NOAA TR NOS 42

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NOAA Technical Report NOS 42

U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
National Ocean Survey

Computational Procedures for the Determination of a Simple Layer Model of the Geopotential From Doppler Observations

BERTOLD U. WITTE

ROCKVILLE, MD.

April 1971

NOAA TECHNICAL REPORTS

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UDC 528.27:528.375:621.3.093.2:629.783

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621.3	Electrical engineering
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Computational Procedures for the Determination of a Simple Layer Model of the Geopotential from Doppler Observations

Bertold U. Witte

ABSTRACT. The geopotential of the earth is represented by the potential of a simple layer distributed over the earth's surface. Density values of this layer for 104 surface elements have been determined from Doppler observations. Computational procedures using this method are presented here. Numerical integration is employed to compute satellite orbits with time sequenced Doppler observations given as input. The geopotential forces acting upon the orbit are described as well as the attraction of the sun and moon, solar radiation pressure, and air drag. The results of the orbit fits are combined in an adjustment to determine the density values of the 104 surface elements.

1. Introduction

The determination of the earth's gravity field from Doppler observations using a simple layer model follows an approach outlined by Koch [1968]. The first results obtained by this method, from optical satellite observations, have been published by Koch and Morrison [1970]. These results showed good agreement with existing solutions, so this approach was applied to other available data.

Many stations of the worldwide satellite triangulation network [Schmid, 1969] are close to the Doppler tracking sites of the U. S. Navy Doppler Tracking Network (TRANET) [Anderle 1965] and connected by local survey to the BC-4 stations of the worldwide triangulation network. In order to combine the results of both systems the Doppler observations are as a first step processed by means of the simple layer potential. Results are given by Koch and Witte [1970], whereas the computational procedures are described here.

2. Observational Equations

Let $\vec{r}$ be the vector of the apparent position of the satellite and

$$\vec{r}^T = [x,y,z] \tag{2-1}$$

where x, y, z are the geocentric rectangular coordinates of the satellite; the origin is the center of mass of the earth; the z-axis is identical to the instantaneous axis of the earth, and the x-axis points at an angle east of the true vernal equinox which equals the precession and nutation in right ascension since 1950. Since the orbit computations extend over arcs not longer than 7 days this coordinate system is a good approximation to an inertial system.

The rectangular coordinates of the station vector (u_s, v_s, w_s) are given in a geocentric coordinate system whose orientation coincides with the worldwide satellite triangulation network; that is, the w-axis points towards the mean pole 1900-1905 and the u-axis towards the intersection of the Greenwich meridian (i.e. the zero meridian of the Bureau International de l'Heure UT1-System) with the equator.

By means of the polar motion as determined by the Bureau de l'Heure and the sidereal angle as defined by the Smithsonian Institution [1966] the station coordinates are rotated into the x,y,z system.

$$\vec{r}_s = \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = S \begin{bmatrix} u_s \\ v_s \\ w_s \end{bmatrix}$$

$$S = \begin{bmatrix} \cos\hat{\theta} & -\sin\hat{\theta} & -a\cos\hat{\theta} - b\sin\hat{\theta} \\ \sin\hat{\theta} & \cos\hat{\theta} & -a\sin\hat{\theta} + b\cos\hat{\theta} \\ a & -b & 1 \end{bmatrix} \qquad (2\text{-}2)$$

where a, b are the components of the polar motion, $\hat{\theta}$ is the sidereal angle, a simple linear expression of UT1, or

$$\hat{\theta} = 0.^{rev}277987616 + 1.^{rev}00273781191\ (MJD-32282.0),$$

MJD is the Modified Julian Day Number as defined by the Smithsonian Institution.

The vector $\vec{r}$ of the apparent position at a time t is a function of the vector $\vec{e}$ of the orbital elements at the epoch t_o, the vector $\vec{\chi}$ of the parameters of the gravity field, the vector of the station coordinates $\vec{r}_s$, the radiation pressure parameter K_R over the observation interval and the drag parameter C_D over the same time interval.

$$\vec{r} = \vec{r}\left(\vec{e}(t_o), \vec{\chi}, \vec{r}_s, K_R(t), C_D(t)\right) \tag{2-3}$$

For the formation of the observation equations the following expression is used, which is derived from the Doppler equation.

$$f = f_b - \frac{f_b}{c}\frac{d}{dt}\left(|\vec{r} - \vec{r}_s|\right) + \delta f_{tro} \tag{2-4}$$

Here f is the calculated frequency, f_b the base frequency, c the velocity of light, which is 299,792,500 m/sec [Fricke et al. 1965], $\frac{d}{dt}|\vec{r} - \vec{r}_s|$ the rate of change of the distance between the satellite and the observation station, and δf_{tro} the tropospheric refraction correction. A detailed description of the model used for tropospheric refraction is given by Witte [1970], but a condensed version is included as chapter 4.

The observed Doppler frequency f_o is obtained by comparing the incoming satellite signal with a reference signal generated from a local precision oscillator and measuring the time required to count a preset number of beats. In order to get the calculated frequency f(eq (2-4)) the base frequency f_b, which is the estimated satellite oscillator frequency, is used. The assumed value of f_b may contain an unknown systematic error which in this report is considered constant over one pass. The observed frequencies are influenced by this systematic error, which is introduced into the adjustment as a bias parameter.

Approximate values for $\vec{e}$, $\vec{\chi}$, $\vec{r}_s$, K_R, C_D are available so that we obtain by means of a Taylor series expansion the following observation equation, where a bias parameter b for the frequency offset of every base frequency for each pass is added

$$\Delta f_i = \sum_{k=1}^{6} \left(\frac{\partial f}{\partial e}\right)_{ik} \Delta e_{ko} + \sum_{\ell=1}^{104} \left(\frac{\partial f}{\partial \chi}\right)_{i\ell} \Delta \chi_\ell + \sum_{m=1}^{3} \left(\frac{\partial f}{\partial r_s}\right)_{imq} \Delta r_{s_{mq}}$$

$$+ \left(\frac{\partial f}{\partial K_R}\right)_i \Delta K_{R_o} + \left(\frac{\partial f}{\partial C_D}\right)_i \Delta C_{D_o} + \left(\frac{\partial f}{\partial b}\right)_{ij} \Delta b_j \qquad (2\text{-}5)$$

i denotes here the ith observation
j denotes here the jth pass
q denotes here the qth station (total of n stations)
k denotes here the kth orbital element
ℓ denotes here the ℓth surface density element
m denotes here the mth coordinate of $\vec{r}_s$
o denotes here the oth orbit
Δf = observed frequency f_o minus calculated frequency f.

3. Determination of Partial Derivatives of the Doppler-Equation

3.1 Evaluation of Derivatives of Doppler Frequency With Respect to Station Coordinates.

For the partial derivative $\frac{\partial f}{\partial r_{sm}}$ using equation (2-4)

$$\frac{\partial f}{\partial r_{sm}} = -\frac{f_b}{c} \frac{\partial(|\vec{r} - \vec{r}_s|^{\cdot})}{\partial r_{sm}} \qquad (3\text{-}1)$$

Here

$$|\vec{r} - \vec{r}_s|^{\cdot} = \frac{d(|\vec{r} - \vec{r}_s|)}{dt}$$

and the slant range is

$$|\vec{r} - \vec{r}_s| = \rho = \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2} \qquad (3\text{-}2)$$

The differentiation of the slant range with respect to the time yields

$$\frac{d\sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}}{dt} =$$

$$\frac{2(x-x_s)\frac{dx}{dt} + 2(y-y_s)\frac{dy}{dt} + 2(z-z_s)\frac{dz}{dt} - 2(x-x_s)\frac{dx_s}{dt} - 2(y-y_s)\frac{dy_s}{dt} - 2(z-z_s)\frac{dz_s}{dt}}{2\sqrt{(x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2}}$$

$$= \frac{(\vec{r} - \vec{r}_s) \cdot (\dot{\vec{r}} - \dot{\vec{r}}_s)}{|\vec{r} - \vec{r}_s|} \qquad (3\text{-}3)$$

Introducing $\hat{\rho}$ as a unit vector the change in the slant range yields with equation (3-3) the following relation

$$|\vec{r} - \vec{r}_s|^{\cdot} = \hat{\rho} \cdot (\dot{\vec{r}} - \dot{\vec{r}}_s) \qquad (3\text{-}4)$$

From this we get for equation (3-1)

$$\frac{\partial f}{\partial r_{sm}} = -\frac{f_b}{c}\left[\frac{\partial\hat{\rho}}{\partial r_{sm}} \cdot (\dot{\vec{r}} - \dot{\vec{r}}_s) - \hat{\rho} \cdot \frac{\partial \dot{\vec{r}}_s}{\partial r_{sm}}\right]$$

where

$$\frac{\partial\hat{\rho}}{\partial r_{sm}} = -\frac{\dfrac{\partial\vec{r}_s}{\partial r_{sm}}}{|\vec{r} - \vec{r}_s|} - \frac{\dfrac{\partial(|\vec{r} - \vec{r}_s|)}{\partial r_{sm}}\ (\vec{r} - \vec{r}_s)}{|\vec{r} - \vec{r}_s|^2}$$

$$\frac{\partial\hat{\rho}}{\partial r_{sm}} = -\frac{\dfrac{\partial\vec{r}_s}{\partial r_{sm}}}{|\vec{r} - \vec{r}_s|} + \frac{\left[\dfrac{\partial\vec{r}_s}{\partial r_{sm}} \cdot (\vec{r} - \vec{r}_s)\right] (\vec{r} - \vec{r}_s)}{|\vec{r} - \vec{r}_s|^3} \qquad (3\text{-}5)$$

with

$$\frac{\partial\vec{r}_s}{\partial x_s} = \begin{bmatrix}1\\0\\0\end{bmatrix} ; \quad \frac{\partial\vec{r}_s}{\partial y_s} = \begin{bmatrix}0\\1\\0\end{bmatrix} ; \quad \frac{\partial\vec{r}_s}{\partial z_s} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

and

$$\frac{\partial\dot{\vec{r}}_s}{\partial x_s} = \begin{bmatrix}0\\-\omega y_s\\0\end{bmatrix} ; \quad \frac{\partial\dot{\vec{r}}_s}{\partial y_s} = \begin{bmatrix}\omega x_s\\0\\0\end{bmatrix} ; \quad \frac{\partial\dot{\vec{r}}_s}{\partial z_s} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

where ω is the angular velocity of the earth.

Finally we get

$$\frac{\partial f}{\partial r_{sm}} = -\frac{f_b}{c}\left[\left(-\frac{\dfrac{\partial\vec{r}_s}{\partial r_{sm}}}{|\vec{r} - \vec{r}_s|} + \frac{\left[\dfrac{\partial\vec{r}_s}{\partial r_{sm}} \cdot (\vec{r} - \vec{r}_s)\right] (\vec{r} - \vec{r}_s)}{|\vec{r} - \vec{r}_s|^3}\right) \cdot (\dot{\vec{r}} - \dot{\vec{r}}_s) - \hat{\rho} \cdot \frac{\partial\dot{\vec{r}}_s}{\partial r_{sm}}\right]$$

(3-6)

3.2 Evaluation of Derivatives of Doppler Frequency with Respect to Parameters Influencing the Orbit.

The derivatives with respect to the parameters e_k, χ_ℓ, K_R and C_D, as used in equation (2-5), are now derived. We set

$$\frac{\partial f}{\partial p_K} = \sum_{m=1}^{3} \frac{\partial f}{\partial X_m} \frac{\partial X_m}{\partial p_K} + \sum_{m=1}^{3} \frac{\partial f}{\partial \dot{X}_m} \frac{\partial \dot{X}_m}{\partial p_K} \tag{3-7}$$

where p_K stands for the above mentioned parameters e_K, χ_ℓ, K_R and C_D; and X_m for the coordinates x, y, z of the apparent position of the satellite. The partial derivatives $\frac{\partial X}{\partial p_K}$ and $\frac{\partial \dot{X}_m}{\partial p_K}$ are computed with the help of variational equations (section 7).

