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THE MEAN AND VARIANCE OF SKYWAVE RADAR SEA-ECHO POWER SPECTRA FOR DISCRETE IONOSPHERIC MULTIPATH

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ABSTRACT

Previous results that give formulas for the mean and variance of skywave radar sea-echo Doppler spectra are transformed from a discrete representation to a general convolution form. While the discrete representation is general enough to represent arbitrary sea-echo spectra and arbitrary ionospheric distortion spectra, the derivation of the variance assumes that both spectra are composed of discrete spectral lines that are narrower than the spectral width of the window (or weighting function) applied to the measured time series and that the separation of the spectral lines is large compared with the spectral width of the window. The formulas are shown to be consistent with corresponding formulas derived from a continuous representation under the same physical assumptions. However, the formula for variance based on discrete spectra differs from that derived from a continuous representation under physical conditions that assume more continuous spectra for both the sea echo and the ionospheric distortion.

1. INTRODUCTION — RELATION TO THE SKYWAVE SEA-STATE RADAR PROJECT

The skywave sea-state radar project is concerned with measuring sea-echo spectra by ionospheric reflections. The advantage of skywave radar is that it increases the coverage area. The disadvantage is that ionospheric motions distort the measured spectra.

When we process the measured spectra to reduce the ionospheric distortion, we need to know the expected mean and variance of the spectra (Georges et al., 1981).

Georges and Jones (1980) calculated these when the ionospheric distortion was represented by discrete multipath. They pointed out that the discrete multipath representation was actually general enough to represent continuous (in the frequency domain) ionospheric distortion, but their formulas for the mean and variance of the power spectra are completely in terms of the discrete representation. The present report transforms their discrete representation formulas into a general convolution form.

2. SUMMARY

The ionosphere distorts skywave sea-echo spectra by convolution in the frequency domain.

$$R(\omega) = S(\omega) * P(\omega) * W(\omega) \tag{2.1}$$

where R is the measured Fourier spectrum (including a window or weighting function), S is the sea-echo Fourier spectrum, P is the ionospheric reflection Fourier spectrum (including both outgoing and returning reflections), and W is the transform of the window or weighting function. The above formula is valid for complex Fourier amplitudes, not power spectral density.

Only in the limit of an infinite ensemble average and under certain statistical assumptions is an analogous expression,

$$\langle\hat{R}(\omega)\rangle = \langle\hat{S}(\omega)\rangle * \langle\hat{P}(\omega)\rangle * \hat{W}(\omega) \ , \tag{2.2}$$

valid for power spectra. Here, ^ denotes power spectral density, and < > denotes an infinite ensemble average.

Under the assumptions that the spectra $\langle\hat{S}(\omega)\rangle$ and $\langle\hat{P}(\omega)\rangle$ consist of discrete spectral lines that are narrower than the spectral width of $\hat{W}(\omega)$, and that the spectral lines are separated by more than the spectral width of $\hat{W}(\omega)$, the variance of the measured power spectrum is shown to be approximately

$$\sigma^2(\hat{R}(\omega)) \approx (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * \sigma^2(\hat{P}(\omega)) +$$

$$+ (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{P}(\omega)>^2 * \sigma^2(\hat{S}(\omega)) +$$

$$+ (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * \sigma^2(\hat{S}(\omega)) * \sigma^2(\hat{P}(\omega)) +$$

$$+ <\hat{R}(\omega)>^2 - (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 \quad (2.3)$$

where T is the sampling period,

$$\sigma^2(\hat{P}(\omega)) \equiv <(\hat{P}(\omega) - <\hat{P}(\omega)>)^2> \quad (2.4)$$

is the (unnormalized) variance of the ionospheric distortion power spectra,

$$\sigma^2(\hat{S}(\omega)) \equiv <(\hat{S}(\omega) - <\hat{S}(\omega)>)^2> \quad (2.5)$$

is the (unnormalized) variance of the sea-echo power spectra, and the convolution with the above variances is a frequency convolution that treats $\sigma^2(\hat{P}(\omega))$ and $\sigma^2(\hat{S}(\omega))$ as functions of ω (that is as spectra).

The first three terms in (2.3) are from the diagonal terms in the square of (2.1). The fourth and fifth terms in (2.3) are from the cross terms in the square of (2.1), and represent phase interference.

For the special case that the sea-echo spectra have unit normalized variance,

$$\sigma^2(\hat{S}(\omega)) = <\hat{S}(\omega)>^2 , \quad (2.6)$$

and where the ionospheric distortion spectra have unit normalized variance,

$$\sigma^2(\hat{P}(\omega)) = <\hat{P}(\omega)>^2 , \quad (2.7)$$

the first, second, third, and fifth terms in (2.3) become equal, so that

$$\sigma^2(\hat{R}(\omega)) = \langle\hat{R}(\omega)\rangle^2 + 2(\frac{2\pi}{T})^2 \hat{W}^2(\omega) * \langle\hat{S}(\omega)\rangle^2 * \langle\hat{P}(\omega)\rangle^2 . \qquad (2.8)$$

The second term in (2.8) is twice the fifth term in (2.3). All of the terms in (2.3) are positive, so the smallest the fifth term can be is zero. The sum of the cross terms (the sum of the fourth and fifth terms in (2.3)) are also positive, so the largest the fifth terms in (2.3) can be is $\langle\hat{R}(\omega)\rangle^2$. Thus, the second term in (2.8) varies from zero to $2\langle\hat{R}(\omega)\rangle^2$, and consequently the normalized variance from (2.8),

$$\frac{\sigma^2(\hat{R}(\omega))}{\langle\hat{R}(\omega)\rangle^2} = 1 + 2\,\frac{(\frac{2\pi}{T})^2 \hat{W}^2(\omega) * \langle\hat{S}(\omega)\rangle^2 * \langle\hat{P}(\omega)\rangle^2}{\langle\hat{R}(\omega)\rangle^2} , \qquad (2.9)$$

varies from one to three.

