Technical Memorandum ERL WPL-86

THE MEAN AND VARIANCE OF SKYWAVE RADAR SEA-ECHO POWER SPECTRA

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noaa NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION / Environmental Research Laboratories

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ABSTRACT

Formulas are derived for the mean and variance of skywave radar sea-echo power spectra. Evaluation of the formulas for reasonable values of the parameters leads to a normalized variance between two and three, in agreement with experiment. The variability of the ionospheric reflection coefficient leads to the greater variance for skywave sea-echo spectra over surface-wave sea-echo spectra (where the normalized variance is usually one).

1. INTRODUCTION — RELATION TO THE SKYWAVE SEA-STATE RADAR PROJECT

When we try to measure sea-echo Doppler spectra with a skywave radar, ionospheric irregularities often distort the received signal so much that we cannot readily extract the sea-state information it contains. To understand and eventually deal with the distortion, it is necessary to model how the ionospheric distortion affects the mean and variance of the measured power spectra. This report develops formulas for the mean and variance of the skywave power spectra under a variety of ionospheric conditions.

This report is one in a series that is investigating the effect of ionospheric distortion on skywave radar sea-echo spectra. Georges and Jones (1980) proposed a convolution model for ionospheric distortion and calculated the mean and variance of skywave power spectra using a discrete model for sea scatter and a discrete multi-path model for the ionospheric distortion. Jones (1981) translated the results of Georges and Jones to a more general representation in which the variance can be more easily estimated. Direct measurements of the ionospheric distortion from ground backscatter (Jones, Riley, and Georges, 1981) indicated that a continuous representation of the ionospheric distortion is more appropriate than is discrete ionospheric multipath. This report derives the mean and variance of skywave power spectra using a continuous model for both the sea scatter and the ionospheric distortion. The calculated normalized variance of from two to three for skywave power spectra agrees with measurements. The discrete model predicted a normalized variance of 1.15 to 1.25. Georges et al., 1981) use the results of the present analysis to evaluate methods for removing ionospheric distortion from skywave sea-echo spectra.

2. SUMMARY

We start with the convolution model for complex Fourier amplitudes (Georges and Jones, 1980)

$$R(\omega) = S(\omega) * P(\omega) * W(\omega)$$

where R is the measured skywave sea-echo spectrum (including a window or weighting function), S is the sea-echo spectrum, P is the ionospheric reflection coefficient (including both outgoing and returning reflections) spectrum, and W is the Fourier transform of a window or weighting function.

Although a corresponding expression for individual power spectra is not valid, a corresponding expression for the infinite ensemble average

$$\langle\hat{R}(\omega)\rangle = \langle\hat{S}(\omega)\rangle * \langle\hat{P}(\omega)\rangle * \hat{W}(\omega)$$

is valid. Here,

$$\hat{R}(\omega) \equiv \frac{2\pi}{T} \left|R(\omega)\right|^2$$

is the power spectral density, T is the length of the time series, and we have corresponding expressions for the other power spectra.

The significance of the above result (as shown by Georges and Jones, 1980, and by Georges et al., 1981) is that, on the average, ionospheric distortion transfers power from the first-order part of the spectrum to the second-order part through convolution with the average ionospheric distortion spectrum $\langle\hat{P}(\omega)\rangle$. The broader the Doppler spectrum of $\langle\hat{P}(\omega)\rangle$, the larger the contamination of second-order power.

VARIANCE

Some of the strategies we use to remove contamination of second-order power depend on the variance of the measured power spectra. Although the incoherent averaging we do to get our average power spectra reduces the variance, the final variance will depend on the variance of the individual skywave spectra. For that reason, we need reasonably good estimates of the variance of skywave spectra. That the theoretical estimates for the variance presented here agree with measurements gives us confidence that our models are correct and that we understand the processes involved in ionospheric distortion of skywave radar sea-echo spectra.

The variance of the power spectra is considerably more complicated to calculate than the mean, even for the infinite ensemble average. For the case where the spectral width of both S and P is larger than that of W, the variance of the measured power spectral density is

$$\sigma^2(\hat{R}(\omega)) \approx \langle\hat{R}(\omega)\rangle^2 + 16\left(\frac{2\pi}{T}\right)^2 \left|W_{1/2}(\omega) * W_{1/2}(\omega)\right|^2 * \langle\hat{S}(\omega)\rangle^2 * \langle\hat{P}(\omega)\rangle^2$$

where

$$W_{1/2}(\omega) \equiv W(2\omega) .$$

To show how spectral width of the various factors affects variance, I approximated both $\hat{W}$ and $\langle\hat{P}\rangle$ by Gaussian functions and approximated $\langle\hat{S}\rangle$ by the sum of two Gaussians (one to represent a first-order line, and one to represent a second-order line). The result for the normalized variance is

$$\frac{\sigma^2(\hat{R}(\omega))}{\langle\hat{R}(\omega)\rangle^2} = 1 + 2a\,\frac{\sqrt{\dfrac{a^2 + c^2 + a_p^2}{a^2 + c^2}} + x^2\sqrt{\dfrac{a^2 + b^2 + a_p^2}{a^2 + b^2}}}{(1 + x)^2\sqrt{a^2 + a_p^2}} ,$$

where

$$x = \frac{A_1}{A_2}\sqrt{\frac{a_p^2 + c^2 + a^2}{a_p^2 + b^2 + a^2}}\,\exp\left(\frac{(\omega - \omega_2)^2}{a_p^2 + c^2 + a^2} - \frac{\omega^2}{a_p^2 + b^2 + a^2}\right);$$

a, b, c, and a_p are spectral widths for the window, the first-order sea echo, the second-order sea echo, and the ionospheric distortion; A_1 is the power in the first-order line; A_2 is the power in the second-order line; ω_2 is the frequency of the second-order line; and the frequency origin is at the first-order line.

Reasonable values for the parameters give a normalized variance between two and three, in agreement with measurement (Jones, Riley, and Georges, 1981).

3. CONVOLUTION MODEL FOR FOURIER AMPLITUDES

Georges and Jones (1980) argue that ionospheric motions distort ionospheric reflection of skywave sea-echo spectra by convolution.

$$R(\omega) = S(\omega) * P(\omega) * W(\omega) \tag{3.1}$$

where R is the measured skywave sea-echo spectrum (including a window or weighting function), S is the sea-echo spectrum, P is the ionospheric reflection coefficient (including both outgoing and returning reflections) spectrum, and W is the Fourier transform of a window or weighting function. Equation (3.1) applies to complex Fourier amplitudes, not to power spectra.

An easy way to see the validity of (3.1) is to consider the time domain. For each pulse or chirp (in the case of FM-CW radar) transmitted, and for each range gate, we can write

$$r = p_1 \; s \; p_2 \; w \tag{3.2}$$

where r is the received amplitude (including a window or weighting function, and after demodulation, which involves a Fourier transform in the case of FM-CW radar), p_1 is the ionospheric reflection coefficient for the outgoing signal, s is the sea-echo backscatter coefficient, p_2 is the ionospheric reflection coefficient for the returning signal, and w is a window or weighting function that we apply to the measurement. A formula like (3.2) applies to each pulse or chirp transmitted. (We keep the range gate fixed.) All of the quantities in (3.2) are complex and will be different for each pulse or chirp because of the time variation of the sea echo and ionospheric reflection and because we purposely change the window or weighting function for each pulse or chirp. To indicate the time variation of (3.2) from pulse to pulse, we write

$$r(t_i) = p_1(t_i)\ s(t_i)\ p_2(t_i)\ w(t_i)\ , \tag{3.3}$$

where the subscript i refers to the ith pulse or chirp. We can consider (3.3) to be a discrete sampling of an ideal continuously varying function,

$$r(t) = p_1(t)\ s(t)\ p_2(t)\ w(t)\ . \tag{3.4}$$

For our analysis, it helps to think that our measurements in (3.3) are really discrete samples of the continuous process indicated in (3.4), but to actually perform the analysis in terms of the continuous process in (3.4).

The product of the outgoing and returning ionospheric reflections p_1 and p_2 in (3.4) (also in (3.2) and (3.3)) can be combined in a single factor:

$$r(t) = s(t)\ p(t)\ w(t)\ . \tag{3.5}$$

Each of the factors in (3.5) is, of course, complex. The Fourier transform of (3.5) is (3.1), where each factor in the convolution in (3.1) is the Fourier transform of the corresponding factor in (3.5).

There is no corresponding expression to (3.1) for power spectra, except on the average. Deriving the corresponding expression for average power and its variance is the subject of this report.

4. MEAN POWER SPECTRA

We want to calculate an expression corresponding to (3.1) for the average power spectrum. The power spectrum is proportional to the absolute square of the Fourier amplitude. Specifically,

$$\hat{R}(\omega) = \frac{2\pi}{T}\ |R(\omega)|^2 \tag{4.1}$$

gives the power spectral density, where T is the length of the sampling interval (usually 102.4 seconds for our measurements). The specific constant factor of proportionality in (4.1) is for convenience. Corresponding expressions to (4.1) hold for the other factors in (3.1). (See Appendix A.)

We will need the absolute square of (3.1). First, we write the convolution in (3.1) explicitly.

$$R(\omega) = \iint_{-\infty}^{\infty} W(\omega-\omega_1)\ S(\omega_1-\omega_2)\ P(\omega_2)\ d\omega_1\ d\omega_2\ . \tag{4.2}$$

We now take (4.2) times its complex conjugate to get the absolute square of (3.1).

$$|R(\omega)|^2 = [\iint_{-\infty}^{\infty} W(\omega-\omega_1)\ S(\omega_1-\omega_2)\ P(\omega_2)\ d\omega_1\ d\omega_2] \times$$

$$\times [\iint_{-\infty}^{\infty} W^*(\omega-\omega_3)\ S^*(\omega_3-\omega_4)\ P^*(\omega_4)\ d\omega_3\ d\omega_4]\ . \tag{4.3}$$

We can rewrite (4.3) as a multiple integral.

$$|R(\omega)|^2 = \iiiint_{-\infty}^{\infty} W(\omega-\omega_1)\ W^*(\omega-\omega_3)\ S(\omega_1-\omega_2)\ S^*(\omega_3-\omega_4)\ P(\omega_2)\ P^*(\omega_4)\ d\omega_1\ d\omega_2\ d\omega_3\ d\omega_4 . \tag{4.4}$$

We assume that S, P, and R are random variables. We are interested in the infinite ensemble average (or expectation value) of (4.4). The window W is, of course, not a random variable, and, because S and P are uncorrelated (see equation (B.8) in Appendix B), we get

$$<|R(\omega)|^2> = \iiiint_{-\infty}^{\infty} \hat{W}(\omega-\omega_1)\ W^*(\omega-\omega_3) <S(\omega_1-\omega_2)\ S^*(\omega_3-\omega_4)>$$

$$<P(\omega_2)\ P^*(\omega_4)>\ d\omega_1\ d\omega_2\ d\omega_3\ d\omega_4 \tag{4.5}$$

for the expectation value of (4.4). We assume that S and P are uncorrelated in frequency. Thus, we can use (B.3) and (B.6) to give

$$<|R(\omega)|^2> = \iiiint_{-\infty}^{\infty} W(\omega-\omega_1)\, W^*(\omega-\omega_3)$$

$$<\hat{S}(\omega_1-\omega_2)>\, \delta(\omega_1-\omega_2-\omega_3+\omega_4)$$

$$<\hat{P}(\omega_2)>\, \delta(\omega_2-\omega_4)\, d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4 \;. \tag{4.6}$$

The presence of the second delta function allows the first delta function to be simplified to $\delta(\omega_1-\omega_3)$. The delta functions allow two of the integrations in (4.6) to be carried out to give

$$<|R(\omega)|^2> = \iint_{-\infty}^{\infty} W(\omega-\omega_1)\, W^*(\omega-\omega_1)\, <\hat{S}(\omega_1-\omega_2)><\hat{P}(\omega_2)>\, d\omega_1\, d\omega_2 \;. \tag{4.7}$$

The double integral in (4.7) is simply a double convolution. Thus, (4.7) is equivalent to

$$<|R(\omega)|^2> = |W(\omega)|^2 * <\hat{S}(\omega)> * <\hat{P}(\omega)> \;. \tag{4.8}$$

We now use (4.1) and the corresponding expression for W from Appendix A. Then (4.8) becomes

$$<\hat{R}(\omega)> = <\hat{S}(\omega)> * <\hat{P}(\omega)> * \hat{W}(\omega) \;. \tag{4.9}$$

Thus, we have derived an expression corresponding to (3.1) for mean power spectral density. We must remember, however, that although (3.1) for complex Fourier amplitudes is always valid, (4.9) for power spectral density is valid only for the infinite ensemble average. The infinite ensemble average of (3.1) is, of course, zero.

