lambda} \frac{\chi r^2 \cos \phi}{r^{*2}} + \frac{\cos \phi}{r^*} \left[r^2 \frac{\partial \chi}{\partial \lambda} + 2r\chi \frac{\partial r}{\partial \lambda} \right] \quad (1.4)$$

$$\begin{aligned} \frac{\partial^2 f}{\partial \phi^2} &= 2 \left(\frac{r}{r^*} \right)^2 \chi \sin \phi \frac{\partial r^*}{\partial \phi} - 2 \frac{r^2}{r^*} \sin \phi \frac{\partial \chi}{\partial \phi} \\ &\quad - 4\chi \frac{r}{r^*} \sin \phi \frac{\partial r}{\partial \phi} - \frac{r^2}{r^*} \chi \cos \phi \\ &\quad + 2\chi \frac{r^2}{r^{*3}} \cos \phi \left(\frac{\partial r^*}{\partial \phi} \right)^2 - \chi \left(\frac{r}{r^*} \right)^2 \cos \phi \frac{\partial^2 r^*}{\partial \phi^2} \\ &\quad - 2 \frac{\cos \phi}{r^{*2}} \frac{\partial r^*}{\partial \phi} \left[r^2 \frac{\partial \chi}{\partial \phi} + 2r\chi \frac{\partial r}{\partial \phi} \right] \\ &\quad + \frac{\cos \phi}{r^*} \left[4r \frac{\partial r}{\partial \phi} \frac{\partial \chi}{\partial \phi} + r^2 \frac{\partial^2 \chi}{\partial \phi^2} + 2\chi \left(\frac{\partial r}{\partial \phi} \right)^2 \right. \\ &\quad \left. + 2r\chi \frac{\partial^2 r}{\partial \phi^2} \right] \quad (1.5) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial \phi \partial \lambda} &= 2\chi \frac{r^2}{r^{*3}} \cos \phi \frac{\partial r^*}{\partial \phi} \frac{\partial r^*}{\partial \lambda} + \chi \left(\frac{r}{r^*} \right)^2 \sin \phi \frac{\partial r^*}{\partial \lambda} \\ &\quad - \chi \left(\frac{r}{r^*} \right)^2 \cos \phi \frac{\partial^2 r^*}{\partial \phi \partial \lambda} - \frac{r}{r^{*2}} \cos \phi \left\{ \left[r \frac{\partial \chi}{\partial \phi} \right. \right. \\ &\quad \left. \left. + 2\chi \frac{\partial r}{\partial \phi} \right] \frac{\partial r^*}{\partial \lambda} + \left[r \frac{\partial \chi}{\partial \lambda} + 2\chi \frac{\partial r}{\partial \lambda} \right] \frac{\partial r^*}{\partial \phi} \right\} \\ &\quad - \frac{r}{r^*} \sin \phi \left[r \frac{\partial \chi}{\partial \lambda} + 2\chi \frac{\partial r}{\partial \lambda} \right] \\ &\quad + \frac{\cos \phi}{r^*} \left[2r \frac{\partial r}{\partial \phi} \frac{\partial \chi}{\partial \lambda} + r^2 \frac{\partial^2 \chi}{\partial \phi \partial \lambda} + 2r\chi \frac{\partial^2 r}{\partial \phi \partial \lambda} \right. \\ &\quad \left. + 2\chi \frac{\partial r}{\partial \phi} \frac{\partial r}{\partial \lambda} + 2r \frac{\partial \chi}{\partial \phi} \frac{\partial r}{\partial \lambda} \right] \quad (1.6) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial \lambda^2} &= 2\chi \frac{r^2}{r^{*3}} \cos \phi \left(\frac{\partial r^*}{\partial \lambda} \right)^2 - \chi \frac{r^2}{r^{*2}} \cos \phi \frac{\partial^2 r^*}{\partial \lambda^2} \\ &\quad - 2 \frac{\cos \phi}{r^{*2}} \frac{\partial r^*}{\partial \lambda} \left[2r\chi \frac{\partial r}{\partial \lambda} + r^2 \frac{\partial \chi}{\partial \lambda} \right] \\ &\quad + \frac{\cos \phi}{r^*} \left[4r \frac{\partial r}{\partial \lambda} \frac{\partial \chi}{\partial \lambda} + r^2 \frac{\partial^2 \chi}{\partial \lambda^2} + 2r\chi \frac{\partial^2 r}{\partial \lambda^2} \right. \\ &\quad \left. + 2\chi \left(\frac{\partial r}{\partial \lambda} \right)^2 \right] \quad (1.7) \end{aligned}$$

$$\mathbf{r}^* = \mathbf{r} - \mathbf{r}_s \quad (1.8)$$

$$\begin{aligned} r^* &= \|\mathbf{r}^*\| \\ &= (\mathbf{r}^* \cdot \mathbf{r}^*)^{1/2} \quad (1.9) \end{aligned}$$

$$\mathbf{r}_s = (x_s, y_s, z_s)^T \quad (1.10)$$

The vector \mathbf{r}_s is the satellite position, \mathbf{r} the coordinates of the increment $d\sigma$ on the reference surface.

$$\mathbf{r} = r(\cos \phi \cos \lambda, \cos \phi \sin \lambda, \sin \phi)^T \quad (1.11)$$

$$\frac{\partial \mathbf{r}^*}{\partial \phi} = \frac{\mathbf{r}^*}{r^*} \cdot \frac{\partial \mathbf{r}}{\partial \phi} \quad (1.12)$$

$$\frac{\partial \mathbf{r}^*}{\partial \lambda} = \frac{\mathbf{r}^*}{r^*} \cdot \frac{\partial \mathbf{r}}{\partial \lambda} \quad (1.13)$$

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \phi} &= r (-\sin \phi \cos \lambda, -\sin \phi \sin \lambda, \cos \phi)^T \\ &+ \frac{\partial r}{\partial \phi} (\cos \phi \cos \lambda, \cos \phi \sin \lambda, \sin \phi)^T \end{aligned} \quad (1.14)$$

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \lambda} &= r (-\cos \phi \sin \lambda, \cos \phi \cos \lambda, 0)^T \\ &+ \frac{\partial r}{\partial \lambda} \frac{\mathbf{r}}{r} \end{aligned} \quad (1.15)$$

$$\frac{\partial^2 \mathbf{r}^*}{\partial \phi^2} = \frac{1}{r^*} \left[r^2 + \mathbf{r}^* \cdot \frac{\partial^2 \mathbf{r}}{\partial \phi^2} - \left(\frac{\partial r^*}{\partial \phi} \right)^2 \right] \quad (1.16)$$

$$\frac{\partial^2 \mathbf{r}^*}{\partial \phi \partial \lambda} = \frac{1}{r^*} \left[\frac{\partial \mathbf{r}}{\partial \phi} \cdot \frac{\partial \mathbf{r}}{\partial \lambda} + \mathbf{r}^* \cdot \frac{\partial^2 \mathbf{r}}{\partial \phi \partial \lambda} - \frac{\partial r^*}{\partial \phi} \frac{\partial r^*}{\partial \lambda} \right] \quad (1.17)$$

$$\frac{\partial^2 \mathbf{r}^*}{\partial \lambda^2} = \frac{1}{r^*} \left[\left(\frac{\partial \mathbf{r}}{\partial \lambda} \right)^2 + \mathbf{r}^* \cdot \frac{\partial^2 \mathbf{r}}{\partial \lambda^2} - \left(\frac{\partial r^*}{\partial \lambda} \right)^2 \right] \quad (1.18)$$

$$\begin{aligned} \frac{\partial^2 \mathbf{r}}{\partial \phi^2} &= -\mathbf{r} + 4 \frac{\mathbf{r}}{r} \frac{\partial^2 r}{\partial \phi^2} \\ &+ 2 \frac{\partial r}{\partial \phi} (-\sin \phi \cos \lambda, -\sin \phi \sin \lambda, \cos \phi)^T \end{aligned} \quad (1.19)$$

$$\begin{aligned} \frac{\partial^2 \mathbf{r}}{\partial \phi \partial \lambda} &= r (\sin \phi \sin \lambda, -\sin \phi \cos \lambda, 0)^T \\ &+ \frac{\partial r}{\partial \phi} (-\cos \phi \sin \lambda, \cos \phi \cos \lambda, 0)^T \end{aligned} \quad (1.20)$$

$$\frac{\partial^2 \mathbf{r}}{\partial \lambda^2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{r} \quad (1.21)$$

Most terms with $\partial^2 r / \partial \lambda^2$ are omitted.

Substitute (1.2) into (1.1) and expand by Taylor's theorem:

$$\begin{aligned} \Delta T &= \int_{-\Delta \phi}^{\Delta \phi} \int_{-\Delta \lambda}^{\Delta \lambda} \left\{ f_0 + \frac{\partial f}{\partial \phi} (\phi - \phi_0) + \frac{\partial f}{\partial \lambda} (\lambda - \lambda_0) \right. \\ &+ \frac{1}{2} \left[\frac{\partial^2 f}{\partial \phi^2} (\phi - \phi_0)^2 + 2 \frac{\partial^2 f}{\partial \phi \partial \lambda} (\phi - \phi_0) (\lambda - \lambda_0) \right. \\ &\left. \left. + \frac{\partial^2 f}{\partial \lambda^2} (\lambda - \lambda_0)^2 \right] + \dots \right\} d\lambda d\phi. \end{aligned} \quad (1.22)$$

All the terms containing *odd* powers of $(\phi - \phi_0)$

or $(\lambda - \lambda_0)$ integrate out to zero, which includes all the third-order terms. Hence, we can find

$$\begin{aligned} \Delta T &= \Delta \phi \Delta \lambda \left[4f_0 + \frac{2}{3} \frac{\partial^2 f}{\partial \phi^2} \Delta \phi^2 + \frac{2}{3} \frac{\partial^2 f}{\partial \lambda^2} \Delta \lambda^2 \right. \\ &\left. + O(\Delta \phi^4, \Delta \lambda^4) \right]. \end{aligned} \quad (1.23)$$

B. The Expressions for the Gravity Vector

The formula

$$\Delta \mathbf{g} = \nabla T \quad (1.24)$$

is fundamental.

Substitute (1.23) and (1.1) into (1.24)

$$\Delta \mathbf{g} = \nabla \sum \Delta T \quad (1.25)$$

$$\begin{aligned} &= \sum \left[4 \nabla f_0 + \frac{2}{3} \nabla f_{\phi\phi} \Delta \phi^2 + \frac{2}{3} \nabla f_{\lambda\lambda} \Delta \lambda^2 \right] \Delta \phi \Delta \lambda \\ &\quad (1.25a) \end{aligned}$$

$$\nabla f_0 = \chi r^2 \cos \phi \frac{\mathbf{r}^*}{r^{*3}} \quad (1.26)$$

The only factor of (1.2) that is not constant with respect to ∇ is $1/r^*$, and ∇ operates on (1.5) and (1.7) similarly. Actually, (1.6) is not needed and (1.3) and (1.4) are only steps to derive (1.5) and (1.7).

Gradients of derivatives of r^*

$$\nabla \left(\frac{\partial r^*}{\partial \phi} \right) = -\frac{1}{r^*} \frac{\partial \mathbf{r}}{\partial \phi} + \frac{\mathbf{r}^*}{r^{*2}} \frac{\partial r^*}{\partial \phi} \quad (1.27)$$

$$\nabla \left(\frac{\partial r^*}{\partial \lambda} \right) = -\frac{1}{r^*} \frac{\partial \mathbf{r}}{\partial \lambda} + \frac{\mathbf{r}^*}{r^{*2}} \frac{\partial r^*}{\partial \lambda} \quad (1.28)$$

$$\begin{aligned} \nabla \left(\frac{\partial^2 r^*}{\partial \phi^2} \right) &= \frac{\mathbf{r}^*}{r^{*2}} \left(\frac{\partial^2 r^*}{\partial \phi^2} \right) - \frac{1}{r^*} \left[\frac{\partial^2 \mathbf{r}}{\partial \phi^2} \right. \\ &\left. + 2 \frac{\partial r^*}{\partial \phi} \nabla \left(\frac{\partial r^*}{\partial \phi} \right) \right] \end{aligned} \quad (1.29)$$

$$\begin{aligned} \nabla \left(\frac{\partial^2 r^*}{\partial \lambda^2} \right) &= \frac{\mathbf{r}^*}{r^{*2}} \left(\frac{\partial^2 r^*}{\partial \lambda^2} \right) - \frac{1}{r^*} \left[\frac{\partial^2 \mathbf{r}}{\partial \lambda^2} \right. \\ &\left. + 2 \frac{\partial r^*}{\partial \lambda} \nabla \left(\frac{\partial r^*}{\partial \lambda} \right) \right] \end{aligned} \quad (1.30)$$

We may factor out seven distinct functions of r^* from $f_{\phi\phi}$ and $f_{\lambda\lambda}$. It is convenient for program coding to write these as elements of 4×4 matrices Φ and Λ .

