Northwest and
Alaska
Fisheries Center

National Marine
Fisheries Service

U.S. DEPARTMENT OF COMMERCE

NWAFC PROCESSED REPORT 83-12

Review and Comparison of Three Methods of Cohort Analysis

July 1983

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by

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July 1983

1.0 INTRODUCTION

Cohort analysis is a descriptive name given to a general class of analytical techniques used by fisheries managers to estimate fishing mortality and population numbers given catch-at-age data. Several methods are available, however each has its own strengths and weaknesses and different methods contain different sources of error. Moreover, application of more than one method to a common data set may give conflicting results. It is not clear which method is best to use under a given set of circumstances since few comparative studies have been carried out. Consequently, confusion exists among scientists as to which method to use and how to interpret the results.

This report describes the reasons why cohort analysis plays an important role in fisheries management, describes the mathematical models in a consistent notation, and compares current methods paying particular attention to solution methods, underlying assumptions, strengths, weaknesses, similarities and differences.

1.1 Etymology

Derzhavin (1922) was perhaps the first to conceive of the idea of applying observed data describing the age structure of a population to catch records in order to calculate the contribution of each cohort to each years total catch. He determined, for each individual cohort the minimum number of fish alive in a reference year by summing the catches removed in future years from the cohorts alive in a given year. Derzhavin did not name his method although Ricker (1971) has called it Derzhavin's Biostatistical Method.

Fry (1949) applied Derzhavin's method to a lake trout fishery and called his method of cohort analysis "virtual population analysis" (VPA). Fry's choice of the title was based on the analogy with the virtual image of the physicist - "... although it is not the real population it is the only one that is seen." (Fry 1957). In this country the name virtual population analysis has commonly been applied to Derzhavin's method, apparently because Gulland (1965) showed that his formulation could be based on a table of virtual populations in the sense of Fry (1949) (i.e. the sum of the fish present in the population that would ultimately appear in the catch). The technique of cohort analysis as currently used in the assessment of exploited fisheries is based on the two equations of the Beverton and Holt (1957) model, thus does not involve virtual populations at all. Nonetheless, VPA has come to be accepted as the traditional name for the method. Pope (1972) introduced an approximation to VPA and unfortunately called it "cohort analysis". Since then the two names have come to be used more or less interchangeably in the scientific literature. Pope is particularly fond of generating new names for modifications of the basic analysis (Seperable VPA, Pope and Shepherd 1982; Legion analysis, Pope and Woolner 1981; Modified cohort analysis, Pope 1979).

I believe considerable confusion exists in this regard. Ambiguity in the naming convention is best exemplified by examining how the two names are used in the scientific literature. Often both names are used at the same time, apparently to avoid confusion (Ulltang 1972, Sims 1982). Other times one name is preferred over the other (Aldenberg 1975, Hoag 1978), or the names are intentionally avoided altogether (Ricker 1971). Clearly this problem needs resolved.

In subsequent discussions the following conventions are employed. Cohort analysis is the name given to the general class of stock assessment techniques that estimate population and fishing mortality from catch-at-age data. Specific methods within the general class will be referenced according to the purveyor of the method, thus VPA is "Fry's method" (I will adhere to the established naming convention used in this country), Gulland's iterative correction to Fry's method is "Gulland's method", and Pope's approximation to Gulland's method is "Pope's method". Subscripted prefixes refer to ages and subscripted suffixes refer to years. Omission of a prefix or suffix indicates the parameter is constant over the missing subscript.

1.2 Classification of Cohort Analysis Methods

Cohort analysis techniques can be broadly classified into two categories; deterministic vs. stochastic and those that use effort data vs. those that do not. Stochastic techniques recognize that some or all of the variables and/or parameters are subject to errors of one kind or another and attempt to rationally allocate those errors. Deterministic techniques assume all of the parameters and/or variables are measured without error. A comparison of selected methods according to this classification scheme is given in Table 1.

Table 1. Classification of common cohort analysis techniques.