Jones (1981) shows that the variance in (2.8) agrees with that calculated from a continuous representation under the same physical assumptions related to discrete spectra for $\langle\hat{S}(\omega)\rangle$ and $\langle\hat{P}(\omega)\rangle$. However, Jones (1981) gives a formula different from (2.8) when $\langle\hat{S}(\omega)\rangle$ and $\langle\hat{P}(\omega)\rangle$ do not consist of discrete spectral lines.

3. THE CONVOLUTION MODEL FOR FOURIER AMPLITUDES

Georges and Jones (1980) show that the ionosphere distorts sea-echo spectra by a convolution in the frequency domain. Their equation (13) is

$$R(\omega) = W(\omega) * S(\omega) * P(\omega) \qquad (3.1)$$

where $R(\omega)$ is the measured Fourier spectrum (complex amplitudes, not power spectral density), $W(\omega)$ is the transform of the window (or weighting function), $S(\omega)$ is the Fourier spectrum of the sea-echo, and $P(\omega)$ is the Fourier spectrum of the product of the outgoing and returning ionospheric reflection coefficients. $P(\omega)$ is a direct measure of the ionospheric distortion.

All of the quantities in the above formula are complex Fourier amplitudes, not power spectral density. Such a simple formula does not hold for power spectra except on the average. Even then, there is a variance associated with use of the formula.

4. MEAN OF THE POWER SPECTRUM

Georges and Jones (1980) derive expressions for the mean and variance of the observed Doppler spectrum in the presence of discrete ionospheric multipath. As they point out, their representation of ionospheric multipath, although discrete, is general enough to represent arbitrary, linear, shift invariant (in the frequency domain) ionospheric distortion. However, the approximations they use restrict their calculations of variance to spectra of S and P that consist of discrete spectral lines narrower than the spectral width of the window W and separated by more than the width of W. Here I transform their results to a general convolution form.

We start with their equation (24), which gives the mean or expected amplitude for the measured Doppler spectrum.

$$<|R(\omega)|^2> = \sum_{k=1}^{K} \sum_{i=1}^{I} <|A_k|^2><|B_i|^2>|W(\omega-\omega_i-\omega_k)|^2 \tag{1}$$

where I have changed notation, and use $R(\omega)$ for their $R''(\omega)$. The brackets < > denote an ideal infinite ensemble average over the same "sea state" and the same "ionospheric state". We assume that the realizations within such an ensemble belong to the same probability distribution. The probability distribution for $R(\omega)$ need not be the same as that for A_k or for B_i, however, and the probability distribution may depend on ω, k, and i.

$R(\omega)$ is the measured Fourier spectrum.

$$S(\omega) = \sum_{k=1}^{K} A_k\, \delta(\omega-\omega_k) \tag{2}$$

is the Fourier spectrum for the backscatter of radar from the ocean for a discrete spectral representation. A_k is complex to include phase.

$$P(\omega) = \sum_{i=1}^{I} B_i \, \delta(\omega-\omega_i) \tag{3}$$

is the Fourier spectrum of the ionospheric transmission response function. It gives the Doppler spectrum shift from the transmitted frequency due to ionospheric motions for two-way ionospheric transmission. B_i is complex to include phase. Each term in (3) is from one ionospheric path in the multipath propagation.

$$W(\omega) = \frac{T}{L} \sum_{\ell=1}^{L} w_\ell \, e^{-i\omega T \frac{\ell-\ell_c}{L}} \tag{4}$$

is the Fourier transform of the window w_ℓ applied to the data.

$$\ell_c = \frac{L+1}{2} \qquad \text{(if the time window is symmetric)} \tag{5a}$$

or

$$\ell_c = \frac{L}{2} \qquad \text{(if the time window is DFT even [Harris, 1978])} \tag{5b}$$

is the center of the time window. $W(\omega)$ is real if the time window is symmetric. It is complex if the time window is DFT even (Harris, 1978). In either case, we assume the time window has the following symmetry.

$$w_\ell = w_{2\ell_c-\ell} \ . \tag{6}$$

Notice that the window $W(\omega)$ in the frequency domain is continuous even though the time window is discrete.

We now begin to translate (1) into a more general expression that does not depend either on the discrete nature of the ionospheric multipath or the assumption of a discrete spectrum for the backscatter from the ocean. In doing this, I try to make the result independent of the discrete nature of our sampling process, and independent as far as possible of the length of the sample. That is, when we calculate the FFT (fast Fourier transform) of the measured time series, we are sampling in the frequency domain. Because the results here should be essentially independent of the details of our sampling process, we will try to ignore, from here on, the discrete nature of the window in the time domain.

Thus, we will treat the window $W(\omega)$ as though it were the Fourier transform of a continuous window function $w(t)$.

$$W(\omega) = \int_{-\infty}^{\infty} w(t)\ e^{-i\omega t}\ dt \tag{7}$$

where

$$w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W(\omega)\ e^{i\omega t}\ d\omega \tag{8}$$

gives the inverse transform. Likewise, we will treat our measured spectrum $R(\omega)$ as though it were the Fourier transform of a continuous function $r(t)$.

$$R(\omega) = \int_{-\infty}^{\infty} r(t)\ e^{-i\omega t}\ dt \tag{9}$$

where

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\omega)\ e^{i\omega t}\ d\omega \tag{10}$$

gives the inverse transform.