We should notice that (4.9) is valid only if S and P are random variables that satisfy certain requirements. For example, (4.9) is not valid when P is a deterministic variable, because the derivation required that P be uncorrelated in frequency. We can easily derive the appropriate expression for the case that P is deterministic. Let

$$Q(\omega) \equiv P(\omega) * W(\omega) \;. \tag{4.10}$$

Then Q is nearly the same as P except that it has been smoothed by the window. Q is a deterministic variable. Then (3.1) becomes

$$R(\omega) = S(\omega) * Q(\omega) \ . \tag{4.11}$$

A derivation similar to (but much simpler than) the one that led to (4.9) gives

$$<\hat{R}(\omega)> = <\hat{S}(\omega)> * \hat{Q}(\omega) \tag{4.12}$$

where

$$\hat{Q}(\omega) = \frac{2\pi}{T} |Q(\omega)|^2 = \frac{2\pi}{T} |W(\omega) * P(\omega)|^2 \tag{4.13}$$

is the power spectral density corresponding to Q.

Equations (4.9) and (4.12) are not in general equal.

5. VARIANCE OF THE POWER SPECTRA

The previous section derived a formula for the average power spectrum. Here we derive formulas for the variance of the power spectrum. Unfortunately, the derivation of formulas for the variance are much more complicated. It does not seem possible to derive a formula for the most general case that is simple enough to give insight into the main effects. It seems to be more useful to consider several special cases, each giving insight into some aspect of the general case, and similarities among the special cases indicating the general properties.

a. Contribution of sea-echo variability

Here we consider the situation where the ionospheric distortion is deterministic, that is, where

$$P(\omega) = <P(\omega)> \ . \tag{5.1}$$

Because both P and W are deterministic, it is useful to combine them in a single function Q, defined in (4.10), that is their convolution. We then begin with (4.11). We write this explicitly as

$$R(\omega) = \int_{-\infty}^{\infty} Q(\omega-\omega_1)\ S(\omega_1)\ d\omega_1\ . \tag{5.2}$$

We take (5.2) times its complex conjugate to get the absolute square of (4.11).

$$|R(\omega)|^2 = \int_{-\infty}^{\infty} Q(\omega-\omega_1)\ S(\omega_1)\ d\omega_1 \int_{-\infty}^{\infty} Q^*(\omega-\omega_2)\ S^*(\omega_2)\ d\omega_2\ . \tag{5.3}$$

We can rewrite (5.3) as a multiple integral.

$$|R(\omega)|^2 = \iint_{-\infty}^{\infty} Q(\omega-\omega_1)\ Q^*(\omega-\omega_2)\ S(\omega_1)\ S^*(\omega_2)\ d\omega_1\ d\omega_2\ . \tag{5.4}$$

Now we square (5.4) and write it as a multiple integral.

$$|R(\omega)|^4 = \iiiint_{-\infty}^{\infty} Q(\omega-\omega_1)\ Q^*(\omega-\omega_2)\ Q(\omega-\omega_3)\ Q^*(\omega-\omega_4)$$

$$S(\omega_1)\ S^*(\omega_2)\ S(\omega_3)\ S^*(\omega_4)\ d\omega_1\ d\omega_2\ d\omega_3\ d\omega_4\ . \tag{5.5}$$

We take the infinite ensemble average of (5.5) to give

$$<|R(\omega)|^4> = \iiiint_{-\infty}^{\infty} Q(\omega-\omega_1)\ Q^*(\omega-\omega_2)\ Q(\omega-\omega_3)\ Q^*(\omega-\omega_4)$$

$$<S(\omega_1)\ S^*(\omega_2)\ S(\omega_3)\ S^*(\omega_4)>\ d\omega_1\ d\omega_2\ d\omega_3\ d\omega_4\ . \tag{5.6}$$

Because S is a zero-mean Gaussian random variable (Barrick and Weber, 1977), we have

$$<S(\omega_1)\ S^*(\omega_2)\ S(\omega_3)\ S^*(\omega_4)> = <S(\omega_1)\ S^*(\omega_2)><S(\omega_3)\ S^*(\omega_4)> +$$

$$+ <S(\omega_1)\ S(\omega_3)><S^*(\omega_2)\ S^*(\omega_4)> + <S(\omega_1)\ S^*(\omega_4)><S^*(\omega_2)\ S(\omega_3)>. \tag{5.7}$$

Because S is uncorrelated in frequency, we can use (B.3), so that (5.7) becomes

$$<S(\omega_1)\ S^*(\omega_2)\ S(\omega_3)\ S^*(\omega_4)> =$$

$$<\hat{S}(\omega_1)><\hat{S}(\omega_3)>\ (\delta(\omega_1-\omega_2)\ \delta(\omega_3-\omega_4)\ +$$

$$+\ \delta(\omega_1-\omega_4)\ \delta(\omega_2-\omega_3))\ . \tag{5.8}$$

Substituting (5.8) into (5.6) gives

$$<|R(\omega)|^4> = \iiiint_{-\infty}^{\infty} Q(\omega-\omega_1)\ Q^*(\omega-\omega_2)\ Q(\omega-\omega_3)\ Q^*(\omega-\omega_4)$$

$$<\hat{S}(\omega_1)><\hat{S}(\omega_3)>\ [\delta(\omega_1-\omega_2)\ \delta(\omega_3-\omega_4) + \delta(\omega_1-\omega_4)\ \delta(\omega_2-\omega_3)]$$

$$d\omega_1\ d\omega_2\ d\omega_3\ d\omega_4\ . \tag{5.9}$$

Integrating over the delta functions gives

$$<|R(\omega)|^4> = 2 \iint_{-\infty}^{\infty} |Q(\omega-\omega_1)|^2\ |Q(\omega-\omega_3)|^2\ <\hat{S}(\omega_1)><\hat{S}(\omega_3)>\ d\omega_1\ d\omega_3\ . \tag{5.10}$$

We see that both terms in (5.9) are equal. In addition, the two integrals in (5.10) can be separated, and each is equal to the same convolution. Thus, (5.10) becomes

$$<|R(\omega)|^4> = 2[<\hat{S}(\omega)> * |Q(\omega)|^2]^2\ . \tag{5.11}$$

Using (4.1) and 4.13) in (5.11) gives

$$<\hat{R}(\omega)^2> = 2[<\hat{S}(\omega)> * \hat{Q}(\omega)]^2\ . \tag{5.12}$$

Comparing with (4.12), we see that

$$<\hat{R}(\omega)^2> = 2<\hat{R}(\omega)>^2\ . \tag{5.13}$$

Thus, we have

$$\sigma^2(\hat{R}(\omega)) = <\hat{R}(\omega)^2> - <\hat{R}(\omega)>^2 = <\hat{R}(\omega)>^2 = [<\hat{S}(\omega)> * \hat{Q}(\omega)]^2\ , \tag{5.14}$$

showing that $\hat{R}$ has unit normalized variance just as $\hat{S}$ does. Because S has unit normalized variance, we have

$$\sigma^2(\hat{S}(\omega)) = \langle\hat{S}(\omega)\rangle^2 \ . \tag{5.15}$$

Thus, we may write (5.14) as

$$\sigma^2(\hat{R}(\omega)) = [\sqrt{\sigma^2(\hat{S}(\omega))} * \hat{Q}(\omega)]^2 \quad , \tag{5.16}$$

which shows directly that the variance of R for this case comes from the variance for S. However, the two are not proportional because of the way the convolution enters.

b. Contribution of ionospheric reflection coefficient variability

While the result for the variance of $\hat{R}$ in (5.14) or (5.16) due to the variance of $\hat{S}$ alone is instructive, it is unrealistic. $\hat{P}$ also has a variance, and measurements so far indicate that the normalized variance is about unity (Jones, Riley, and Georges, 1981), as it is for $\hat{S}$

Here, we consider the combined contributions of the variances of $\hat{S}$ and $\hat{P}$ to that of $\hat{R}$. We allow P to have a variance, but no average ionospheric distortion. (We postpone a more general case until the next subsection because it is so complicated.) That is, we consider

$$\hat{P}(\omega) = \hat{P}\ \delta(\omega) \quad , \tag{5.17}$$

so that (4.9) becomes

$$\langle\hat{R}(\omega)\rangle = \langle\hat{S}(\omega)\rangle * \hat{W}(\omega)\ \langle\hat{P}\rangle \ . \tag{5.18}$$

In (5.18), the mean value of the power spectrum R is just what it would be (except for a constant factor) without any ionospheric effect. Thus, it is clear here that we are considering the variability of the ionosphere without any average ionospheric distortion.

I will now use an Ansatz to get an approximate result. I have already pointed out that

$$\hat{R}(\omega) = \hat{S}(\omega) * \hat{P}(\omega) * \hat{W}(\omega) \tag{5.19}$$

is not correct, except on the average, as in (4.9). However, when we substitute (5.17) into (5.19), we get

$$\hat{R}(\omega) = \hat{S}(\omega) * \hat{W}(\omega)\ \hat{P}\ , \tag{5.20}$$

showing that $\hat{P}$ contributes variability but not distortion to individual spectra. To be more accurate, we shall use

$$\hat{R}(\omega) = \frac{2\pi}{T}\ [W(\omega) * S(\omega)]^2\ \hat{P} \tag{5.21}$$

instead of (5.20), although there is not a great deal of difference because the spectrum of the window W is so narrow.