	Deterministic	Stochastic
Do not use effort	Fry (1949)/Gulland (1965); Pope (1972); Pope and Shepherd (1982)	Doubleday (1976); Fournier and Archibald (1982) (may be modified to include effort)
Use effort	Beverton and Holt (1957)	Gray (1977) Paloheimo (1961,1980)

2.0 THE NEED FOR COHORT ANALYSIS

Early fisheries management relied primarily on theoretical advances made in the 1940's and 1950's by Ricker and Beverton and Holt. Management regulations developed from these advances in fishery science were based on the assumptions that catch per unit of effort (CPUE) could be used as an index of relative abundance in the assessment of total mortality. Specific regulations such as fishing net mesh size regulations worked well during the era when fishing fleets were relatively unchanging with respect to their design, fishing patterns and efficiency. Rapid changes in fishing technology and increases in fishing effort in the late 1960's and 1970's increased variability in the fisheries and caused a decline in the production of numerous stocks. In multispecies fisheries the inability to separate directed effort from total effort introduced ambiguities into the estimates of effort which made it difficult to maintain time series of consistent CPUE estimates. Interpretation of CPUE data became further complicated by the changing nature of fisheries and variability in the availability of the target species. As a result of these problems, concern began to be expressed as to the effectiveness of the regulatory mechanisms mentioned earlier. At the same time the need developed to describe stock numbers in absolute numbers rather than by a relative index which had variable calibration between stock areas. Furthermore, in fisheries where partially recruited age classes contributed a significant part to the overall catch, estimates of fishing mortality on these groups critically needed to be included in management regulations.

These problems contributed directly to the development of theory to estimate fishing mortality from catch and age data without reliance on CPUE.

3.0 COHORT ANALYSIS - FRY'S METHOD

3.1 The Model

Cohort analysis relies on two equations commonly encountered in fishery population dynamics; the catch equation of Baranov (1918) which expresses catch rate in numbers instead of weight

$$aCy = aNy \frac{aFy}{aZy} (1 - \exp(-aZy)) \qquad [1]$$

where
- y = year (y=1,...,Y)
- a = age of a cohort (a=1,...,A)
- aCy = catch in numbers of age a in year y
- aNy = numbers of age a animals in the beginning of year y
- aFy = fishing mortality on age a in year y
- aZy = total mortality on age a in year y (aZy = M + aFy)
- M = natural mortality

and the exponential survival model

$$a{+}1Ny{+}1 = aNy \exp(-aZy). \qquad [2]$$

Equations [1] and [2] can be combined together to give

$$\frac{a{+}1Ny{+}1}{aCy} = \frac{(aFy + M) \exp(-(aFy + M))}{aFy\ (1 - \exp(-(aFy + M)))} \qquad [3]$$

3.2 Parameter Estimation Procedure

Parameters are estimated separately for each year. If aCy, aFy, and

M are known, then [1] and [2] can be manipulated together (iterated) in a backwards or hindcast mode to yield estimates of $_aN_y$ and $_aF_y$ for all past years of life of the cohort in the following manner. Using the best estimate of terminal fishing mortality $_aF_y$ (where a=A) and observed catch $_aC_y$ (a=A), [1] is used to solve for $_aN_y$ (a=A), then [3] is used to solve for $_{a-1}F_{y-1}$ (a=A), finally [2] can be used to estimate $_{a-1}N_{y-1}$ (a=A), and so on until the youngest cohort is done. The mechanics of sequential computation of these two equations was described by Ricker (1948) and the method was popularized by Murphy (1965) and Gulland (1965). Instead of iteratively solving the two simultaneous equations [1] and [2] Pope (1972) simplified the procedure somewhat by introducing a discrete approximation to the continuous exponential survival model [2]

$$\exp(M/2) = \frac{(_aF_y + M)\ (1 - \exp(-_aF_y)}{_aF_y\ (1 - \exp(-_aZ_y))} \qquad [4]$$

This assumes all fish are caught midway through the year. Note that in Pope's method the estimate of $_aN_y$

$$_aN_y = {}_{a+1}N_{y+1} \exp(M) + {}_aC_y \exp(M/2) \qquad [5]$$

is obtained first and then $_aF_y$ is obtained directly from

$$_aF_y = \ln\left(\frac{_aN_y}{_{a+1}N_{y+1}}\right) - M \qquad [6]$$

just the reverse order of Fry's method.