$$\Phi_{11} = 1/r^* \quad (1.31a)$$

$$\Phi_{12} = \frac{1}{r^{*2}} \frac{\partial r^*}{\partial \phi} \quad (1.31b)$$

$$\Phi_{13} = \frac{1}{r^{*2}} \frac{\partial^2 r^*}{\partial \phi^2} \quad (1.31c)$$

$$\Phi_{14} = \frac{1}{r^{*3}} \left(\frac{\partial r^*}{\partial \phi} \right)^2 \quad (1.31d)$$

$$\Lambda_{11} = \Phi_{11} \quad (1.32a)$$

$$\Lambda_{12} = \frac{1}{r^{*2}} \frac{\partial r^*}{\partial \lambda} \quad (1.32b)$$

$$\Lambda_{13} = \frac{1}{r^{*2}} \frac{\partial^2 r^*}{\partial \lambda^2} \quad (1.32c)$$

$$\Lambda_{14} = \frac{1}{r^{*3}} \left(\frac{\partial r^*}{\partial \lambda} \right)^2 \quad (1.32d)$$

Then we can write

$$(f_{\phi\phi}, \nabla f_{\phi\phi})^T = \mathbf{\Phi} \boldsymbol{\alpha} \quad (1.33)$$

$$(f_{\lambda\lambda}, \nabla f_{\lambda\lambda})^T = \mathbf{\Lambda} \boldsymbol{\beta}. \quad (1.34)$$

with $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ being 4-vectors:

$$\begin{aligned} \alpha_1 = & -2 \frac{\partial \chi}{\partial \phi} r^2 \sin \phi - 4\chi r \sin \phi \frac{\partial r}{\partial \phi} - \chi r^2 \cos \phi \\ & + \cos \phi \left[4r \frac{\partial r}{\partial \phi} \frac{\partial \chi}{\partial \phi} + r^2 \frac{\partial^2 \chi}{\partial \phi^2} + 2\chi \left(\frac{\partial r}{\partial \phi} \right)^2 \right. \\ & \left. + 2r\chi \frac{\partial^2 r}{\partial \phi^2} \right] \quad (1.35a) \end{aligned}$$

$$\alpha_2 = 2\chi r^2 \sin \phi - 2 \cos \phi \left[r^2 \frac{\partial \chi}{\partial \phi} + 2r\chi \frac{\partial r}{\partial \phi} \right] \quad (1.35b)$$

$$\alpha_3 = -\chi r^2 \cos \phi \quad (1.35c)$$

$$\alpha_4 = 2\chi r^2 \cos \phi \quad (1.35d)$$

$$\begin{aligned} \beta_1 = & \cos \phi \left[4r \frac{\partial r}{\partial \lambda} \frac{\partial \chi}{\partial \lambda} + r^2 \frac{\partial^2 \chi}{\partial \lambda^2} + 2r\chi \frac{\partial^2 r}{\partial \lambda^2} \right. \\ & \left. + 2\chi \left(\frac{\partial r}{\partial \lambda} \right)^2 \right] \quad (1.36a) \end{aligned}$$

$$\beta_2 = 2 \cos \phi \left[2r\chi \frac{\partial r}{\partial \lambda} + r^2 \frac{\partial \chi}{\partial \lambda} \right] \quad (1.36b)$$

$$\beta_3 = \alpha_3 \quad (1.36c)$$

$$\beta_4 = \alpha_4 \quad (1.36d)$$

The remaining rows of $\mathbf{\Phi}$ and $\mathbf{\Lambda}$ are obtained by applying the operator ∇ to the first row of each

$$(\Phi_{21}, \Phi_{31}, \Phi_{41})^T = \nabla \left(\frac{1}{r^*} \right) = \frac{\mathbf{1}^*}{r^{*3}} \quad (1.37a)$$

$$(\Lambda_{21}, \Lambda_{31}, \Lambda_{41}) = (\Phi_{21}, \Phi_{31}, \Phi_{41}) \quad (1.38a)$$

$$\begin{aligned} (\Phi_{22}, \Phi_{32}, \Phi_{42})^T = & \frac{2}{r^*} \frac{\partial r^*}{\partial \phi} \nabla \left(\frac{1}{r^*} \right) \\ & + \frac{1}{r^{*2}} \nabla \left(\frac{\partial r^*}{\partial \phi} \right) \quad (1.37b) \end{aligned}$$

$$\begin{aligned} (\Lambda_{22}, \Lambda_{32}, \Lambda_{42})^T = & \frac{2}{r^*} \frac{\partial r^*}{\partial \lambda} \nabla \left(\frac{1}{r^*} \right) \\ & + \frac{1}{r^{*2}} \nabla \left(\frac{\partial r^*}{\partial \lambda} \right) \quad (1.38b) \end{aligned}$$

$$\begin{aligned} (\Phi_{23}, \Phi_{33}, \Phi_{43})^T = & \frac{2}{r^*} \frac{\partial^2 r^*}{\partial \phi^2} \nabla \left(\frac{1}{r^*} \right) \\ & + \frac{1}{r^{*2}} \nabla \left(\frac{\partial^2 r^*}{\partial \phi^2} \right) \quad (1.37c) \end{aligned}$$

$$\begin{aligned} (\Lambda_{23}, \Lambda_{33}, \Lambda_{43})^T = & \frac{2}{r^*} \frac{\partial^2 r^*}{\partial \lambda^2} \nabla \left(\frac{1}{r^*} \right) \\ & + \frac{1}{r^{*2}} \nabla \left(\frac{\partial^2 r^*}{\partial \lambda^2} \right) \quad (1.38c) \end{aligned}$$

$$\begin{aligned} (\Phi_{24}, \Phi_{34}, \Phi_{44})^T = & \frac{3}{r^{*2}} \left(\frac{\partial r^*}{\partial \phi} \right)^2 \nabla \left(\frac{1}{r^*} \right) \\ & + \frac{2}{r^{*3}} \frac{\partial r^*}{\partial \phi} \nabla \left(\frac{\partial r^*}{\partial \phi} \right) \quad (1.37d) \end{aligned}$$

$$\begin{aligned} (\Lambda_{24}, \Lambda_{34}, \Lambda_{44})^T = & \frac{3}{r^{*2}} \left(\frac{\partial r^*}{\partial \lambda} \right) \nabla \left(\frac{1}{r^*} \right) \\ & + \frac{2}{r^{*3}} \left(\frac{\partial r^*}{\partial \lambda} \right) \nabla \left(\frac{\partial r^*}{\partial \lambda} \right). \quad (1.38d) \end{aligned}$$

C. Special Procedure for the Cosine Factor in (1.2) for Large Block Sizes.

For larger block sizes one may wish to use an approximation higher than second order to $\cos \phi$ in (1.2).

Let

$$f = H \cos \phi \quad (1.39)$$

$$H = \frac{\chi r^2}{r^*} \quad (1.40)$$

The expansion in Taylor series is now done as

$$\begin{aligned} f = & \cos \phi \{ H_0 + (\phi - \phi_0) H_\phi + (\lambda - \lambda_0) H_\lambda \\ & + \frac{1}{2} [(\phi - \phi_0)^2 H_{\phi\phi} + 2H_{\phi\lambda} (\phi - \phi_0)(\lambda - \lambda_0) \\ & + (\lambda - \lambda_0)^2 H_{\lambda\lambda}] + \dots \} \end{aligned} \quad (1.41)$$

Integrating the zero-order term

$$\begin{aligned} \iint H_0 \cos \phi d\phi d\lambda &= 2\Delta\lambda H_0 \int_{-\Delta\phi}^{\Delta\phi} \cos \phi d\phi \\ &= 4\Delta\lambda H_0 \cos \phi_0 \sin \Delta\phi \end{aligned} \quad (1.42)$$

Comparing this to (1.23) we can observe $f_0 = H_0 \cos \phi_0$ and $\sin \Delta\phi$ is fairly small (less than $7^{1/2}^\circ$), we can use

$$\sin \Delta\phi = \Delta\phi - \frac{1}{6} \Delta\phi^3 + \frac{\Delta\phi^5}{120} - \frac{\Delta\phi^7}{5040} + \dots \quad (1.43)$$

Since

$$\sin \Delta\phi \approx \Delta\phi - \frac{\Delta\phi^3}{6} \quad (1.43)$$

was assumed in (1.22), we can add the remaining terms of (1.43) to get a correction to (1.23)

$$\delta_1(\Delta T) = \frac{\Delta\phi^5 \Delta\lambda f_0}{30} \left(1 - \frac{\Delta\phi^2}{42} \right) \quad (1.44)$$

None of the *factors* in the correction depends on the satellite distance r^* : they depend only on the block size used.

Another correction of the same order as (1.44) may be obtained from the consideration of the first-order terms of (1.41). The term containing $(\lambda - \lambda_0)H_\lambda$ integrates out to zero, but $\frac{1}{2}H_\phi(\phi - \phi_0) \cos \phi$ does not. Hence, we may write

$$\begin{aligned} \Delta T_1 &= \frac{1}{2} \iint H_\phi (\phi - \phi_0) \cos \phi d\phi d\lambda \\ &= \Delta\lambda H_\phi \left\{ \int_{\phi_0 - \Delta\phi}^{\phi_0 + \Delta\phi} \phi \cos \phi d\phi \right. \\ &\quad \left. - \phi_0 \int_{\phi_0 - \Delta\phi}^{\phi_0 + \Delta\phi} \cos \phi d\phi \right\} \end{aligned} \quad (1.45)$$

The integrals in (1.45) are standard forms and simplify to

$$\Delta T_1 = 2\Delta\lambda H_\phi \sin \phi_0 (\Delta\phi \cos \Delta\phi - \sin \Delta\phi) \quad (1.46)$$

The trigonometric functions of $\Delta\phi$ in (1.46) can be expanded in Taylor series: the terms in $\Delta\phi$ drop out:

$$\begin{aligned} \Delta T_1 &= -\frac{2}{3} \Delta\lambda \Delta\phi^3 H_\phi \sin \phi_0 \\ &\quad \times \left[1 - \frac{1}{10} \Delta\phi^2 + O(\Delta\phi^4) \right] \end{aligned} \quad (1.47)$$

Since the first term of (1.47) is already included in (1.23), the correction is

$$\delta_2(\Delta T) = \frac{1}{15} \Delta\lambda \Delta\phi^5 H_\phi \sin \phi_0 \quad (1.48)$$

All other corrections would be of even higher order and have been neglected since they will not improve the accuracy of the computation.

D. Corrections to the Gravity Vector

Applying the operator ∇ to (1.44) yields

$$\nabla[\delta_1(\Delta T)] = \frac{\Delta\phi^5 \Delta\lambda}{30} \left(1 - \frac{\Delta\phi^2}{42} \right) \nabla f_0 \quad (1.49)$$

For (1.48) the result is

$$\nabla[\delta_2(\Delta T)] = \frac{\Delta\phi^5 \Delta\lambda}{15} \sin \phi_0 \nabla H_\phi \quad (1.50)$$

$$H_\phi = -\frac{\chi r^2}{r^{*2}} \frac{\partial r^*}{\partial \phi}$$

$$= -\chi r^2 \Phi_{12} \quad (1.51)$$

$$= \Delta\lambda H_\phi = -\chi r^2 (\Phi_{21}, \Phi_{31}, \Phi_{41})^T \quad (1.52)$$

2. THE METHOD OF SINGULARITY-MATCHING

A. Computing the Potential

Instead of (1.2) we use

$$\Delta T = G \iint \frac{f}{\sqrt{g}} d\lambda d\phi \quad (2.1)$$

where f is now defined as

$$f = \chi r^2 \cos \phi \quad (2.2)$$

and

$$g = r^{*2} \quad (2.3)$$

To compute the integral (2.1) expand f and g in Taylor series about (ϕ_0, λ_0)

$$f = f_0 + f_\phi \Delta\phi + f_\lambda \Delta\lambda + \frac{1}{2} [f_{\phi\phi} \Delta\phi^2 + 2f_{\phi\lambda} \Delta\phi \Delta\lambda + f_{\lambda\lambda} \Delta\lambda^2] + \dots \quad (2.4)$$

$$g = g_0 + g_\phi \Delta\phi + g_\lambda \Delta\lambda + \frac{1}{2} [g_{\phi\phi} \Delta\phi^2 + 2g_{\phi\lambda} \Delta\phi \Delta\lambda + g_{\lambda\lambda} \Delta\lambda^2] + \dots \quad (2.5)$$