The derivation of the Doppler frequency in equation (3-7) with respect to the apparent position of the satellite leads to

$$\frac{\partial f}{\partial X_m} = - \frac{f_b}{c} \frac{\partial (|\vec{r} - \vec{r}_s|^{\cdot})}{\partial X_m} \tag{3-8}$$

Applying (3-4)

$$\frac{\partial f}{\partial X_m} = - \frac{f_b}{c} \frac{\partial (\hat{\rho} \cdot (\dot{\vec{r}} - \dot{\vec{r}}_s))}{\partial X_m} \tag{3-9}$$

$$\frac{\partial f}{\partial X_m} = - \frac{f_b}{c} \left(\frac{\partial \hat{\rho}}{\partial X_m} \cdot (\dot{\vec{r}} - \dot{\vec{r}}_s) + \frac{\partial \dot{\vec{r}}}{\partial X_m} \cdot \hat{\rho} \right) \tag{3-10}$$

Analogous to equation (3-5) and with $\frac{\partial \dot{\vec{r}}}{\partial X_m} = 0$ we get

$$\frac{\partial f}{\partial X_m} = - \frac{f_b}{c} \left[\left(\frac{\frac{\partial \vec{r}}{\partial X_m}}{|\vec{r} - \vec{r}_s|} - \frac{\left[\frac{\partial \vec{r}}{\partial X_m} \cdot (\vec{r} - \vec{r}_s)\right] (\vec{r} - \vec{r}_s)}{|\vec{r} - \vec{r}_s|^3} \right) \cdot (\dot{\vec{r}} - \dot{\vec{r}}_s) \right] \tag{3-11}$$

The differentiation of the Doppler frequency with respect to the velocity $\dot{X}_m$, where $\dot{X}_m$ stands for the coordinates $\dot{x}$, $\dot{y}$, $\dot{z}$ leads to

$$\frac{\partial f}{\partial \dot{X}_m} = - \frac{f_b}{c} \frac{\partial (|\vec{r} - \vec{r}_s|^{\cdot})}{\partial \dot{X}_m} \tag{3-12}$$

$$= - \frac{f_b}{c} \frac{\partial (\hat{\rho} \cdot (\dot{\vec{r}} - \dot{\vec{r}}_s))}{\partial \dot{X}_m} \tag{3-13}$$

$$= - \frac{f_b}{c} \left[\hat{\rho} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{X}_m} \right] \tag{3-14}$$

For the components of the derivative $\frac{\partial \vec{r}}{\partial X_m}$ and $\frac{\partial \dot{\vec{r}}}{\partial \dot{X}_m}$ we use

$$\frac{\partial \vec{r}}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad \frac{\partial \vec{r}}{\partial y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; \quad \frac{\partial \vec{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3-15}$$

and

$$\frac{\partial \dot{\vec{r}}}{\partial \dot{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad \frac{\partial \dot{\vec{r}}}{\partial \dot{y}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; \quad \frac{\partial \dot{\vec{r}}}{\partial \dot{z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3-16}$$

3.3 Evaluation of the Derivative for the Bias Parameter

From equation (2-4) we get for the partial derivative of the Doppler frequency with respect to the base frequency f_b

$$\frac{\partial f}{\partial f_b} = 1 - \frac{1}{c} |\vec{r} - \vec{r}_s|^{\cdot} \tag{3-17}$$

where $|\vec{r} - \vec{r}_s|^{\cdot}$ is obtained from equation (3-3).

4. Tropospheric Refraction Model

For the computation of the correction term δf_{tro} in equation (2-4) a tropospheric refraction model derived by the Naval Weapons Laboratory is used. It is preferred to models developed by Hopfield [1963 and 1969] because the mean error of unit weight resulting from the use of this formula, which will be derived below, is somewhat less than that obtained with the Hopfield models [Witte 1971]

The range error Δs caused by the tropospheric refraction may be described by the following equation

$$\Delta s = \int_{tro} n \, ds - \int_{g} ds \qquad (4-1)$$

where n is the index of refraction for radio waves and varies along the signal path. The first integral is taken along the signal path through the troposphere and the second along the geometric path. If both ways are set equal, only an error of second order occurs

$$\Delta s = \int_{g} (n-1) \, ds \qquad (4-2)$$

The contribution of the tropospheric refraction to the Doppler shift of the signal received from a passing satellite is given by (compare also Hopfield 1963)

$$\delta f_{tro} = - \frac{f_b}{c} \frac{d(\Delta s)}{dt} \qquad (4-3)$$

In equation (4-3) Δs is differentiated with respect to the time t. For the approximation of a flat earth we get with equation (4-2) and (4-3)

$$\delta f_{tro} = - \frac{f_b}{c} \frac{d}{dt} \left(\frac{\int_o^H (n-1)\, dh}{\cos Z} \right) \qquad (4-4)$$

The angle Z is the zenith distance and H the height of the satellite above the earth. After integration and differentiation we get for (4-4)

$$\delta f_{tro} = - \frac{f_b}{c} \left(\frac{\sin Z}{\cos^2 Z} \dot{Z}\, H\, (\bar{n} - 1) \right) \qquad (4-5)$$

where $\bar{n}$ is the mean refractive index and $\dot{Z}$ the change of the zenith distance with time. In order to solve equation (4-5) the following relations are used (see also fig. 1)

1) The component of velocity along the direction of the unit vector $\hat{q}$, perpendicular to the vector $\vec{r} - \vec{r}_s = \vec{\rho}$, and in the plane of $\vec{\rho}$ and $\vec{H}$, is $\hat{q} \cdot \dot{\vec{\rho}}$

2) The component of velocity along $\hat{q}$ can also be expressed in a spherical coordinate system centered at the observer as $|\vec{r} - \vec{r}_s|\ \dot{Z}$ or $\rho \dot{Z}$. Thus

$$\rho \dot{Z} = \dot{\vec{\rho}} \cdot \hat{q} \qquad (4-6)$$

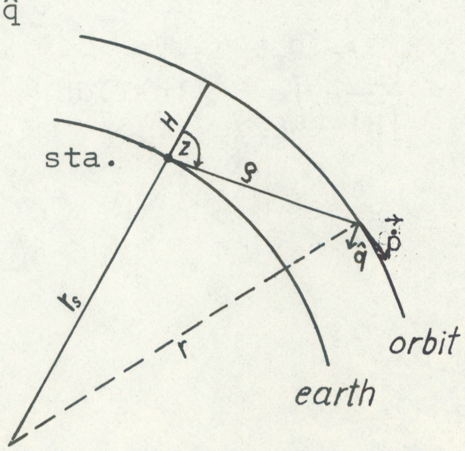

Figure 1. Diagram to illustrate eq.(4-6)

The unit vector $\hat{q}$ is given by the following relation

$$\hat{q} = \frac{\hat{\rho}\times(\hat{\rho}\times H)}{|\hat{\rho}\times(\hat{\rho}\times\hat{H})|} \tag{4-7}$$

where

$$|\hat{\rho}\times(\hat{\rho}\times\hat{H})| = \sin Z \tag{4-8}$$

$\dot{\vec{\rho}}$ in equation (4-6) can be rewritten as

$$\dot{\vec{\rho}} = \dot{\vec{r}} - \dot{\vec{r}}_s \tag{4-9}$$

and

$$\rho \approx \frac{|\vec{H}|}{\cos Z} \tag{4-10}$$

Inserting these expressions (4-7), (4-8), (4-9), (4-10) in equation (4-6) leads to

$$\frac{|\vec{H}|}{\cos Z}\dot{Z} = (\dot{\vec{r}} - \dot{\vec{r}}_s)\cdot\frac{\hat{\rho}\times(\hat{\rho}\times\hat{H})}{\sin Z} \tag{4-11}$$

and substituting $\hat{\rho}\times(\hat{\rho}\times\hat{H}) = (\hat{\rho}\cdot\hat{H})\hat{\rho} - (\hat{\rho}\cdot\hat{\rho})\hat{H} = \hat{\rho}\cos Z - \hat{H}$

into (4-11), equation (4-5) becomes

$$\delta f_{tro} = -\frac{f_b}{c}(\dot{\vec{r}} - \dot{\vec{r}}_s)\cdot(\hat{\rho} - \frac{\hat{H}}{\cos Z})(\overline{n} - 1) \tag{4-12}$$

We replace $(\overline{n} - 1)$ by

$$\overline{n} - 1 = \frac{1}{|\vec{H}|}\int_{h_{stat}}^{h_{tro}}(n-1)dh \approx \frac{\Delta N_{tro} - N_{stat}}{(\vec{r} - \vec{r}_s)\cdot\hat{H}} \tag{4-13}$$

h_{stat} = height of station

h_{tro} = height of troposphere

ΔN_{tro} = total change in refractive index from sea level up to the top of the troposphere scaled to height.

ΔN_{stat} = change in refractive index from sea level up to the height of the station scaled to height.

ΔN_{stat} is determined by means of an exponential model similar to that described by [Bean et. al. 1966].

$$\Delta N_{stat} = C_o \left(1-e^{-C_1 h_{stat}}\right) + \frac{C_2}{\cos^2 Z} \qquad (4\text{-}14)$$

In order to get flat observations corrected, the term $\frac{C_2}{\cos^2 Z}$ is added. Observations with Z>85° are deleted, because the computed corrections indicate that for this data the model used is overcorrecting.

Combining equation (4-13) and (4-12) leads to the results given in the following expression

$$\delta f_{tro} = - \frac{f_b}{c} \left\{ |\vec{r} - \vec{r}_s|^{\cdot} - \frac{\dot{\vec{r}} \cdot \dot{\vec{r}}_s}{|\vec{r}_s| \cos Z} \right\} \frac{\Delta N_{tro} - \Delta N_{stat}}{\rho \cos Z} \qquad (4\text{-}15)$$

For the constants C_o, C_1, C_2 and ΔN_{tro} the following numerical values, which are used by the Naval Weapons Laboratory (NWL)*, have shown good results

$C_o = 0.0025525$

$C_1 = 0.1238$

$C_2 = 0.657 \times 10^{-5}$

$\Delta N_{tro} = 0.0023$

These numbers were used for the computation of the tropospheric correction model as given by equation (4-15).

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*Private communication from R. J. Anderle [1970].

5. Representation of the Geopotential

The gravitational potential W of the earth is divided into the normal potential U, which is taken as known, and the disturbing potential T, which will be determined by the approach used by Koch [1968],

$$W = U + T \tag{5-1}$$

with

$$U = \frac{kM}{r}\left[1 + \sum_{n=2}^{7} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n \overline{P}_{nm}(\sin\phi) \cdot \left(\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda\right)\right] + \frac{1}{2}\omega^2 r^2 \cos^2\phi \tag{5-2}$$

r, ϕ, λ are spherical coordinates in an earth-fixed coordinate system, k is the gravitational constant, M the mass of the earth, a the mean equatorial radius, $\overline{P}_{nm}$ the fully normalized associated Legendre function of degree m and order n, and ω the angular velocity of the earth. $\overline{C}_{nm}$ and $\overline{S}_{nm}$ are fully normalized harmonic coefficients. Their values are taken from the solution of Anderle [1965]. The coefficients $\overline{C}_{2,1}$ and $\overline{S}_{2,1}$ have been deleted because of the chosen coordinate system, defined above. The centrifugal term in (5-2) is omitted for points outside the earth.

The disturbing potential in equation (5-1) is represented as the potential of a simple layer with the density $\chi(\phi,\lambda)$

distributed over the surface E of the earth

$$T = \iint_E \frac{\chi(\phi,\lambda)}{s} dE \tag{5-3}$$

s being the distance between the point at which T is computed and a variable point on E. To evaluate this integral, the earth's surface is divided into elements on which the density of the simple layer is assumed to be constant so that the integral in (5-3) can be replaced by a summation. 104 surface elements ΔE_i are chosen here. These are bordered by parallels and meridians and approximate the size of a 20° x 20° area at the equator.

$$T = \sum_{\ell=1}^{104} \chi_\ell \iint_{\Delta E_\ell} \frac{dE}{s_\ell} \tag{5-4}$$

The integral over the surface elements ΔE_ℓ in (5-4) is computed numerically by splitting ΔE_ℓ into 4 subdivisions and assuming the kernel of the integral computed for the midpoint of the subdivision to be constant over the subdivision. To compute the distance s between a satellite and the midpoint of the subdivision of ΔE_i it is necessary to determine the coordinates of these midpoints. For this purpose the equipotential surface U(eq 5-2) is defined as an ellipsoid U_o. U_o is computed from the equation which holds for the surface of a level ellipsoid

$$U_o = \frac{kM}{(a^2 - b^2)^{1/2}} \text{ arc tan } \frac{(a^2 - b^2)^{1/2}}{b} + \frac{1}{3} \omega^2 a^2 \qquad (5\text{-}5)$$

All the parameters of (5-5) are available from (5-2) except b. We obtain b from b = a(1-f), and f from the value of $\overline{C}_{20}$ in (5-2) in the following manner. The relation between $\overline{C}_{20}$ and f is given by [Heiskanen and Moritz 1967, p. 110].

$$\overline{C}_{20} = - J_2/\sqrt{5}$$

$$J_2 = \frac{1}{3} \frac{e'^2}{1+e'^2} \left(1 - \frac{2}{15} \frac{me'}{q_o} \right) \qquad (5\text{-}6)$$

with $$q_o = \frac{1}{2}\left[\left(1 + \frac{3}{e'^2}\right) \text{arc tan } e' - \frac{3}{e'}\right]$$

and $$m = \frac{\omega^2 a^2}{kM\sqrt{1+e'^2}}$$

$$e'^2 = \frac{1}{(1-f)^2} - 1$$

e' is obtained from (5-6) as a function of J_2 by the Newton-Raphson method of successive approximation. For this, the derivative $\frac{dJ_2}{de'}$ is needed

$$\frac{dJ_2}{de'} = \frac{1}{3}\left(1 - \frac{2}{15}\frac{me'}{q_o}\right)\left(\frac{2e'}{1+e'^2} - \frac{2e'^3}{(1+e'^2)^2}\right) - \frac{2}{45}\frac{e'^2}{1+e'^2}\left[\frac{m}{q_o} - \frac{e'^2}{q_o}\frac{\omega^2 a^3}{kM\sqrt{(1+e'^2)^3}} - \right.$$

$$\left. \frac{me'}{2q_o^2}\left(\left(1+\frac{3}{e'^2}\right)\frac{1}{1+e'^2} - \frac{6}{e'^3} \text{arc tan } e' + \frac{3}{e'^2}\right)\right] \qquad (5\text{-}7)$$

Now all parameters for the computation of U_o (eq (5-5)) are known and therefore the radius vector of the reference surface can be computed in setting $U_o = U$. Solving equation (5-2) for r we get

$$r = \frac{kM}{U_o} \left[1 + \sum_{n=2}^{7} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n \overline{P}_{nm} (\sin\phi) \times (\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda) \right] + \frac{\omega^2 r^2 \cos^2\phi}{2U_o} \qquad (5\text{-}8)$$

Equation (5-8) is determined for the earth-fixed coordinates of each midpoint by direct iteration for the reference surface, where the initial value for r is obtained for a given latitude ϕ on an ellipsoid of revolution with semi-axes a and b. To these values of the radius vector the topographic heights above sea level, published by Kaula et al. [1966], were added. Employing the given values ϕ, λ of the midpoints, we can compute the corresponding u, v, w coordinates, and then the x, y, z coordinates of the midpoints by means of the rotation matrix (2-2).