We now let

$$\hat{S}(\omega) = <\hat{S}(\omega)> + \Delta\hat{S}(\omega) \tag{5.22}$$

and

$$\hat{P} = <\hat{P}> + \Delta\hat{P}\ . \tag{5.23}$$

Clearly, we have

$$<(\Delta\hat{S}(\omega))^2> = <(\hat{S}(\omega) - <\hat{S}(\omega)>)^2> \equiv \sigma^2(\hat{S}(\omega)) \tag{5.24}$$

and

$$<(\Delta\hat{P})^2> = <(\hat{P} - <\hat{P}>)^2 \equiv \sigma^2(\hat{P})\ . \tag{5.25}$$

If we define

$$U(\omega) = W(\omega) * S(\omega)\ , \tag{5.26}$$

then

$$\hat{U}(\omega) = \frac{2\pi}{T}\ |W(\omega) * S(\omega)|^2 \tag{5.27}$$

gives the corresponding power spectral density. We can use (4.12) for the special case of $Q(\omega) = W(\omega)$ to see that

$$\langle\hat{U}(\omega)\rangle = \langle\hat{S}(\omega)\rangle * \hat{W}(\omega) \ . \tag{5.28}$$

We can use (5.14) for the special case of $Q(\omega) = W(\omega)$ to see that

$$\sigma^2(\hat{U}(\omega)) = [\langle\hat{S}(\omega)\rangle * \hat{W}(\omega)]^2 \ . \tag{5.29}$$

Using (5.27), we can write (5.21) as

$$\hat{R}(\omega) = \hat{U}(\omega)\ \hat{P} \ . \tag{5.30}$$

We now let

$$\hat{U}(\omega) = \langle\hat{U}(\omega)\rangle + \Delta\hat{U}(\omega) \ . \tag{5.31}$$

Thus,

$$(\Delta\hat{U}(\omega))^2 = \langle(\hat{U}(\omega) - \langle\hat{U}(\omega)\rangle)^2\rangle \equiv \sigma^2(\hat{U}(\omega)) \ . \tag{5.32}$$

Substituting (5.31) and (5.23) into (5.30) gives

$$\hat{R}(\omega) = \langle\hat{U}(\omega)\rangle\ \langle\hat{P}\rangle + \langle\hat{U}(\omega)\rangle\ \Delta\hat{P} + \langle\hat{P}\rangle\ \Delta\hat{U}(\omega) + \Delta\hat{U}(\omega)\ \Delta\hat{P} \ . \tag{5.33}$$

If we take the infinite ensemble average of (5.33), and use (5.28), we clearly get (5.18), in agreement with previous results.

To get the variance of $\hat{R}$, we need to square (5.33) and take the infinite ensemble average. If we square (5.33), we will get 8 terms. The infinite ensemble average of half of those terms will be zero. The average of all of the terms that are linear in $\Delta\hat{P}$ or $\Delta\hat{U}$ will be zero. Keeping only the nonzero terms, we can write the infinite ensemble average of the square of (5.33) as

$$\langle\hat{R}(\omega)^2\rangle = \langle\hat{U}(\omega)\rangle^2\ \langle\hat{P}\rangle^2 + \langle\hat{U}(\omega)\rangle^2\ \langle(\Delta\hat{P})^2\rangle +$$

$$+ \langle\hat{P}\rangle^2\ \langle(\Delta\hat{U}(\omega))^2\rangle + \langle(\Delta\hat{U}(\omega))^2\rangle\ \langle(\Delta\hat{P})^2\rangle \ . \tag{5.34}$$

Using

$$\sigma^2(\hat{R}(\omega)) = <\hat{R}(\omega)^2> - <\hat{R}(\omega)>^2 = <\hat{R}(\omega)^2> - <\hat{U}(\omega)>^2 <\hat{P}>^2 \tag{5.35}$$

and (5.25) and (5.32) in (5.34) gives

$$\sigma^2(\hat{R}(\omega)) = <\hat{U}(\omega)>^2 \sigma^2(\hat{P}) + <\hat{P}>^2 \sigma^2(\hat{U}(\omega)) + \sigma^2(\hat{U}(\omega)) \sigma^2(\hat{P}) . \tag{5.36}$$

Using (5.28) and (5.29) in (5.36) gives

$$\sigma^2(\hat{R}(\omega)) = [<\hat{S}(\omega)> * \hat{W}(\omega)]^2 \sigma^2(\hat{P}) + <\hat{P}>^2 [<\hat{S}(\omega)> * \hat{W}(\omega)]^2 + + [<\hat{S}(\omega)> * \hat{W}(\omega)]^2 \sigma^2(\hat{P}) . \tag{5.37}$$

There is a common factor in the three terms on the right of (5.37) because U has unit normalized variance. (Compare (5.28) with (5.29).) Thus, (5.37) can be written

$$\sigma^2(\hat{R}(\omega)) = [<\hat{S}(\omega)> * \hat{W}(\omega)]^2 [<\hat{P}>^2 + 2 \sigma^2(\hat{P})] . \tag{5.38}$$

If P also has unit normalized variance,

$$\sigma^2(\hat{P}) = <\hat{P}>^2 , \tag{5.39}$$

as our measurements indicate (Jones, Riley, and Georges, 1981), then (5.38) becomes

$$\sigma^2(\hat{R}(\omega)) = 3[<\hat{S}(\omega)> * \hat{W}(\omega)]^2 <\hat{P}>^2 = 3<\hat{R}(\omega)>^2 \tag{5.40}$$

where we have used (5.18). Thus, we would expect the normalized variance of $\hat{R}$ to be about three whenever the ionospheric distortion is negligible. Measurements of the variance of $\hat{R}$ show variances that vary between two and three (Jones, Riley, and Georges, 1981).

c. Contribution of phase interference (cross terms in the convolution)

So far in this section we calculated the variance or $\hat{R}$ in two cases in which the convolution of S and P did not enter in an essential way in the variance of $\hat{R}$.

In section 5a, the assumption that P was deterministic led to a unit normalized variance for $\hat{R}$. (This disagrees with measured variances for $\hat{R}$ of between two and three (Jones, Riley, and Georges, 1981).) In section 5b, we assumed that $\hat{P}$ was random (eventually with unit normalized variance), but gave no ionospheric distortion of the measured spectra. This gave a normalized variance of three for $\hat{R}$.

Here, we consider the more general case where P is both random and gives ionospheric distortion. Phase interference (as represented by squaring the convolution of S and P in (3.1)) will play a significant role and will make the calculations much more complicated. Even though we restrict both S and P to being Gaussian random variables at the outset, the resulting formulas are so complicated that it is necessary to make further approximations to get answers simple enough to be useful.

The restriction that S and P are Gaussian random variables implies that $\hat{S}$ and $\hat{P}$ have unit normalized variance, in agreement with measurement (Barrick and Snider, 1977; Jones, Riley, and Georges, 1981). As we shall see, the further approximations and special cases yield a normalized variance for $\hat{R}$ that varies between two and three, in agreement with measurements (Jones, Riley, and Georges, 1981).

The variance calculations in section 5a are not a special case of the calculations here because, there, $\hat{P}$ had zero variance; here, $\hat{P}$ has unit normalized variance. The variance calculations in section 5b are a special case of those here, and we can make appropriate comparisons.

We begin with the standard formula for variance.

$$\sigma^2(\hat{R}(\omega)) = \langle \hat{R}(\omega)^2 \rangle - \langle \hat{R}(\omega) \rangle^2 . \tag{5.41}$$

Equation (4.9) gives the second term in (5.41). We need to calculate the first term here.

Equation (4.1) shows that we first need to calculate the absolute square of (3.1), and this is given in (4.4). Here, we need to square (4.4). We can do that as an 8-fold multiple integral if we do not repeat any integration variables. Thus, the square of (4.4) is

$$|R(\omega)|^4 = \int_{-\infty}^{\infty} W(\omega-\omega_1)\ W^*(\omega-\omega_3)\ S(\omega_1-\omega_2)\ S^*(\omega_3-\omega_4)\ P(\omega_2)\ P^*(\omega_4)$$

$$W(\omega-\omega_5)\ W^*(\omega-\omega_7)\ S(\omega_5-\omega_6)\ S^*(\omega_7-\omega_8)\ P(\omega_6)\ P^*(\omega_8)\ d^8\omega \qquad (5.42)$$

where I have used the shorthand notation

$$d^8\omega \equiv d\omega_1\ d\omega_2\ d\omega_3\ d\omega_4\ d\omega_5\ d\omega_6\ d\omega_7\ d\omega_8 \quad . \qquad (5.43)$$

We take the infinite ensemble average of (5.42) to give

$$<|R(\omega)|^4> = \int_{-\infty}^{\infty} W(\omega-\omega_1)\ W^*(\omega-\omega_3)\ W(\omega-\omega_5)\ W^*(\omega-\omega_7)$$

$$<S(\omega_1-\omega_2)\ S^*(\omega_3-\omega_4)\ S(\omega_5-\omega_6)\ S^*(\omega_7-\omega_8)>$$

$$<P(\omega_2)\ P^*(\omega_4)\ P(\omega_6)\ P^*(\omega_8)>\ d^8\omega, \qquad (5.44)$$

where we have used (B.8) (that S and P are uncorrelated). Because P is a zero-mean Gaussian random variable, we have (Barrick and Weber, 1977)

$$<P(\omega_2)\ P^*(\omega_4)\ P(\omega_6)\ P^*(\omega_8)> =$$

$$= <P(\omega_2)\ P^*(\omega_4)>\ <P(\omega_6)\ P^*(\omega_8)> +$$

$$+ <P(\omega_2)\ P(\omega_6)>\ <P^*(\omega_4)\ P^*(\omega_8)> +$$

$$+ <P(\omega_2)\ P^*(\omega_8)>\ <P^*(\omega_4)\ P(\omega_6)> \qquad (5.45)$$

in analogy to (5.7). Because P is uncorrelated in frequency, we can use (B.6) to give

$$<P(\omega_2)\ P^*(\omega_4)\ P(\omega_6)\ P^*(\omega_8)> =$$

$$= <\hat{P}(\omega_2)>\ <\hat{P}(\omega_6)>\ \delta(\omega_2-\omega_4)\ \delta(\omega_6-\omega_8) +$$

$$+ <\hat{P}(\omega_2)>\ <\hat{P}(\omega_6)>\ \delta(\omega_2-\omega_8)\ \delta(\omega_4-\omega_6)\ . \qquad (5.46)$$

Substituting (5.8) and (5.46) into (5.44) gives

$$<|R(\omega)|^4> = \int_{-\infty}^{\infty} W(\omega-\omega_1)\ W^*(\omega-\omega_2)\ W(\omega-\omega_5)\ W^*(\omega-\omega_7)$$

$$<\hat{S}(\omega_1-\omega_2)>\ <\hat{S}(\omega_5-\omega_6)>\ <\hat{P}(\omega_2)>\ <\hat{P}(\omega_6)>$$

$$[\delta(\omega_1-\omega_2-\omega_3+\omega_4)\ \delta(\omega_5-\omega_6-\omega_7+\omega_8)\ \delta(\omega_2-\omega_4)\ \delta(\omega_6-\omega_8)\ +$$

$$+\ \delta(\omega_1-\omega_2-\omega_3+\omega_4)\ \delta(\omega_5-\omega_6-\omega_7+\omega_8)\ \delta(\omega_2-\omega_8)\ \delta(\omega_4-\omega_6)\ +$$

$$+\ \delta(\omega_1-\omega_2-\omega_7+\omega_8)\ \delta(\omega_3-\omega_4-\omega_5+\omega_6)\ \delta(\omega_3-\omega_4)\ \delta(\omega_6-\omega_8)\ +$$

$$+\ \delta(\omega_1-\omega_2-\omega_7+\omega_8)\ \delta(\omega_3-\omega_4-\omega_5+\omega_6)\ \delta(\omega_2-\omega_8)\ \delta(\omega_4-\omega_6)]\ d^8\omega\ . \tag{5.47}$$