3.3 Assumptions

The asssumption underlying cohort analysis are generally those of catch equation and the exponential survival model. These are

(1) All removals from the population are accounted for in the catch except for losses due to natural mortality;

(2) All fish in the stock become available to the fishery at some time in their life;

(3) Catches are aged without error;

(4) Natural mortality is constant over age and year; and

(5) A relatively large part of the total removals are due to fishing. This implies that in an intensive fishery the numbers caught will represent a substantial portion of the total loss. Thus information

on numbers caught provide useful information about total removals.

3.4 Advantages

Advantages of cohort analysis are

(1) The method is extremely easy to carry out;

(2) The method is independent of errors associated with measures of CPUE;

(3) No assumptions are required regarding catchability or vulnerability;

(4) Estimates of F can be used to test more effectively the proportionality of F to effort (i.e. F=qf) and the validity of CPUE data (Garrod 1976, Hyman et al. 1980);

(5) It is very valuable in understanding a fishery in a historic sense, for explaining its population dynamics, and is potentially of great value in showing up large, and possible detrimental, changes in fishing mortality soon after they have happened;

(6) Results are insensitive to errors in the estimated or assumed value of terminal F (Jones 1961). This is especially true when the ratio F/Z is in the range 0.5 - 1.0 (i.e. fishing accounts for about 50% or more of the total deaths (Jones 1981)) or cumulative Z over the life of a cohort is greater than Z (Pope 1972);

(7) Errors in the estimates of aNy and aFy caused by random fluctuations in M (when M assumed constant) are likely to be small when M fluctuates moderately (Ulltang 1977, Pope 1979), although this would tend to be more severe on older animals since they occur in relatively smaller numbers. Agger et al. (1973) estimated that the bias in F would be 25% if M is known with a mean error of 0.1.

(8) Results of cohort analysis are relatively insensitive to seasonal trends in M and F (Ulltang 1977); and

(9) Effects of unevenly distributed catches (i.e. the intra-year frequency distribution of catches is not constant) on the relative error in estimates of aNy are not severe unless M is large and/or F is high (Sims 1982).

3.5 Disadvantages

The disadvantages of cohort analysis are

(1) The method is not stochastic, that is, it does not consider the form of the observational errors which gave rise to the observed catch data. Since the number of parameters equals the number of data points, there is no measure of the variability about the parameter estimates nor a measure of the amount of variation in the data explained by the model;

(2) The method does not do a very good job at predicting the current

situation of the fishery since the estimate of the current population is only as good as the current estimate or guess of the terminal fishing mortality. Older cohorts are highly sensitive to errors in estimates of this parameter. The problem is less serious in fisheries that have a very high and constant exploitation rate because the rate of convergence with age is a function of the mortality rate;

(3) The assumption of constant natural mortality is extremely strong. Several factors such as disease or predation which most likely vary with age and year contribute to M;

(4) Relative strength of strong and weak cohorts will be biased if M varies with cohort strength (Ulltang 1977);

(5) If trends exist in natural mortality (say decreasing with age) bias in parameter estimates results since actual increases in natural mortality (when M was erroneously assumed to be constant) would show up as increasing fishing mortality (Ulltang 1977);

(6) Aging errors are not considered.

4.0 COHORT ANALYSIS - DOUBLEDAY'S METHOD

The model of Doubleday (1976) is based on equations [1] and [2] which are nonlinear in F.

4.1 The Model

In Doubleday's model instantaneous fishing mortality is a product of two terms, availability which changes only with age and effective effort (fishing intensity) which changes only with year. The model consists of two equations. The first is derived by taking logarithms of [1] and substituting exp(aV + Ey) for aFy

$$\ln(aCy) = \ln(rNy{-}a{+}r) - (a-r)M + (aV + Ey) - \sum_{i=r}^{a-1} [\exp(iV + Ey{-}a{+}i)] -$$

$$\ln(\exp(aV + Ey) + M) + \ln(1 - \exp(-\exp(aV + Ey) - M)) + aWy \qquad [7]$$

where aV - log (base e) of availability at age a,
Ey - log (base e) of effective effort multiplier (fishing intensity) in year y,
aFy - fishing mortality at age a in year y (aFy = exp(aV + Ey)),
aWy - sampling error in observing ln(aCy). Assumed to be indentically and independently distributed with zero mean and constant variance for all ages and years,
r - age when year class aCy enter table of catches, and other parameters are as explained before.