6. Partial Derivatives of the Potential W

The derivatives of the force acting upon the satellite with respect to the position of the satellite are needed in the solution of the variational equations (sec. 7) for the orbit computation. These derivatives are

$$\frac{\partial^2 W}{\partial x^2}, \frac{\partial^2 W}{\partial x \partial y}, \frac{\partial^2 W}{\partial x \partial z}, \frac{\partial^2 W}{\partial y^2}, \frac{\partial^2 W}{\partial y \partial z}, \text{ and } \frac{\partial^2 W}{\partial z^2} .$$

The gradient of the potential U is used for the computation of the reference orbit. The approximate density values are set equal to zero. Therefore the gradient of the potential T will not be computed here.

6.1 Gradient of the Potential U

Using the chain rule method for the partial derivatives of U, where $U = U(r, \phi, \lambda)$, we find

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial x} \quad (6\text{-}1)$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial y} \quad (6\text{-}2)$$

$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial z} \quad (6\text{-}3)$$

The differentiation of equation (5-2) with respect to r yields

$$\frac{\partial U}{\partial r} = -\frac{kM}{r^2}\left[1 + \sum_{n=2}^{7}\sum_{m=0}^{n} (n+1)\left(\frac{a}{r}\right)^n \overline{P}_{nm}(\sin\phi) \times (\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda)\right] + \omega^2 r \cos^2\phi \quad (6\text{-}4)$$

With $\dfrac{d\,\overline{P}_{nm}(\sin\phi)}{d\phi} = -m \tan\phi\, \overline{P}_{nm}(\sin\phi) + \overline{P}_{n,m+1}(\sin\phi)$

we obtain

$$\frac{\partial U}{\partial \phi} = - \frac{kM}{r}\left[\sum_{n=2}^{7} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n \left(m \tan\phi\, \overline{P}_{nm} (\sin\phi) - \overline{P}_{n,m+1} (\sin\phi)\right) \right.$$

$$\left. (\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda)\right] - \omega^2 r^2 \sin\phi \cos\phi \qquad (6\text{-}5)$$

and

$$\frac{\partial U}{\partial \lambda} = - \frac{kM}{r}\left[\sum_{n=2}^{7} \sum_{m=0}^{n} m \left(\frac{a}{r}\right)^n \overline{P}_{nm} (\sin\phi)(\overline{C}_{nm} \sin m\lambda - \overline{S}_{nm} \cos m\lambda)\right] \qquad (6\text{-}6)$$

As mentioned earlier the centrifugal terms in (6-4) and (6-5) should be omitted for points outside the earth. The other partial derivatives--equations (6-1) - (6-3)--are found with $r = \sqrt{x^2 + y^2 + z^2}$, $\tan\phi = \frac{z}{\sqrt{x^2 + y^2}}$, and $\tan\lambda = \frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r} \qquad (6\text{-}7)$$

$$\frac{\partial \phi}{\partial x} = - \frac{xz}{r^2\sqrt{x^2 + y^2}}, \quad \frac{\partial \phi}{\partial y} = \frac{-yz}{r^2\sqrt{x^2 + y^2}}, \quad \frac{\partial \phi}{\partial z} = \frac{\sqrt{x^2 + y^2}}{r^2} \qquad (6\text{-}8)$$

$$\frac{\partial \lambda}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial \lambda}{\partial y} = \frac{x}{x^2 + y^2}, \quad \frac{\partial \lambda}{\partial z} = 0 \qquad (6\text{-}9)$$

6.2 Second Derivatives of the Potential U

The second derivatives with respect to x, y z of the potential U yield (using equations (6-1) - (6-3)) the following expressions

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial^2 U}{\partial \phi^2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{\partial^2 U}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2$$

$$+ 2\frac{\partial^2 U}{\partial r\cdot\partial \phi}\frac{\partial r}{\partial x}\frac{\partial \phi}{\partial x} + 2\frac{\partial^2 U}{\partial r\cdot\partial \lambda}\frac{\partial r}{\partial x}\frac{\partial \lambda}{\partial x} + 2\frac{\partial^2 U}{\partial \phi\cdot\partial \lambda}\frac{\partial \phi}{\partial x}\frac{\partial \lambda}{\partial x}$$

$$+ \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial U}{\partial \phi}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial x^2} \qquad (6\text{-}10)$$

$$\frac{\partial^2 U}{\partial x\partial y} = \frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial x}\frac{\partial r}{\partial y} + \frac{\partial^2 U}{\partial \phi^2}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial^2 U}{\partial \lambda^2}\frac{\partial \lambda}{\partial x}\frac{\partial \lambda}{\partial y}$$

$$+ \frac{\partial^2 U}{\partial r\partial \phi}\left(\frac{\partial r}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial r}{\partial y}\frac{\partial \phi}{\partial x}\right) + \frac{\partial^2 U}{\partial r\partial \lambda}\left(\frac{\partial r}{\partial x}\frac{\partial \lambda}{\partial y} + \frac{\partial r}{\partial y}\frac{\partial \lambda}{\partial x}\right)$$

$$+ \frac{\partial^2 U}{\partial \lambda\partial \phi}\left(\frac{\partial \lambda}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \lambda}{\partial y}\frac{\partial \phi}{\partial x}\right) + \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x\partial y} + \frac{\partial U}{\partial \phi}\frac{\partial^2 \phi}{\partial x\partial y} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial x\partial y} \qquad (6\text{-}11)$$

$$\frac{\partial^2 U}{\partial x\partial z} = \frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial x}\frac{\partial r}{\partial z} + \frac{\partial^2 U}{\partial \phi^2}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial z} + \frac{\partial^2 U}{\partial \lambda^2}\frac{\partial \lambda}{\partial x}\frac{\partial \lambda}{\partial z}$$

$$+ \frac{\partial^2 U}{\partial r\partial \phi}\left(\frac{\partial r}{\partial x}\frac{\partial \phi}{\partial z} + \frac{\partial r}{\partial z}\frac{\partial \phi}{\partial x}\right) + \frac{\partial^2 U}{\partial r\partial \lambda}\left(\frac{\partial r}{\partial x}\frac{\partial \lambda}{\partial z} + \frac{\partial r}{\partial z}\frac{\partial \lambda}{\partial x}\right)$$

$$+ \frac{\partial^2 U}{\partial \lambda\partial \phi}\left(\frac{\partial \lambda}{\partial x}\frac{\partial \phi}{\partial z} + \frac{\partial \lambda}{\partial z}\frac{\partial \phi}{\partial x}\right) + \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x\partial z} + \frac{\partial U}{\partial \phi}\frac{\partial^2 \phi}{\partial x\partial z} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial x\partial z} \qquad (6\text{-}12)$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{\partial^2 U}{\partial \phi^2}\left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{\partial^2 U}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial y}\right)^2$$

$$+ 2\frac{\partial^2 U}{\partial r\partial \phi}\frac{\partial r}{\partial y}\frac{\partial \phi}{\partial y} + 2\frac{\partial^2 U}{\partial r\partial \lambda}\frac{\partial r}{\partial y}\frac{\partial \lambda}{\partial y} + 2\frac{\partial^2 U}{\partial \lambda\partial \phi}\frac{\partial \lambda}{\partial y}\frac{\partial \phi}{\partial y}$$

$$+ \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial U}{\partial \phi}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial y^2} \quad (6\text{-}13)$$

$$\frac{\partial^2 U}{\partial y \partial z} = \frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial y}\frac{\partial r}{\partial z} + \frac{\partial^2 U}{\partial \phi^2}\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial z} + \frac{\partial^2 U}{\partial \lambda^2}\frac{\partial \lambda}{\partial y}\frac{\partial \lambda}{\partial z}$$

$$+ \frac{\partial^2 U}{\partial r \partial \phi}\left(\frac{\partial r}{\partial y}\frac{\partial \phi}{\partial z} + \frac{\partial r}{\partial z}\frac{\partial \phi}{\partial y}\right) + \frac{\partial^2 U}{\partial r \partial \lambda}\left(\frac{\partial r}{\partial y}\frac{\partial \lambda}{\partial z} + \frac{\partial r}{\partial z}\frac{\partial \lambda}{\partial y}\right)$$

$$+ \frac{\partial^2 U}{\partial \lambda \partial \phi}\left(\frac{\partial \lambda}{\partial y}\frac{\partial \phi}{\partial z} + \frac{\partial \lambda}{\partial z}\frac{\partial \phi}{\partial y}\right) + \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial y \partial z} + \frac{\partial U}{\partial \phi}\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial y \partial z} \quad (6\text{-}14)$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial r^2}\left(\frac{\partial r}{\partial z}\right)^2 + \frac{\partial^2 U}{\partial \phi^2}\left(\frac{\partial \phi}{\partial z}\right)^2 + \frac{\partial^2 U}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial z}\right)^2$$

$$+ 2\frac{\partial^2 U}{\partial r \partial \phi}\frac{\partial r}{\partial z}\frac{\partial \phi}{\partial z} + 2\frac{\partial^2 U}{\partial r \partial \lambda}\frac{\partial r}{\partial z}\frac{\partial \lambda}{\partial z} + 2\frac{\partial^2 U}{\partial \lambda \partial \phi}\frac{\partial \lambda}{\partial z}\frac{\partial \phi}{\partial z}$$

$$+ \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial z^2} + \frac{\partial U}{\partial \phi}\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial z^2} \quad (6\text{-}15)$$

Now the partial derivatives in equations (6-10) - (6-15) are derived. With equation (6-4) we find

$$\frac{\partial^2 U}{\partial r^2} = \frac{2kM}{r^3}\left[1 + \frac{1}{2}\sum_{n=2}^{7}\sum_{m=0}^{n}(n+1)(n+2)\left(\frac{a}{r}\right)^n \overline{P}_{nm}(\sin\phi)\right.$$

$$\left.(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda)\right] \quad (6\text{-}16)$$

$$\frac{\partial^2 U}{\partial r \partial \phi} = -\frac{kM}{r^2}\left[\sum_{n=2}^{7}\sum_{m=0}^{n}(n+1)\left(\frac{a}{r}\right)^n\left(\overline{P}_{n,m+1}(\sin\phi) - m\tan\phi\,\overline{P}_{nm}(\sin\phi)\right)\right.$$

$$\left.(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda)\right] \quad (6\text{-}17)$$

$$\frac{\partial^2 U}{\partial r \partial \lambda} = - \frac{kM}{r^2} \left[\sum_{n=2}^{7} \sum_{m=0}^{n} (n+1) \left(\frac{a}{r}\right)^n \overline{P}_{nm} (\sin\phi)(-\overline{C}_{nm} \sin m\lambda \right.$$

$$\left. + \overline{S}_{nm} \cos m\lambda) \right] \qquad (6\text{-}18)$$

$$\frac{\partial^2 U}{\partial \phi^2} = - \frac{kM}{r} \left[\sum_{n=2}^{7} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n \left(\frac{m}{\cos^2\phi} \overline{P}_{nm} (\sin \phi) + m \tan \phi \right. \right.$$

$$(\overline{P}_{n,m+1} (\sin \phi) - m \tan \phi \, \overline{P}_{nm} (\sin \phi)) + (m+1) \tan \phi$$

$$\left. \overline{P}_{n,m+1} (\sin \phi) - \overline{P}_{n,m+2} (\sin \phi) \right) (\overline{C}_{nm} \cos m\lambda$$

$$\left. + \overline{S}_{nm} \sin m\lambda) \right] \qquad (6\text{-}19)$$

$$\frac{\partial^2 U}{\partial \phi \partial \lambda} = - \frac{kM}{r} \left[\sum_{n=2}^{7} \sum_{m=0}^{n} m \left(\frac{a}{r}\right)^n (m \tan \phi \, \overline{P}_{nm} (\sin \phi) \right.$$

$$\left. - \overline{P}_{n,m+1} (\sin \phi))(-\overline{C}_{nm} \sin m\lambda + \overline{S}_{nm} \cos m\lambda) \right] \qquad (6\text{-}20)$$

$$\frac{\partial^2 U}{\partial \lambda^2} = - \frac{kM}{r} \left[\sum_{n=2}^{7} \sum_{m=0}^{n} m^2 \left(\frac{a}{r}\right)^n \overline{P}_{nm} (\sin \phi)(\overline{C}_{nm} \cos m\lambda \right.$$

$$\left. + \overline{S}_{nm} \sin m\lambda) \right] \qquad (6\text{-}21)$$

And for the other terms in equations (6-10) - (6-15) we get

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{x^2}{r^3} \; ; \; \frac{\partial^2 r}{\partial x \partial y} = - \frac{xy}{r^3} \; ; \; \frac{\partial^2 r}{\partial x \partial z} = - \frac{xz}{r^3} \qquad (6\text{-}22)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{z}{r^2 \sqrt{x^2+y^2}} \left(- 1 + \frac{2x^2}{r^2} + \frac{x^2}{x^2+y^2} \right)$$