Substituting (2.4) and (2.5) into (2.1) leads to

$$\Delta T = \iint \frac{A + B\Delta\lambda + C\Delta\lambda^2}{\sqrt{D + E\Delta\lambda + F\Delta\lambda^2}} d\lambda d\phi \quad (2.6)$$

with

$$A = f_0 + f_\phi \Delta\phi + \frac{1}{2} f_{\phi\phi} \Delta\phi^2. \quad (2.7a)$$

$$B = f_\lambda + f_{\phi\lambda} \Delta\phi, \quad (2.7b)$$

$$C = \frac{1}{2} f_{\lambda\lambda}. \quad (2.7c)$$

$$D = g_0 + g_\phi \Delta\phi + \frac{1}{2} g_{\phi\phi} \Delta\phi^2, \quad (2.7d)$$

$$E = g_\lambda + g_{\phi\lambda} \Delta\phi. \quad (2.7e)$$

$$F = \frac{1}{2} g_{\lambda\lambda}, \quad (2.7f)$$

$$f_0 = \chi r^2 \cos \phi. \quad (2.8)$$

$$f_\phi = \frac{\partial \chi}{\partial \phi} r^2 \cos \phi + 2r \frac{\partial r}{\partial \phi} \chi \cos \phi - \chi r^2 \sin \phi. \quad (2.9)$$

$$f_{\phi\phi} = \cos \phi \left[r^2 \frac{\partial^2 \chi}{\partial \phi^2} + 4 \frac{\partial r}{\partial \phi} \frac{\partial \chi}{\partial \phi} + 2r \chi \frac{\partial^2 r}{\partial \phi^2} + 2\chi \left(\frac{\partial r}{\partial \phi} \right)^2 \right] - \sin \phi \left[2r^2 \frac{\partial \chi}{\partial \phi} - 4r \chi \frac{\partial r}{\partial \phi} \right] - f_0. \quad (2.10)$$

$$f_\lambda = \cos \phi \left[2r \chi \frac{\partial r}{\partial \lambda} + r^2 \frac{\partial \chi}{\partial \lambda} \right]. \quad (2.11)$$

$$f_{\lambda\lambda} = \cos \phi \left[2\chi \left(\frac{\partial r}{\partial \lambda} \right)^2 + 2r \chi \frac{\partial^2 r}{\partial \lambda^2} + 4r \frac{\partial r}{\partial \lambda} \frac{\partial \chi}{\partial \lambda} + r^2 \frac{\partial^2 \chi}{\partial \lambda^2} \right]. \quad (2.12)$$

$$f_{\lambda\phi} = -\sin \phi \left[2r \chi \frac{\partial r}{\partial \lambda} + r^2 \frac{\partial \chi}{\partial \lambda} \right] \quad (2.13)$$

$$+ \cos \phi \left[2\chi \frac{\partial r}{\partial \phi} \frac{\partial r}{\partial \lambda} + 2r \left(\chi \frac{\partial^2 r}{\partial \phi \partial \lambda} + \frac{\partial r}{\partial \lambda} \frac{\partial \chi}{\partial \phi} + \frac{\partial r}{\partial \phi} \frac{\partial \chi}{\partial \lambda} \right) + r^2 \frac{\partial^2 \chi}{\partial \lambda \partial \phi} \right]. \quad (2.14)$$

$$g_0 = r^{*2}, \quad (2.14)$$

$$g_\phi = 2r^* \frac{\partial r^*}{\partial \phi}, \quad (2.15)$$

$$g_\lambda = 2r^* \frac{\partial r^*}{\partial \lambda}, \quad (2.16)$$

$$g_{\phi\phi} = 2 \left[r^{*2} \frac{\partial^2 r^*}{\partial \phi^2} + \left(\frac{\partial r^*}{\partial \phi} \right)^2 \right], \quad (2.17)$$

$$g_{\phi\lambda} = 2 \left[\frac{\partial r^*}{\partial \lambda} \frac{\partial r^*}{\partial \phi} + r^* \frac{\partial^2 r^*}{\partial \phi \partial \lambda} \right]. \quad (2.18)$$

and

$$g_{\lambda\lambda} = 2 \left[\left(\frac{\partial \mathbf{r}}{\partial \lambda} \right)^2 + \mathbf{r}^* \cdot \frac{\partial^2 \mathbf{r}}{\partial \lambda^2} \right], \quad (2.19)$$

where (1.17) is applied. The derivatives of r^* and \mathbf{r} are obtained from (1.12), (1.13), (1.14), (1.15), (1.16) and (1.20).

The computational strategy will be to integrate (2.6) analytically with respect to λ . This will eliminate the improper condition from the integral. The integration with respect to ϕ then can be done numerically in a completely straightforward manner. The possibility of doing some or all of the integration with respect to ϕ by analytic means will be treated elsewhere. Special procedures for very low altitudes also will be treated elsewhere. These formulas will be suitable for satellite altitudes, typically 200 km or more.

We now define

$$(\Delta T)_i = \int_{\lambda_0 - \Delta\lambda}^{\lambda_0 + \Delta\lambda} (A_i + B_i \Delta\lambda + C_i \Delta\lambda^2) \times (D_i + E_i \Delta\lambda + F_i \Delta\lambda^2)^{-1/2} d\lambda \quad (2.20)$$

where the subscript i indicates A, B , etc. are evaluated at sequentially spaced intervals of ϕ

$$\begin{aligned}\phi_0 - \Delta\phi = \phi_1 < \phi_2 < \phi_3 \dots < \phi_{n-1} < \phi_n \\ &= \phi_0 + \Delta\phi \\ \phi_{i+1} - \phi_i &= \delta\phi_i\end{aligned}$$

so that (2.6) can be computed from (2.19) by numerical quadrature formulas. For simplicity the subscripts will be omitted in the detailed formulas that follow.

Let us adopt the abbreviation

$$X = F\Delta\lambda^2 + E\Delta\lambda + D \quad (2.21)$$

and also drop the Δ symbols, so we can write

$$\begin{aligned}X &= F\lambda^2 + E\lambda + D \\ (\Delta T)_i &= \int (A + B\lambda + C\lambda^2)X^{-1/2}d\lambda. \quad (2.22)\end{aligned}$$

All this does is move the working origin to the "center" of each equal-area block. Now we can apply standard forms to obtain

$$\begin{aligned}(\Delta T)_i &= A \int \frac{d\lambda}{\sqrt{X}} + B \int \frac{\lambda d\lambda}{\sqrt{X}} + C \int \frac{\lambda^2 d\lambda}{\sqrt{X}} \\ &= A \int \frac{d\lambda}{\sqrt{X}} + B \left[\frac{\sqrt{X}}{F} - \frac{E}{2F} \int \frac{d\lambda}{\sqrt{X}} \right] \\ &\quad + C \left[\left(\frac{\lambda}{2F} - \frac{3E}{4F^2} \right) \sqrt{X} \right. \\ &\quad \left. + \frac{3E^2 - 4DF}{8F^2} \int \frac{d\lambda}{\sqrt{X}} \right] \quad (2.23)\end{aligned}$$

(Peirce and Foster 1956, eq. 174 and 177).

By collecting terms we obtain

$$(\Delta T)_i = \frac{B\sqrt{X}}{F} + C \left[\frac{\lambda}{2F} - \frac{3E}{4F^2} \right] \sqrt{X} + Q \int \frac{d\lambda}{\sqrt{X}} \quad (2.24)$$

with

$$Q = A - \frac{BE}{2F} + \frac{(3E^2 - 4DF)C}{8F^2}. \quad (2.25)$$

The expression $\int X^{-1/2} d\lambda$ can be considered to define a function with two poles; one at each root of X . Hence, for all values of D, E, F it can be expressed as a single elementary function of a complex variable, at least when we exclude negative

values of the square root. Since we have the special case that D, E, F are all real and that the result is real and the integral is taken only on the real line, it is more efficient computationally to express $\int X^{-1/2} d\lambda$ as different real functions of a real variable and use the one designated by appropriate functions of D, E, F . Using the appropriate complex function would about double the amount of computation.

The auxiliary functions needed are F and

$$q = 4DF - E^2. \quad (2.26)$$

The three useful basic forms are

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{F}} \sinh^{-1} \left(\frac{2F\lambda + E}{\sqrt{q}} \right), \quad (2.27)$$

$$F > 0, q > 0;$$

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{-F}} \sin^{-1} \left(\frac{2F\lambda + E}{\sqrt{-q}} \right), \quad (2.28)$$

$$F < 0, q < 0,$$

and

$$\int \frac{d\lambda}{\sqrt{X}} = \frac{1}{\sqrt{F}} \log \left(\sqrt{X} + \lambda \sqrt{F} + \frac{E}{2\sqrt{F}} \right), \quad (2.29)$$

$$F > 0.$$

Form (2.29) is very handy for evaluating definite integrals, since the difference of logarithms is the logarithm of the quotient of the arguments

$$\log x - \log y = \log \frac{x}{y}, \quad (2.30)$$

hence

$$\log [\kappa(\lambda)]_{-\Delta\lambda}^{\Delta\lambda} = \log \left[\frac{\kappa(\Delta\lambda)}{\kappa(-\Delta\lambda)} \right], \quad (2.31)$$

where $\kappa(\lambda)$ is a function of λ . Similar, convenient forms can be obtained for (2.27) and (2.28) by applying appropriate identities.

The inverse hyperbolic sine may be expressed in logarithms as

$$\sinh^{-1} x = \log (x + \sqrt{x^2 + 1}), \quad (2.32)$$

so (2.31) may be applied to evaluate a definite integral of the form (2.27). Definite integrals of the form (2.28) may be evaluated with the identity

$$\begin{aligned}\sin^{-1} x - \sin^{-1} y &= \sin^{-1} [x\sqrt{1-y^2} - y\sqrt{1-x^2}] \\ &= \tan^{-1} \left[\frac{x\sqrt{1-y^2} - y\sqrt{1-x^2}}{xy + \sqrt{1-x^2}\sqrt{1-y^2}} \right].\end{aligned} \quad (2.33)$$

In some cases $F=0$, or F may be so small that (2.27), (2.28), and (2.29) may cause numerical problems. If E is not similarly small the approximation

$$\Delta T \approx \int \int \frac{A + B\Delta\lambda + C\Delta\lambda^2}{\sqrt{D + E\Delta\lambda}} d\lambda d\phi \quad (2.34)$$

may be used in place of (2.6). The analog of (2.23) is, then (Peirce and Foster, 1956, eq. 100, 101, 102):

$$(\Delta T)_i = \sqrt{D + E\lambda} \left[\frac{2A}{E} - \frac{2B(2D - E\lambda)}{3E^2} + \frac{2C(8D^2 - 4DE\lambda + 3E^2\lambda^2)}{15E^3} \right] \Big|_{-\Delta\lambda}^{\Delta\lambda} \quad (2.35)$$

Both $E\Delta\lambda$ and $F\Delta\lambda^2$ may be small compared to D , so that the binomial series is the most accurate method to use.

$$(D + E\lambda + F\lambda^2)^{-1/2} = D^{-1/2} \left[1 - \frac{E\lambda + F\lambda^2}{2D} + \frac{3}{8} \left(\frac{E\lambda + F\lambda^2}{D} \right)^2 - \frac{5}{16} \left(\frac{E\lambda + F\lambda^2}{D} \right)^3 + \dots \right]. \quad (2.36)$$

Computing $(\Delta T)_i$ is done by multiplying and integrating power series in the usual way

$$\int_{-\Delta\lambda}^{\Delta\lambda} (A + B\lambda + C\lambda^2) X^{-1/2} d\lambda = D^{-1/2} \left\{ 2A\Delta\lambda + \frac{2}{3} \Delta\lambda^3 \left[C - \frac{BE}{2D} + \frac{3AE^2}{D^2} - \frac{AF}{2D} \right] + \frac{\Delta\lambda^5}{5} \left[\frac{3CE^2}{4D^2} + \frac{3AF^2}{4D^2} + \frac{3BEF}{2D^2} - \frac{CF}{D} \right] + \frac{3}{28} \Delta\lambda^7 \frac{CF^2}{D^2} + O(\Delta\lambda^9) \right\}. \quad (2.37)$$

The use of binomial series might be feasible over an entire block, but in some cases E and F will be small only for a limited band in latitude and the expansion of ΔT would not be accurate if done in terms of ϕ as well as λ .

B. Gravity From the Singularity-Matching Method

Let us determine the gradients of the factors in (2.24):

$$\nabla Q = -\frac{C}{2F} \nabla D + \left(\frac{3EC}{4F^2} - \frac{B}{2F} \right) \nabla E + \left(\frac{BE + CD}{2F^2} - \frac{3CE^2}{4F^3} \right) \nabla F \quad (2.38)$$

$$\nabla D = -2\mathbf{r}^* - 2 \frac{\partial \mathbf{r}}{\partial \phi} \Delta\phi - \frac{\partial^2 \mathbf{r}}{\partial \phi^2} \Delta\phi^2 \quad (2.39)$$

$$\nabla E = -2 \frac{\partial \mathbf{r}}{\partial \lambda} - 2 \frac{\partial^2 \mathbf{r}}{\partial \phi \partial \lambda} \Delta\phi \quad (2.40)$$

$$\nabla F = -\frac{\partial^2 \mathbf{r}}{\partial \lambda^2}. \quad (2.41)$$

The derivatives of \mathbf{r} are given in (1.14), (1.15), (1.19), (1.20), and (1.21).