The delta functions in the brackets in (5.47) can be rewritten as

$$\delta(\omega_1-\omega_2)\ \delta(\omega_5-\omega_7)\ \delta(\omega_2-\omega_4)\ \delta(\omega_6-\omega_8)\ +$$

$$\delta(\omega_1-\omega_2-\omega_3+\omega_4)\ \delta(\omega_5-\omega_4-\omega_7+\omega_2)\ \delta(\omega_2-\omega_8)\ \delta(\omega_4-\omega_6)\ +$$

$$\delta(\omega_1-\omega_2-\omega_7+\omega_8)\ \delta(\omega_3-\omega_4-\omega_5+\omega_6)\ \delta(\omega_2-\omega_4)\ \delta(\omega_6-\omega_8)\ +$$

$$\delta(\omega_1-\omega_7)\ \delta(\omega_3-\omega_5)\ \delta(\omega_2-\omega_8)\ \delta(\omega_4-\omega_6)\ . \tag{5.48}$$

Substituting (5.48) into (5.47), integrating over the delta functions, and using (4.1) and the corresponding expression for W from Appendix A gives

$$<\hat{R}(\omega)^2> = \int_{-\infty}^{\infty} \hat{W}(\omega-\omega_1)\ \hat{W}(\omega-\omega_5)\ <\hat{S}(\omega_1-\omega_2)>\ <\hat{S}(\omega_5-\omega_6)>\ <\hat{P}(\omega_2)>$$

$$<\hat{P}(\omega_6)>\ d\omega_1\ d\omega_2\ d\omega_5\ d\omega_6 + (\frac{2\pi}{T})^2 \int_{-\infty}^{\infty} W(\omega-\omega_1)\ W^*(\omega-\omega_3)\ W(\omega-\omega_5)$$

$$W(\omega-\omega_5+\omega_3-\omega_1)\ <\hat{S}(\omega_1-\omega_2)>\ <\hat{S}(\omega_5-\omega_3-\omega_2+\omega_1)>\ <\hat{P}(\omega_2)>$$

$$<\hat{P}(\omega_2-\omega_1+\omega_3)>\ d\omega_1\ d\omega_2\ d\omega_3\ d\omega_5 + (\frac{2\pi}{T})^2 \int_{-\infty}^{\infty} W(\omega-\omega_1)\ W^*(\omega-\omega_3)$$

$$W(\omega-\omega_5)\ W^*(\omega-\omega_1+\omega_3-\omega_5)\ <\hat{S}(\omega_1-\omega_2)>\ <\hat{S}(\omega_3-\omega_2)>\ <\hat{P}(\omega_2)>$$

$$<\hat{P}(\omega_2-\omega_3+\omega_5)>\ d\omega_1\ d\omega_2\ d\omega_3\ d\omega_5 + \int_{-\infty}^{\infty} \hat{W}(\omega-\omega_1)\ \hat{W}(\omega-\omega_3)$$

$$<\hat{S}(\omega_1-\omega_2)>\ <\hat{S}(\omega_3-\omega_4)>\ <\hat{P}(\omega_2)>\ <\hat{P}(\omega_4)>\ d\omega_1\ d\omega_2\ d\omega_3\ d\omega_4\ . \qquad (5.49)$$

The first and fourth terms in (5.49) are clearly equal. In these two terms the fourth-order integrations can be separated into the product of two second-order integrations that are clearly double convolutions. In fact, we recognize each of these double convolutions as being equal to (4.9).

The second and third terms in (5.49) are also equal. This can be seen by changing to the new variable

$$x = -\ \omega_3 + \omega_1 + \omega_5 \qquad \text{instead of} \qquad \omega_3$$

in the third term. Thus, we get for (5.49)

$$<\hat{R}(\omega)^2> = 2<\hat{R}(\omega)>^2 + 2(\frac{2\pi}{T})^2 \int_{-\infty}^{\infty} W(\omega-\omega_1)\ W^*(\omega-\omega_3)\ W(\omega-\omega_5)$$

$$W^*(\omega-\omega_5+\omega_3-\omega_1)\ <\hat{S}(\omega_1-\omega_2)>\ <\hat{S}(\omega_5-\omega_3-\omega_2+\omega_1)>$$

$$<\hat{P}(\omega_2)>\ <\hat{P}(\omega_2-\omega_1+\omega_3)>\ d\omega_1\ d\omega_2\ d\omega_3\ d\omega_5\ . \qquad (5.50)$$

There is not much we can do to evaluate the quadruple integral in (5.50) without making some approximations. However, we can put it in more symmetric form with the variable change

$$\Delta = \frac{1}{2}\omega_1 \qquad -\frac{1}{2}\omega_3 \tag{5.51a}$$

$$\delta = \qquad -\frac{1}{2}\omega_3 + \frac{1}{2}\omega_5 \tag{5.51b}$$

$$\tilde{\omega} = \omega - \frac{1}{2}\omega_1 \qquad -\frac{1}{2}\omega_5 \tag{5.51c}$$

$$x = -\frac{1}{2}\omega_1 + \omega_2 + \frac{1}{2}\omega_3 \tag{5.51d}$$

The Jacobian of the transformation in (5.51) shows that

$$d\omega_1\, d\omega_2\, d\omega_3\, d\omega_5 = 4\, d\Delta\, d\delta\, d\tilde{\omega}\, dx\ . \tag{5.52}$$

Substituting (5.50) into (5.41), and using the transformation (5.51) gives

$$\sigma^2(\hat{R}(\omega)) = \langle\hat{R}(\omega)\rangle^2 + 8\left(\frac{2\pi}{T}\right)^2 \int_{-\infty}^{\infty} W(\tilde{\omega}-\Delta+\delta)\, W^*(\tilde{\omega}+\Delta+\delta)\, W(\tilde{\omega}+\Delta-\delta)\, W^*(\tilde{\omega}-\Delta-\delta)$$

$$\langle\hat{S}(\omega-\tilde{\omega}-x-\delta)\rangle\, \langle\hat{S}(\omega-\tilde{\omega}-x+\delta)\rangle\, \langle\hat{P}(x+\Delta)\rangle\, \langle\hat{P}(x-\Delta)\rangle\, dx\, d\tilde{\omega}\, d\delta\, d\Delta\ . \tag{5.53}$$

This is about as far as we can go without making some approximations and considering special cases. However, it is useful to point out some properties of the integration variables in (5.53). Because the window spectrum W is so narrow (about two FFT bins), the integrand in (5.53) will be small except when Δ and δ are small. The approximations we shall soon make will be to neglect these parameters in the appropriate places. The parameters $\tilde{\omega}$ and x are convolution integration variables when appropriate approximations have been made. The kind of convolution will depend on the kind of approximation.

A negligible amount of ionospheric Doppler spreading

Appendix C approximates the integral in (5.53) when the spectral width of P is smaller than that of the window W and when both those spectral widths are smaller than that of S. The result is

$$\sigma^2(\hat{R}(\omega)) \approx \langle\hat{R}(\omega)\rangle^2 + 32\ \hat{W}_{1/2}(\omega) * \hat{W}_{1/2}(\omega) * \langle\hat{S}(\omega)\rangle^2 * \langle\hat{P}_{1/2}(\omega)\rangle * \langle\hat{P}_{1/2}(\omega)\rangle \qquad (5.54)$$

where $\hat{W}_{1/2}(\omega)$ and $\hat{P}_{1/2}(\omega)$ are half-width spectra defined in Appendix A.

This is not a realistic case. Our measurements have never shown a case where the spectral width of P was not larger than that of W (Jones, Riley, and Georges, 1981). It is certainly not normal. I include it here only for consistency checks.

Section 5d evaluates (5.54) under the assumption that the spectra W, S, and P have Gaussian shapes. This leads to a normalized variance of 3 for $\hat{R}$, in agreement with section 5b.

A normal amount of ionospheric Doppler spreading

Appendix D approximates the integral in (5.53) when the spectral width of the window W is smaller than that of either S or P. The result is

$$\sigma^2(\hat{R}(\omega)) \approx \langle\hat{R}(\omega)\rangle^2 + 16\left(\frac{2\pi}{T}\right)^2 \left|W_{1/2}(\omega) * W_{1/2}(\omega)\right|^2 * \langle\hat{S}(\omega)\rangle^2 * \langle\hat{P}(\omega)\rangle^2 \qquad (5.55)$$

where $W_{1/2}(\omega)$ is a half-width window spectrum defined in Appendix A.

This is a realistic approximation. Our measurements have always shown the spectral width of P to be wider than W (Jones, Riley, and Georges, 1981). Although the spectral width of the first-order part of S may often be about the same as that of W, the spectral width of the second-order part of S is usually greater than that of W.

Section 5d evaluates (5.55) under the assumption that the spectra W, S, and P have Gaussian shapes.

d. For Gaussian-shaped spectra

Appendix E shows that the multiple integral in (5.53) simplifies considerably when a Gaussian window

$$W(\omega) = \exp(-\omega^2/2a^2)/a\sqrt{2\pi} \tag{5.56}$$

is used. The result is

$$\sigma^2(\hat{R}(\omega)) = \langle\hat{R}(\omega)\rangle^2 + 4\pi a^2 \hat{W}^2(\omega) * \langle\tilde{S}_W(\omega)\rangle^2 * \langle\tilde{P}_W(\omega)\rangle^2 \tag{5.57}$$

where

$$\hat{W}(\omega) = \frac{2\pi}{T} |W(\omega)|^2 = \exp(-\omega^2/a^2)/T\ a^2 \tag{5.58}$$

is the power spectrum of the window,

$$\langle\tilde{S}_W(\omega)\rangle^2 \equiv \frac{1}{a}\sqrt{2/\pi} \int_{-\infty}^{\infty} \exp(-2\ \delta^2/a^2)\ \langle\hat{S}(\omega-\delta)\rangle\ \langle\hat{S}(\omega+\delta)\rangle\ d\delta \tag{5.59}$$

is the square of an effective sea-echo power spectrum, and

$$\langle\tilde{P}_W(\omega)\rangle^2 \equiv \frac{1}{a}\sqrt{2/\pi} \int_{-\infty}^{\infty} \exp(-2\ \Delta^2/a^2)\ \langle\hat{P}(\omega-\Delta)\rangle\ \langle\hat{P}(\omega+\Delta)\rangle\ d\Delta \tag{5.60}$$

is the square of an effective ionospheric reflection power spectrum.

The above results depend explicitly on the spectrum of the window having a Gaussian shape. It is very unlikely that the specific form of (5.57) is general enough to apply to other shapes of window spectra. However, the Blackman-Harris window (Harris, 1978) that we use has a shape near the peak that looks very much like a Gaussian. It is straightforward to choose a value of a so that the Gaussian-shaped window matches a Blackman-Harris window at its peak. Appendix B gives such a value.

When the spectral width of the window is much less than that of S and of P, then $\tilde{S}_W = \hat{S}$ and $\tilde{P}_W = \hat{P}$, and (5.57) reduces to (5.55), as we expect. On the other hand, when the spectral width of P is much smaller than that of the window W, then (5.57) reduces to (5.54), as we expect.