In our measurements, we sample $r(t)$ (which includes the window), and perform an FFT to give sampled values of $R(\omega)$. We then calculate $|R(\omega)|^2$ to give something proportional to power spectral density. To find out the proportionality factor, we start with Rayleigh's theorem (Bracewell, 1965, page 112)

$$\int_{-\infty}^{\infty} r(t)\ r^*(t)\ dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\omega)\ R^*(\omega)\ d\omega\ . \tag{11}$$

Because of the window and/or the finite length of our measuring period, the integral on the left has no contribution outside of the measuring period. Thus, we can rewrite (11) as

$$\int_{-T/2}^{T/2} r(t)\ r^*(t)\ dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\omega)\ R^*(\omega)\ d\omega \tag{12}$$

where T is the length of the measuring period. We assume that the measured echoes (without the window) are, on the average, independent of time. That is, they are statistically stationary. (We know this to be true for measuring periods that are not so short that the statistics are meaningless and for periods that are not so long that the statistics have changed.) Thus, if we vary the length of the measuring period, but keep the shape of the window the same (but keep the length of the window proportional to the length of the measuring period), then the integral on the left of (12) will, on the average, be proportional to the length of the measuring period T. To get a quantity that is independent of the length of the measuring period, we divide both sides of (12) by T.

$$\frac{1}{T}\int_{-T/2}^{T/2} r(t)\ r^*(t)\ dt = \frac{1}{2\pi T}\int_{-\infty}^{\infty} R(\omega)\ R^*(\omega)\ d\omega\ . \tag{13}$$

The quantity of the left of (13) is on the average independent of T, and therefore, the quantity on the right must also be independent of T.

It is convenient to define the left side of (13) to be proportional to the frequency integral of the power spectral density $\hat{R}(\omega)$ because that would lead to a measure of the power spectral density that is, on the average, independent of the length of the measuring period T. Thus, we define

$$\frac{1}{T}\int_{-T/2}^{T/2} r(t)\ r^*(t)\ dt \equiv \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \hat{R}(\omega)\ d\omega \tag{14}$$

where the numerical factor is for convenience later. Comparing (14) with (13) gives

$$\hat{R}(\omega) = \frac{2\pi}{T}\ R(\omega)\ R^*(\omega) = \frac{2\pi}{T}\ |R(\omega)|^2 \tag{15}$$

for the power spectral density. Similar arguments allow us to define

$$\hat{W}(\omega) = \frac{2\pi}{T}\ W(\omega)\ W^*(\omega) = \frac{2\pi}{T}\ |W(\omega)|^2 \tag{16}$$

for the power spectral density for the window.

If we try to write the corresponding expression to (15) or (16) for $S(\omega)$ from (2), we get products of delta functions. That clearly will not do, so we calculate the inverse Fourier transform of (2) to give

$$s(t) = \frac{1}{2\pi} \sum_{k=1}^{K} A_k e^{i\omega_k t} . \tag{17}$$

We now consider

$$\int_{-\infty}^{\infty} s(t)\, s^*(t)\, dt . \tag{18}$$

Unlike the left side of (11), (18) has no window. Thus, if the backscatter from the sea is, on the average, independent of time, then the integral in (18) diverges. On the average, the integral will be proportional to the length of the measuring interval T for finite measuring intervals. It is thus more appropriate to consider, instead of (18), a finite integration time and to divide by the integration period to get a quantity that is independent on the average of the integration period. Thus, we define, in analogy to (14)

$$\frac{1}{T} \int_{-T/2}^{T/2} s(t)\, s^*(t)\, dt \equiv \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \hat{S}(\omega)\, d\omega \tag{19}$$

a quantity proportional to the frequency integral of the power spectral density $\hat{S}(\omega)$. Substituting (17) into (19) gives

$$\frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1}{2\pi}\right)^2 \sum_{k=1}^{K} \sum_{k'=1}^{K} |A_k A_{k'}^*| \, e^{i(\phi_k - \phi_{k'}) + i(\omega_k - \omega_{k'})t} dt = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \hat{S}(\omega)\, d\omega \tag{20}$$

where ϕ_k is the phase of A_k.

The variables A_k, ϕ_k, and $\hat{S}(\omega)$ are stochastic variables. We are interested in the infinite ensemble average of (20). We assume that the statistics of the A_k and ϕ_k are independent for different k. Thus, the infinite ensemble average of any term on the left of (20) for which $k' \neq k$ will be zero. Thus

$$\frac{1}{T} \sum_{k=1}^{K} \langle |A_k|^2 \rangle \int_{-T/2}^{T/2} dt = \int_{-\infty}^{\infty} \langle \hat{S}(\omega) \rangle\, d\omega . \tag{21}$$

This gives

$$\sum_{k=1}^{K} \langle |A_k|^2 \rangle = \int_{-\infty}^{\infty} \langle \hat{S}(\omega) \rangle\, d\omega \tag{22}$$

which clearly leads to

$$\langle\hat{S}(\omega)\rangle = \sum_{k=1}^{K} \langle|A_k|^2\rangle\, \delta(\omega-\omega_k) \tag{23}$$

for the power spectral density.

We can use similar arguments starting with (3) to give

$$\langle\hat{P}(\omega)\rangle = \sum_{i=1}^{I} \langle|B_i|^2\rangle\, \delta(\omega-\omega_i) \tag{24}$$

for the infinite ensemble average of the power spectral density $\hat{P}(\omega)$ corresponding to the ionospheric transmission response function $P(\omega)$.