The gradient of (2.24) is given by

$$\nabla(\Delta T)_i = \nabla \left\{ \frac{B\sqrt{X}}{F} + C \left[\frac{\lambda}{2F} - \frac{3E}{4F^2} \right] \sqrt{X} \right\} + (\nabla Q) \int \frac{d\lambda}{\sqrt{X}} + Q \nabla \int \frac{d\lambda}{\sqrt{X}}. \quad (2.42)$$

The first part of (2.42) is written symbolically because for the case of constant density blocks on an oblate spheroid $B=C=0$; moreover, for that case $\nabla Q=0$. Then (2.42) reduces to

$$\nabla(\Delta T)_i = Q \nabla \int \frac{d\lambda}{\sqrt{X}}. \quad (2.42A)$$

If (2.27) or (2.28) is used, it is convenient to define

$$Y = (2F\lambda + E) |q|^{-1/2} \quad (2.43)$$

and obtain

$$\nabla \int \frac{d\lambda}{\sqrt{X}} = -\frac{1}{2} F^{-3/2} (\nabla F) \sinh^{-1} Y + [F(1 + Y^2)]^{-1/2} \nabla Y. \quad (2.44)$$

or

$$\nabla \int \frac{d\lambda}{\sqrt{X}} = \frac{1}{2} (-F)^{-3/2} (\nabla F) \sin^{-1} Y + [-F(1 - Y^2)]^{-1/2} \nabla Y, \quad (2.45)$$

with

$$\nabla Y = |q|^{-1/2} (2\lambda \nabla F + \nabla E) - \frac{1}{2} (2F\lambda + E) |q|^{-3/2} \nabla |q| \quad (2.46)$$

$$\nabla |q| = \frac{q}{|q|} \nabla q \quad (2.47)$$

$$\nabla q = 4(F\nabla D + D\nabla F) - 2E\nabla E. \quad (2.48)$$

If (2.29) is used, define

$$Z = X^{1/2} + \lambda F^{1/2} + 2EF^{-1/2} \quad (2.49)$$

and obtain

$$\nabla \int \frac{d\lambda}{\sqrt{X}} = -\frac{1}{2} F^{-3/2} (\nabla F) \log Z + \frac{\nabla Z}{\sqrt{FZ}} \quad (2.50)$$

with

$$\begin{aligned} \nabla Z &= \frac{1}{2} X^{-1/2} \nabla X + \frac{\lambda}{2} F^{-1/2} \nabla F \\ &\quad + 2F^{-1/2} \nabla E - EF^{-3/2} \nabla F \end{aligned} \quad (2.51)$$

$$\nabla X = \nabla D + \lambda \nabla E + \lambda^2 \nabla F. \quad (2.52)$$

Where $B=C=0$, (2.35) simplifies to

$$(\Delta T)_i = \frac{2A}{E} \sqrt{D+E\lambda} \quad (2.53)$$

and

$$\begin{aligned} \nabla (\Delta T)_i &= -\frac{2A\nabla E}{E^2} \sqrt{D+E\lambda} \\ &\quad + \frac{A}{E} (\lambda \nabla E) (D+E\lambda)^{-1/2}. \end{aligned} \quad (2.54)$$

For $B=C=0$ (2.37) simplifies to

$$\int_{-\Delta\lambda}^{\Delta\lambda} AX^{-1/2} d\lambda = D^{-1/2} W \quad (2.55)$$

$$W = 2A\Delta\lambda + \frac{2}{3} \Delta\lambda^3 \left[\frac{3AE^2}{D^2} - \frac{AF}{2D} \right] + \frac{3}{20} \Delta\lambda^5 \frac{AF^2}{D^2} \quad (2.56)$$

$$\nabla \int_{-\Delta\lambda}^{\Delta\lambda} AX^{-1/2} d\lambda = D^{-1/2} \nabla W - \frac{1}{2} W D^{-3/2} \nabla D \quad (2.57)$$

$$\begin{aligned} \nabla W &= \frac{2}{3} \Delta\lambda^3 \left[6A \frac{E}{D} \nabla(E/D) - \frac{A}{2} \nabla(F/D) \right] \\ &\quad + \frac{3}{10} \Delta\lambda^5 \frac{AF}{D} \nabla(F/D) \end{aligned} \quad (2.58)$$

$$\nabla(E/D) = (D\nabla E - E\nabla D)/D^2 \quad (2.59)$$

$$\nabla(F/D) = (D\nabla F - F\nabla D)/D^2. \quad (2.60)$$

3. POINT MASS AND NUMERICAL CUBATURE ALGORITHMS

The simplest possible numerical cubature scheme was used by Koch and Morrison (1970) and Koch and Witte (1971). The constant density blocks were divided into four sub-blocks and the distance r^* from the sub-block to the satellite was used; each sub-block was weighted by its area. Obviously, this is the same as using four point masses. Any conventional numerical cubature would correspond to some array of point masses.

One sets up the array of sampling points in the ϕ, λ coordinate system and applies the numerical cubature formula to

$$T = \iint \frac{\chi d\sigma}{r^*} \quad (3.1)$$

to obtain

$$\Delta T_n = \sum_{i=1}^N \frac{w_i \chi(\phi_i, \lambda_i) \cos \phi_i \Delta\phi \Delta\lambda}{\sqrt{(x_i - x_s)^2 + (y_i - y_s)^2 + (z_i - z_s)^2}} \quad (3.2)$$

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = r_i(\phi_i, \lambda_i) \begin{pmatrix} \cos \lambda_i \cos \phi_i \\ \sin \lambda_i \cos \phi_i \\ \sin \phi_i \end{pmatrix} \quad (3.3)$$

$$T = \sum_n \Delta T_n, \quad (3.4)$$

where w_i are the cubature formula weights.

For the formula actually used, $N=1$, and $w_1 \Delta\phi \Delta\lambda \cos \phi_i =$ area of the block or sub-block. Hence, (3.2) simplifies to

$$\Delta T_n = A_n \chi_n / r_n^* \quad (3.5)$$

and

$$\nabla (\Delta T_n) = A_n \chi_n \nabla (1/r_n^*) \quad (3.6)$$

with

$$\nabla (1/r_n^*) = \frac{1}{r_n^{*3}} \begin{pmatrix} x_n - x_s \\ y_n - y_s \\ z_n - z_s \end{pmatrix}. \quad (3.7)$$

4. ANNOTATED COMPUTER LISTING

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```
PROGRAM          DETOUR
C
C...             PROGRAM DETOUR<INPUT,OUTPUT>
C               PROGRAM DETOUR IS THE DRIVING ROUTINE.
5              COMMON/GECON/ALFA(4)
              COMMON/LYNX/BETA(21)
              COMMON/DIFCOR/GAMMA(7)
              COMMON/CONST/EPSILON(180)
              COMMON/XDT/ZETA(5)
10             COMMON/VRBL/ETA(12)
              COMMON/EARTH/THETA(1640)
              COMMON/KEYSYM/IOTA(182)
              COMMON/QUAD/KAPPA(25)
              COMMON/SCRICH/LAMBDA(6)
15             COMMON/POINTS/MU(300)
              COMMON/AXEL/NU(5)
              CALL SETUP
              CALL TTEST
              STOP6600
              END
20
```


SUBROUTINE TTEST

SUBROUTINE TTEST

TTEST COMPUTES THE POTENTIAL AND ATTRACTION OF A POINT MASS AND
 CALLS DTDX TO COMPUTE THEM FOR THE APPROXIMATE DISTRIBUTION ON AN
 OBLATE SPHEROID.

COMMON/GEOCOM/DYR, SR, ECCSQ, XMU
 COMMON/LYNX/X(6), UVM(6), ROTRAT, PI, PI0180, TWOPI, FOURPI,
 1 THETA, SINTH, COSTH, FLWJD
 COMMON/DIFCOR/COSB, SINB, RAD, RADXY, RAD2, RADXY2, RAD3
 COMMON/AXEL/IGRAV, DX(3), ITEM
 COMMON/VRBL/XS(3), RTM(3), J
 COMMON/XDT/TT, DTDX(3), FACTOR
 DIMENSION UTOPIA(3), UE(4), UEG(3), UES(3), UED(3)
 EQUIVALENCE (YS,XS(2)), (ZS,XS(3)), (UE(2),UEG(1)), (UTOPIA,DX)
 KOUNT = 0

96 CONTINUE

KOUNT = 0
 100 FORMAT(F16.9, 4X, 3F15.8)
 READ TIME, ASTRONOMICAL LATITUDE, LONGITUDE, RADIAL COORDINATE.
 10 IF(TIME.LT.0.0) CALL EXIT
 IF(TIME.LT.0.0) CALL EXIT

CONVERT ANGLES TO RADIAN.
 PHIZ=PHI*0.017453292519943
 XLONGZ=XLONGS*0.017453292519943
 COSFI=COS(PHIZ)
 RS = RS*DYR
 RS3 = RS**3

COMPUTE EARTH-FIXED COORDINATES.
 XS = RS*COSFI*COS(XLONGZ)
 YS = RS*COSFI*SIN(XLONGZ)
 SINFI = SIN(PHIZ)
 ZS = RS*SINFI

UVM(1) = XS
 UVM(2) = YS
 X(3) = ZS

FACTOR = 1.0
 CALL SECOND(TT1)
 CALL DTDX
 VALUES FROM SUBROUTINE DTDX.

POTENTIAL
 ATTRACTION COMPONENTS.

ANGLE IS THE
 CALL SECOND(TT2)
 TEA = TT2 - TT1
 GEE2 = DTDX(1)**2 + DTDX(2)**2 + DTDX(3)**2
 GEE = SQRT(GEE2)
 IF(KOUNT.NE.0) GO TO 22

PRINT 101
 PRINT 104

22 CONTINUE

101 FORMAT(1H1,10X,1HR,8X,3HPHI,7X,6HLAMBDA,11X,1HU,16X,2HUX,16X,2HUY,
 1 16X,2HUZ,16X, 1HG)

SUBROUTINE TTEST

```

102 FORMAT(4X,F10.6,2F12.6,5E18.10 )
106 FORMAT(1X,E13.6,2F12.6,5E18.10 )
IF(RS.GT.999.9999995) GO TO 25
PRINT 102 ,RS, PHIS, XLONGS, TT, DTDX(1), DTDX(2), DTDX(3), GEE
GO TO 26
25 PRINT 106 ,RS, PHIS, XLONGS, TT, DTDX(1), DTDX(2), DTDX(3), GEE
26 CONTINUE
KOUNT=KOUNT + 1
    
```

C COMPUTE VALUES FOR A POINT MASS OR UNIFORM SPHERICAL DISTRIBUTION.

```

C... UTAH
C... POTENTIAL
C... ATTRACTION COMPONENTS
UTAH = XMU/RS
UTOPIA(1) = -(XMU*XS(1))/RS3
UTOPIA(2) = -(XMU*YS)/RS3
UTOPIA(3) = -(XMU*ZS)/RS3
GP2 = UTOPIA(1)**2 + UTOPIA(2)**2 + UTOPIA(3)**2
GEEP = SQRT(GP2)
PRINT 103, UTAH, (UTOPIA(KKK),KKK=1,3), GEEP
103 FORMAT(38X,5E18.10 )
KOUNT = KOUNT + 1
    
```

C COMPUTE TRUNCATION ERRORS OF VALUES FROM DTTDX.

```

C... UE(1) = UTAH - TT
UE(2) = UTOPIA(1) - DTDX(1)
UE(3) = UTOPIA(2) - DTDX(2)
UE(4) = UTOPIA(3) - DTDX(3)
GE = GEEP - GEE
PRINT 103, (UE(KKK), KKK = 1, 4), GE
KOUNT = KOUNT + 1
IF(GEE.LE.1.0E-42) GO TO 10
ERROR2 = UEG(1)**2 + UEG(2)**2 + UEG(3)**2
ERROR = SQRT(ERROR2)
DO 30 IVY = 1, 3
UES(IVY) = (UTOPIA(IVY)/GEEP + DTDX(IVY)/GEE)/2.0
UED(IVY) = (UTOPIA(IVY)/GEEP - DTDX(IVY)/GEE)/2.0
    
```

30 CONTINUE

```

ANUM = UED(1)**2 + UED(2)**2 + UED(3)**2
ADEN = UES(1)**2 + UES(2)**2 + UES(3)**2
ANGLE = UED(1)**2 + UED(2)**2 + UED(3)**2
ANGLE IN DEGREES BETWEEN THE THEORETICAL AND
COMPUTED ATTRACTION VECTORS.
    