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{xyz}{r^2\sqrt{x^2+y^2}}\left(\frac{2}{r^2} + \frac{1}{x^2+y^2}\right) ; \frac{\partial^2\phi}{\partial x\partial z} = \frac{-x}{r^2\sqrt{x^2+y^2}} + \frac{2xz^2}{r^4\sqrt{x^2+y^2}} \quad (6\text{-}23)$$

$$\frac{\partial^2\lambda}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} ; \frac{\partial^2\lambda}{\partial x\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} ; \frac{\partial^2\lambda}{\partial x\partial z} = 0 \quad (6\text{-}24)$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{y^2}{r^3} ; \frac{\partial^2 r}{\partial y\partial z} = -\frac{yz}{r^3} \quad (6\text{-}25)$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{z}{r^2\sqrt{x^2+y^2}}\left(\frac{2y^2}{r^2} - 1 + \frac{y^2}{x^2+y^2}\right)$$

$$\frac{\partial^2\phi}{\partial y\partial z} = -\frac{y}{r^2\sqrt{x^2+y^2}} + \frac{2yz^2}{r^4\sqrt{x^2+y^2}} \quad (6\text{-}26)$$

$$\frac{\partial^2\lambda}{\partial y^2} = -\frac{2xy}{(x^2+y^2)^2} ; \frac{\partial^2\lambda}{\partial y\partial z} = 0 \quad (6\text{-}27)$$

$$\frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{z^2}{r^3} ; \frac{\partial^2\phi}{\partial z^2} = -\frac{2\sqrt{x^2+y^2}\, z}{r^4} ; \frac{\partial^2\lambda}{\partial z^2} = 0 \quad (6\text{-}28)$$

This method of computing the partials of U was found by Gulick [1970] to be the most efficient compared to two other alternatives. Subroutines which compute these partials based on this and other methods may be found in this reference along with a detailed discussion of comparative efficiencies and a complete derivation of expressions given in equations (6-10) - (6-28).

7. Variational Equations

The position of a satellite can be described as a function of the parameters of the earth's gravity field plus non-conservative forces F_c

$$\ddot{\vec{r}} = \nabla W(\vec{r},\vec{\chi},t) + F_c(\vec{r},\dot{\vec{r}}) \tag{7-1}$$

For the time being the non-conservative forces F_c will not be considered here so that we have

$$\ddot{\vec{r}} = \nabla W(\vec{r},\vec{\chi},t) \tag{7-2}$$

For the solution of this differential equation a numerical integration procedure, as described in the next section is needed. In order to do an orbit adjustment, additional equations, called variational equations, are necessary for the determination of the partial derivatives $\frac{\partial X_m}{\partial p_k}$ and $\frac{\partial \dot{X}_m}{\partial p_K}$ given in equation (3-7). To make the notation more concise, we form state vectors $\vec{r}_{st} = (\vec{r},\dot{\vec{r}})$ and $\vec{r}_{o_{st}} = (\vec{r}_o,\dot{\vec{r}}_o)$. $\vec{r}_{st}$ consists of the quantities X_m, $\dot{X}_m$. $\vec{r}_{o_{st}}$ constitutes a subset of the parameters p_k. Here $\vec{r}_o$ is the initial position vector and $\dot{\vec{r}}_o$ the vector of the initial velocity; $\vec{r}_o$ and $\dot{\vec{r}}_o$ are equivalent to the six orbital elements represented by e_k in equation (2-5). Using the state vector notation we get for eq (7-2)

$$\dot{\vec{r}}_{st} = (\dot{\vec{r}},\nabla W) = G_s(\vec{r}_{st},\vec{\chi},t) \tag{7-3}$$

The position of a satellite is now given as a solution to the differential equation (7-3) with the initial state vector as boundary condition. If we form the time derivative of the partial of $\vec{r}_{st}$ with respect to the parameters $\vec{p}_K$ we get

$$\frac{d\left(\dfrac{\partial \vec{r}_{st}}{\partial \vec{p}_K}\right)}{dt} = \frac{\partial \vec{\dot{r}}_{st}}{\partial \vec{p}_K} \tag{7-4}$$

The numerical integration procedure will be applied to (7-4) in order to obtain the required partial derivatives. First we will determine the above mentioned time derivative of the partial of $\vec{r}_{st}$ with respect to the initial state vector and get, with eq (7-3),

$$\frac{d\left(\dfrac{\partial \vec{r}_{st}}{\partial \vec{r}_{o\,st}}\right)}{dt} = \frac{\partial \vec{\dot{r}}_{st}}{\partial \vec{r}_{st}} \frac{\partial \vec{r}_{st}}{\partial \vec{r}_{o\,st}} + \frac{\partial \vec{\dot{r}}_{st}}{\partial \vec{\chi}} \frac{\partial \vec{\chi}}{\partial \vec{r}_{o\,st}} + \frac{\partial \vec{\dot{r}}_{st}}{\partial t} \frac{\partial t}{\partial \vec{r}_{o\,st}}$$

Note that the last two terms are zero, because χ and t do not depend on the initial state vector. So we finally get

$$\frac{d\left(\dfrac{\partial \vec{r}_{st}}{\partial \vec{r}_{o\,st}}\right)}{dt} = \frac{\partial \vec{\dot{r}}_{st}}{\partial \vec{r}_{st}} \frac{\partial \vec{r}_{st}}{\partial \vec{r}_{o\,st}} \tag{7-5}$$

Using the rectangular coordinate system x, y, z, defined in section 2, we get in matrix notation for the first factor in eq (7-5)

$$\frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{r}_{st}} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \dot{y}} & \frac{\partial \dot{x}}{\partial \dot{z}} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} & \frac{\partial \dot{y}}{\partial \dot{x}} & \frac{\partial \dot{y}}{\partial \dot{y}} & \frac{\partial \dot{y}}{\partial \dot{z}} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial \dot{x}} & \frac{\partial \dot{z}}{\partial \dot{y}} & \frac{\partial \dot{z}}{\partial \dot{z}} \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial y} & \frac{\partial \ddot{x}}{\partial z} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial \dot{y}} & \frac{\partial \ddot{x}}{\partial \dot{z}} \\ \frac{\partial \ddot{y}}{\partial x} & \frac{\partial \ddot{y}}{\partial y} & \frac{\partial \ddot{y}}{\partial z} & \frac{\partial \ddot{y}}{\partial \dot{x}} & \frac{\partial \ddot{y}}{\partial \dot{y}} & \frac{\partial \ddot{y}}{\partial \dot{z}} \\ \frac{\partial \ddot{z}}{\partial x} & \frac{\partial \ddot{z}}{\partial y} & \frac{\partial \ddot{z}}{\partial z} & \frac{\partial \ddot{z}}{\partial \dot{x}} & \frac{\partial \ddot{z}}{\partial \dot{y}} & \frac{\partial \ddot{z}}{\partial \dot{z}} \end{bmatrix} \tag{7-6}$$

If we partition this matrix, we will get the following structure

$$\frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{r}_{st}} = \begin{bmatrix} 0 & I \\ \frac{\partial^2 W}{\partial X_i \, \partial X_j} & \frac{\partial \ddot{X}_m}{\partial \dot{X}_m} \end{bmatrix} \tag{7-7}$$

where 0 is a 3×3 zero matrix and I is a 3×3 identity matrix (since $\vec{r}$ and $\dot{\vec{r}}$ are the independent parameters of the differential equation (7-3)), the lower left hand portion arises from differentiation of (7-5) with $\frac{\partial^2 W}{\partial X_i \partial X_j}$ as a shorthand notation for this part, and the lower right hand portion is 0 if drag is not present or is neglected; terms from drag would occur in the entire lower half of the matrix if drag were present. If a simple radiation pressure model is included, a Dirac delta function term occurs in the lower left portion of the matrix; for a very complex model with re-radiation an impulselike component would still appear.

In setting up the variational equations for the solution of $\vec{\chi}$ and $\vec{r}_{o_{st}}$, the nonconservative forces F_c are not being applied. Hence, we set the term $\frac{\partial \ddot{X}_m}{\partial \dot{X}_m}$ in equation (7-7) to zero, and get instead of (7-7)

$$\frac{\partial \vec{r}_{st}}{\partial \vec{r}_{st}} = \begin{bmatrix} 0 & I \\ \frac{\partial^2 W}{\partial X_i \partial X_j} & 0 \end{bmatrix} \tag{7-8}$$

The differentiation of the state vector $\vec{r}_{st}$ with respect to the initial state vector $\vec{r}_{o_{st}}$ yields

$$\frac{\partial \vec{r}_{st}}{\partial \vec{r}_{o_{st}}} = \begin{bmatrix} \frac{\partial x}{\partial x_o} & \frac{\partial x}{\partial y_o} & \frac{\partial x}{\partial z_o} & \frac{\partial x}{\partial \dot{x}_o} & \frac{\partial x}{\partial \dot{y}_o} & \frac{\partial x}{\partial \dot{z}_o} \\ \frac{\partial y}{\partial x_o} & \frac{\partial y}{\partial y_o} & \frac{\partial y}{\partial z_o} & \frac{\partial y}{\partial \dot{x}_o} & \frac{\partial y}{\partial \dot{y}_o} & \frac{\partial y}{\partial \dot{z}_o} \\ \frac{\partial z}{\partial x_o} & \frac{\partial z}{\partial y_o} & \frac{\partial z}{\partial z_o} & \frac{\partial z}{\partial \dot{x}_o} & \frac{\partial z}{\partial \dot{y}_o} & \frac{\partial z}{\partial \dot{z}_o} \\ \frac{\partial \dot{x}}{\partial x_o} & \frac{\partial \dot{x}}{\partial y_o} & \frac{\partial \dot{x}}{\partial z_o} & \frac{\partial \dot{x}}{\partial \dot{x}_o} & \frac{\partial \dot{x}}{\partial \dot{y}_o} & \frac{\partial \dot{x}}{\partial \dot{z}_o} \\ \frac{\partial \dot{y}}{\partial x_o} & \frac{\partial \dot{y}}{\partial y_o} & \frac{\partial \dot{y}}{\partial z_o} & \frac{\partial \dot{y}}{\partial \dot{x}_o} & \frac{\partial \dot{y}}{\partial \dot{y}_o} & \frac{\partial \dot{y}}{\partial \dot{z}_o} \\ \frac{\partial \dot{z}}{\partial x_o} & \frac{\partial \dot{z}}{\partial y_o} & \frac{\partial \dot{z}}{\partial z_o} & \frac{\partial \dot{z}}{\partial \dot{x}_o} & \frac{\partial \dot{z}}{\partial \dot{y}_o} & \frac{\partial \dot{z}}{\partial \dot{z}_o} \end{bmatrix} \tag{7-9}$$

For the start of the integration procedure this matrix has the following structure

$$\frac{\partial \vec{r}_{st}}{\partial \vec{r}_{o\,st}} = \begin{bmatrix} I & 0 \\ & \\ 0 & I \end{bmatrix} \tag{7-10}$$