Now we can calculate (1). It looks a little like a double convolution, except that the frequencies are not necessarily evenly spaced. Therefore, we start with

$$\hat{W}(\omega) * \langle\hat{S}(\omega)\rangle \equiv \int_{-\infty}^{\infty} \hat{W}(\omega-\omega') \langle\hat{S}(\omega')\rangle\, d\omega' \quad . \tag{25}$$

Substituting (16) and (23) into (25) and integrating over the delta function gives

$$\hat{W}(\omega) * \langle\hat{S}(\omega)\rangle = \frac{2\pi}{T} \sum_{k=1}^{K} \langle|A_k|^2\rangle\, |W(\omega-\omega_k)|^2 \quad . \tag{26}$$

Continuing further, we have

$$\hat{W}(\omega) * \langle\hat{S}(\omega)\rangle * \langle\hat{P}(\omega)\rangle \equiv \int_{-\infty}^{\infty} \hat{W}(\omega-\omega') * \langle\hat{S}(\omega-\omega')\rangle\langle\hat{P}(\omega')\rangle d\omega' \quad . \tag{27}$$

Substituting (26) and (24) into (27) and integrating over the delta function gives

$$\hat{W}(\omega) * \langle\hat{S}(\omega)\rangle * \langle\hat{P}(\omega)\rangle = \frac{2\pi}{T} \sum_{k=1}^{K} \sum_{i=1}^{I} \langle|A_k|^2\rangle\langle|B_i|^2\rangle|W(\omega-\omega_i-\omega_k)|^2 \ . \tag{28}$$

Now we notice that (28) is the same as (1) except for a factor. We multiply both sides of (1) by $2\pi/T$, and use (15) and (28) to give

$$\langle\hat{R}(\omega)\rangle = \hat{W}(\omega) * \langle\hat{S}(\omega)\rangle * \langle\hat{P}(\omega)\rangle \tag{29}$$

which is the result we want. (29) is equivalent to (1), although (29) is more general. We have not shown that (29) is true in general, but only that it is true when $S(\omega)$ has the form given in (2) and $P(\omega)$ has the form given in (3).

Equation (29) gives the mean of the observed Doppler spectrum. Now, we want to transform equation (27) of Georges and Jones (1980) to give the variance of the observed Doppler spectrum.

5. VARIANCE OF THE POWER SPECTRUM

The variance in the observed Doppler spectrum (equation (27) of Georges and Jones (1980)) is

$$\sigma^2(|R(\omega)|^2) = \sum_{k=1}^{K} \sum_{i=1}^{I} |W_{ik}|^4 \{<|A_k|^4><|B_i|^4> - <|A_k|^2>^2<|B_i|^2>^2\}$$

$$+ \sum_{k\neq k'} \sum_{i\neq i'} <|A_k|^2><|A_{k'}|^2><|B_i|^2><|B_{i'}|^2>|W_{ik}|^2|W_{i'k'}|^2 \qquad (30)$$

where

$$W_{ik} = W(\omega-\omega_i-\omega_k) \qquad (31)$$

and I have replaced

$$2<|A_k|^2>^2 \qquad \text{by} \qquad <|A_k|^4> \qquad (32)$$

and dropped the incorrect factor of 1/2 from the second term in equation (27) of Georges and Jones (1980). The first term of (30) is the same as the first term of Georges and Jones' equation (26). Their equation (26) is more general than their equation (27) because the latter assumes a unit normalized variance for A_k. Thus, (30) allows arbitrary variance for both A_k and B_k.

Georges and Jones point out that the first term in (30) arises from the diagonal terms in the square of the convolution in (3.1), and that the second term in (30) arises from cross terms (or phase interference terms) in the square of that convolution.

We can rewrite (30) by using

$$<|A_k|^4> = <|A_k|^2>^2 + \sigma^2(|A_k|^2) \tag{33}$$

and

$$<|B_i|^4> = <|B_i|^2>^2 + \sigma^2(|B_i|^2) \ . \tag{34}$$

We can also rewrite the double sum in the second term of (30) as

$$\sum_{k,k'} \sum_{i,i'} - \sum_{k'=k=1}^{K} \sum_{i \neq i'} - \sum_{k \neq k'} \sum_{i'=i=1}^{K} - \sum_{k'=k=1}^{K} \sum_{i'=i=1}^{I} \ . \tag{35}$$

However the middle two terms in (35) will be small because $W(\omega)$ is small except for ω small, and if the frequencies ω_k and $\omega_{k'}$ are separated by more than $2\pi/T$, (and similarly for ω_i and $\omega_{i'}$) then there will be no values of i, i', k, k' in those two terms where both W_{ik} and $W_{i'k'}$ will be significant. Georges and Jones (1980) used similar arguments to derive their equation (27).

Thus, we may rewrite (30) as

$$\sigma^2(|R(\omega)|^2) = \sum_{k=1}^{K} \sum_{i=1}^{I} |W_{ik}|^4 \ [<|A_k|^2>^2 \ \sigma^2(|B_i|^2) +$$

$$+ <|B_i|^2>^2 \ \sigma^2(|A_k|^2) + \sigma^2(|A_k|^2) \ \sigma^2(|B_i|^2)] +$$

$$+ \sum_{k,k'} \sum_{i,i'} |W_{ik}|^2 |W_{i'k'}|^2 \ <|A_k|^2><|A_{k'}|^2><|B_i|^2><|B_{i'}|^2> +$$

$$- \sum_{k=1}^{K} \sum_{i=1}^{I} |W_{ik}|^4 <|A_k|^2>^2 \ <|B_i|^2>^2 \ . \tag{36}$$

We notice that the sum over i' and k' in the next to last term of (36) is independent of the sum over i and k. Thus, we can change the multiple sum into a product. Each factor in the product is the same as (1). Thus, we can rewrite (36) as

$$\sigma^2(|R(\omega)|^2) = \sum_{k=1}^{K} \sum_{i=1}^{I} |W_{ik}|^4 \, [<|A_k|^2>^2 \, \sigma^2(|B_i|^2) +$$

$$+ <|B_i|^2>^2 \, \sigma^2(|A_k|^2) + \sigma^2(|A_k|^2) \, \sigma^2(|B_i|^2)] +$$

$$+ <|R(\omega)|^2>^2 - \sum_{k=1}^{K} \sum_{i=1}^{I} |W_{ik}|^4 <|A_k|^2>^2 <|B_i|^2>^2 \ . \qquad (37)$$

The first term in (37) arises from the diagonal terms in the square of (3.1). The last two terms in (37) arise from the cross terms in the square of (3.1). To continue transforming (37), we need to derive expressions analogous to (23) and (24) for some of the quantities in (37). We start with $<|A_k|^2>^2$. We expect that this will somehow be proportional to $<\hat{S}(\omega)>^2$.