```

C ANGLE = 114.5915590261*ATAN(SQRT(ANUM/ADEN))

```

PRINT 105, ERROR, ANGLE, TEA
KOUNT = KOUNT + 2
105 FORMAT(38X,9HABS(DG) = ,E13.5,5X,12HDEFLECTION = ,F13.9,5H DEG. ,
1 5X,7HTIME = ,F8.4,5H SEC. /)
104 FORMAT(/)
IF(KOUNT .GT.52) KOUNT = 0
GO TO 10
END
    
```

100

105

SUBROUTINE SETUP

```

SUBROUTINE SETUP
COMMON/GEON/DYR, SR, ECCSQ, XMU
COMMON/LYMX/X(6), UVM(6), ROTRAT, PI, PIO180, TWOPI, FOURPI,
1 THETA, SINTH, COSTH, FLMD
COMMON/DIFCOR/COSB, SINB, RAD, RADXY, RAD2, RADXY2, RAD3
COMMON/CONST/ D1(36), D2(36), D22(36), D11(36), D21(36)
COMMON/XDT/ TT, DTDX(3)
COMMON/VRBL/ XS(3), RTM(3,3)
COMMON/EARTH/ RHO(1640)
COMMON/KEYSYM/KEY(36), KEY(36), KEYSUM(37), KEESUM(37), WTK(36)
COMMON/QUAD/INTGT,INTL, DEEPhi, DEEFEE(10), DEFESQ(10)
COMMON/SCRIPTCH/US(3), U(3)
COMMON/POINTS/FCPHI(36,2), FCLAM(2552,2), R(36), DR(36), D2R(36)
COMMON/AXEL/IGRAV,DX(3), ITEM
DIMENSION PHISPH(18)
N.B. THE CDC 6600 FORTRAN IV ALLOWS ONE TO LOAD COMMON BY DATA.
C.... PHISPH
C.... DATA
1 0.0872226457377, 0.1761387651605, 0.2609487747858, 0.3477560061365,
2 0.4320564016823, 0.5198194379129, 0.6063482369929, 0.6984854081328,
3 0.7845763350315, 0.8788890739735, 0.9595073402986, 1.0507850152541,
4 1.1245264580253, 1.2120202733884, 1.2899116186106, 1.37244955970138,
5 1.4717630250231, 1.5707963267949/
R
C.... DATA
1 0.9966552494476, 0.9967202763778, 0.9968349994554, 0.9969770143458,
2 0.9971594289735, 0.9973678447199, 0.9976067771352, 0.9978763272499,
3 0.9981642186147, 0.9984662417476, 0.9987602423835, 0.9990411766387,
4 0.9992933418563, 0.9995134867237, 0.9996975056559, 0.9998416344681,
5 0.9999416897678, 0.9999935212250/
DR
C.... DATA
1 -.0003293204460, -.0009760124276, -.0015372659667, -.0019913712656,
2 -.0024055254852, -.0027483404743, -.0030235933640, -.0032257790967,
3 -.0033320203234, -.0033373476021, -.0032347008642, -.003029666146229,
4 -.0027354104516, -.0023632173291, -.0019227104544, -.0014240619733,
5 -.0008775797852, -.0002948443230/
D2R
C.... DATA
1 0.0033173064115, 0.0031892505111, 0.0029639096906, 0.0026838183208,
2 0.002329744340, 0.0019107279252, 0.0014363152502, 0.0009005506196,
3 0.000326575432, -.0002770035489, -.0008660974843, -.0014306000230,
4 -.0019384104682, -.0023828167453, -.0027547886644, -.0030467097636,
5 -.0032493581603, -.003354246541/
DATA (FOURPI = 12.5663706143559173)
THE KEY SYSTEM FOR REFERENCING THE BLOCKS AND SUB-BLOCKS.
C.... DATA KEY/4,12,16,20,28,40,40,52,52,60,60,64,64,68,68,72,72
1 72,72,68,68,64,64,60,60,52,52,40,40,28,28,20,20,16,16,4/
DATA KEY/18,6,6,3,2,2,2,2,1,1,1,1,1,1,1,1,1
1 1,1,2,2,2,2,3,4,6,18/
DATA KEYSUM/0,4,16,32,52,80,108,148,188,240,292,352,412,476,540,
1 608,676,748,820,892,964,1032,1100,1164,1228,1288,1348,1400,
21452,1492,1552,1560,1588,1608,1624,1636,1640/
DATA KEESUM/0,72,144,208,268,324,380,460,540,644,748,808,868,932

```


SUBROUTINE SETUP

```

1 ,996,1064,1132,1204,1276,1348,1420,1488,1556,1620,1684
2 ,1744,1804,1908,2012,2092,2172,2228,2284,2344,2408,2480,2552/
C... THE EQUATION NUMBERS IN THE MATHEMATICAL DESCRIPTION ARE GIVEN IN
C COLUMNS 72.....80

```

```

60 C... GREENWICH HOUR ANGLE, SET TO 0.0 IN THIS TEST.

```

```

65 C... NUMBER OF POINTS IN NEWTON-COTES INTEGRATION.

```

```

70 C...
    THETA = 0.0
    SINTH = 0.0
    COSTH = 1.0
    XMU = 1.0
    INTGT = 7
    INTL = INTGT - 1
    XINTL = FLOAT(INTL)
    DYR = 1.0
    PI = FOURPI/4.0
    TWOPI = FOURPI/2.0

```

```

75 C... COMPUTE THE AREA FOR ANY SUB-BLOCK IN ALL 36 ZONES.
    DO 62 IY = 1, 36
    WTK(IY) = 12.538290662704/(1640.*FLOAT(KEEY(IY)))

```

```

80 C...
    DO 60 IY = 1, 18
    IY = IY - 1
    PHIBRS = 0.5*PHISPH(IDX)
    D1(IY) = 0.5*PHISPH(IDX)
    IDS = 37 - IY
    R(IDS) = R(IY)
    DR(IDS) = -DR(IY)
    D2R(IDS) = D2R(IY)
    IF (IDX.GT.1) D1(IY) = D1(IY) - 0.5*PHISPH(IDX-1)
    IF (IDX.GT.1) PHIBRS = PHIBRS + 0.5*PHISPH(IDX-1)
    FCPHI(IY,1) = COS(PHIBRS)
    FCPHI(IY,2) = SIN(PHIBRS)
    FCPHI(IDS,1) = FCPHI(IY,1)
    FCPHI(IDS,2) = -FCPHI(IY,2)
    D1(IY) = D1(IY)**2
    IWTOP = KEY(IY)*KEEY(IY)
    D2(IY) = PI/IWTOP
    D22(IY) = D2(IY)**2
    D21(IY) = D1(IY)*D2(IY)
    D1(IDS) = D1(IY)
    D2(IDS) = D2(IY)
    D11(IDS) = D11(IY)
    D21(IDS) = D21(IY)
    D22(IDS) = D22(IY)
    DO 60 IW = 1, IWTOP
    ARGL = FLOAT(2*IW - 1)*D2(IY)
    IDXL = KEESUM(IY) + IW
    IDXL2 = KEESUM(36-IY) - IW + 1
    FCLAM(IDXL,1) = COS(ARGL)
    FCLAM(IDXL,2) = SIN(ARGL)
    FCLAM(IDXL2,1) = FCLAM(IDXL,1)

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85 C...

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90 C...

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95 C...

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100 C...

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105 C...

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110 C...

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CDC 6600 FTN V3.0-324 OPT=1 03/26/76 13.05.18.

SUBROUTINE DTTDX

SUBROUTINE DTTDX

C DTTDX COMPUTES THE POTENTIAL AND ATTRACTION DUE TO A SURFACE
 C DENSITY ON AN OBLATE SPHEROID. THE DENSITY IS PARAMETERIZED BY THE
 C USE OF 1640 EQUAL AREA BLOCKS, EACH ABOUT 5 DEGREES ON A SIDE.

C--- COMMON/GEOCOM/DYR, SR, ECCSQ, XMU
 COMMON/CONST/ D1(36), D2(36), D22(36), D11(36), D21(36)
 COMMON/LYNX/X(6), UVM(6), ROTRAT, PI, PI0180, TWOPI, FOURPI,
 1 THETA, SINTH, COSTH, FLNJD
 COMMON/DIFCOR/COSB, SINB, RAD, RADXY, RAD2, RADY2, RAD3
 COMMON/EARTH/ RHO(1640)
 COMMON/KEYSYM/ KEY(36), KEYSUM(37), KEESUM(37), WTK(36)
 COMMON/XDT/TFCN(4), FACTOR
 COMMON/VRBL/ XS(3), RTM(3,3)
 COMMON/QUAD/INTGT, INTL, DEEPHI, DEEFEE(10), DEFESQ(10)
 COMMON/SCRCH/US(3), U(3)
 COMMON/POINTS/FCPHI(36,2), FCLAM(2552,2), R(36), DR(36), D2R(36)

C--- DIMENSION DSRP(3), DSRPL(3), DSR2P(3), DSR2L(3), FPHI(4), FZERO(4)
 DIMENSION FCOMP(4,2), PL(4,4,2), ALBT(4,2), GRTQ(3), RSTAR(3)
 DIMENSION FFF(10), D(4), E(4), F(4), D2RDL(3), GARGG(2,3)
 DIMENSION DRDFI(3), DRDLM(3), D2RDF2(3), D2RDL2(3), ALPHA(6,10)
 DIMENSION GTOTAL(4), FDF(10,4), GRAD(2,3), GARG(2,3), GROQQ(3)
 DIMENSION GRADF(10,3), GRT(2,3), GRRTF(3), GRASH(2,3), GRTTF(3)
 C--- EQUIVALENCE(PL(1,2,1),R2RF), (PL(1,2,2),R2RL), (FONE,AF1)
 EQUIVALENCE(FFF(1),FDF(1,1)), (GRADF(1,1),FDF(1,2)), (GO,RSTAR2)
 EQUIVALENCE(PL(1,3,1),R2RF), (PL(1,3,2),R2RL), (FO,FB)
 EQUIVALENCE(A,ABC,Q), (F(1),G22A), (U(3),W)

C COEFFICIENTS FOR NEWTON-COTES NUMERICAL QUADRATURE
 C--- DATA ALPHA/1.0,4*0.0,1.0,1.0,1.0,3*0.0,2.0,1.0,2.0,3*0.0,3.0,
 1 3.0,9.0,3*0.0,8.0,14.0,64.0,12.0,2*0.0,45.0
 2 95.0,375.0,250.0,2*0.0,288.0,41.0,216.0,27.0,136.0,0.0,140.0,6,7
 3 5257.0,25039.0,9561.0,20923.0,0.0,17280.0
 4 3956.0,23552.0,-5712.0,41984.0,-9080.0,14175.0
 5 25713.0,141669.0,9720.0,174096.0,52002.0,89600.0
 DATA ALBT(1,2) = 0.0, (ALBT(2,2) = 0.0)
 DATA (ORDM(3)=0.0), (D2RDL2(3)=0.0), (F(4)=0.0)
 DATA (D2RDL(3)=0.0)

C THE EQUATION NUMBERS IN THE MATHEMATICAL DESCRIPTION ARE GIVEN IN
 C COLUMNS 72.....80

C INTGTY = INTGT + 1
 C--- INTACT = INTGTY / 2
 TAG2 = DYR
 TAG = 0.25*DYR
 C--- ALPHER = FLOAT(INTGT-1)*ALPHA(6,INTGT)
 US
 US(1) = UVM(1)
 US(2) = UVM(2)
 US(3) = X(3)
 DO 101 NO = 1, 4


```

TECN(NO) = 0.0
101 CONTINUE
C... CYCLE THROUGH THE 36 ZONES.
DO 999 I = 1, 36
  JTOP = KEY(I)
  KTOP = KEY(I)
  FAKTOR = (2.0*D1(I))/ALPHER
  DEFI = (2.0*D1(I))/FLOAT(INTGT-1)
DO 105 II = 1, INTGT
  DEEFEE(II) = -D1(I) + FLOAT(II-1)*DEFI
  DEFESQ(II) = DEEFEE(II)**2
105 CONTINUE
DELLAM = D2(1)
DLAM2 = D22(1)
COSPHI = FCPHI(1,1)
SINPHI = FCPHI(1,2)
R1 = DR(1)
R2 = DR(1)
TWR2 = R2 + R2
ARE = R(1)
RCOSFI = ARE*COSPHI
RCF2 = RCOSFI**2
ARE2 = ARE**2
C... COORDINATES OF CENTER OF BLOCK
U(3) = ARE*SINPHI
RSTAR(3) = U(3) - US(3)
RST3SQ = RSTAR(3)**2
C... CYCLE OVER THE BLOCKS IN A ZONE.
DO 999 J = 1, JTOP
  JRHO = KEYSUM(I) + J
  JRHO = KEYSUM(I) + (J-1)*KEYE(I)
  RHX = RHO(JRHO)
C... CYCLE OVER THE SUB-BLOCKS WITHIN A BLOCK.
DO 999 K = 1, KTOP
  JLAM = JLAMO + K
  COSLAM = FCLAM(JLAM, 1)
  SINLAM = FCLAM(JLAM, 2)
  U(1) = RCOSFI*COSLAM
  U(2) = RCOSFI*SINLAM
  RSTAR(1) = U(1) - US(1)
  RSTAR(2) = U(2) - US(2)
  RSTAR2 = RSTAR(1)**2 + RSTAR(2)**2 + RST3SQ
  RST = SQRT(RSTAR2)
  RST1=1.0/RST
  RST2=1.0/RSTAR2
  RST3 = RST1*RST2
C... IF(RST.GT.TAG2) USE THE ONE-POINT NUMERICAL CUBATURE.
  IF(RST.GT.TAG2) GO TO 701
  RST3 = 3.0 * RST2
  CFCL = COSPHI*COSLAM
  CFSL = COSPHI*SINLAM
  DRDFI(1) = -W*COSLAM + R1*CFCL
  DRDFI(2) = -W*SINLAM + R1*CFSL

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(1.11)
(1.8)

(1.11)
(1.11)
(1.8)
(1.8)

(1.14)
(1.14)