To see how (5.57) behaves in the transition regions, where the various spectral widths may be comparable, Appendix F evaluates (5.57) when both $\langle\hat{S}\rangle$ and $\langle\hat{P}\rangle$ are Gaussian shaped. In particular, we represent $\langle\hat{S}\rangle$ as the sum of a first-order Gaussian and a second-order Gaussian.

$$\hat{S}(\omega) = \frac{A_1}{b\sqrt{\pi}} \exp(-\omega^2/b^2) + \frac{A_2}{c\sqrt{\pi}} \exp(-(\omega-\omega_2)^2/c^2) \tag{5.61}$$

where I have arbitrarily chosen zero frequency at the first-order peak. $<\hat{P}>$ is represented by a single Gaussian-shaped spectrum.

$$<\hat{P}(\omega)> = \frac{1}{a_p\sqrt{\pi}} \exp(-\omega^2/a_p^2) \ . \tag{5.62}$$

Then Appendix F shows that the normalized variance is approximately

$$\frac{\sigma^2(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} \approx 1 + 2a \frac{\sqrt{\dfrac{a_p^2 + c^2 + a^2}{c^2 + a^2}} + x^2\sqrt{\dfrac{a_p^2 + b^2 + a^2}{b^2 + a^2}}}{(1 + x)^2 \sqrt{a_p^2 + a^2}} \ , \tag{5.63}$$

where

$$x = \frac{A_1}{A_2} \sqrt{\frac{a_p^2 + c^2 + a^2}{a_p^2 + b^2 + a^2}} \exp\left(- \frac{\omega^2}{a_p^2 + b^2 + a^2} + \frac{(\omega-\omega_2)^2}{a_p^2 + c^2 + a^2}\right). \tag{5.64}$$

In (5.63), x is the only quantity that varies with frequency. When b and c (the spectral widths of the first-order and second-order sea-echo lines) are nearly equal, (5.63) does not vary much with x. However, we will consider the limits of its variability.

Whenever x is small, (5.63) becomes

$$\frac{\sigma^2(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 1 + \frac{2a}{\sqrt{c^2 + a^2}} \sqrt{\frac{a_p^2 + c^2 + a^2}{a_p^2 + a^2}} \ . \tag{5.65}$$

Small x corresponds to the second-order part of the spectrum when the ionospheric contamination is small. Small ionospheric contamination means that a_p is small. For small a_p, (5.65) becomes

$$\frac{\sigma^2(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 3 \tag{5.66}$$

in agreement with (5.40), as we expect.

When x is large, (5.63) becomes

$$\frac{\sigma^2(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 1 + \frac{2a}{\sqrt{b^2 + a^2}} \sqrt{\frac{a_p^2 + b^2 + a^2}{a_p^2 + a^2}} \quad . \tag{5.67}$$

Large x corresponds to the first-order part of the spectrum, or it can correspond to the second-order part of the spectrum for normal ionospheric contamination ($a_p >> a$, $a_p >> b$, $a_p >> c$). Taking a_p large in (5.67) gives

$$\frac{\sigma^2(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 1 + \frac{2a}{\sqrt{b^2 + a^2}} \tag{5.68}$$

for the normalized variance for normal ionospheric contamination. For the values of a and b given in Appendix B, (5.68) becomes

$$\frac{\sigma^2(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 2.64 \tag{5.69}$$

for a normal amount of ionospheric contamination.

There is a value of x for which (5.63) is a minimum. That minimum value is

$$\frac{\sigma^2_{min}(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 1 + \left[\frac{2a}{\sqrt{a_p^2 + a^2}} \sqrt{\frac{a_p^2 + c^2 + a^2}{c^2 + a^2}} \sqrt{\frac{a_p^2 + b^2 + a^2}{b^2 + a^2}}\right] \left[\frac{\sqrt{\frac{a_p^2 + c^2 + a^2}{c^2 + a^2}} + \sqrt{\frac{a_p^2 + b^2 + a^2}{b^2 + a^2}}}{\frac{a_p^2 + c^2 + a^2}{c^2 + a^2} + \frac{a_p^2 + b^2 + a^2}{b^2 + a^2}}\right] \quad . \tag{5.70}$$

For small a_p, (5.70) becomes

$$\frac{\sigma^2_{min}(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 3 \quad , \tag{5.71}$$

in agreement with (5.40), as we expect. For a normal amount of ionospheric contamination ($a_p >> a$, $a_p >> b$, $a_p >> c$), (5.70) becomes

$$\frac{\sigma^2_{min}(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 1 + 2a\,\frac{\sqrt{a^2 + b^2} + \sqrt{a^2 + c^2}}{2a^2 + b^2 + c^2} \quad . \tag{5.72}$$

For the values of a, b, and c (a = 1.14, b = 0.8, c = 1.3 to 1.9) in Appendix B, (5.72) gives

$$\frac{\sigma^2_{min}(\hat{R}(\omega))}{<\hat{R}(\omega)>^2} = 2.2 \text{ to } 2.4 \ . \tag{5.73}$$

We see that the predicted normalized variance of R varies from about 2.2 to 3, in approximate agreement with measured variances of R between about 2 and 3 (Jones, Riley, and Georges, 1981).

6. COMPARISON WITH CALCULATIONS BASED ON A DISCRETE REPRESENTATION OF THE SPECTRA

Georges and Jones (1980) calculate the mean and variance of $\hat{R}$ under the assumptions that both S and P can be represented by a sum of discrete terms. They correctly point out that such a representation is general enough to represent arbitrary spectral variation for S and P. Their results, however, are in terms of the parameters in their discrete representation, making it difficult to compare with the results presented here. However, Jones (1981) has transformed their results to a form that can be directly compared with the results calculated here.

Jones (1981) gets

$$<\hat{R}(\omega)> = <\hat{S}(\omega)> * <\hat{P}(\omega)> * \hat{W}(\omega) \tag{6.1}$$

for the mean power spectrum, in agreement with (4.9).

For the variance, Jones (1981) gets

$$\sigma^2(\hat{R}(\omega)) = (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * \sigma^2(\hat{P}(\omega))$$

$$+ (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{P}(\omega)>^2 * \sigma^2(\hat{S}(\omega))$$

$$+ (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * \sigma^2(\hat{S}(\omega)) * \sigma^2(\hat{P}(\omega))$$

$$+ <\hat{R}(\omega)>^2 - (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 \quad . \tag{6.2}$$

Georges and Jones (1980) point out that the first three terms in (6.2) represent contributions from the variability of the sea-echo backscatter coefficient and the ionospheric reflection coefficient, whereas the last two terms in (6.2) represent a contribution from phase interference (or cross terms in the square of the convolution in (3.1)). Notice the similarity of the sum of the first three terms in (6.2) with (5.37), which neglects the phase interference.

The statistical assumptions that led to (6.2) are less restrictive than those used to derive corresponding expressions here. In particular, (B.4) and (B.7) assume that $\hat{S}(\omega)$ and $\hat{P}(\omega)$ have unit normalized variance. To make further comparisons, it is therefore useful to apply those restrictions to (6.2). Substituting (B.4) and (B.7) into (6.2) gives

$$\sigma^2(\hat{R}(\omega)) = (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 +$$

$$+ (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 +$$

$$+ (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 +$$

$$+ <\hat{R}(\omega)>^2 - (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 \quad . \tag{6.3}$$

Before making further comparisons, it is useful to consider some of the properties of (6.3). The first three terms arise from diagonal terms in the square of (3.1). The last two terms arise from cross terms in the square of (3.1). None of the first four terms can be negative. In addition, the sum of the last two terms can never be negative. Because each of the first three terms and (minus) the last term are equal, each of these terms is restricted to vary between zero and $<\hat{R}(\omega)>^2$. Thus, the normalized variance in (6.3) can vary only between one and three. (This property is shared by (5.54), (5.55), (5.57), (5.63).) An interesting property of (6.3) (and the other equations just mentioned) is that although the sum of the last two terms (the cross terms) contributes a positive variance, the **smaller** the sum of the last two terms the higher the total variance.

We can rewrite (6.3) as

$$\sigma^2(\hat{R}(\omega)) = <\hat{R}(\omega)>^2 + 2(\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 . \qquad (6.4)$$

If we compare (6.4) with (5.57) using $\tilde{S}_W = \hat{S}$, $\tilde{P}_W = \hat{P}$, and a = 1.14 FFT bins (from Appendix B), we see that the second term on the right of (6.4) is about a factor of 8 smaller than that in (5.57). Thus, the normalized variance of 2.2 to 3 for (5.57) becomes 1.15 to 1.25 for (6.4) Although the discrete representation of the spectra for $S(\omega)$ and $P(\omega)$ used by Georges and Jones (1980) is general enough to represent arbitrary spectral variations, they made approximations in the derivation of the variance in (6.2) that effectively limited the applicability to spectra composed of discrete spectral lines. In particular, they assumed that both $S(\omega)$ and $P(\omega)$ are composed of a sum of spectral lines whose spectral widths are narrower than the spectral width of the window $W(\omega)$ and whose separation is larger than the spectral width of $W(\omega)$.

Jones (1981) evaluated (5.59) and (5.60) under the above conditions to give

$$<\tilde{S}_W(\omega)>^2 = \frac{1}{a\sqrt{2\pi}} (\frac{2\pi}{T}) <\hat{S}(\omega)>^2 \qquad (6.5)$$

and

$$<\tilde{P}_W(\omega)>^2 = \frac{1}{a\sqrt{2\pi}} (\frac{2\pi}{T}) <\hat{P}(\omega)>^2 . \qquad (6.6)$$

Substituting (6.5) and (6.6) into (5.57) gives (6.4), which shows that the discrete representation gives identical results with the continuous representation under the same physical assumptions.

Our measurements of ground backscatter (Jones et al., 1981) show that a continuous representation for $\langle \hat{P}(\omega) \rangle$ is more realistic than is a sum of discrete spectral lines. Thus, the variance given by (5.55) or (5.63) is probably more realistic than that given by (6.2) or (6.4).

We see that whereas the variance calculated by Georges and Jones (1980) is more general than the present calculation because it is not restricted to unit normalized variance for $\hat{S}(\omega)$ and $\hat{P}(\omega)$, it is less general than the present calculation in that it is restricted to spectra for $\langle \hat{S}(\omega) \rangle$ and $\langle \hat{P}(\omega) \rangle$ that are composed of effectively discrete spectral lines.

7. ACKNOWLEDGMENTS

I thank Dr. T.M. Georges for useful discussions. I thank Dr. Georges and the editorial staff of Publication Services for their comments after reading the manuscript. The superb typing job you see before you is the work of Ms. Mildred Birchfield.

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APPENDIX A. DEFINITIONS

$s(t)$ is the time series for backscatter from the sea.

$S(\omega)$ is the Fourier transform of $s(t)$.

$\hat{S}(\omega) = \frac{2\pi}{T} |S(\omega)|^2$ is the power spectrum corresponding to $S(\omega)$, where

T is the length of the measured time series $s(t)$.

$\langle\hat{S}(\omega)\rangle$ is the infinite ensemble average of $\hat{S}(\omega)$. The brackets $\langle\ \rangle$ denote an ideal infinite ensemble average over the same "sea state" and the same "ionospheric state." We assume that the realizations within such an ensemble belong to the same probability distribution.