For the time change of the derivatives of the state vector $\vec{r}_{st}$ with respect to $\vec{\chi}$ we get, analogous to (7-5),

$$\frac{d\left(\frac{\partial \vec{r}_{st}}{\partial \vec{\chi}}\right)}{dt} = \frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{r}_{st}} \frac{\partial \vec{r}_{st}}{\partial \vec{\chi}} + \frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{\chi}} \tag{7-11}$$

with

$$\frac{\partial \vec{r}_{st}}{\partial \vec{\chi}} = \begin{bmatrix} \frac{\partial x}{\partial \chi_1} & \frac{\partial x}{\partial \chi_2} & \cdot\;\cdot\;\cdot & \frac{\partial x}{\partial \chi_{104}} \\ \frac{\partial y}{\partial \chi_1} & \frac{\partial y}{\partial \chi_2} & \cdot\;\cdot\;\cdot & \frac{\partial y}{\partial \chi_{104}} \\ \frac{\partial z}{\partial \chi_1} & \frac{\partial z}{\partial \chi_2} & \cdot\;\cdot\;\cdot & \frac{\partial z}{\partial \chi_{104}} \\ \frac{\partial \dot{x}}{\partial \chi_1} & \frac{\partial \dot{x}}{\partial \chi_2} & \cdot\;\cdot\;\cdot & \frac{\partial \dot{x}}{\partial \chi_{104}} \\ \frac{\partial \dot{y}}{\partial \chi_1} & \frac{\partial \dot{y}}{\partial \chi_2} & \cdot\;\cdot\;\cdot & \frac{\partial \dot{y}}{\partial \chi_{104}} \\ \frac{\partial \dot{z}}{\partial \chi_1} & \frac{\partial \dot{z}}{\partial \chi_2} & \cdot\;\cdot\;\cdot & \frac{\partial \dot{z}}{\partial \chi_{104}} \end{bmatrix} \tag{7-12}$$

$$\frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{\chi}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{\partial \ddot{x}}{\partial \chi_1} & \frac{\partial \ddot{x}}{\partial \chi_2} & , \cdots & \frac{\partial \ddot{x}}{\partial \chi_{104}} \\ \frac{\partial \ddot{y}}{\partial \chi_1} & \frac{\partial \ddot{y}}{\partial \chi_2} & , \cdots & \frac{\partial \ddot{y}}{\partial \chi_{104}} \\ \frac{\partial \ddot{z}}{\partial \chi_1} & \frac{\partial \ddot{z}}{\partial \chi_2} & , \cdots & \frac{\partial \ddot{z}}{\partial \chi_{104}} \end{bmatrix} \qquad (7\text{-}13)$$

and $\frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{r}_{st}}$ supplied by (7-8).

In (7-11), $\frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{\chi}}$ represents the explicit instantaneous dependence of velocity and acceleration on the gravity field. Since instantaneous velocity is an independent variable, representing, with the instantaneous position the instantaneous state of the satellite, the top half of (7-13) is null. On the other hand, $\frac{\partial \vec{r}_{st}}{\partial \vec{\chi}}$ represents the dependence of the state vector (position and velocity) on the gravity field, considering the state vector as obtained by integrating the accleration over time; hence the terms of (7-12) are not necessarily zero, although the lower half of (7-12) literally appears to be the same as the upper half of (7-13).

Initial conditions for the integration are $\left.\frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{\chi}_\ell}\right|_o = \begin{bmatrix} 0 \end{bmatrix}$ (7-14)

For the elements of matrix (7-13) we find

$$\frac{\partial \ddot{x}}{\partial \chi_\ell} = \frac{\partial}{\partial \chi_\ell} \left(\frac{\partial T}{\partial x}\right) \qquad (7\text{-}15)$$

$$\frac{\partial \ddot{y}}{\partial \chi} = \frac{\partial}{\partial \chi_\ell} \left(\frac{\partial T}{\partial y}\right) \qquad (7\text{-}16)$$

$$\frac{\partial \ddot{z}}{\partial \chi_\ell} = \frac{\partial}{\partial \chi_\ell} \left(\frac{\partial T}{\partial z}\right) \qquad (7\text{-}17)$$

where T is the disturbing potential as defined in eq. (5-3). Using the distance $s = \sqrt{(x - x_\ell)^2 + (y - y_\ell)^2 + (z - z_\ell)^2}$ between the satellite and the midpoint of the ℓth surface element we get for the partial derivatives of T with respect to x, y, z from eq (5-4)

$$\frac{\partial T}{\partial x} = \sum_{\ell=1}^{104} \chi_\ell \iint_{\Delta E_\ell} - \frac{x-x_\ell}{s^3}\, dE \tag{7-18}$$

$$\frac{\partial T}{\partial y} = \sum_{\ell=1}^{104} \chi_\ell \iint_{\Delta E_\ell} - \frac{y-y_\ell}{s^3}\, dE \tag{7-19}$$

$$\frac{\partial T}{\partial z} = \sum_{\ell=1}^{104} \chi_\ell \iint_{\Delta E_\ell} - \frac{z-z_\ell}{s^3}\, dE \tag{7-20}$$

For the equations (7-15) - (7-17) we now get

$$\frac{\partial}{\partial \chi_\ell}\left(\frac{\partial T}{\partial x}\right) = \iint_{\Delta E_\ell} - \frac{x-x_\ell}{s^3}\, dE \tag{7-21}$$

$$\frac{\partial}{\partial \chi_\ell}\left(\frac{\partial T}{\partial y}\right) = \iint_{\Delta E_\ell} - \frac{y-y_\ell}{s^3}\, dE \tag{7-22}$$

$$\frac{\partial}{\partial \chi_\ell}\left(\frac{\partial T}{\partial z}\right) = \iint_{\Delta E_\ell} - \frac{z-z_\ell}{s^3}\, dE \tag{7-23}$$

8. Orbit Integration Procedures

The equations of motion and the variational equations of the satellite expressed in the rectangular coordinate system defined in sec. 2 are integrated numerically with an 0.8 minute time step. The twelfth-order Cowell-Störmer multistep process is used for the positions and the tenth-order predictor corrector formulas of Adams-Bashforth and Adams-Moulton respectively are used for the velocities [Henrici 1962, pp. 191-198, 291-294]. The accompanying flow-chart shows the structure of the above mentioned procedures.

The derivatives with respect to the initial state vector and to the unknown density values (i.e., eq (7-5) and (7-11)) are obtained numerically by integrating the variational equations. In this integration the corrector is solved explicitly without predicting, using the linearity of the variational equations [Riley et al. 1967].

These procedures require the use of a starting process, because the first time steps cannot be computed without knowing previous values. Gill's modification of the Runge-Kutta integration method [Romanelli 1960] is therefore employed for the computation at these time steps. The methods mentioned are now explained.

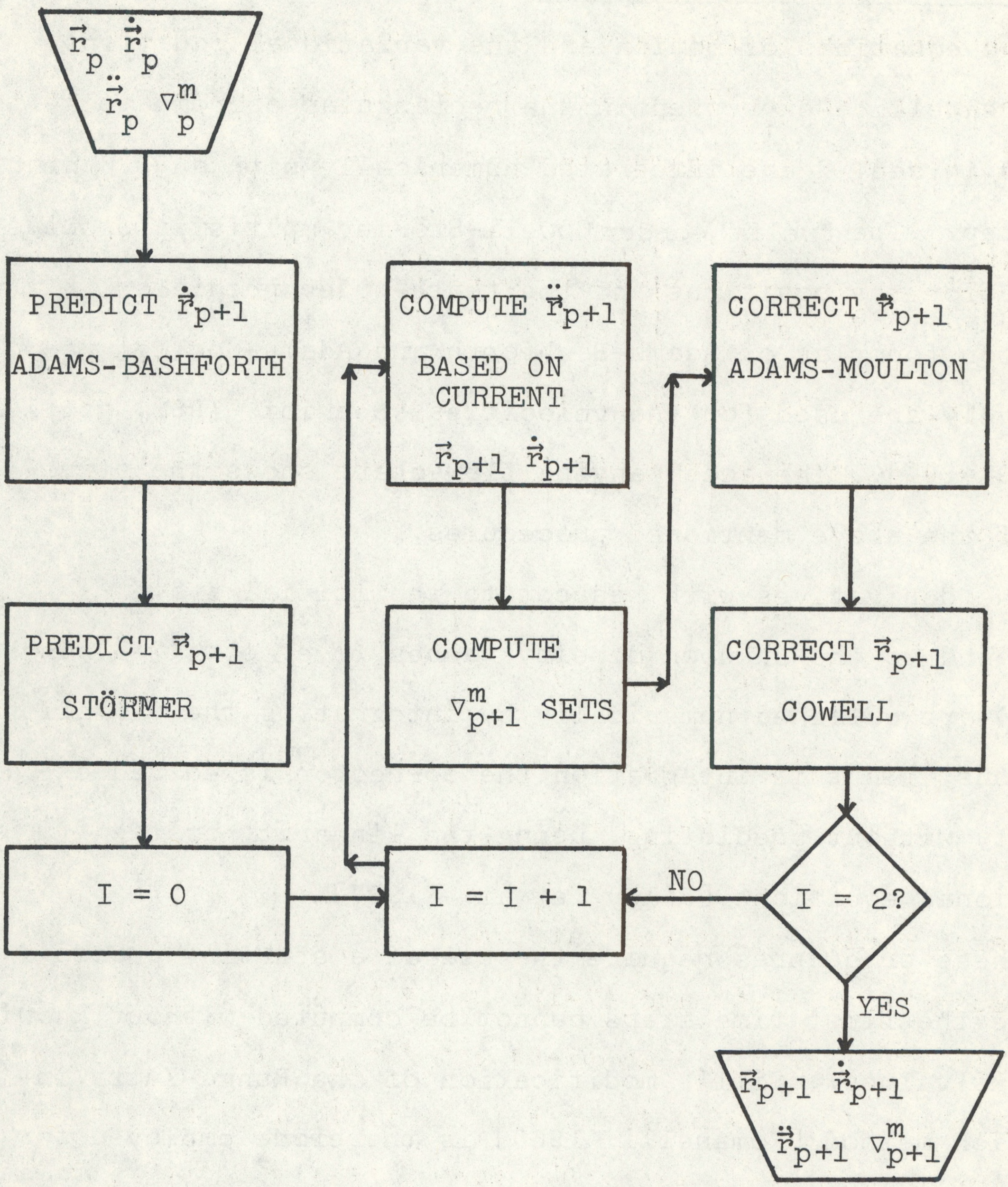


Figure 2. Integration process for the computation of position, velocity, and acceleration

8.1 Runge-Kutta-Gill Method

The totality of the variational equations is, as has been shown in section 7, a 6×(6+104) matrix with $\vec{\alpha}=\vec{\alpha}(\vec{r}_{o\,st},\vec{\chi})$, where $\vec{\alpha}$ is a subset of parameters of $\vec{p}_k$

$$\frac{\partial(\vec{r}_{st})}{\partial(\vec{\alpha})} = \frac{\partial(\vec{r}_{st})}{\partial(\vec{r}_{o\,st},\vec{\chi})} = \left[\frac{\partial\vec{r}_{st}}{\partial\vec{r}_{o\,st}}, \frac{\partial\vec{r}_{st}}{\partial\vec{\chi}}\right] \tag{8-1}$$

We now apply the Runge-Kutta-Gill integration formulas to determine $\partial(\vec{r}_{st})/\partial(\vec{\alpha})$ from its time derivatives, given by (7-5) and (7-11)

$$\left.\begin{aligned} K_p &= t\,\frac{d}{dt}\,Y_p \\ R_{p+1} &= \beta_p(K_p - \delta_p\,Q_p) \\ Y_{p+1} &= Y_p + R_{p+1} \\ Q_{p+1} &= Q_p + 3R_{p+1} + \varepsilon_p\,K_p \end{aligned}\right\} \quad p = 0,1,2,3 \tag{8-2}$$

After each cycle an intermediate value for $d/dt[\partial(\vec{r}_{st})/\partial(\vec{\alpha})]$ is computed from (7-5) and (7-11) using Y_p for $\partial(\vec{r}_{st})/\partial(\vec{\alpha})$ (i.e. for $\partial(\vec{r}_{st})/\partial(\vec{r}_{o\,st})$ in (7-5) and $\partial(\vec{r}_{st})/\partial(\vec{\chi})$ in (7-11)). The value of $\partial(\vec{r}_{st})/\partial(\vec{\alpha})$ for $t + \Delta t$ is then Y_4. The initial value for $Q_p = Q_o$ is a 6×(6+104) zero-matrix and for $Y_p = Y_o$ the analytical derivatives at the starting time t_o as given in section 7.

$\beta_{p+1}, \delta_p, \varepsilon_p$ are

p	β_p	δ_p	ε_p
0	1/2	2	-1/2
1	$1 - \sqrt{1/2}$	1	$\sqrt{1/2} - 1$
2	$1 + \sqrt{1/2}$	1	$-1 - \sqrt{1/2}$
3	1/6	2	-1/2

(8-3)

This method is also used for the computation of the starting values for the components of the velocity and position-vectors with Y_o now as the initial state vector as obtained for the different satellites from the Smithsonian Institution. Q_o again is a zero vector. For both computations a time step Δt of 0.1 minute is used only in the starting procedure.