We start with the transform of $<\hat{S}(\omega)>$. Call this

$$\hat{s}(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} <\hat{S}(\omega)> e^{i\omega t} \, d\omega \ . \qquad (38)$$

Substituting (23) into (37) gives

$$\hat{s}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=1}^{K} <|A_k|^2> \delta(\omega-\omega_k) e^{i\omega t} \, d\omega \ . \qquad (39)$$

Integrating over the delta functions gives

$$\hat{s}(t) = \frac{1}{2\pi} \sum_{k=1}^{K} <|A_k|^2> e^{i\omega_k t} \ . \qquad (40)$$

The complex conjugate of (40) is

$$\hat{s}^*(t) = \frac{1}{2\pi} \sum_{k'=1}^{K} <|A_{k'}|^2 \, e^{-i\omega_{k'} t} \ . \qquad (41)$$

Rayleigh's theorem (Bracewell, 1965, page 112) is

$$\int_{-\infty}^{\infty} \hat{s}(t) \, \hat{s}^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} <\hat{S}(\omega)>^2 \, d\omega \ . \qquad (42)$$

Substituting (40) and (41) into (42) gives

$$\left(\frac{1}{2\pi}\right)^2 \sum_{k=1}^{K} \sum_{k'=1}^{K} <|A_k|^2><|A_{k'}|^2> \int_{-\infty}^{\infty} e^{i(\omega_k-\omega_{k'})t} \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} <\hat{S}(\omega)>^2 \, d\omega \; . \tag{43}$$

The integral on the left of (43) is proportional to a Dirac delta function $\delta(\omega_k-\omega_{k'})$. It is not particularly useful to evaluate the integral as such because of the double sum, but it does show that only the k'=k terms in the sum on the left contribute. Thus, we can neglect the k'≠k terms in (43) to give

$$\frac{1}{2\pi} \sum_{k=1}^{K} <|A_k|^2>^2 \int_{-\infty}^{\infty} dt = \int_{-\infty}^{\infty} <\hat{S}(\omega)>^2 \, d\omega \; . \tag{44}$$

The integral on the left diverges as expected, but if we consider a finite interval T, and divide by T, we get a quantity that is independent of the interval T.

$$\sum_{k=1}^{K} <|A_k|^2>^2 \frac{1}{T} \int_{-T/2}^{T/2} dt = \frac{2\pi}{T} \int_{-\infty}^{\infty} <\hat{S}(\omega)>^2 \, d\omega \; . \tag{45}$$

Clearly, $<\hat{S}(\omega)>^2$ diverges as T approaches infinity, whereas $<\hat{S}(\omega)>^2/T$ is roughly independent of T. Thus, for measurements over a finite time interval T, we can consider

$$\frac{<\hat{S}(\omega)>^2}{T} \tag{46}$$

to be an estimate of

$$\lim_{T\to\infty} \frac{<\hat{S}(\omega)>^2}{T} \tag{47}$$

where T in (46) and (45) is the actual length of our measurement interval in our experiment.

From (45), we have

$$\sum_{k=1}^{K} <|A_k|^2>^2 \, \delta(\omega-\omega_k) = \frac{2\pi}{T} <\hat{S}(\omega)>^2 \; . \tag{48}$$

This is the expression we need to use in (37). Similarly, we have

$$\sum_{i=1}^{I} <|B_i|^2>^2 \, \delta(\omega-\omega_i) = \frac{2\pi}{T} <\hat{P}(\omega)>^2 \tag{49}$$

as the corresponding expression for the ionospheric transmission response function $P(\omega)$.

Now we must find corresponding expressions for the variances $\sigma^2(|A_k|^2)$ and $\sigma^2(|B_i|^2)$ in (37). We suspect that

$$\sigma^2(\hat{S}(\omega)) = <\hat{S}(\omega)^2> - <\hat{S}(\omega)>^2 \tag{51}$$

is somehow proportional to

$$\sum_{k=1}^{K} \sigma^2(|A_k|^2) = \sum_{k=1}^{K} (<|A_k|^4> - <|A_k|^2>^2) \quad . \tag{52}$$

Equation (48) already gives us an expression for the second term in (51). To calculate the first term, we need an expression for $\hat{S}(\omega)$. Equation (23) suggests that we might get the correct expression by simply removing the "expectation value brackets" from (23). To give further evidence for this, we consider using equations like (15) or (16) for $\hat{S}(\omega)$. That is, we consider

$$\hat{S}(\omega) \overset{?}{=} \frac{2\pi|S(\omega)|^2}{T} \quad . \tag{53}$$

Substituting (2) into (53) gives

$$\hat{S}(\omega) \overset{?}{=} \frac{2\pi}{T} \sum_{k=1}^{K} \sum_{k'=1}^{K} A_k A_{k'}^* \delta(\omega-\omega_k) \delta(\omega-\omega_{k'}) \quad . \tag{54}$$

The product of delta functions in (54) shows that only the k'=k terms in the double sum contribute. Thus, (54) becomes

$$\hat{S}(\omega) \overset{?}{=} \frac{2\pi}{T} \sum_{k=1}^{K} |A_k|^2 \delta^2(\omega-\omega_k) \quad . \tag{55}$$

The square of the delta function in (55) makes it useless in calculations, but it at least shows that the $k' \neq k$ terms do not contribute to $\hat{S}(\omega)$. Thus, it seems reasonable to remove the "expectation value brackets" from (23) to give

$$\hat{S}(\omega) = \sum_{k=1}^{K} |A_k|^2 \delta(\omega-\omega_k) \quad . \tag{56}$$

Equation (56) has the same form as (2), that is, a sum of delta functions. We now want the expectation value of the square of (56) just as we earlier wanted the expectation value of the square of (2). To see that more clearly, we can consider that (53) is a correct equation, but simply that the resulting expression (55) is not usable. Thus, from (53) and (23), we have

$$<|S(\omega)|^2> = \frac{T}{2\pi} \sum_{k=1}^{K} <|A_k|^2> \delta(\omega-\omega_k) \tag{57}$$

which we can interpret as the expectation value of the square of (2).