$w(t)$ is a window that we apply to the measured time series before performing a Fourier transform.

$W(\omega)$ is the Fourier transform of $w(t)$. It is complex because $w(t)$ is not symmetric.

$\hat{W}(\omega) = \frac{2\pi}{T} |W(\omega)|^2$ is the power spectrum corresponding to $W(\omega)$.

$p(t)$ is the time series corresponding to two-way ionospheric transmission between the radar and some fixed spot (or patch) on the ground or ocean.

$P(\omega)$ is the Fourier transform of $p(t)$. It includes Doppler spreading from ionospheric multipath. It does not include Doppler spreading from movement of the scattering point on the ground or ocean. If the ionosphere were a perfect reflector, P would be a delta function.

$\hat{P}(\omega) = \frac{2\pi}{T} |P(\omega)|^2$ is the power spectrum corresponding to $P(\omega)$.

$\langle\hat{P}(\omega)\rangle$ is the infinite ensemble average of $\hat{P}(\omega)$.

$r(t) = w(t)s(t)p(t)$ is the measured skywave sea-echo.

$R(\omega)^{\dagger} = W(\omega)*S(\omega)*P(\omega)$ is the Fourier transform of $r(t)$.

$*$ denotes convolution.

$\hat{R}(\omega) = \frac{2\pi}{T} |R(\omega)|^2$ is the power spectrum corresponding to $R(\omega)$.

$\langle\hat{R}(\omega)\rangle$ is the infinite ensemble average of $\hat{R}(\omega)$.

$\sigma^2(\hat{R}(\omega))$ is the variance of $\hat{R}(\omega)$.

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† Our variable $R(\omega)$ was called $R''(\omega)$ by Georges and Jones (1980).

Half-width spectra

In the course of the derivations, it was necessary to define certain "half-width spectra" so that some of the convolution formulas would be unambiguous. These are spectra that have the same shape, but have only half of the spectral width.

$$W_{1/2}(\omega) \equiv W(2\omega) \tag{A.1}$$

for the half-width Fourier-spectrum window.

$$\hat{W}_{1/2}(\omega) \equiv \hat{W}(2\omega) = \frac{2\pi}{T} \left|W(2\omega)\right|^2 \tag{A.2}$$

for the half-width power spectrum of the window.

$$\hat{P}_{1/2}(\omega) \equiv \hat{P}(2\omega) = \frac{2\pi}{T} \left|P(2\omega)\right|^2 \tag{A.3}$$

for the half-width power spectrum of the ionospheric reflection coefficient.

APPENDIX B. ASSUMPTIONS

1. The convolution model (Georges and Jones, 1980)

$$R(\omega) = W(\omega) * S(\omega) * P(\omega) \ . \tag{B.1}$$

2. $S(\omega)$ is a Gaussian random variable. This approximation is supported by theory (Weber and Barrick 1977; Barrick and Weber, 1977) and measurement (Barrick and Snider, 1977).

3. $S(\omega)$ has zero mean.

$$\langle S(\omega)\rangle = 0 \ . \tag{B.2}$$

This is supported by theory and experiment (as above).

4. $S(\omega)$ is uncorrelated in frequency. That is,

$$\langle S(\omega_1)\ S^*(\omega_2)\rangle = \langle \hat{S}(\omega_1)\rangle\ \delta(\omega_1 - \omega_2) = \langle \hat{S}(\omega_2)\rangle\ \delta(\omega_1 - \omega_2) \ ; \tag{B.3}$$

also

$$\langle S(\omega_1)\ S(\omega_2)\rangle = \langle S^*(\omega_1)\ S^*(\omega_2)\rangle = 0 \ .$$

This is only an approximation. It is the continuous frequency representation of a corresponding expression involving Kronecker deltas used by Weber and Barrick (1977). Whereas (B.3) indicates that adjacent frequencies are uncorrelated no matter how close they are (in seeming contradiction to what we measure), convolution with a window gives a correlation frequency of about one or two FFT frequency bins, in agreement with experiment. (We never measure $S(\omega)$; even in the absence of ionospheric distortion, we measure $W(\omega)*S(\omega)$.)

5. $\hat{S}(\omega)$ has unit normalized variance. That is,

$$\sigma^2(\hat{S}(\omega)) = \langle \hat{S}(\omega)^2\rangle - \langle \hat{S}(\omega)\rangle^2 = \langle \hat{S}(\omega)\rangle^2 \ . \tag{B.4}$$

This follows from assumptions 2 and 3 above (Barrick and Weber, 1977; Barrick and Snider, 1977). It is supported by theory and experiment (as above).

6. A continuous function of frequency is a more realistic representation for $P(\omega)$ than is discrete multipath. Our measurements in October 1980 (Jones, Riley, and Georges, 1981) show this. Although the discrete multipath representation of Georges and Jones (1980) is general enough to represent continuous Doppler spreading, a continuous representation is more convenient.

7. $P(\omega)$ is a Gaussian random variable. This is consistent with our analysis of the October 1980 measurements (Jones, Riley, and Georges, 1981).

8. $P(\omega)$ has zero mean. That is,

$$\langle P(\omega)\rangle = 0 \ . \tag{B.5}$$

We assume that analysis of the October 1980 measurements will confirm this.

9. $P(\omega)$ is uncorrelated in frequency. That is,

$$\langle P(\omega_1)\ P^*(\omega_2)\rangle = \langle \hat{P}(\omega_1)\rangle\ \delta(\omega_1 - \omega_2) = \langle \hat{P}(\omega_2)\rangle\ \delta(\omega_1 - \omega_2) \ ; \tag{B.6}$$

also

$$\langle P(\omega_1)\ P(\omega_2)\rangle = \langle P^*(\omega_1)\ P^*(\omega_2)\rangle = 0 \ .$$

Similar comments apply here as they did to (B.3). We hope that analysis of our October 1980 measurements will confirm that the cross correlation of $P(\omega)*W(\omega)$ is consistent with (B.6).

10. $\hat{P}(\omega)$ has unit normalized variance. That is,

$$\sigma^2(\hat{P}(\omega)) = \langle \hat{P}(\omega)^2\rangle - \langle \hat{P}(\omega)\rangle^2 = \langle \hat{P}(\omega)\rangle^2 \ . \tag{B.7}$$

This follows from assumptions 7 and 8. Our analysis of the October 1980 measurements (Jones, Riley, and Georges, 1981) shows a higher variance for ground backscatter, but sea echoes are consistent with this. The larger variance from ground backscatter is probably from the ground rather than the ionosphere.

11. $S(\omega)$ and $P(\omega)$ are uncorrelated. That is,

$$\langle f(S(\omega))\ g(P(\omega))\rangle = \langle f(S(\omega))\rangle\langle g(P(\omega))\rangle \qquad \text{(B.8)}$$

where f and g are arbitrary functions. The following special case

$$\langle S(\omega)\ P(\omega)\rangle = \langle S(\omega)\ P^*(\omega)\rangle = \langle S^*(\omega)\ P(\omega)\rangle = 0 \qquad \text{(B.9)}$$

follows from (B.2) and (B.5). This is an assumption. We can see no reason why they should be correlated.

12. We assume that only $\hat{R}(\omega)$ has useful information, and that the phase of $R(\omega)$ is randomly distributed and thus has no useful information. This is an assumption. We do not know whether our measurements are consistent with this assumption.

13. The 3-dB bandwidth of the Blackman-Harris window we use is 1.9 FFT bins (Harris, 1978). This corresponds to a value of <u>a</u> for the Gaussian window of 1.14 FFT bins.

14. Representative 10-dB bandwidths for the first- and second-order spectral lines of $\langle \hat{S}(\omega)\rangle$ are 2.4 FFT bins and 4 to 5.6 FFT bins respectively. These correspond to b and c for the Gaussian approximation of 0.8 FFT bins and 1.3 to 1.9 FFT bins, respectively.

15. The effective number (for estimating degrees of freedom) of first-order FFT bins ($= b\sqrt{\pi}$) is 1.4 bins. (Same as "equivalent width" measured in FFT bins.)

16. The effective number (for estimating degrees of freedom) of second-order FFT bins ($= 2c\sqrt{\pi}$) is 4.6 to 6.7 bins.

17. Typical 10-dB bandwidths for $\langle \hat{P}(\omega)\rangle$ are 7 to 27 FFT bins. These correspond to a_p for the Gaussian approximation of 2.3 to 8.9 FFT bins (Jones, Riley, and Georges, 1981; Georges et al., 1981).

APPENDIX C. APPROXIMATION TO THE MULTIPLE INTEGRAL IN (5.53) WHEN THE IONOSPHERIC DOPPLER SPREADING IS NEGLIGIBLE

Here, we consider the case where the spectral width of P is much smaller than that of the window W, which in turn is much smaller than that of S. (Small spectral width for P corresponds to small ionospheric Doppler spreading.)

If the spectral width of P is much smaller than that of W, then the shape of P will dominate over that of W in the Δ integration in (5.53). Thus, we may replace W everyplace it occurs by its value when Δ = zero (where P has its peak).

Similarly, if the spectral width of W is much smaller than that of S, then the shape of W will dominate over that of S in the δ integration in (5.53). Thus we may replace S everywhere it occurs by its value when δ = zero (where W has its peak).

With the above two considerations in mind, we may approximate the integral in (5.53) by

$$\int_{-\infty}^{\infty} W(\tilde{\omega}+\delta)\ W^*(\tilde{\omega}+\delta)\ W(\tilde{\omega}-\delta)\ W^*(\tilde{\omega}-\delta)$$

$$\langle\hat{S}(\omega-\tilde{\omega}-x)\rangle\ \langle\hat{S}(\omega-\tilde{\omega}-x)\rangle$$

$$\langle\hat{P}(x+\Delta)\rangle\ \langle\hat{P}(x-\Delta)\rangle\ dx\ d\tilde{\omega}\ d\delta\ d\Delta\ . \qquad \text{(C.1)}$$

This allows us to separate the multiple integral into factors.

$$\iint_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |W(\tilde{\omega}+\delta)|^2\ |W(\tilde{\omega}-\delta)|^2\ d\delta\right] \langle\hat{S}(\omega-\tilde{\omega}-x)\rangle^2$$

$$\left[\int_{-\infty}^{\infty} \langle\hat{P}(x+\Delta)\rangle\ \langle\hat{P}(x-\Delta)\rangle\ d\Delta\right] dx\ d\tilde{\omega}\ . \qquad \text{(C.2)}$$

We will consider the integral inside the first pair of brackets in (C.2) first. Using the definition of power spectral density from Appendix A,

$$\hat{W}(\omega) \equiv \frac{2\pi}{T}\ |W(\omega)|^2\ , \qquad \text{(C.3)}$$

we can write the integral inside the first pair of brackets in (C.2) as

$$\left(\frac{T}{2\pi}\right)^2 \int_{-\infty}^{\infty} \hat{W}(\tilde{\omega}+\delta)\ \hat{W}(\tilde{\omega}-\delta)\ d\delta\ . \tag{C.4}$$