8.2 The Adams-Bashforth Method

Here we have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_{p+1} - \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_{p} \cong \int_{t_p}^{t_{p+1}} P(t)dt = \Delta t \sum_{m=0}^{q} \gamma_m \nabla^m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}_{p} \qquad (8\text{-}4)$$

where the constants γ_m are independent and will be calculated numerically below. Δt is the time step; q is the number of steps, which in this case is 10; and $P(t)$ is the interpolating polynomial over the interval $[t_{p-q}, t_p]$.

From the Runge-Kutta-Gill method initial values for the state vector are supplied. Using these values

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_q , \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{q-1} , \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{q-2} ; \ldots \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_0$$ are computed

from the equation of motion so that the differences $\nabla^m \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_p$

can be calculated from the relations

$$\nabla \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_p = \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_p - \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p-1}$$

$$\nabla^m \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_p = \nabla \left(\nabla^{m-1} \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_p \right) \qquad (8\text{-}5)$$

The expression on the right hand side of equation (8-4) is thus known, and $\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{p+1}$ can be calculated for $p \geq q$. After using the corrector formula, explained in sec. 8.3, the index p is increased by 1, and the same formula is used to calculate $\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{p+2}$, etc. With the help of the following recurrence formula, which is derived in Henrici [1962, p. 193], numerical values for γ_m were calculated

$$\gamma_m + \frac{1}{2}\gamma_{m-1} + \frac{1}{3}\gamma_{m-2} + \ldots + \frac{1}{m+1}\gamma_{m-q} = 1, \; m = 0,1,2, \ldots$$

8.3 The Adams-Moulton Method

Whereas the Adams-Bashforth method is a predictor formula the Adams-Moulton method is a corrector formula. Here we have

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{p+1} - \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{p} = \int_{t_p}^{t_{p+1}} P(t)dt = \Delta t \sum_{m=0}^{q} \gamma_m^* \nabla^m \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p+1} \qquad (8\text{-}6)$$

where P(t) is the interpolating polynomial over the interval $[t_{p-q+1}, t_{p+1}]$. The recurrence formula for γ_m^* is similar to that one for γ_m and can also be found in Henrici [1962] on page 194. Formula (8-6) is used like the Adams-Bashforth formula (8-4) except that now only the values $\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p}$, $\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p-1}$ ... $\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p-q+1}$ are known. It is employed to redetermine $\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{p+1}$ in an iterative procedure, where an approximate value $\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p+1}$ has been obtained using the results of the predictor formulas (Adams-Bashforth and Cowell-Störmer)

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{p+1}^{(1)} = \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{p} + \Delta t \sum_{m=0}^{q} \gamma_m^* \nabla^m \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p+1}^{(0)} \qquad (8\text{-}7)$$

Calculating $\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p+1}^{(1)}$ and re-evaluating the differences, a better values is then obtained for $\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}_{p+1}$.

8.4 The Störmer Method

By integrating a differential equation of the second order y"(x) = f(x,y,(x)) twice, we obtain finally the formulas for the computation of the position vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{p+1} - 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{p} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{p-1} = \Delta t^2 \sum_{m=0}^{q} \sigma_m \nabla^m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}_{p} \qquad (8\text{-}8)$$

where again the above used symbols are defined as before.
σ_m is obtained with the help of the following recurrence formula

$$\sigma_m = 1 - \frac{2}{3} h_2 \, \sigma_{m-1} - \frac{2}{4} h_3 \sigma_{m-2} - \, . \, . \, . \, . - \frac{2}{m+2} h_{m+1} \sigma_o \, ; m=1,2,\ldots$$

with $h_m = 1 + \frac{1}{2} + \, . \, . \, . + \frac{1}{m}$; m = 1,2 ... .

Formula (8-8) is used in the same manner as the Adams-Bashforth method.

8.5 The Cowell Method

Here we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{p} - 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{p-1} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{p-2} = \Delta t^2 \sum_{m=0}^{q} \sigma_m^* \nabla^m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}_{p} \qquad (8\text{-}9)$$

where $\sigma_m^* = - \frac{2}{3} h_2 \, \sigma_{m-1}^* - \frac{2}{4} h_3 \sigma_{m-2}^* - \ldots - \frac{2}{m+2} h_{m+1} \, \sigma_o^*$, m = 1,2, ...

8.6 Integrating Procedure for Variational Equations

The system of variational equations (7-5), (7-11) can be written as

$$\dot{R} = SR + T \tag{8-10}$$

where $R = \left[\frac{\partial \vec{r}_{st}}{\partial \vec{r}_{o\,st}}, \frac{\partial \vec{r}_{st}}{\partial \vec{\chi}}\right]$ (8-11)

and is a 6×(6+104) matrix;

$S = \frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{r}_{st}}$, and is 6×6 ;

and $T = \left[0 \ , \ \frac{\partial \dot{\vec{r}}_{st}}{\partial \vec{\chi}}\right]$

and is also 6×(6+104).

The equations (8-10) are integrated using the qth order Adams-Moulton formula. At the $(p+1)_{st}$ step,

$$R_{p+1} = R_p + \Delta t \sum_{\rho=0}^{q} \beta^*_{q\rho} \dot{R}_{p+1-\rho} \tag{8-12}$$

where $\beta^*_{qp} = (-1)^\rho \quad \binom{\rho}{\rho} \gamma^*_\rho + \binom{\rho+1}{\rho} \gamma^*_{\rho+1} + \ldots + \binom{q}{\rho} \gamma^*_q$

$\rho = 0,1 \ldots, q.$

$q = 0,1 \ldots .$

Assuming that R and $\dot{R}$ up to the pth step are known from previous applications of this procedure (or at the start from initial values obtained by the Runge-Kutta-Gill method), we see that the only quantity not known on the right-hand side is $\dot{R}_{p+1}$. Let us rewrite (8-12) as

$$R_{p+1} = R_p + \Delta t \sum_{\rho=1}^{q} \beta^*_{q\rho} \dot{R}_{p+1-\rho} + \Delta t\, \beta^*_{qo} \dot{R}_{p+1}$$

Substituting for $\dot{R}_{p+1}$ from (8-10), we obtain

$$R_{p+1} = R_p + \Delta t \sum_{\rho=1}^{q} \beta^*_{q\rho} \dot{R}_{p+1-\rho} + \Delta t\, \beta^*_{qo} SR_{p+1} + \Delta t\, \beta^*_{qo} T \ ,$$

and solving for R_{p+1} ,

$$R_{p+1} = \left[I - \Delta t\, \beta^*_{qo} S \right]^{-1} \left\{ R_p + \Delta t \left[\beta^*_{qo} T + \sum_{\rho=1}^{q} \beta^*_{q\rho} \dot{R}_{p+1-\rho} \right] \right\} .$$

The matrix in brackets to be inverted is 6×6, and the matrix in braces is 6×(6+104). Following the computation of R_{p+1}, $\dot{R}_{p+1}$ can be obtained directly from (8-10) for use in the next step.

The actual computational procedure breaks the matrix R into components as shown in (8-11) and follows the outline given by Riley et al. [1967].

8.7 The Lagrangian Interpolation Method.

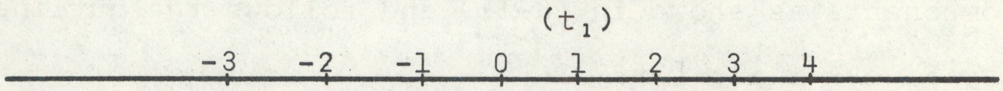
To interpolate between the time steps, Lagrange's interpolation for equidistant abscissas as given by Henrici [1964] on pp. 201 - 202 is applied. Because an equal time step Δt is used, we can assume that the points, where the values (f_k) of the positions, velocities and variational equations are given, are equally spaced. Here we have

$$p(s) = \sum_{k=m}^{n} \ell_k(s)\, f_k$$

where

$$\ell_k(s) = \prod_{\substack{q=m \\ q \neq k}}^{n} \frac{s-q}{k-q}$$

This representation of the Lagrangian polynomial p(s) is independent of the size of Δt. The functions $\ell_k(s)$, which Henrici calls the normalized Lagrangian interpolation coefficients, depend only on s (the relative location of the observation time t with respect to next lowest time t_k) and on the integers m and n, which are the bounds of the set of interpolating points. It turned out that the results for n=4 and m = -3 were sufficiently accurate.



Using this method only the values f_k for eight time steps and the value t_1 need be saved. The value t_1 is incremented by Δt for each step in the integration process, and the time-sequenced observations are processed as soon as the time of the next observation (t_{obs}) is exceeded or equaled by the current t_1. We then have

$$s = 1 - (t_1 - t_{obs})\Delta t$$

The f_k are stored in a circular fashion so that the latest values replace the former values for f_{-3}, and we need only monitor the index of the current location of f_{-3} in the eight place array.

9. Forces Besides the Earth's Gravity Field Acting Upon a Satellite

In addition to the perturbations caused by the variations of the earth's gravitational field as observed in the previous chapters, satellite orbits will also be perturbed appreciably by the gravitational attractions of the sun and moon, the radiation pressure of the sun, and the drag due to the atmosphere.

9.1 Attraction of the Sun and the Moon

All the effects of lunar and solar gravitation upon an artificial satellite may be expressed by adding additional terms for the potential field through which the satellite travels. If $\vec{r}_2$ denotes the position of the sun or moon and

M_2 its mass, the disturbing function may be written [Brouwer and Clemence, 1961, p. 465]

$$\Delta F = GM_2 \left(\frac{1}{|\vec{r}_2 - \vec{r}|} - \frac{\vec{r}\cdot\vec{r}_2}{r_2^3} \right) \qquad (9\text{-}1)$$

The first term gives the potential due to the disturbing body. The second term arises from choosing the center of mass of the earth rather than the center of mass of the entire system of bodies as the origin [Danby 1962].
If we differentiate (9-1) with respect to X_m, where X_m stands for the coordinates x,y,z, we get

$$\frac{\partial \Delta F}{\partial X_m} = GM_2 \left(- \frac{X_m}{|\vec{r}_2 - \vec{r}|^3} + \xi_m \left(\frac{1}{|\vec{r}_2 - \vec{r}|^3} - \frac{1}{r_2^3} \right) \right) \qquad (9\text{-}2)$$

with $r_2 = r_2 \ (\xi_1, \xi_2, \xi_3)$, where ξ_1, ξ_2 lie in the equator with ξ_1 pointing toward the vernal equinox. If $\frac{1}{r_2^3}$ is factored out we get

$$\frac{\partial \Delta F}{\partial X_m} = \frac{GM_2}{r_2^3} [-\xi_m + (\xi_m - X_m) \frac{r_2^3}{|\vec{r}_2 - \vec{r}|^3}] \qquad (9\text{-}3)$$

with

$$\frac{r_2^3}{|\vec{r}_2 - \vec{r}|^3} = [1 + (\frac{r}{r_2})^2 - \frac{2\vec{r}\cdot\vec{r}_2}{r\, r_2} (\frac{r}{r_2})]^{-\frac{3}{2}} \ . \qquad (9\text{-}4)$$

If $(\frac{r}{r_2})^2 - \frac{2\vec{r}\cdot\vec{r}_2}{r\,r_2}(\frac{r}{r_2}) \leq 10^{-3}$, equation (9-4) will then be developed in a series so that we get

$$\frac{r_2^3}{|\vec{r}_2 - \vec{r}|^3} \approx 1 - \frac{3}{2}[(\frac{r}{r_2})^2 - \frac{2\vec{r}\cdot\vec{r}_2}{r\,r_2}(\frac{r}{r_2})] + \frac{15}{8}[(\frac{r}{r_2})^2 - \frac{2\vec{r}\cdot\vec{r}_2}{r\,r_2}(\frac{r}{r_2})]^2$$

$$- \frac{35}{16}[(\frac{r}{r_2})^2 - \frac{2\vec{r}\cdot\vec{r}_2}{r\,r_2}(\frac{r}{r_2})]^3 + \frac{315}{128}[(\frac{r}{r_2})^2 - \frac{2\vec{r}\cdot\vec{r}_2}{r\,r_2}(\frac{r}{r_2})]^4 \qquad (9\text{-}5)$$

The quantity $\vec{r}\cdot\vec{r}_2/(r\,r_2)$ is the product of the normal vectors pointing to the satellite and perturbing body. The scale of the perturbing body's orbit is given by a_2 and is needed in a few of the small terms, where $r_2 = a_2\,(1 - e\cos E)$. The series (9-5) is then applied to (9-3) to obtain the difference

$$- \xi_m + \xi_m \frac{r_2^3}{|\vec{r}_2 - \vec{r}|^3}\,.$$

The coefficient GM/r_2^3 can be deduced from Kepler's third law. Applying this law to the orbit of the moon and the sun, where the subscript E indicates the earth, M the moon, and S the sun we find

$$G(M_E + M_M) = n_M^2 a_M^3$$

$$G(M_E + M_M) = n_M^2 r_M^3 (1 - e_M\cos E_M)^{-3} \qquad (9\text{-}6)$$

with n_M, e_M, E_M being elements of the moon's orbit where n = mean motion, e = eccentricity, E = eccentric anomaly. r_M stands for the radius of the moon's orbit. Hence,

$$\frac{GM_M}{r_M^3} = \frac{M_M}{M_E + M_M}\, n_M^2\,(1 - e_M\cos E_M)^{-3} \qquad (9\text{-}7)$$

For the sun

$$\frac{GM_S}{r_S^3} = \frac{M_S}{M_S + M_E} n_S^2 (1 - e_S \cos E_S)^{-3} ; \qquad (9-8)$$

n_S, e_S, E_S, r_S are the corresponding quantities for the sun's orbit referred to the earth; the ratio $M_S/(M_S + M_M)$ is very nearly 1.