The analogous formula for (56) is

$$<\hat{S}(\omega)^2> = \frac{T}{2\pi} \sum_{k=1}^{K} <|A_k|^4> \delta(\omega-\omega_k) \; . \tag{58}$$

Now we have enough information to get the proportionality factor between (51) and (52). Substituting (58) and (48) into (51) gives

$$\sigma^2(\hat{S}(\omega)) = \frac{T}{2\pi} \sum_{k=1}^{K} (<|A_k|^4> - <|A_k|^2>^2) \; \delta(\omega-\omega_k) \; . \tag{59}$$

Comparing (59) with (52) gives

$$\sigma^2(\hat{S}(\omega)) = \frac{T}{2\pi} \sum_{k=1}^{K} \sigma^2(|A_k|^2) \; \delta(\omega-\omega_k) \; . \tag{60}$$

Similarly,

$$\sigma^2(\hat{P}(\omega)) = \frac{T}{2\pi} \sum_{i=1}^{I} \sigma^2(|B_i|^2) \; \delta(\omega-\omega_i) \; . \tag{61}$$

From (16), we have

$$\hat{W}(\omega)^2 = \left(\frac{2\pi}{T}\right)^2 |W(\omega)|^4 \; . \tag{62}$$

The double sums in (37) are of the same form as that in (1). Thus, we can convert that double sum to a double convolution in the same way we did in (28). Thus, using (48), (49), (60), (61), and (62) in (37), we get

$$\sigma^2(|R(\omega)|^2) = \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * \sigma^2(\hat{P}(\omega)) +$$

$$+ \hat{W}^2(\omega) * <\hat{P}(\omega)>^2 * \sigma^2(\hat{S}(\omega)) +$$

$$+ \hat{W}^2(\omega) * \sigma^2(\hat{S}(\omega)) * \sigma^2(\hat{P}(\omega)) +$$

$$+ <|R(\omega)|^2>^2 - \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 \ . \tag{63}$$

We also have

$$\sigma^2(|R(\omega)|^2) = <|R(\omega)|^4> - <|R(\omega)|^2>^2 \tag{64}$$

and

$$\sigma^2(\hat{R}(\omega)) = <(\hat{R}(\omega))^2> - <\hat{R}(\omega)>^2 \ . \tag{65}$$

Equation (15) gives

$$<\hat{R}(\omega)> = \frac{2\pi}{T} <|R(\omega)|^2> \ . \tag{66}$$

The square of (15) gives

$$<(\hat{R}(\omega))^2> = \left(\frac{2\pi}{T}\right)^2 <|R(\omega)|^4> \ . \tag{67}$$

From (64) through (67), we get

$$\sigma^2(\hat{R}(\omega)) = \left(\frac{2\pi}{T}\right)^2 \sigma^2(|R(\omega)|^2) \ . \tag{68}$$

Using (68) and (66) in (63) gives

$$\sigma^2(\hat{R}(\omega)) = (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * \sigma^2(\hat{P}(\omega)) +$$

$$+ (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{P}(\omega)>^2 * \sigma^2(\hat{S}(\omega)) +$$

$$+ (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * \sigma^2(\hat{S}(\omega)) * \sigma^2(\hat{P}(\omega)) +$$

$$+ <\hat{R}(\omega)>^2 - (\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 . \quad (69)$$

Equation (69) is the approximate equivalent of equation (27) of Georges and Jones (1980). This report does not constitute a derivation of its validity in general, but only for the form of $S(\omega)$ and $P(\omega)$ given in (2) and (3). The first three terms in (69) arise from the diagonal terms in the square of (3.1). The last two terms arise from the cross terms in the square of (3.1).

6. COMPARISON WITH CALCULATIONS BASED ON A CONTINUOUS REPRESENTATION OF THE SPECTRA

Jones (1981) calculates the mean and variance of skywave radar sea-echo power spectra starting from a continuous representation of all of the spectra involved. His results are more general than those presented here in terms of the spectral characteristics, but less general in terms of the statistical properties of the spectra. It is obviously a useful check of consistency to compare the results for those special cases where the two methods should agree.

For the mean of the power spectrum, Jones (1981) gets the same result as in (29) here. To compare variance, we must consider special cases.

Jones' (1981) results apply only when the spectra for $\hat{S}(\omega)$ and $\hat{P}(\omega)$ have normalized unit variance, that is, when

$$\sigma^2(\hat{S}(\omega)) = <\hat{S}(\omega)>^2 \quad (70)$$

and

$$\sigma^2(\hat{P}(\omega)) = <\hat{P}(\omega)>^2 . \quad (71)$$

Substituting (70) and (71) into (69) gives

$$\sigma^2(\hat{R}(\omega)) = <\hat{R}(\omega)>^2 + 2(\frac{2\pi}{T})^2 \hat{W}^2(\omega) * <\hat{S}(\omega)>^2 * <\hat{P}(\omega)>^2 . \qquad (72)$$

We must compare (72) with the variance calculated by Jones (1981).