```

115 DRDFI(3) = RCOSFI + R1*SINLAM
    DRDLM(1) = -U(2)
    DRDLM(2) = U(1)
    DRDLM(3) = 0.0
    D2RDF2(1) = -U(1) + TWOR2*CFCL
    D2RDF2(2) = -U(2) + TWOR2*CFSL
    D2RDF2(3) = -U(3) + TWOR2*SINPHI
    D2RDL2(1) = -U(1)
    D2RDL2(2) = -U(2)
    DRFC = RSTAR(1)*DRDFI(1) + RSTAR(2)*DRDFI(2) + RSTAR(3)*DRDFI(3)
    G1 = DRFC + DRFC
    DRSDFI = RST1*DRFC
    DRFC2 = -RSTAR(1)*U(2) + RSTAR(2)*U(1)
    G2 = DRFC2 + DRFC2
    DRSDLM = RST1*DRFC2
    GAA = RSTAR(1)*U(1) + RSTAR(2)*U(2)
    G22A = RCF2 - GAA
    G11A = ARE2 - GAA -M*RSTAR(3)
    FA = RHX*ARE2
    FB = FA*COSPHI
    FC = FA*SINPHI
    TWOR = ARE+ARE
    FOUR = TWOR + TWOR
    IF(RST.GT.TAG) USE THE TAYLOR SERIES METHOD.
    IF(RST.GT.TAG) GO TO 601
120 SINGULARITY-MATCHING METHOD.
    D2RDPL(1) = -DRDFI(2)
    D2RDPL(2) = DRDFI(1)
    D2RDPL(3) = 0.0
    DOTT1 = -RSTAR(1)*DRDFI(2) + RSTAR(2)*DRDFI(1)
    DOTT2 = -DRDFI(1)*U(2) + DRDFI(2)*U(1)
    DRSDDB = RST1*(DOTT1 + DOTT2 - DRSDFI*DRSDLM)
    G21 = 2.0*(DRSDFI*DRSDLM + RST*DRSDDB)
    FO = RHO(I,J)*ARE2*COSPHI
    F1 = COSPHI*(TWOR*RHX*DR(I) - FC
    F11 = F11 + SINPHI*FOUR*DR(I)*RHX
    F11 = F11 - FB
    F11A = 0.5*F11
    CYCLE THROUGH THE NUMERICAL QUADRATURE.
    DO 445 II = 1, INTGT
    A = FO + F1*DEEFEE(II) + F11A*DEFESQ(II)
    D(1) = G0 + G1*DEEFEE(II) + G11A*DEFESQ(II)
    E(1) = G2 + G21*DEEFEE(II)
    F(1) = G22/2.0
    ESQ = E(1)**2
    DO 441 IB = 2, 4
    IBL = IB - 1
    D(IB) = -(2.0*RSTAR(1BL) + 2.0*DRDFI(1BL)*DEEFEE(II)
    + D2RDF2(1BL)*DEFESQ(II))
    E(1B) = -2.0*(DRDLM(1BL) + D2RDPL(1BL)*DEEFEE(II))
    F(1B) = -D2RDL2(1BL)
160
165 C 441 CONTINUE

```

(1.14)
(1.15)
(1.15)
(1.15)
(1.19)
(1.19)
(1.21)
(1.21)
(1.14)
(2.15)
(2.16)
(2.16)
(2.16)
(2.17)
(2.17)
(2.17)
(1.20)
(1.20)
(1.20)
(1.17)
(2.18)
(2.8)
(2.9)
(2.10)
(2.10)
(2.10)
(2.7A)
(2.7D)
(2.7E)
(2.7F)
(2.39)
(2.40)
(2.41)


```

170 F(2)=U(1)
    F(3)=U(2)
    1010 FORMAT(9E13.5)
    AD1 = ABS(D(1))
    IF(AE1.GT.0.001*AD1) GO TO 443
    AE1 = ABS(E(1))
    IF(AE1.GT.10000.0*AE1.AND.AE1.GT.0.01*AD1) GO TO 439

```

```

175 C
    C... THIS SECTION USES THE BINOMIAL SERIES.
    DHF = 1.0/SQRT(D(1))
    DL2 = DELLAM**2
    DL3 = DELLAM*DL2
    DLS = DL3*DL2
    BIGED = E(1)/D(1)
    ED2 = F(1)/D(1)
    FED2 = FED**2

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```

185 DL323 = 0.6666666666667*DL3
    FCT2 = A*(2.0*DELLAM + DL323*(0.375*ED2 - 0.5*FED) + 0.15*DLS*FED2)
    DSQ = D(1)**2
    FFF(II) = DHF*FCT2
    DO 438 IB = 1, 3
    IBP = IB + 1
    EDPR = (D(1)*E(1BP)-E(1)*D(1BP))/DSQ
    FEDPR = (D(1)*F(1BP) - F(1)*D(1BP))/DSQ
    GRADF(II,IB) = (-0.5*D(1BP)*DHF*FCT2)/D(1)

```

```

195 1 + DHF*(DL323*(0.75*A*BIGED*EDPR - 0.5*A*FEDPR)
    2 + 0.30*A*FED*FEDPR*DLS)
    438 CONTINUE
    GO TO 445

```

```

200 C... THIS SECTION ASSUMES F = 0.
    439 CONTINUE
    1001 FORMAT(3I5,6E16.7)
    FFF(II) = 0.0
    GRADF(II,1) = 0.0
    GRADF(II,2) = 0.0
    GRADF(II,3) = 0.0
    ECUBE = ESQ*E(1)
    DO 442 IBIS = 1, 2
    FCT1 = 2*IBIS-3
    FCT2 = -FCT1
    DLL = FC2*DELLAM
    FCT1 = SQRT(D(1) + E(1)*DLL)

```

```

210 ABC = A
    FCT2 = (ABC+ABC)/E(1)
    FFF(II) = FFF(II) + FCT1*FCT2*FCT2
    DO 442 IB = 1, 3
    IBP = IB + 1
    EDP = E(1)*D(1BP)
    DEP = D(1)*E(1BP)
    DFCT1 = (0.5*(D(1BP) + E(1BP)*DLL))/FCT1
    DFCT2 = -(ABC+ABC)*E(1BP)/ESQ
    GRADF(II,IB) = GRADF(II,IB) + FCT2*(FCT1*DFCT2 + FCT2*DFCT1)

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```

215
220

```



```

442 CONTINUE
GO TO 445

443 CONTINUE
FSQ = F(1)**2
FCUBE = F(1)*FSQ
Q = A
FONE = ABS(F(1))
FONERT = SQRT(FONE)
RTF = SIGN(FONERT, F(1))
QRTF = Q/RTF
QQ = 4.0*D(1)*F(1) - ESQ
F32 = FONE*FONERT
FF32 = 1.0/FF32
QQQ = ABS(QQ)
RTQ = SQRT(QQQ)
HALF = SIGN(Q.5, F(1))
TWO RTQ = 2.0*RTQ
DO 1445 IB = 1, 3
IBP = IB + 1
GRQIB = 4.0*(D(IBP)*F(1) + D(1)*F(IBP)) - 2.0*(E(1)*E(IBP))
GRQQQ(IB) = GRQIB*SIGN(1.0, QQ)
GRRTF(IB) = (HALF*F(IBP))/FONERT
GRRTQ(IB) = GRQQQ(IB)/TWO RTQ
GRRTF(IB) = - (0.5*Q*F(IBP))/F32

1445 CONTINUE
RADO = D(1) + F(1)*DLAM2
ED = E(1)*DELLAM
RAD1 = RADO + ED
RAD2 = RADO - ED
ARGO = 2.0*(1)*DELLAM
ARG1 = (E(1) + ARGO)/RTQ
ARG2 = (E(1) - ARGO)/RTQ
DO 1446 IB = 1, 3
IBP = IB + 1
GRD = D(IBP) + F(IBP)*DLAM2
EDD = F(IBP)*DELLAM
GRAD(1, IB) = GRD + EDD
GRAD(2, IB) = GRD - EDD
GARGO = 2.0*(F(IBP)*DELLAM
GARG(1, IB) = (RTQ*(E(IBP)+GARGO) - (E(1)+ARGO)*GRRTQ(IB))/QQQ
GARG(2, IB) = (RTQ*(E(IBP)-GARGO) - (E(1)-ARGO)*GRRTQ(IB))/QQQ

1446 CONTINUE
RT1 = SQRT(RAD1)
RT2 = SQRT(RAD2)
RT01 = 0.5/RT1
RT02 = 0.5/RT2
DO 1447 IB = 1, 3
GRT(1, IB) = RT01*GRAD(1, IB)
GRT(2, IB) = RT02*GRAD(2, IB)

1447 CONTINUE
C...
IF(QQ.LT.0.0) GO TO 446
BARG1 = SQRT(1.0 + ARG1**2)

```

(2.25)

(2.26)

(2.43)
(2.43)

(2.46)
(2.46)

280 BARG2 = SQRT(1.0 + ARG2**2)
 DASH = ALOG((ARG1 + BARG1)/(ARG2 + BARG2))
 448 CONTINUE
 FARG1 = 1.0/BARG1
 FARG2 = 1.0/BARG2
 DO 1448 IB = 1, 3
 GRASH(1,IB) = FARG1*GARG(1,IB)
 GRASH(2,IB) = FARG2*GARG(2,IB)
 1448 CONTINUE
 GO TO 447

(2.32)

C...
 446 CONTINUE
 IF(ABS(ARG1).LE.1.0.AND.ABS(ARG2).LE.1.0) GO TO 449
 IF(F(1).GT.0.0) GO TO 452
 453 CONTINUE
 1002 FORMAT(6X,6E20.10)
 PRINT 1002, QQ, D(1), E(1), F(1)
 GO TO 475
 452 CONTINUE
 A1 = 2.0*(RTF*RT1 + F(1)*DELLAM) + E(1)
 A2 = 2.0*(RTF*RT2 - F(1)*DELLAM) + E(1)
 TEST = A1/A2
 IF(TEST.GT.0.0) GO TO 451
 PRINT 1003, A1,A2
 1003 FORMAT(3X, 4HARGG , 2E20.12)
 GO TO 453
 C...
 451 CONTINUE
 DASH = ALOG(TEST)
 DO 454 IB = 1, 3
 IBP = IB + 1
 FIDL = F(IBP)*DELLAM
 GARGG(1,IB) = 2.0*(GRRTF(IB)*RT1 + RTF*GRT(1,IB) + FIDL)+E(IBP)
 GARGG(2,IB) = 2.0*(GRRTF(IB)*RT2 + RTF*GRT(2,IB) - FIDL)+E(IBP)
 GRASH(1,IB) = GARGG(1,IB)/A1
 GRASH(2,IB) = GARGG(2,IB)/A2
 454 CONTINUE
 GO TO 447

(2.31)

C...
 449 CONTINUE
 BARG1 = SQRT(1.0 - ARG1**2)
 BARG2 = SQRT(1.0 - ARG2**2)
 XNUM = ARG1*BARG2 - ARG2*BARG1
 DENO = ARG1*ARG2 + BARG1*BARG2
 DASH = ATAN2(XNUM,DENO)
 GO TO 448

(2.33)

C...
 447 CONTINUE
 FFF(11) = QRTF*DASH
 DO 444 IB = 1, 3
 GRADF(11,IB) = GRORTF(11,IB) + FFF(11)
 444 CONTINUE
 C 445 CONTINUE

(2.24)

(2.44)

(2.45)

C 445 CONTINUE


```

C... NEWTON-COTES NUMERICAL QUADRATURE.
DO 476 IB = 1, 4
DO 476 NI = 1, INTACT
INTMI = INTGT - NI
TFCN(IB) = TFCN(1B) + ALPHA(NI, INTGT)*(FDF(NI, 1B) + FDF(INTMI, 1B))
335 1 *FAKTOR
476 CONTINUE
GO TO 999

C... THE TAYLOR SERIES METHOD.
C
601 CONTINUE
DR2FC = G11A - DRSDFI**2
DR2FC2 = G22A - DRSDLM**2
DRS2DF = RST1*DR2FC
DRS2DL = RST1*DR2FC2
DELPHI = D1(I)
DLP = D21(I)
DPH12 = D11(I)
ZWOLF = 12.0 + (0.1 - 0.02380952381*DPHI2)*DPHI2**2
DLP3 = DLP/3.0
340 COMPUTE F.
C...
FZERO = RST1*FB
DO 490 IJK = 2, 4
ICY = IJK - 1
FZERO(IJK) = RST3*STAR(ICY)
PL(IJK, 1) = FZERO(IJK)
FZERO(IJK, 2) = FB*FZERO(IJK)
350 CONTINUE
490 COMPUTE SECOND DERIVATIVES OF F.
C...
ALBT(1, 1) = COSPHI*(RHX+RHX)*( R1**2 + ARE* R2 ) - FB
1 -FOUR*SIMPFI*RHX*R1
FPHIB = TWOR*RT*RHX
ALBT21 = FC - COSPHI*FPHIB
ALBT(2, 1) = ALBT21 + ALBT21
ALBT(3, 1) = -FB
ALBT(4, 1) = FB + FB
ALBT(1, 2) = 0.0
ALBT(1, 2) = 0.0
ALBT(3, 2) = -FB
ALBT(4, 2) = ALBT(4, 1)
365 DRPSQ = DRSDFI**2
DRDLSQ = DRSDLM**2
R2RF = RST2*DRSDFI
R2RL = RST2*DRSDLM
R2RF = RST2*DRS2DF
R2RL = RST2*DRS2DL
DO 440 IVY = 1, 3
DSRP(IVY) = -RST1*DRDFI(IVY) + R2RF*RSTAR(IVY)
DSRL(IVY) = -RST1*DRDLM(IVY) + R2RL*RSTAR(IVY)
DSR2P(IVY) = R2RF*RSTAR(IVY) - RST1*(D2RDF2(IVY)
375
380
385

```