We now use the definition of the half-width power spectrum of the window in (A.2) in (C.4) to give

$$\left(\frac{T}{2\pi}\right)^2 \int_{-\infty}^{\infty} \hat{W}_{1/2}\left(\frac{\tilde{\omega}}{2} + \frac{\delta}{2}\right)\ \hat{W}_{1/2}\left(\frac{\tilde{\omega}}{2} - \frac{\delta}{2}\right)\ d\delta. \tag{C.5}$$

We change variables of integration in (C.5) to

$$Y = \frac{\tilde{\omega}}{2} + \frac{\delta}{2} \tag{C.6}$$

so that (C.5) becomes

$$2\left(\frac{T}{2\pi}\right)^2 \int_{-\infty}^{\infty} \hat{W}_{1/2}(Y)\ \hat{W}_{1/2}(\tilde{\omega}-Y)\ dY\ . \tag{C.7}$$

This is clearly a convolution, which we can write as

$$2\left(\frac{T}{2\pi}\right)^2 \hat{W}_{1/2}(\tilde{\omega}) * \hat{W}_{1/2}(\tilde{\omega})\ . \tag{C.8}$$

The integral in the second pair of brackets in (C.2) has the same form as the integral in (C.4). Thus, we can write it in analogy with (C.8) as

$$2\langle\hat{P}_{1/2}(x)\rangle * \langle\hat{P}_{1/2}(x)\rangle \tag{C.9}$$

where the half-width spectrum $\hat{P}_{1/2}$ is defined in (A.3). Substituting (C.8) and (C.9) for the integrals in the brackets in (C.2) gives

$$\iint_{-\infty}^{\infty} \left[2\left(\frac{T}{2\pi}\right)^2 \hat{W}_{1/2}(\tilde{\omega}) * \hat{W}_{1/2}(\tilde{\omega})\right] \langle\hat{S}(\omega-\tilde{\omega}-x)\rangle^2$$

$$\left[2\langle\hat{P}_{1/2}(x)\rangle * \langle\hat{P}_{1/2}(x)\rangle\right]\ dx\ d\tilde{\omega}\ . \tag{C.10}$$

Clearly, the integral in (C.10) is a double convolution. Thus, (C.10) is

$$4\left(\frac{T}{2\pi}\right)^2 \left[\hat{W}_{1/2}(\omega) * \hat{W}_{1/2}(\omega)\right] * \langle\hat{S}(\omega)\rangle^2 * \left[\langle\hat{P}_{1/2}(\omega)\rangle * \langle\hat{P}_{1/2}(\omega)\rangle\right] . \qquad (C.11)$$

Finally, substituting (C.11) for the integral in (5.53) gives

$$\sigma^2(\hat{R}(\omega)) = \langle\hat{R}(\omega)\rangle^2 + 32\ \hat{W}_{1/2}(\omega) * \hat{W}_{1/2}(\omega) * \langle\hat{S}(\omega)\rangle^2 * \langle\hat{P}_{1/2}(\omega)\rangle * \langle\hat{P}_{1/2}(\omega)\rangle. \qquad (C.12)$$

APPENDIX D. APPROXIMATION TO THE MULTIPLE INTEGRAL IN (5.53) FOR A NORMAL AMOUNT OF IONOSPHERIC DOPPLER SPREADING

Measurements (Jones, Riley, and Georges, 1981) show that for a normal amount of Doppler spreading the spectral width of P is larger than that for the window W. Under those conditions, the shape of W will dominate over the shape of P in the Δ integration in the integral in (5.53). Thus, we may replace P everywhere it occurs by its value when Δ = zero (where W has its peak).

Similarly (as in Appendix C), we may replace S everywhere it occurs by its value for δ = zero (where W has its peak).

With the above two considerations in mind, we may approximate the integral in (5.53) by

$$\int_{-\infty}^{\infty} W(\tilde{\omega}-\Delta+\delta)\ W^*(\tilde{\omega}+\Delta+\delta)\ W(\tilde{\omega}+\Delta-\delta)\ W^*(\tilde{\omega}-\Delta-\delta)$$

$$\langle\hat{S}(\omega-\tilde{\omega}-x)\rangle\ \langle\hat{S}(\omega-\tilde{\omega}-x)\rangle\ \langle\hat{P}(x)\rangle\ \langle\hat{P}(x)\rangle\ dx\ d\tilde{\omega}\ d\delta\ d\Delta\ . \quad \text{(D.1)}$$

This allows us to separate the multiple integral into factors.

$$\iint_{-\infty}^{\infty} \left[\iint_{-\infty}^{\infty} W(\tilde{\omega}-\Delta+\delta)\ W^*(\tilde{\omega}+\Delta+\delta)\ W(\tilde{\omega}+\Delta-\delta)\ W^*(\tilde{\omega}-\Delta-\delta)\ d\Delta\ d\delta\right]$$

$$\langle\hat{S}(\omega-\tilde{\omega}-x)\rangle^2\ \langle\hat{P}(x)\rangle^2\ dx\ d\tilde{\omega}\ . \quad \text{(D.2)}$$

We consider first the double integral in brackets in (D.2).

$$\iint_{-\infty}^{\infty} W(\tilde{\omega}-\Delta+\delta)\ W^*(\tilde{\omega}+\Delta+\delta)\ W(\tilde{\omega}+\Delta-\delta)\ W^*(\tilde{\omega}-\Delta-\delta)\ d\Delta\ d\delta\ . \quad \text{(D.3)}$$

We next make a change in integration variables to

$$\delta_1 = 1/2(\tilde{\omega}-\Delta+\delta)$$

$$\delta_2 = 1/2(\tilde{\omega}+\Delta+\delta)\ . \quad \text{(D.4)}$$

Thus, (D.3) becomes

$$2 \iint_{-\infty}^{\infty} W(2\delta_1)\, W^*(2\delta_2)\, W(2\tilde{\omega}-2\delta_1)\, W^*(2\tilde{\omega}-2\delta_2)\, d\delta_1\, d\delta_2 \; . \tag{D.5}$$

We notice now that (D.5) can be separated into the product of two integrals. In addition, we can write (D.5) in terms of the half-width window spectra defined in (A.1). Thus (D.5) becomes

$$2\left[\int_{-\infty}^{\infty} W_{1/2}(\delta_1)\, W_{1/2}(\tilde{\omega}-\delta_1)\, d\delta_1 \right]\left[\int_{-\infty}^{\infty} W^*_{1/2}(\delta_2)\, W^*_{1/2}(\tilde{\omega}-\delta_2)\, d\delta_2 \right] \; . \tag{D.6}$$

Each of the integrals in (D.6) is a convolution, and the second integral is the complex conjugate of the first. Therefore, (D.6) is

$$2\left| W_{1/2}(\tilde{\omega}) * W_{1/2}(\tilde{\omega}) \right|^2 \; . \tag{D.7}$$

Substituting (D.7) for the quantity in the brackets in (D.2) gives

$$\iint_{-\infty}^{\infty} \left[2\left| W_{1/2}(\tilde{\omega}) * W_{1/2}(\tilde{\omega}) \right|^2 \right] \langle \hat{S}(\omega-\tilde{\omega}-x) \rangle^2 \langle \hat{P}(x) \rangle^2 \, dx \, d\tilde{\omega} \; . \tag{D.8}$$

The double integral in (D.8) is a double convolution. Thus, (D.8) becomes

$$2\left| W_{1/2}(\omega) * W_{1/2}(\omega) \right|^2 * \langle \hat{S}(\omega) \rangle^2 * \langle \hat{P}(\omega) \rangle^2 \; . \tag{D.9}$$

Substituting (D.9) for the multiple integral in (5.53) gives

$$\sigma^2(\hat{R}(\omega)) = \langle \hat{R}(\omega) \rangle^2 + 16 \left(\frac{2\pi}{T}\right)^2 \left| W_{1/2}(\omega) * W_{1/2}(\omega) \right|^2 * \langle \hat{S}(\omega) \rangle^2 * \langle \hat{P}(\omega) \rangle^2 \; . \tag{D.10}$$

APPENDIX E. EVALUATION OF THE MULTIPLE INTEGRAL IN (5.53) FOR A GAUSSIAN-SHAPED WINDOW SPECTRUM

If we consider the Gaussian-shaped window spectrum defined in (5.56), then

$$W(\tilde{\omega}-\Delta+\delta)\ W^*(\tilde{\omega}+\Delta+\delta)\ W(\tilde{\omega}+\Delta-\delta)\ W^*(\tilde{\omega}-\Delta-\delta)$$

$$= \frac{1}{(2\pi)^2\ a^4}\ \exp(-\ 2\ \frac{\tilde{\omega}^2}{a^2} - 2\ \frac{\delta^2}{a^2} - 2\ \frac{\Delta^2}{a^2})\ . \tag{E.1}$$

From (E.1), we see that the integral in (5.53) is

$$\frac{1}{(2\pi)^2\ a^4} \int_{-\infty}^{\infty} \exp(-\ 2\ \frac{\tilde{\omega}^2}{a^2})\ \exp(-\ 2\ \frac{\delta^2}{a^2})\ \exp(-\ 2\ \frac{\Delta^2}{a^2})$$

$$<\hat{S}(\omega-\tilde{\omega}-x-\delta)>\ <\hat{S}(\omega-\tilde{\omega}-x+\delta)>$$

$$<\hat{P}(x+\Delta)>\ <\hat{P}(x-\Delta)>\ dx\ d\tilde{\omega}\ d\delta\ d\Delta\ . \tag{E.2}$$

The multiple integral in (E.2) is partly separable to give

$$\frac{1}{(2\pi)^2\ a^4} \iint_{-\infty}^{\infty} \exp(-\ 2\ \frac{\tilde{\omega}^2}{a^2})\ [\int_{-\infty}^{\infty} \exp(-\ 2\ \frac{\delta^2}{a^2})\ <\hat{S}(\omega-\tilde{\omega}-x-\delta)>\ <\hat{S}(\omega-\tilde{\omega}-x+\delta)\ d\delta]$$

$$[\int_{-\infty}^{\infty} \exp(-\ 2\ \frac{\Delta^2}{a^2})\ <\hat{P}(x+\Delta)>\ <\hat{P}(x-\Delta)>\ d\Delta]\ dx\ d\tilde{\omega}. \tag{E.3}$$

Equation (5.59) gives an expression for the integral in the first pair of brackets in (E.3). Equation (5.60) gives an expression for the integral in the second pair of brackets in (E.3). Making these substitutions gives for (E.3)

$$\frac{1}{8\ \pi\ a^2} \iint_{-\infty}^{\infty} \exp(-\ 2\ \frac{\tilde{\omega}^2}{a^2})\ <\tilde{S}_W(\omega-\tilde{\omega}-x)>^2\ <\tilde{P}_W(x)>^2\ dx\ d\tilde{\omega}\ . \tag{E.4}$$

Using (5.58) allows us to write (E.4) as

$$\frac{T^2\ a^2}{8\pi} \iint_{-\infty}^{\infty} (\hat{W}(\tilde{\omega}))^2\ <\tilde{S}_W(\omega-\tilde{\omega}-x)>^2\ <\tilde{P}_W(x)>^2\ dx\ d\tilde{\omega}\ . \tag{E.5}$$

We see that the double integral in (E.5) is a double convolution. Thus, (E.5) becomes

$$\frac{T^2 a^2}{8\pi} \hat{W}^2(\omega) * \langle\tilde{S}_W(\omega)\rangle^2 * \langle\tilde{P}_W(\omega)\rangle^2 \quad . \tag{E.6}$$

Substituting (E.6) for the multiple integral in (5.53) gives

$$\sigma^2(\hat{R}(\omega)) = \langle\hat{R}(\omega)\rangle^2 + 4\pi a^2 \hat{W}^2(\omega) * \langle\tilde{S}_W(\omega)\rangle^2 * \langle\tilde{P}_W(\omega)\rangle^2 \ . \tag{E.7}$$

APPENDIX F. VARIANCE FOR GAUSSIAN-SHAPED SPECTRA

In this Appendix, we evaluate the variance of the power spectra in (5.57) when $<\hat{S}>$, $<\hat{P}>$, and $\hat{W}$ are the Gaussian-shaped spectra given in (5.61), (5.62), and (5.58). That involves two steps. The first is the evaluation of the integrals in (5.59) and (5.60). The second is the evaluation of the convolution in (5.57).