In order to get the inertial coordinates ξ_1, ξ_2, ξ_3 (eq (9-3)) for the moon and sun, respectively, 3 rotations have to be applied

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = P_1(-\varepsilon)P_3(-\Omega)P_1(-i)P_3(-\omega) \begin{bmatrix} (\cos E - e)/(1-e \cos E) \\ \sqrt{1-e^2} \sin E/(1-e \cos E) \\ 0 \end{bmatrix} \qquad (9-9)$$

with

$$P_1(-\varepsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix}$$

$$P_3(-\Omega) = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_1(-i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}$$

$$P_3(-\omega) = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\varepsilon = 23° \; 26' \; 44''84$

Ω, ω, i are given for the moon by

$\Omega_M = 12° \; 06' \; 46''05 - 0°0529538652T$
$\omega_M = 196° \; 43' \; 52''316 + 0.1643580025T$
$\cos i_M = 0.995970322$
$\sin i_M = 0.089683648$
with T = MJD = 33282.0

For the sun, $i_S = 0$.

Hence, $P_1(-i) = I$, and $P_3(-\Omega)$ and $P_3(-\omega)$ may be combined into one transformation employing $\tilde{\omega} = \Omega_S + \omega_S$.

$\Omega_S + \omega_S = \tilde{\omega}_S = 282°04' \; 45''92 + 0°0000470684T$

For the solution of equations (9-7), (9-8) values for e and E are also needed:

$e_S = 0.01677194$

$e_M = 0.054900489$

With the help of m = E - e sin E values for E are computed by iteration. The mean anomaly m is found by

$m_S = 358° \; 0' \; 2''42 + 0°9856002647T$

$m_M = 215° \; 31' \; 53''26 + 13°0649924490T$

The orbital elements given here for the sun and the moon have been computed for 1950.0 from values given in Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac [1960]. Only secular terms have been considered here, i.e., the orbits of the sun and moon have been approximated by rotating ellipses.

9.2 Direct Solar Radiation Pressure

The force exerted by sunlight pressure is proportional to the solar flux and to the area of the satellite as projected on a plane perpendicular to the direction of the flux. According to Shapiro [1963] the acceleration is

$$\vec{Q}_{sp} = \left(\frac{K_R A}{M_{sa}}\right) \left(\frac{I}{c}\right) \frac{\vec{r}_S}{r_S} \qquad (9\text{-}10)$$

where $\vec{r}_S$ is the vector from the earth to the sun, as in section 9.1 , I is known as the solar flux in the vicinity of the earth, c is the speed of light, (A/M_{sa}) is the area-to-mass ratio of the satellite, and K_R is a function $K_R(\vec{r},\vec{r}_S)$ whose value is 0 or 2 depending on the location of the satellite in its orbit. In the space cylinder formed by the earth's shadow there is a sharp cutoff of the force for any point within the cylinder. Here the value $K_R = 0$ is used. For the other part of the orbit the value 2, which expresses the reflection characteristics of the satellite, is used.

Since the earth's orbit is elliptical, the solar flux will vary during the year:

$$I = I_o \left(\frac{a_e}{r_E}\right)^2 \qquad (9\text{-}11)$$

where a_e is the semimajor axis of the earth's orbit, and r_E is the earth-sun distance. The solar constant, I_o, which represents the mean rate at which enery is received at a point on the earth, is approximately 2 cal/cm^2 sec. and remains quite constant. Because the arcs used are not longer than 7 days equation (9-11) was not applied.

To decide whether the satellite is in sunlight or in shadow the inner product of the position vectors of the satellite and the sun is computed. ζ is the angle between these two position vectors. The satellite is in shadow, if the following conditions are fulfilled (see also fig. 3)

$$1)\quad \cos\zeta = \frac{\vec{r}\cdot\vec{r}_S}{|\vec{r}|\;|\vec{r}_S|} < 0$$

$$2)\quad |\vec{r}|\,\sin\zeta < R_{mean}$$

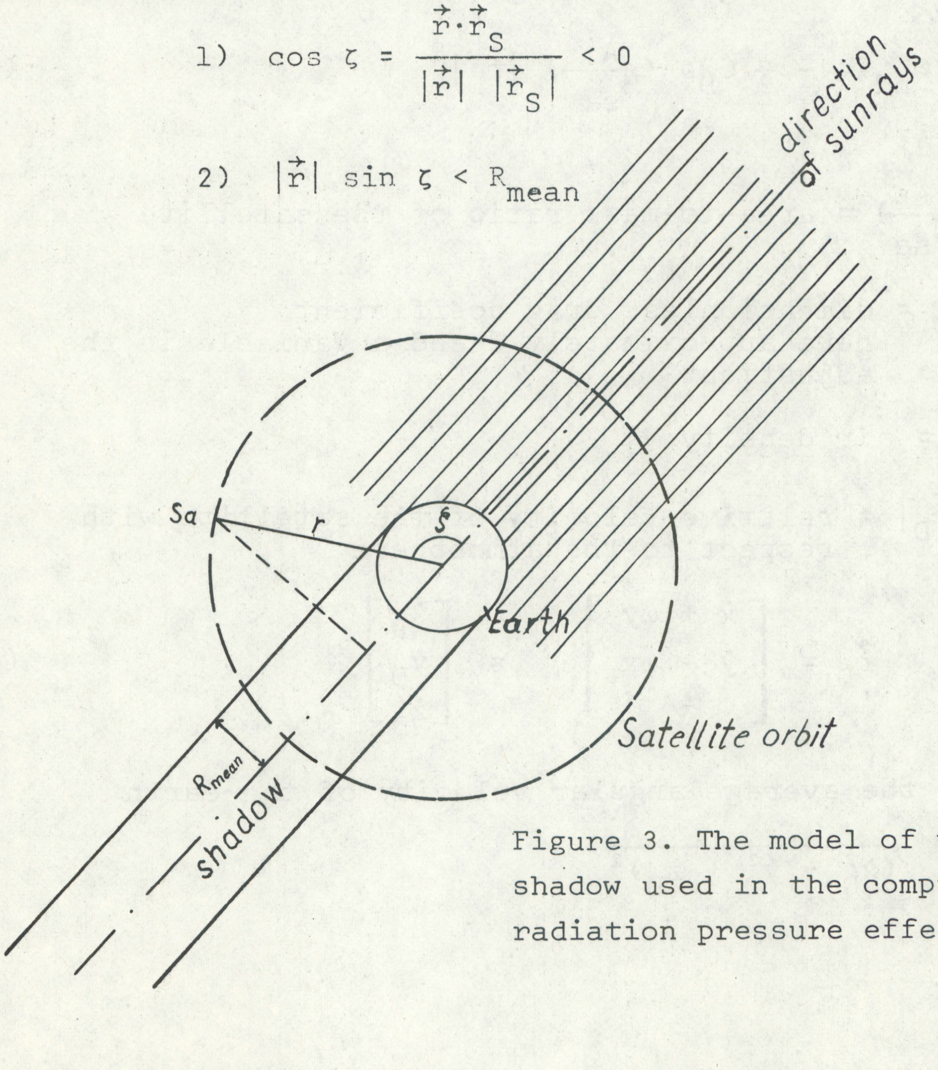


Figure 3. The model of the earth's shadow used in the computation of radiation pressure effects

The components of eq (9-10) are transformed to the ξ_1, ξ_2, ξ_3 coordinate system as defined in section 9.1, i.e., the same rotations are used as for the computations of the attraction of the sun. The partial $\frac{\partial f}{\partial K_R}$ as necessary for eq (2-5) is numerically integrated in the same manner as the variational equations.

9.3 Atmospheric Drag

Most discussions of air drag in the current literature on artificial satellites begin with the equation for the magnitude of the drag force (e.g., Shapiro 1963)

$$Q_{nd} = -\frac{1}{2} C_D \rho \left(\frac{A}{M_{sa}}\right) |\dot{\vec{r}}_D|^2 \qquad (9\text{-}17)$$

with

$\left(\frac{A}{M_{sa}}\right)$ = area-to-mass ratio of the satellite

C_D = dimensionless drag coefficient
here approximately 2 and a variable in the adjustment

ρ = air density

$|\dot{\vec{r}}_D|$ = relative velocity of the satellite with respect to the atmosphere

$$\dot{\vec{r}}_D = \begin{bmatrix} \dot{x} + \omega y \\ \dot{y} - \omega x \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{x}_D \\ \dot{y}_D \\ \dot{z}_D \end{bmatrix} \qquad (9\text{-}18)$$

where ω is the average angular velocity of the earth

and $|\dot{\vec{r}}_D| = \sqrt{(\dot{x}_D^2 + \dot{y}_D^2 + \dot{z}_D)^2}$

The drag resistance is then

$$\vec{Q}_{nd} = -\frac{1}{2} C_D \rho \left(\frac{A}{M_{sa}}\right) |\dot{\vec{r}}_D| \begin{bmatrix} \dot{x}_D \\ \dot{y}_D \\ \dot{z}_D \end{bmatrix} \qquad (9\text{-}19)$$

The air density ρ is given as a function decreasing with height

$$\rho = \rho_o \cdot \exp\left[-(h - 8\cdot 10^5)\Delta\rho\right] , \qquad (9\text{-}20)$$

h is the height of the satellite in meters obtained from the following formula

$$h = |\vec{r}| \left(1 - \frac{R_{maj}}{\sqrt{(r^2 + E_s z^2)}} \right) \qquad (9\text{-}21)$$

with R_{maj} = semimajor axis

and $E_s = \frac{R_{maj}^2 - R_{min}^2}{R_{min}^2}$,

R_{min} = minor axis ,

ρ_o in equation (9-20) is a value for the air density at a height of 800km and is taken from the U.S. Standard Atmosphere Supplements, 1966.

The value $8\cdot 10^5$ is incorporated because of the chosen ρ_o and the unit of meters for h. Δ_ρ is the inverse of a quantity known as the "scale height" and is computed by

$$\Delta\rho = \ln\left(\frac{\rho_1}{\rho_o}\right)/(2\cdot 10^5) \qquad (9\text{-}22)$$

where ρ_1 is a value for the air density at a height of 1000km and is also taken from the U. S. Standard Atmosphere Supplements. The denominator $2 \cdot 10^5$ gives the height difference between ρ_o and ρ_1 in meters.

In order to get the values for ρ_o and ρ_1 the exospheric temperature for the observation time has to be determined. For this purpose the solar flux, the monthly solar flux, and the geomagnetic index A_p for this period are taken from the World Data Center at Boulder, Colorado. The formulas used here can be found in the U. S. Standard Atmosphere Supplements. First the variation in the solar cycle is taken into account

$$\overline{T}_o = 362 + 3.60\, \overline{F}_{10.7} \tag{9-23}$$

where $\overline{F}_{10.7}$ is the 10.7cm solar flux in units of 10^{-22} watts/m^2/cycle/sec averaged over three solar rotations. The variation within one solar rotation yields

$$T_o' = \overline{T}_o + 1.8\,(F_{10.7} - \overline{F}_{10.7}) \tag{9-24}$$

where $F_{10.7}$ is the daily solar flux averaged over the observation time of one week. Now the semiannual variation is supplied

$$T_o = T_o' + f(d)\, \overline{F}_{10.7} \tag{9-25}$$

where

$$f(d) = \left(0.37 + 0.14 \sin(2\pi \frac{d - 151}{365})\right) \sin(4\pi \frac{d - 59}{365})$$

with d = number of days elapsed since January 1 of each year.

The correction for the diurnal variation was also supplied

$$T = 1.1 \times T_o$$

This expression is a very simplified version of the equation for the diurnal variation found in the U. S. Standard Atmosphere Supplements.

Finally the variation with geomagnetic activity yields

$$T_\infty = T + \Delta T \qquad (9\text{-}26)$$

where $\Delta T = A_p + 100\ (1 - \exp(-0.08\ A_p))$

The geomagnetic index A_p is averaged over the observation time.