```

390      1 DSRL2(IVY) = +2.0*DRSDFI*DSRP(IVY)
      1      + R2RL*RSTAR(IVY) - RST1*(D2RDL2(IVY)
      1      +2.0*DRSOLM*DSRL(IVY))
440 CONTINUE
C...  OTHER PL(1, -) IN EQUIVALENCE STATEMENTS.
      PL(1, 1) = RST1
      PL(1, 4, 1) = RST3*DRDPSQ
      PL(1, 1, 2) = RST1
      PL(1, 4, 2) = RST3*DRDLSQ
395 C      TWORS3 = RST3 + RST3
      FPHI(1) = -RST2*DRSDFI*FA + RST1*FPHIB
C
      DO 460 IVY = 1, 3
      IVORY = IVY+1
      PLRST = (RST1+RST1)*PL(IVORY, 1, 1)
      PL23 = RST3*PL(IVORY, 1, 1)
      PL(IVORY, 2, 1) = PLRST*DRSDFI + RST2*DSRP(IVY)
      FPHI(IVORY) = -PL(IVORY, 2, 1)*FA + PL(IVORY, 1, 1)*FPHIB
      PL(IVORY, 2, 2) = PLRST*DRSOLM + RST2*DSRL(IVY)
      PL(IVORY, 3, 1) = PLRST*DRSDFI + RST2*DSRP(IVY)
      PL(IVORY, 3, 2) = PLRST*DRSDFI + RST2*DSRP(IVY)
      PL(IVORY, 4, 1) = PL23*DRDPSQ + TWORS3*DRSDFI*DSRP(IVY)
      PL(IVORY, 4, 2) = PL23*DRDLSQ + TWORS3*DRSOLM*DSRL(IVY)
460 CONTINUE
C
      DO 410 JJ = 1, 2
      DO 410 II = 1, 4
      FCOMP(II, JJ) = 0.0
      DO 410 KK = 1, 4
      FCOMP(II, JJ) = FCOMP(II, JJ) + PL(II, KK, JJ)*ALBT(KK, JJ)
410 CONTINUE
C
      SD2 = 0.2*SINPHI*DPHI2
      DO 470 IVORY = 1, 4
      FCOMP(IVORY, 1) = FCOMP(IVORY, 1) + SD2 * FPHI(IVORY)
470 CONTINUE
C
      DO 510 N = 1, 4
      TFCN(N) = TFCN(N) + (ZWOLF*FZERO(N) + 2.0*(FCOMP(N, 1)*DPHI2 +
      1 FCOMP(N, 2)*DLAM2))*DLDP3
510 CONTINUE
C
      GO TO 999
C...  THE ONE-POINT NUMERICAL CUBATURE.
C
      701 CONTINUE
      RHXQ = WTK(1)*RHX
      TFCN(1) = TFCN(1) + RST1*RHXQ
      RHXQ3 = RST3*RHXQ
      DO 799 N = 2, 4
      TFCN(N) = TFCN(N) + RHXQ3*RSTAR(N-1)
799 CONTINUE

```

(1.31A)
(1.31D)
(1.32A)
(1.32D)

(1.37B)
(1.38B)
(1.37C)
(1.38C)
(1.37D)
(1.38D)

(1.33)
(1.34)

(1.48)
(1.50)

(1.23)
(1.25A)

(3.5)

(3.6)

SUBROUTINE DTTDX

799 CONTINUE

C 999 CONTINUE

C 475 CONTINUE

ROTATE ATTRACTION VECTOR INTO INERTIAL SYSTEM.

TFCN(1) = FACTOR*TFCN(1)

TFX = FACTOR*(COSH*TFCN(2) - SINH*TFCN(3))

TFY = FACTOR*(SINH*TFCN(2) + COSH*TFCN(3))

TFCN(4) = FACTOR*TFCN(4)

TFCN(2) = TFX

TFCN(3) = TFY

RETURN

END

445

450

R	PHI	LAMBDA	J	UX	UY	UZ	U	V	W	X	Y	Z
1.047000	0.000000	282.000000		-.1899349292E+00	.8935121503E+00	.1574606630E-04	.9132415645E+00	-.1896642801E+00	-.1896642801E+00	-.1896642801E+00	-.1896642801E+00	-.1896642801E+00
				-.1896642801E+00	.8923002828E+00	0.	.9122348019E+00	-.2706490710E-03	-.1211867453E-02	-.1574606630E-04	-.1241656575E-02	-.4630 SEC.
				DEFLECTION =	-.001271737 DEG.			ABS(DG) =		TIME =		
1.047000	1.000000	282.000000		-.1898629909E+00	.8931430484E+00	-.1605287805E-01	.9132415645E+00	-.1898629909E+00	-.1898629909E+00	-.1898629909E+00	-.1898629909E+00	-.1898629909E+00
				-.1898629909E+00	.8921607416E+00	-.4774268068E-01	.9122348019E+00	-.2725976440E-03	-.9786672731E-03	.1321855156E-03	-.1006762601E-02	-.4620 SEC.
				DEFLECTION =	-.007291959 DEG.			ABS(DG) =		TIME =		
1.047000	3.000000	282.000000		-.1896645347E+00	.8922345418E+00	-.4806076307E-01	.9134357943E+00	-.1896645347E+00	-.1896645347E+00	-.1896645347E+00	-.1896645347E+00	-.1896645347E+00
				-.1896645347E+00	.8910774163E+00	-.4774268068E-01	.9122348019E+00	-.2701828953E-03	-.1157125552E-02	.3180823963E-03	-.1200992374E-02	-.4610 SEC.
				DEFLECTION =	-.016055118 DEG.			ABS(DG) =		TIME =		
1.047000	5.000000	282.000000		-.1892415366E+00	.8901938100E+00	-.7976767724E-01	.9135755365E+00	-.1892415366E+00	-.1892415366E+00	-.1892415366E+00	-.1892415366E+00	-.1892415366E+00
				-.1892415366E+00	.8899048109E+00	-.7950650172E-01	.9122348019E+00	-.2701564790E-03	-.128899190E-02	.2611755169E-03	-.1340734585E-02	-.4710 SEC.
				DEFLECTION =	-.009214519 DEG.			ABS(DG) =		TIME =		
1.047000	7.000000	282.000000		-.1885097708E+00	.8867737083E+00	-.1113487487E+00	.9134013834E+00	-.1885097708E+00	-.1885097708E+00	-.1885097708E+00	-.1885097708E+00	-.1885097708E+00
				-.1885097708E+00	.8856492118E+00	-.1111734563E+00	.9122348019E+00	-.2606849782E-03	-.1124496449E-02	.1752923876E-03	-.1166581461E-02	-.4630 SEC.
				DEFLECTION =	-.002432695 DEG.			ABS(DG) =		TIME =		
1.047000	10.000000	282.000000		-.1870989664E+00	.8800540748E+00	-.1589166314E+00	.9136496558E+00	-.1870989664E+00	-.1870989664E+00	-.1870989664E+00	-.1870989664E+00	-.1870989664E+00
				-.1870989664E+00	.8787442363E+00	-.1584079110E+00	.9122348019E+00	-.2251396782E-03	-.3161128670E-03	.5087204862E-03	-.1414853851E-02	-.4760 SEC.
				DEFLECTION =	-.016908847 DEG.			ABS(DG) =		TIME =		
1.047000	13.000000	282.000000		-.1850291933E+00	.8704011291E+00	-.2059924226E+00	.9133820633E+00	-.1850291933E+00	-.1850291933E+00	-.1850291933E+00	-.1850291933E+00	-.1850291933E+00
				-.1850291933E+00	.8694306844E+00	-.2052081805E+00	.9122348019E+00	-.2257829270E-03	-.2259623984E-03	.7842420937E-03	-.1147261337E-02	-.4700 SEC.
				DEFLECTION =	-.033897945 DEG.			ABS(DG) =		TIME =		
1.047000	20.000000	282.000000		-.1785044555E+00	.8396098783E+00	-.3130607728E+00	.9136824590E+00	-.1785044555E+00	-.1785044555E+00	-.1785044555E+00	-.1785044555E+00	-.1785044555E+00
				-.1785044555E+00	.8384879913E+00	-.3120026777E+00	.9122348019E+00	-.1662548208E-03	-.2783310838E-03	.1058095090E-02	-.1447657041E-02	-.4850 SEC.
				DEFLECTION =	-.037652564 DEG.			ABS(DG) =		TIME =		
1.047000	25.000000	282.000000		-.1721154338E+00	.8095428369E+00	-.3868174942E+00	.9135705233E+00	-.1721154338E+00	-.1721154338E+00	-.1721154338E+00	-.1721154338E+00	-.1721154338E+00
				-.1721154338E+00	.8086986947E+00	-.3852270863E+00	.9122348019E+00	-.1089707988E-03	-.2212197879E-03	.1290407889E-02	-.1335721369E-02	-.5000 SEC.
				DEFLECTION =	-.050308338 DEG.			ABS(DG) =		TIME =		
1.047000	30.000000	282.000000		-.1643755672E+00	.7734608546E+00	-.4574909097E+00	.9135419822E+00	-.1643755672E+00	-.1643755672E+00	-.1643755672E+00	-.1643755672E+00	-.1643755672E+00
				-.1643755672E+00	.7727571212E+00	-.4561174010E+00	.9122348019E+00	-.4838399800E-04	-.1214824892E-03	.1373508704E-02	-.1307180224E-02	-.5370 SEC.
				DEFLECTION =	-.052180359 DEG.			ABS(DG) =		TIME =		
1.047000	35.000000	282.000000		-.1554306789E+00	.7312474790E+00	-.5247049054E+00	.9133337520E+00	-.1554306789E+00	-.1554306789E+00	-.1554306789E+00	-.1554306789E+00	-.1554306789E+00
				-.1554306789E+00	.7309296008E+00	-.5233363868E+00	.9122348019E+00	-.5028700539E-04	-.6679616983E-04	.1468518590E-02	-.1108950039E-02	-.5910 SEC.
				DEFLECTION =	-.063775226 DEG.			ABS(DG) =		TIME =		

R	PHI	LAMBDA	U	UX	UY	UZ	G
1.047000	40.000000	282.000000	.9549420197E+00 .9551098376E+00 .1678179317E-03 ABS(DG) = .10078E-02	-.1451996431E+00 -.1452912678E+00 -.9162476483E-04 DEFLECTION =	-.5872671690E+00 -.5863722278E+00 -.8939412133E-03 TIME = .6170 SEC. .0611779804 DEG.	-.9124535524E+00 -.9122348019E+00 -.2187504356E-03	
1.047000	42.500000	282.000000	.9548897742E+00 .9551098376E+00 .2200634500E-03 ABS(DG) = .93468E-03	-.1397081448E+00 -.1398351753E+00 -.127030517E-03 DEFLECTION =	-.6169933404E+00 -.6162968992E+00 -.6964411372E-03 TIME = .6290 SEC. -.058700702 DEG.	-.9122461950E+00 -.9122348019E+00 -.1139302720E-04	
1.047000	50.000000	282.000000	.9547385079E+00 .9551098376E+00 .3713297776E-03 ABS(DG) = .88375E-03	-.1217693489E+00 -.1219138492E+00 -.1445002973E-03 DEFLECTION =	-.6993569949E+00 -.6988124008E+00 -.5445940496E-03 TIME = .6710 SEC. -.055475835 DEG.	-.9122050290E+00 -.9122348019E+00 -.2977297521E-04	
1.047000	55.000000	282.000000	.9546314808E+00 .9551098376E+00 .4783568742E-03 ABS(DG) = .12423E-02	-.1085360928E+00 -.1087869619E+00 -.2508690416E-03 DEFLECTION =	-.7474644917E+00 -.7472590029E+00 -.2054888364E-03 TIME = .6920 SEC. .070478566 DEG.	-.9117010919E+00 -.9122348019E+00 -.537100253E-03	
1.047000	60.000000	282.000000	.9545382730E+00 .9551098376E+00 .5715646044E-03 ABS(DG) = .11410E-02	-.9459744085E-01 -.9483244005E-01 -.2346991941E-03 DEFLECTION =	-.7897769920E+00 -.7900185127E+00 -.2415206949E-03 TIME = .6960 SEC. .053117523 DEG.	-.9114684514E+00 -.9122348019E+00 -.7663505326E-03	
1.047000	65.000000	282.000000	.9544440205E+00 .9551098376E+00 .6658171646E-03 ABS(DG) = .16783E-02	-.7984358748E-01 -.8015538837E-01 -.312008880E-03 DEFLECTION =	-.8259975776E+00 -.8267655040E+00 -.7679275050E-03 TIME = .7340 SEC. .064655576 DEG.	-.9109087432E+00 -.9122348019E+00 -.1326058716E-02	
1.047000	70.000000	282.000000	.9543680277E+00 .9551098376E+00 .7418098938E-03 ABS(DG) = .18704E-02	-.6459879024E-01 -.6486900426E-01 -.2702140260E-03 DEFLECTION =	-.8558917993E+00 -.8572203118E+00 -.1328512526E-02 TIME = .7750 SEC. .047255752 DEG.	-.9103564646E+00 -.9122348019E+00 -.1698337346E-02	
1.047000	75.000000	282.000000	.9543053191E+00 .9551098376E+00 .8045185603E-03 ABS(DG) = .21131E-02	-.4886218349E-01 -.4908872786E-01 -.226543782E-03 DEFLECTION =	-.8793487317E+00 -.8811511548E+00 -.1802423169E-02 TIME = .8690 SEC. .0327697291 DEG.	-.9102085298E+00 -.9122348019E+00 -.2026272160E-02	
1.047000	80.000000	282.000000	.9542591384E+00 .9551098376E+00 .8506992514E-03 ABS(DG) = .24094E-02	-.3275407560E-01 -.3293485661E-01 -.1807810112E-03 DEFLECTION =	-.8961330819E+00 -.8983759055E+00 -.2242823592E-02 TIME = 1.0420 SEC. .030065763 DEG.	-.9098733176E+00 -.9122348019E+00 -.2361484344E-02	