We begin by using (5.61) to get

$$\hat{S}(\omega+\delta)\ \hat{S}(\omega-\delta) = \frac{A_1^{\ 2}}{b^2\pi} \exp(-\frac{2\omega^2}{b^2} - \frac{2\delta^2}{b^2}) + \frac{A_2^{\ 2}}{c^2\pi} \exp(-\frac{2(\omega-\omega_2)^2}{c^2} - \frac{2\delta^2}{c^2})$$

$$+ \frac{A_1\ A_2}{b\ c\ \pi} \exp(-\frac{(\omega-\delta)^2}{b^2} - \frac{(\omega+\delta-\omega_2)^2}{c^2}) + \frac{A_1\ A_2}{b\ c\ \pi} \exp(-\frac{(\omega+\delta)^2}{b^2} - \frac{(\omega-\delta-\omega_2)^2}{c^2}). \quad (F.1)$$

The integrand in (5.59) will be small except when δ is small. When δ is small, the third and fourth terms in (F.1) will be small whenever

$$\omega_2^{\ 2} >> b^2 + c^2 , \quad (F.2)$$

that is, whenever the first- and second-order lines of the undistorted sea-echo spectrum are well separated (which is the normal situation). Thus, we may neglect the third and fourth terms in (F.1). Substituting the remaining part of (F.1) into (5.59) gives

$$<\tilde{S}_W(\omega)>^2 = \frac{1}{a} \sqrt{\frac{2}{\pi}} \frac{A_1^{\ 2}}{b^2\pi} \exp(-\ 2\ \frac{\omega^2}{b^2}) \int_{-\infty}^{\infty} \exp(-2\ \frac{\delta^2}{a^2} - 2\ \frac{\delta^2}{b^2})\ d\delta$$

$$+ \frac{1}{a} \sqrt{\frac{2}{\pi}}\ \frac{A_2^{\ 2}}{c^2\pi}\ \exp(-\ 2\frac{(\omega-\omega_2)^2}{c^2})\ \int_{-\infty}^{\infty} \exp(-2\ \frac{\delta^2}{a^2} - 2\ \frac{\delta^2}{c^2})\ d\delta. \quad (F.3)$$

Evaluating the integrals in (F.3) gives

$$<\tilde{S}_W(\omega)>^2 = \frac{A_1^2}{b\,\pi} \frac{1}{\sqrt{a^2+b^2}} \exp(-2\,\frac{\omega^2}{b^2}) + \frac{A_2^2}{c\,\pi} \frac{1}{\sqrt{a^2+c^2}} \exp(-\,2\,\frac{(\omega-\omega_2)^2}{c^2})\ . \qquad (F.4)$$

For a check, we will consider the case where the first- and second-order lines of the sea-echo spectrum are much wider than the spectral width of the window. That is, where

$$b^2 >> a^2 \qquad (F.5)$$

$$c^2 >> a^2\ .$$

Then (F.4) becomes

$$<\tilde{S}_W(\omega)>^2 = \frac{A_1^2}{b^2\pi} \exp(-2\,\frac{\omega^2}{b^2}) + \frac{A_2^2}{c^2\pi} \exp(-2\,\frac{(\omega-\omega_2)^2}{c^2}) \approx <\hat{S}(\omega)>^2 \qquad (F.6)$$

where the last equality comes from neglecting the cross term in the square of (5.61). This is related to neglecting the third and fourth terms in (F.1).

Continuing on, we substitute (5.62) into (5.60) to give

$$<\tilde{P}_W(\omega)>^2 = \frac{1}{a} \sqrt{\frac{2}{\pi}}\ \frac{1}{a_P^2\pi} \exp(-\,2\,\frac{\omega^2}{a_P^2}) \int_{-\infty}^{\infty} \exp(-2\,\frac{\Delta^2}{a^2}) \exp(-2\,\frac{\Delta^2}{a_P^2})\, d\Delta. \qquad (F.7)$$

Evaluating the integral in (F.7) gives

$$<\tilde{P}_W(\omega)>^2 = \frac{1}{a_P\,\pi\sqrt{a^2+a_P^2}} \exp(-2\,\frac{\omega^2}{a_P^2}) = \frac{a_P}{\sqrt{a^2+a_P^2}} <\hat{P}(\omega)>^2 \qquad (F.8)$$

where the last equality comes from using (5.62).

Now we need to evaluate the double convolution in (5.57). Evaluating a convolution is easy for Gaussian functions, because if the Gaussians have unit area, then the convolution will also have unit area, and the square of the spectral widths will add. We first need to express each of the convolution factors in (5.57) as the product of a constant times a unit area Gaussian. The square of (5.58) is

$$\hat{W}^2(\omega) = \frac{\sqrt{\pi}}{a^3\, T^2\, \sqrt{2}} \left[\frac{\sqrt{2}}{a\sqrt{\pi}} \exp(-2\, \frac{\omega^2}{a^2}) \right] , \qquad (F.9)$$

where the factor in brackets is a unit area Gaussian. Equation (F.4) is equivalent to

$$\langle \tilde{S}_W(\omega) \rangle^2 = \frac{A_1^{\,2}}{\sqrt{2\pi}\, \sqrt{a^2+b^2}} \left[\frac{\sqrt{2}}{b\sqrt{\pi}} \exp(-2\, \frac{\omega^2}{b^2}) \right]$$

$$+ \frac{A_2^{\,2}}{\sqrt{2\pi}\, \sqrt{a^2+c^2}} \left[\frac{\sqrt{2}}{c\sqrt{\pi}} \exp(-2\, \frac{(\omega-\omega_2)^2}{c^2}) \right] , \qquad (F.10)$$

where the factors in brackets are unit area Gaussians. Equation (F.8) is equivalent to

$$\langle \tilde{P}_W(\omega) \rangle^2 = \frac{1}{\sqrt{2\pi}\, \sqrt{a^2+a_P^{\,2}}} \left[\frac{\sqrt{2}}{a_P\sqrt{\pi}} \exp(-2\, \frac{\omega^2}{a_P^2}) \right] , \qquad (F.11)$$

where the factor in brackets is a unit area Gaussian. Now we can directly evaluate the double convolution in (5.57) to give

$$\sigma^2(\hat{R}(\omega)) = \langle\hat{R}(\omega)\rangle^2 + 4\,\pi\,a^2 \frac{\sqrt{\pi}}{a^3\,T^2\,\sqrt{2}} \frac{1}{\sqrt{2\pi}\,\sqrt{a^2+a_P^2}}$$

$$\left[\frac{A_1^{\,2}}{\sqrt{2\pi}\,\sqrt{a^2+b^2}} \frac{\sqrt{2}}{\sqrt{\pi}\,\sqrt{a^2+b^2+a_P^2}} \exp(-2\,\frac{\omega^2}{a^2+b^2+a_P^2}) + \right.$$

$$\left. + \frac{A_2^{\,2}}{\sqrt{2\pi}\,\sqrt{a^2+c^2}} \frac{\sqrt{2}}{\sqrt{\pi}\,\sqrt{a^2+c^2+a_P^2}} \exp(-2\,\frac{(\omega-\omega_2)^2}{a^2+c^2+a_P^2}\,)\right] . \qquad \text{(F.12)}$$

We can now use (G.3) in (F.12) and divide by $\langle\hat{R}(\omega)\rangle^2$ to give the normalized variance as

$$\frac{\sigma^2(\hat{R}(\omega))}{\langle\hat{R}(\omega)\rangle^2} = 1 + 2a\,\frac{\sqrt{\dfrac{a^2+c^2+a_P^2}{a^2+c^2}} + x^2\sqrt{\dfrac{a^2+b^2+a_P^2}{a^2+b^2}}}{(1+x)^2\,\sqrt{a^2+a_P^2}} \qquad \text{(F.13)}$$

where x is given by (G.4).

APPENDIX G. MEAN POWER FOR GAUSSIAN-SHAPED SPECTRA

In this Appendix, we evaluate the mean power spectrum for $\langle\hat{R}\rangle$ given by (4.9) when $\langle\hat{S}\rangle$, $\langle\hat{P}\rangle$, and $\hat{W}$ are the Gaussian-shaped spectra given in (5.61), (5.62), and (5.58).

In evaluating the double convolution in (4.9), we will be taking the convolution of three Gaussian functions. It is easy to do that if we remember that if the Gaussian functions have unit area, then the convolution is also a Gaussian with unit area, and that Gaussian has a width whose square is the sum of the squares of the Gaussians in the convolution. Each of the Gaussian functions in (5.61) has unit area, as does the Gaussian in (5.62). The Gaussian function in (5.58) does not have unit area, but we can rewrite (5.58) in terms of a Gaussian that does have unit area. Thus, (5.58) is

$$\hat{W}(\omega) = \frac{\sqrt{\pi}}{Ta}\left[\frac{\exp(-\omega^2/a^2)}{a\sqrt{\pi}}\right] \quad . \tag{G.1}$$

The Gaussian function in the brackets in (G.1) has unit area. The other factor will simply remain as a factor in the evaluation of (4.9).

With this in mind, we can evaluate (4.9) directly to give

$$\langle\hat{R}(\omega)\rangle = \hat{W}(\omega) * \langle\hat{S}(\omega)\rangle * \langle\hat{P}(\omega)\rangle$$

$$= \frac{\sqrt{\pi}}{Ta}\left[\frac{A_1 \exp\left(-\frac{\omega^2}{a_P^2+b^2+a^2}\right)}{\sqrt{a_P^2+b^2+a^2}\;\sqrt{\pi}} + \frac{A_2 \exp\left(-\frac{(\omega-\omega_2)^2}{a_P^2+c^2+a^2}\right)}{\sqrt{a_P^2+c^2+a^2}\;\sqrt{\pi}}\right] \quad . \tag{G.2}$$

It will be useful to write (G.2) in the form

$$\langle\hat{R}(\omega)\rangle = \frac{1}{Ta} \frac{A_2 \exp\left(-\frac{(\omega-\omega_2)^2}{a_P^2+c^2+a^2}\right)}{\sqrt{a_P^2+c^2+a^2}} (1 + x) \tag{G.3}$$

where

$$x = \frac{A_1}{A_2} \sqrt{\frac{a_P^2+c^2+a^2}{a_P^2+b^2+a^2}} \exp\left(\frac{(\omega-\omega_2)^2}{a_P^2+c^2+a^2} - \frac{\omega^2}{a_P^2+b^2+a^2} \right) . \tag{G.4}$$