With the value for the exospheric temperature T determined from equation (9-26), values for the air densities ρ_o and ρ_1 are taken out of the tables 6.1, 6.2, 6.3 given in the U. S. Standard Atmosphere Supplements.

Values for the air density computed with equation (9-20) are compared with values given by Jacchia [1970] and found to be acceptably in agreement over the range of the height of a satellite orbit.

In order to obtain the partial $\frac{\partial f}{\partial C_D}$ in eq (2-5) the partial derivative of Q_{nd} with respect to C_D is numerically integrated in the same manner as the variational equations.

10. Adjustment of Observations

Rewriting equation (2-5) and using a vector-matrix notation we get

$$\Delta\vec{f} = A\,\Delta\vec{e} + B\,\Delta\vec{\chi} + D\,\Delta\vec{r}_s + E\,\Delta\vec{o} + F\,\Delta\vec{b}^{*} \qquad (10\text{-}1)$$

with

$$\Delta\vec{e} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{z} \end{bmatrix} \qquad A = \left(\frac{\partial f}{\partial e}\right)_{ik} \qquad k = 1,6$$

$\Delta\vec{e}$ being a (6 x 1) vector and A a matrix of (g x 6) elements, where g indicates the total number of observation equations per arc.

$$\Delta\vec{\chi} = \begin{bmatrix} \Delta\chi_1 \\ \vdots \\ \Delta\chi_\ell \\ \vdots \\ \Delta\chi_{104} \end{bmatrix} \qquad B = \left(\frac{\partial f}{\partial \chi}\right)_{i\ell} \qquad \ell = 1,104$$

---

*The program provides an option to do the adjustment with the parameters for air drag and radiation pressure (C_D and K_R) as constants. In that case equation (10-1) is solved without $\Delta\vec{o}$.

with B being a matrix of (g x 104) elements

$$\Delta\vec{r}_s = \begin{bmatrix} \Delta r_{s_{11}} \\ \Delta r_{s_{12}} \\ \Delta r_{s_{13}} \\ \Delta r_{s_{21}} \\ \vdots \\ \Delta r_{s_{n_1}} \\ \Delta r_{s_{n_2}} \\ \Delta r_{s_{n_3}} \end{bmatrix} = \begin{bmatrix} \Delta x_{s_1} \\ \Delta y_{s_1} \\ \Delta z_{s_1} \\ \Delta x_{s_2} \\ \vdots \\ \Delta x_{s_n} \\ \Delta y_{s_n} \\ \Delta z_{s_n} \end{bmatrix} \qquad D = \left(\frac{\partial f}{\partial r_s}\right)_{i\sigma}$$

$$\sigma = 1, 3n$$

D being a matrix of (g x 3n) elements with $\sigma = 3(q - 1) + m$ where, as explained on page 5, n stands for the total number of stations, m denotes the mth coordinate of $\vec{r}_s$, and q indicates the qth station. Note that this matrix has blocks of zeros.

$$\Delta\vec{o} = \begin{bmatrix} \Delta K_R \\ \Delta C_D \end{bmatrix} \qquad E = \left(\left(\frac{\partial f}{\partial K_R}\right)_i , \left(\frac{\partial f}{\partial C_D}\right)_i \right)$$

with E a (g x 2) matrix

$$\Delta\vec{b} = \begin{bmatrix} \Delta b_1 \\ \vdots \\ \Delta b_j \\ \vdots \\ \Delta b_t \end{bmatrix} \qquad F = \left(\frac{\partial f}{\partial b}\right)_{ij} \qquad j = 1, t$$

$\Delta\vec{b}$ being a vector of t elements, where t stands for the total number of passes per arc and F a (g x t) matrix. Note that F has blocks of zeros.

Equation (10-1) may be written as

$$[F, G, H] \begin{bmatrix} \Delta\vec{b} \\ \Delta\vec{\nu} \\ \Delta\vec{\eta} \end{bmatrix} = \vec{\ell} + \vec{v} \tag{10-2}$$

Here $\vec{\ell}$ is a vector equivalent to $\Delta\vec{f}$ in equation (10-1) and $\vec{v}$ the vector of the residuals where

$$G = [A,E] \ , \quad H = [D,B]$$

$$\Delta\vec{\nu} = \begin{bmatrix} \Delta\vec{e} \\ \Delta\vec{o} \end{bmatrix} \ , \text{ and } \Delta\vec{\eta} = \begin{bmatrix} \Delta\vec{r}_s \\ \Delta\vec{\chi} \end{bmatrix} .$$

By means of the covariance matrix Σ_ℓ associated with the observations we get the normal equations

$$\begin{bmatrix} F^T\Sigma_\ell^{-1}F & F^T\Sigma_\ell^{-1}G & F^T\Sigma_\ell^{-1}H \\ G^T\Sigma_\ell^{-1}F & G^T\Sigma_\ell^{-1}G & G^T\Sigma_\ell^{-1}H \\ H^T\Sigma_\ell^{-1}F & H^T\Sigma_\ell^{-1}G & H^T\Sigma_\ell^{-1}H \end{bmatrix} \begin{bmatrix} \Delta\vec{b} \\ \Delta\vec{\nu} \\ \Delta\vec{\eta} \end{bmatrix} = \begin{bmatrix} F^T\Sigma_\ell^{-1}\ell \\ G^T\Sigma_\ell^{-1}\ell \\ H^T\Sigma_\ell^{-1}\ell \end{bmatrix} \tag{10-3}$$

In the formation and solution of the least squares normal equations, the unknowns fall into three groups. The three groups are: bias parameters (one Δb for each pass), orbital parameters (orbital elements $\vec{e}$, plus air drag C_D, and radiation pressure K_R), and surface parameters (station coordinates $\vec{r}_s$ and density values χ).

The normal equations formed from all observation equations having the form of (2-5) contain all three groups of unknowns; a bias parameter for every pass, a set of orbital parameters for every orbit, and one set of station coordinates and densities. Rather than directly form and solve these large normal equations, we eliminate the bias and orbital parameters at the earliest possible stage. Only the reduced normal equations associated with the station coordinates and densities, the parameters of main interest, are accumulated over all orbits.

The elimination of the bias and orbital parameters is accomplished by an application of Gaussian elimination that takes account of the special structure of the normal equations. Generally, the elimination of X_1 from

$$\begin{bmatrix} M & P \\ P^T & Q \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad (10\text{-}4)$$

produces reduced normals for X_2 of the form

$$(Q - P^T M^{-1} P)\, X_2 = U_2 - P^T M^{-1} U_1 \;. \tag{10-5}$$

If desired, X_1 can be recovered from a "back solution"

$$X_1 = M^{-1}(U_1 - P\, X_2) \;. \tag{10-6}$$

If M is "block diagonal", that is, of the form

$$M = \begin{bmatrix} m_{11} & 0 & 0 & \\ 0 & m_{22} & 0 & \\ 0 & 0 & m_{33} & \\ & & & \ddots \; m_{nn} \end{bmatrix} \tag{10-7}$$

with a corresponding partitioning of P and U_1,

$$P = \begin{bmatrix} m_{1c} \\ m_{2c} \\ m_{3c} \\ \vdots \\ m_{nc} \end{bmatrix} \;, \qquad U_1 = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

the reduced normals become;

$$\left[Q - \sum_{j=1}^{n} m_{jc}^T\, m_{jj}^{-1}\, m_{jc} \right] X_2 = U_2 - \sum_{j=1}^{n} m_{jj}^T\, m_{jj}^{-1}\, u_j \;. \tag{10-8}$$

Clearly, each term of the summation can be accumulated as soon as m_{jj}, m_{jc}, and u_j are available; the terms can be accumulated in any order.

In equation (10-3), $F^T \Sigma_\ell^{-1} F$, the submatrix of the normal equation associated with the bias parameters $\Delta\vec{b}$, and $G^T\Sigma_\ell^{-1}G$, the submatrix of the normal equations associated with the orbital parameters $\Delta\vec{v}$, has a block diagonal form which is retained after the elimination of $\Delta\vec{b}$. The scheme described above is applied in succession to eliminate first the bias parameters and then the orbital parameters. The contribution to the reduced normals arising from the elimination of the jth bias parameter is computed during the formation of the partial normals arising from the jth pass.

After all observations involving the 1st orbit have been processed, the accumulated reduced normal equations (reduced by the elimination of all bias parameters) are solved for preliminary values of corrections to the orbital parameters of the 1st orbit (orbital elements, air drag and radiation pressure) and surface parameters (station coordinates and densities).

The normal equations are then reduced a second time to eliminate the orbital parameters. These reduced normal equations, along with the solution for the densities and station coordinates, are stored on tape for introduction during the formation of normals from the second orbit according to formula (10-8).

During the processing of the observations from the oth orbit, the reduced partial normals from this orbit (bias parameters eliminated) are accumulated, added to the fully reduced normals carried forward from the (o-1)th orbit, and solved for the oth preliminary values of orbital (for the oth orbit) and surface parameters. Then the oth orbital parameters are eliminated to produce reduced normals to be carried forward, along with the oth preliminary values, to the (0+1)st solution. The final solution for station coordinates and density values is obtained after all orbits are processed.

In fact, the accumulation involved in the formation of reduced normals, as above, is already an intrinsic part of most Gaussian elimination programs, such as the Gauss-Jordan used here. Retrieval of the reduced normals from such a program requires only minor modifications.

The results of the adjustment for the density values χ_ℓ are now taken to compute normalized coefficients $\overline{C}_{nm}$ and $\overline{S}_{nm}$ in order to compare these values with existing satellite results. If $\overline{C}_{nmu}$ and $\overline{S}_{nmu}$ denote the harmonic coefficients introduced in eq (5-2) to define the normal potential U, we obtain [Koch 1968]

$$\overline{C}_{nm} = \overline{C}_{nmu} + \frac{1}{(2n+1)kM\,a^n} \sum_{\ell=1}^{104} \chi_\ell \iint_{\Delta E_\ell} r^n\, \overline{P}_{nm}(\sin\phi) \cos m\lambda \, dE \quad (10\text{-}9)$$

$$\overline{S}_{nm} = \overline{S}_{nmu} + \frac{1}{(2n+1)kM\,a^n} \sum_{\ell=1}^{104} \chi_\ell \iint_{\Delta E_\ell} r^n\, \overline{P}_{nm}(\sin\phi) \sin m\lambda \, dE \quad (10\text{-}10)$$

The integral over the surface elements ΔE_ℓ in ((10-9) and (10-10)) is solved numerically by dividing ΔE_ℓ into 9 subdivisions.

The origin of the earth-fixed coordinate system and of the orbital system--as mentioned earlier--is the center of mass of the earth. Hence the harmonic coefficients $\overline{C}_{10}$, $\overline{C}_{11}$, and $\overline{S}_{11}$ must equal zero. Furthermore $\overline{C}_{21}$ and $\overline{S}_{21}$ should be small in comparison to the rest of the harmonic coefficients since during the orbit computations the z-axis effectively coincides with the rotational axis of the earth. To insure this, constraints in the form of observation equations with small variances (see Koch and Pope 1969) are set up according to (eq (10-9) and (10-10)) and their contribution is added to the normal equations. Five such constraint equations are used, setting each of $\overline{C}_{10}$, $\overline{C}_{11}$, $\overline{S}_{11}$, $\overline{C}_{21}$, and $\overline{S}_{21}$ equal to zero. As an option, C_{oo} can also be constrained.

In addition to these constraints the longitude of one station was held fixed in the solution to prevent a singularity which would arise if all the station longitudes and the right ascension of each orbit were all unknown. We get the following condition equation

$$\text{arc tan } \frac{y + \Delta y}{x + \Delta x} = \text{arc tan } \frac{y}{x}$$

or by linearization

$$\frac{1}{1 + (\frac{y}{x})^2} \left(\frac{\Delta y}{x} - \frac{y \Delta x}{x^2} \right) = 0 \; . \qquad (10\text{-}11)$$

For some of the stations, which have been moved during the observation time of different orbits, the surveyed distances between the old and new location are introduced as constraints.

The differences in the coordinates ($\Delta u, \Delta v, \Delta w$) are weighted based on a mean square root error according to the obtained accuracy and observation equations with these differences are added.

$$\begin{bmatrix} u_{s_m} \\ v_{s_m} \\ w_{s_m} \end{bmatrix} + \begin{bmatrix} u_{s_n} \\ v_{s_n} \\ w_{s_n} \end{bmatrix} + \begin{bmatrix} \Delta u_{mn} \\ \Delta v_{mn} \\ \Delta w_{mn} \end{bmatrix} = 0 \qquad (10\text{-}12)$$

Acknowledgment

The author wishes to thank the director of the NOS Geodetic Research and Development Laboratory, Dr. Hellmut H. Schmid, and the members of this organization for their generous help, especially Mr. Bernard H. Chovitz, Mr. Lewis J. Gulick, Dr. Karl R. Koch, Mr. Foster Morrison, and Mr. Allen J. Pope.

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