1.047000	85.000000	282.000000	.954235236E+00 .9551098376E+00 .8746020534E-03 ABS(DG) = .21471E-02	-.1641113147E-01 -.1653033120E-01 -.119197375E-03 DEFLECTION =	-.9067004925E+00 -.9087654731E+00 -.2062980598E-02 TIME = 1.1050 SEC. .026002464 DEG.	-.9101279006E+00 -.9122348019E+00 -.2106901356E-02	
1.047000	89.500000	282.000000	.9542215221E+00 .9551098376E+00 .8883155598E-03 ABS(DG) = .24248E-02	-.1662218806E-02 -.1655112073E-02 -.71067352864E-05 DEFLECTION =	-.9097753062E+00 -.9122000668E+00 -.2424760585E-02 TIME = 1.1020 SEC. .002244425 DEG.	-.9098102585E+00 -.9122348019E+00 -.2424543406E-02	

R	PHI	LAMBDA	U	UX	UY	UZ
1.047000	90.000000	282.000000	.9542214621E+00	-.1215291363E-13	-.2068808914E-04	-.9099154999E+00
			.9551098376E+00	-.6412930969E-14	.3017046813E-13	-.9123348019E+00
			.8883755077E-03	.5739982662E-14	.2068808917E-04	-.2319302046E-02
			ABS(DG) = .23194E-02	DEFLECTION =	.001302693 DEG.	TIME = .8580 SEC.
1.300000	90.000000	282.000000	.7687186783E+00	-.6038053189E-14	-.1583409184E-05	-.5906188548E+00
			.7692307692E+00	-.4159711624E-14	.1956990455E-13	-.5917159763E+00
			.5120909246E-03	.1878341564E-14	.1583409204E-05	-.1097121523E-02
			ABS(DG) = .10971E-02	DEFLECTION =	.000153606 DEG.	TIME = .5450 SEC.
2.100000	90.000000	282.000000	.4761396001E+00	-.2008311306E-14	.4913608507E-14	-.2266959797E+00
			.4761904762E+00	-.1594084500E-14	.7499577936E-14	-.2267573696E+00
			.5087604768E-04	.4142268063E-15	.2585969428E-14	-.6138990277E-04
			ABS(DG) = .61390E-04	DEFLECTION =	.000000000 DEG.	TIME = .1760 SEC.
1.047000	-5.000000	282.000000	.9553735285E+00	-.1891920583E+00	.8899675267E+00	.7977411629E-01
			.9551098376E+00	-.1889425503E+00	.8899048109E+00	.7950650172E-01
			-.2636908899E-03	.2495082468E-03	-.1062715862E-02	-.2676145671E-03
			ABS(DG) = .11239E-02	DEFLECTION =	.0108954275 DEG.	TIME = .4740 SEC.
1.047000	-15.000000	282.000000	.9553065394E+00	-.1833806908E+00	.8625417741E+00	.7367588020E+00
			.9551098376E+00	-.1832016265E+00	.8618958880E+00	.7361037403E+00
			-.1967017515E-03	.1790643409E-03	-.6458860952E-03	-.6550616524E-03
			ABS(DG) = .93720E-03	DEFLECTION =	.028954022 DEG.	TIME = .4830 SEC.
1.047000	-35.000000	282.000000	.9550313708E+00	-.1552911152E+00	.7306276860E+00	.5239962691E+00
			.9551098376E+00	-.153638828E+00	.7309296008E+00	.5232363868E+00
			.7846679773E-04	.7276756043E-04	.3019147822E-03	-.7598623188E-03
			ABS(DG) = .82088E-03	DEFLECTION =	.050271152 DEG.	TIME = .6070 SEC.
1.047000	-60.000000	282.000000	.9545172423E+00	-.9453919930E-01	.448277443E+00	.7892317325E+00
			.9551098376E+00	-.9483214005E-01	.4461501414E+00	.7900185127E+00
			.5925953065E-03	-.2929407507E-03	.1322397103E-02	.7867802144E-03
			ABS(DG) = .15664E-02	DEFLECTION =	.049041122 DEG.	TIME = .7010 SEC.
1.047000	-75.000000	282.000000	.9542991670E+00	-.4887031010E-01	.2299256970E+00	.8792846522E+00
			.9551098376E+00	-.4908872786E-01	.2309443071E+00	.8811511548E+00
			.8106706502E-03	-.2184177653E-03	.1018610193E-02	.1866502623E-02
			ABS(DG) = .21375E-02	DEFLECTION =	.032934941 DEG.	TIME = .8890 SEC.
1.047000	-90.000000	282.000000	.9542214621E+00	-.1020711293E-13	.2068808899E-04	.9099154999E+00
			.9551098376E+00	-.6412930969E-14	.3017046813E-13	.9122348019E+00
			.8883755077E-03	.3794181963E-14	-.2068808896E-04	-.2319302044E-02
			ABS(DG) = .23194E-02	DEFLECTION =	.001302693 DEG.	TIME = .8770 SEC.
1.047000	60.000000	267.000000	.9545383120E+00	-.2381718256E-01	.4543644167E+00	-.7897632181E+00
			.9551098376E+00	-.2387134034E-01	.4554923079E+00	-.7900185127E+00
			.5715256194E-03	.541577695E-04	.1127891242E-02	-.2552946414E-03
			ABS(DG) = .11577E-02	DEFLECTION =	.053449624 DEG.	TIME = .7180 SEC.
1.047000	60.000000	285.000000	.954538308E+00	-.117750464E+00	.4394809862E+00	-.7897606414E+00
			.9551098376E+00	-.1180518702E+00	.4405755774E+00	-.7900185127E+00
			.5713067922E-03	-.2768237699E-03	.1094581196E-02	-.2578712998E-03
			ABS(DG) = .11581E-02	DEFLECTION =	.053303703 DEG.	TIME = .7200 SEC.

R	PHI	LAMBDA	U	UX	UY	UZ	ABS(DG)
1.047000	60.000000	288.000000	.9545377853E+00 .9551098376E+00 .5720522923E-03 .11517E-02	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	.4327274578E+00 .4337934264E+00 .1065988552E-02 .053066631 DEG.	-.7897621706E+00 -.7900185127E+00 -.2563420592E-03 TIME =	.9114518218E+00 .9122348019E+00 .7829801712E-03 .7050 SEC.
1.157000	90.000000	282.000000	.8636207841E+00 .8643042351E+00 .0834510325E-03 .15740E-02	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	-.5716817902E-03 -.270635594E-13 -.7470218108E+00 -.1574023535E-02 -.5716817927E-03 .000439400 DEG.	-.7454477874E+00 -.7470218108E+00 -.1574023535E-02 TIME =	.7454477875E+00 .7470218108E+00 .1574023535E-02 .7660 SEC.
1.157000	89.968750	282.000000	.8636207878E+00 .8643042351E+00 .0834472535E-03 .15740E-02	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	3.9120100653E-03 3.985336933E-03 7.970672679E-03 7.332686713E-03 -.000499216 DEG.	-.7454474204E+00 -.7470213664E+00 -.1573945970E-02 TIME =	.7454478206E+00 .7470218108E+00 .1573990206E-02 .8670 SEC.
1.157000	89.937500	282.000000	.8636207921E+00 .8643042351E+00 .0834429732E-03 .15740E-02	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	7.881188805E-03 7.970672679E-03 8.948387397E-03 -.000559276 DEG.	-.7454471793E+00 -.7470213664E+00 -.1573945970E-02 TIME =	.7454479342E+00 .7470218108E+00 .1573954702E-02 .9510 SEC.
1.157000	89.875000	282.000000	.8636208022E+00 .8643042351E+00 .0834329097E-03 .15739E-02	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	1.581954331E-02 1.594133587E-02 1.217925608E-04 .000679857 DEG.	-.7454461793E+00 -.747020330E+00 -.1573853739E-02 TIME =	.7454479342E+00 .7470218108E+00 .1573876555E-02 .9720 SEC.
1.157000	86.000000	282.000000	.8636256330E+00 .8643042351E+00 .0786021279E-03 .15349E-02	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	5.076682985E-01 5.097088739E-01 2.040573316E-03 .007822239 DEG.	-.7436813946E+00 -.7452021032E+00 -.152070865E-02 TIME =	.7454902789E+00 .7470218108E+00 .1531531853E-02 .9300 SEC.
1.157000	70.000000	282.000000	.8637264227E+00 .8643042351E+00 .5778123667E-03 .13813E-02	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	2.490608405E+00 2.499132951E+00 8.524546073E-03 .034744355 DEG.	-.7008990323E+00 -.7019708032E+00 -.1071850904E-02 TIME =	.7457167190E+00 .7470218108E+00 .1305091795E-02 .6540 SEC.
1.157000	55.000000	282.000000	.8639184090E+00 .8643042351E+00 .3838260440E-03 .10421E-02	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	4.181311024E+00 4.191109208E+00 9.798183986E-03 .050331429 DEG.	-.6116367792E+00 -.6119244434E+00 -.287664226E-03 TIME =	.7462119331E+00 .7470218108E+00 .8098776973E-03 .5990 SEC.
1.157000	42.600000	282.000000	.8641179683E+00 .8643042351E+00 .1862667965E-03 .74776E-03	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	5.371878026E+00 5.378643690E+00 6.765663621E-03 .051985640 DEG.	-.5059256242E+00 -.505641126E+00 -.2843116435E-03 TIME =	.7467056720E+00 .7470218108E+00 .3161388046E-03 .5520 SEC.
1.157000	30.000000	282.000000	.8642950633E+00 .8643042351E+00 .9171793643E-05 .57618E-03	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	6.326488209E+00 6.328026771E+00 1.5388471473E-03 .042836000 DEG.	-.3740652856E+00 -.3733109054E+00 -.5543802244E-03 TIME =	.7471632578E+00 .7470218108E+00 -.1414469711E-03 .4640 SEC.
1.157000	15.000000	282.000000	.8644221913E+00 .8643042351E+00 -.1179562159E-03 .61046E-03	DEFLECTION = DEFLECTION = DEFLECTION = DEFLECTION =	7.062041149E+00 7.057996752E+00 -4.044396896E-03 .025087131 DEG.	-.1937929918E+00 -.1933634717E+00 -.4493200384E-03 TIME =	.7475371774E+00 .7470218108E+00 -.5153665618E-03 .4180 SEC.

TABLES

END

R	PHI	LAMBDA	U	UX	UY	UZ	G
1.157000	5.000000	282.000000	.8644664093E+00 .8643042351E+00 -.1621741881E-03 ABS(DG) = .64486E-03	-.1548474989E+00 -.1547235489E+00 .1239499465E-03 DEFLECTION =	.7285280310E+00 .7279170670E+00 -.6109640207E-03 DEFLECTION = .008438638 DEG.	-.6527216420E-01 -.6510724077E-01 .1649234299E-03 TIME = .4160 SEC.	.7476572044E+00 .7470218108E+00 -.6353936041E-03 .4160 SEC.
1.157000	0.000000	282.000000	.8644713549E+00 .8643042351E+00 -.1671198113E-03 ABS(DG) = .64607E-03	-.1554443930E+00 -.1553145678E+00 .129825273E-03 DEFLECTION =	.7313304747E+00 .7306975919E+00 -.632882773E-03 DEFLECTION = .000427121 DEG.	.3153818584E-05 -0. -.3153818584E-05 TIME = .4180 SEC.	.7476678558E+00 .7470218108E+00 -.6460449696E-03 .4180 SEC.
1.157000	-5.000000	282.000000	.8644655987E+00 .8643042351E+00 -.1613633649E-03 ABS(DG) = .63585E-03	-.1548450956E+00 -.1547235489E+00 .121546687E-03 DEFLECTION =	.7285174891E+00 .7279170670E+00 -.6004221316E-03 DEFLECTION = .008925603 DEG.	.6527759575E-01 .6510724077E-01 -.1703549767E-03 TIME = .4140 SEC.	.7476669087E+00 .7470218108E+00 -.6250979062E-03 .4140 SEC.
.100000E+04	65.000000	282.000000	.1000004413E-02 .100000000E-02 -.4413349347E-08 ABS(DG) = .44125E-11	-.878676696E-07 -.878672737E-07 .3875885018E-12 DEFLECTION =	.4133848622E-06 .4133850387E-06 -.1823460699E-11 DEFLECTION = .000000038 DEG.	-.9063117864E-06 -.9063077870E-06 .3999364557E-11 TIME = .1790 SEC.	.1000004412E-05 .1000000000E-05 -.4412499310E-11 .1790 SEC.

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