For a Gaussian-shaped window

$$W(\omega) = \frac{\exp(-\frac{\omega^2}{2a^2})}{a\sqrt{2\pi}} \qquad (73)$$

that has the power spectrum

$$\hat{W}(\omega) \equiv \frac{2\pi}{T} |W(\omega)|^2 = \frac{\exp(-\omega^2/a^2)}{T a^2} , \qquad (74)$$

Jones (1981) gets

$$\sigma^2(\hat{R}(\omega)) = <\hat{R}(\omega)>^2 + 4\pi a^2 \hat{W}^2(\omega) * <\tilde{S}_w(\omega)>^2 * <\tilde{P}_w(\omega)>^2 , \qquad (75)$$

where

$$<\tilde{S}_w(\omega)>^2 \equiv \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \exp(-2\frac{\delta^2}{a^2}) <\hat{S}(\omega-\delta)><\hat{S}(\omega+\delta)> d\delta \qquad (76)$$

and

$$<\tilde{P}_w(\omega)>^2 \equiv \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \exp(-2\frac{\Delta^2}{a^2}) <\hat{P}(\omega-\Delta)><\hat{P}(\omega+\Delta)> d\Delta . \qquad (77)$$

The Blackman-Harris window (Harris, 1978) that we use has a shape near the peak that is very similar to a Gaussian. Thus, the case of a Gaussian-shaped window has practical applications.

If we substitute (23) into (76), we get

$$<\tilde{S}_w(\omega)>^2 = \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \exp(-2\delta^2/a^2) \sum_{k=1}^{K} \sum_{k'=1}^{K} <|A_k|^2> \delta(\omega-\delta-\omega_k)$$

$$<|A_{k'}|^2> \delta(\omega+\delta-\omega_{k'}) \, d\delta . \qquad (78)$$

The product of the two delta functions in (78) is equivalent to

$$\delta(\omega-\delta-\omega_k)\ \delta(2\delta+\omega_k-\omega_{k'}) \tag{79}$$

which in turn is equivalent to

$$\delta(\omega - \frac{\omega_k}{2} - \frac{\omega_{k'}}{2})\ \delta(2\delta+ \omega_k-\omega_{k'}) \ . \tag{80}$$

The parameter a in (73) is the spectral width of the window $W(\omega)$. That the spectral lines in $S(\omega)$ are separated by more than the spectral width of $W(\omega)$ means that

$$|\omega_k - \omega_{k'}| >> a \text{ for } k \neq k' \ . \tag{81}$$

Because the value of a delta function is zero unless the argument is zero, (81) requires that the second delta function in (80) be zero unless

$$2|\delta| >> a \text{ for } k \neq k' \ . \tag{82}$$

However, the exponential in (78) will be small under the conditions of (82), and we may neglect the contribution to the integral under those conditions. Thus, the terms in the double sum in (78) for which $k \neq k'$ are negligible. Thus, we may replace (78) by

$$<\tilde{S}_w(\omega)>^2 = \frac{1}{a}\sqrt{\frac{2}{\pi}} \sum_{k=1}^{K} <|A_k|^2>^2 \delta(\omega-\omega_k) \int_{-\infty}^{\infty} \exp(-2\delta^2/a^2)\ \delta(2\delta)\ d\delta \ . \tag{83}$$

Integrating (83) gives

$$<\tilde{S}_w(\omega)>^2 = \frac{1}{a\sqrt{2\pi}} \sum_{k=1}^{K} <|A_k|^2>^2 \delta(\omega-\omega_k) \ . \tag{84}$$

Substituting (48) into (84) gives

$$<\tilde{S}_w(\omega)>^2 = \frac{1}{a\sqrt{2\pi}} \frac{2\pi}{T} <\hat{S}(\omega)>^2 \ . \tag{85}$$

Similar considerations give

$$\langle \tilde{P}_w(\omega)\rangle^2 = \frac{1}{a\sqrt{2\pi}} \; \frac{2\pi}{T} \; \langle \hat{P}(\omega)\rangle^2 \; . \tag{86}$$

Substituting (85) and (86) into (75) gives

$$\sigma^2(\hat{R}(\omega)) = \langle \hat{R}(\omega)\rangle^2 + 2(\frac{2\pi}{T})^2 \; \hat{W}^2(\omega) \; * \; \langle \hat{S}(\omega)\rangle^2 \; * \; \langle \hat{P}(\omega)\rangle^2 \; , \tag{87}$$

in agreement with (72). Thus, the discrete representation and the continuous representation give the same results for variance under the same physical assumptions (that is, for effectively discrete spectra).

However, Jones (1981) shows that when the spectral widths of $\hat{S}(\omega)$ and $\hat{P}(\omega)$ are larger (rather than smaller) than the spectral width of $\hat{W}(\omega)$ (as our measurements show them to be (Jones et al., 1981)), (76) and (77) reduce to

$$\langle \tilde{S}_w(\omega)\rangle^2 = \langle \hat{S}(\omega)\rangle^2 \tag{88}$$

and

$$\langle \tilde{P}_w(\omega)\rangle^2 = \langle \hat{P}(\omega)\rangle^2 \; . \tag{89}$$

Substituting (88) and (89) into (75) gives

$$\sigma^2(\hat{R}(\omega)) = \langle \hat{R}(\omega)\rangle^2 + 4\pi a^2 \; \hat{W}^2(\omega) \; * \; \langle \hat{S}(\omega)\rangle^2 \; * \; \langle \hat{P}(\omega)\rangle^2 \; . \tag{90}$$

Our measurements (Jones et al., 1981) show that the variance in (90) based on broad spectra for $S(\omega)$ and $P(\omega)$ is probably more realistic than the variance in (72) or (87) based on discrete spectra.

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APPENDIX A. DEFINITIONS

$s(t)$ is the time series for backscatter from the sea.

$S(\omega)$ is the Fourier transform of $s(t)$.

$\hat{S}(\omega) = \frac{2\pi}{T} |S(\omega)|^2$ is the power spectrum corresponding to $S(\omega)$, where

T is the length of the measured time series $s(t)$.

$<\hat{S}(\omega)>$ is the infinite ensemble average of $\hat{S}(\omega)$. The brackets < > denote an ideal infinite ensemble average over the same "sea state" and the same "ionospheric state." We assume that the realizations within such an ensemble belong to the same probability distribution.

$w(t)$ is a window that we apply to the measured time series before performing a Fourier transform.

$W(\omega)$ is the Fourier transform of $w(t)$. It is complex because $w(t)$ is not symmetric.

$\hat{W}(\omega) = \frac{2\pi}{T} |W(\omega)|^2$ is the power spectrum corresponding to $W(\omega)$.

$p(t)$ is the time series corresponding to two-way ionospheric transmission between the radar and some fixed spot (or patch) on the ground or ocean.

$P(\omega)$ is the Fourier transform of $p(t)$. It includes Doppler spreading from ionospheric multipath. It does not include Doppler spreading from movement of the scattering point on the ground or ocean. If the ionosphere were a perfect reflector, P would be a delta function.

$\hat{P}(\omega) = \frac{2\pi}{T} |P(\omega)|^2$ is the power spectrum corresponding to $P(\omega)$.

$<\hat{P}(\omega)>$ is the infinite ensemble average of $\hat{P}(\omega)$.

$r(t) = w(t)s(t)p(t)$ is the measured skywave sea-echo.

$R(\omega)^{\dagger} = W(\omega)*S(\omega)*P(\omega)$ is the Fourier transform of $r(t)$.

* denotes convolution.

$\hat{R}(\omega) = \frac{2\pi}{T} |R(\omega)|^2$ is the power spectrum corresponding to $R(\omega)$.

$<\hat{R}(\omega)>$ is the infinite ensemble average of $\hat{R}(\omega)$.

$\sigma^2(\hat{R}(\omega))$ is the variance of $\hat{R}(\omega)$.

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† Our variable $R(\omega)$ was called $R''(\omega)$ by Georges and Jones (1980).

APPENDIX B. ASSUMPTIONS

1. The convolution model (Georges and Jones, 1980)

$$R(\omega) = W(\omega) * S(\omega) * P(\omega) \ . \tag{B.1}$$

2. $S(\omega)$ has zero mean.

$$\langle S(\omega) \rangle = 0 \ . \tag{B.2}$$

This is supported by theory (Weber and Barrick, 1977; Barrick and Weber, 1977) and measurement (Barrick and Snider, 1977).

3. $S(\omega)$ is uncorrelated in frequency. That is

$$\langle S(\omega_1)\ S^*(\omega_2) \rangle = \langle \hat{S}(\omega_1) \rangle\ \delta(\omega_1 - \omega_2) = \langle \hat{S}(\omega_2) \rangle\ \delta(\omega_1 - \omega_2) \tag{B.3}$$

also

$$\langle S(\omega_1)\ S(\omega_2) \rangle = \langle S^*(\omega_1)\ S^*(\omega_2) \rangle = 0 \ .$$

This is only an approximation. It is the continuous frequency representation of a corresponding expression

$$\begin{aligned} \langle A_k\ A^*_{k'} \rangle &= \langle |A_k|^2 \rangle\ \delta_{k\ k'} \\ \langle A_k\ A_{k'} \rangle &= \langle A^*_k\ S^*_{k'} \rangle = 0 \end{aligned} \tag{B.4}$$

involving Kronecker deltas used by Weber and Barrick (1977). While (B.3) indicates that adjacent frequencies are uncorrelated no matter how close they are (in seeming contradiction to what we measure), convolution with a window gives a correlation frequency of about one or two FFT frequency bins, in agreement with experiment. (We never measure $S(\omega)$; even in the absence of ionospheric distortion, we measure $W(\omega)*S(\omega)$.)

4. A continuous function of frequency is a more realistic representation for $P(\omega)$ than is discrete multipath. Our measurements in October, 1980 (Jones et al., 1981) show this. Although the discrete multipath representation of Georges and Jones (1980) is general enough to represent continuous Doppler spreading, a continuous representation is more convenient to get accurate results.

5. $P(\omega)$ has zero mean. That is

$$<P(\omega)> = 0 \ . \tag{B.5}$$

We assume that analysis of the October 1980 measurements will confirm this.

6. $P(\omega)$ is uncorrelated in frequency. That is

$$<P(\omega_1)\ P^*(\omega_2)> = <\hat{P}(\omega_1)>\ \delta(\omega_1 - \omega_2) = <\hat{P}(\omega_2)>\ \delta(\omega_1 - \omega_2) \tag{B.6}$$

also

$$<P(\omega_1)\ P(\omega_2)> = <P^*(\omega_1)\ P^*(\omega_2)> = 0 \ .$$

Similar comments apply here as they did to (B.3). We hope that analysis of our October 1980 measurements will confirm that the cross correlation of $P(\omega)*W(\omega)$ is consistent with (B.6). The expressions corresponding to (B.6) for the discrete representation are

$$\begin{aligned} &<B_i\ B^*_{i'}> = <|B_i|^2>\ \delta_{i\,i'} \\ &<B_i\ B_{i'}> = <B^*_i\ B^*_{i'}> = 0 \ . \end{aligned} \tag{B.7}$$

7. $S(\omega)$ and $P(\omega)$ are uncorrelated. That is

$$<f(S(\omega))\ g(P(\omega))> = <f(S(\omega))><g(P(\omega))> \tag{B.8}$$

where f and g are arbitrary functions. The following special case,

$$<S(\omega)\ P(\omega)> = <S(\omega)\ P^*(\omega)> = <S^*(\omega)\ P(\omega)> = 0, \tag{B.9}$$

follows from (B.2) and (B.5). This is an assumption. We can see no reason why they should be correlated.

8. We assume that only $\hat{R}(\omega)$ has useful information, and that the phase of $R(\omega)$ is randomly distributed and thus has no useful information. This is an assumption. We do not know whether our measurements are consistent with this assumption.

9. The spectra $\langle\hat{S}(\omega)\rangle$ and $\langle\hat{P}(\omega)\rangle$ consist of discrete spectral lines whose width is small compared with the spectral width of the window $W(\omega)$, and whose separation is large compared with the spectral width of the window $W(\omega)$.