Equation [7] is nearly linear in the range 0.01 <= F <= 2.72. A second equation is used to provide starting values for the iterative procedure. He uses a model of logarithms of catch ratios

$$\ln\left(\frac{{}_aC_y}{{}_{a+1}C_{y+1}}\right) =$$

$$\ln({}_aV + E_y) - \ln(\exp({}_aV + E_y) + M) + \ln(1 - \exp(-\exp({}_aV + E_y) - M)) +$$

$$(\exp({}_aV + E_y) + M) - ({}_{a+1}V + E_{y+1}) + \ln(\exp({}_{a+1}V + E_{y+1}) + M) -$$

$$\ln(1 - \exp(-\exp(V_{a+1} + E_{y+1}) - M)) + {}_aW_y \qquad [8]$$

4.2 Parameter Estimation Procedure

Doubleday uses a process of iterative linear approximation and estimation (linearization) to obtain least squares estimates of the parameters. This is a method belonging to the general class of gradient methods, that is the algorithm uses measurements of the slope of the function to be minimized as an indication of the direction towards the minimum. The most general gradient approach is Newton's method which is based on the concept of taking a nonlinear function of several parameters and expanding the function in a Taylor series and keeping only the second order terms. The objective function only contains the unknown parameters, which depend explicitly on the model equations, which in turn depend on the parameters. Therefore, to compute derivatives of the objective function, derivatives with respect to the model equations must be determined, and then these must be differentiated with respect to the model parameters. When the number of equations is greater than one and the number of parameters per equation is greater than one, the system can be expressed in matrix notation as follows

$$f(\underline{X}) = f(\underline{X}') + \bar{J}\,(\underline{X} - \underline{X}') + 1/2\ \bar{H}\,(\underline{X} - \underline{X}')^2$$

where $\underline{X}$ - a vector of parameter values $(x1, x2, \ldots, xn)$

$\underline{X}'$ - an estimate of the parameter vector

$\bar{J}$ - the Jacobian matrix; a matrix of first partial derivatives with respect to the model equations and the parameter vector evaluated at $\underline{X}'$

$\bar{H}$ - the Hessian matrix; a matrix of second partial derivatives with respect to the model equations and the parameter vector evaluated at $\underline{X}'$

A problem with the Newton method is that evaluation of the partial derivatives can be complicated in complex objective functions and the method does not work when the Hessian matrix is not positive definite.

Several alternative methods have been devised to overcome these difficulties. Three are suggested by Draper and Smith (1966). These are linearization (sometimes called Gauss's method), steepest descent, and Marquardt's method. The linearization method described by Bard (1974; p 96) is similar to Newton's method, yet has the advantage that the second derivatives of the model equations are eliminated when the Hessian matrix is being computed.

The linearization method proceeds as follows. First a Taylor series expansion of the nonlinear equations is carried out and only the first partial derivatives are retained. The original nonlinear equations are now represented by two terms, a vector of mean responses (the original nonlinear equation evaluated at the parameter estimates) and a Jacobian matrix of linear approximations. The Jacobian matrix is linear with respect to the difference between the current parameter vector and the predicted parameter vector. An initial guess of the parameter vector is obtained by fixing the effort parameter for the last year and the availability for the oldest cohort, and then applying the linearization procedure to [8]. In this way convergence is rapid and arbitrary starting values can be used. Next the Taylor series expansion of [7] is calculated, and the Jacobian matrix is used to estimate a revised estimate of the parameter vector by applying linear least squares theory. The revised parameter vector, which is averaged with the previous parameter vector, is inserted into the Taylor expansion and this iterative process is continued until the solution converges. The variances/covariance matrix is available since the Jacobian is used to approximate the Hessian from the relation $H = 2 J'J$, where H is the Hessian matrix, J is the Jacobian matrix, and J' is the transpose of the Jacobian. The approximation is considered adequate when the residuals are small and the parameter vector is in the vicinity of the minimum.

4.3 Objective Function

The objective function is the sum of squared differences of the observed data minus the two terms of the Taylor series expansion mentioned above.

4.4 Assumptions

(1) A more restrictive assumption regarding fishing mortality proposes that fishing mortality can be expressed as the product of availability and effective effort;

(2) Availability is constant within age;

(3) Effective effort is constant within year;

(4) Natural mortality is known and independent of year and age effects; and

(5) Random errors in $\ln({}_aC_y)$ are distributed with zero mean and constant variance and are independent of observed catches ${}_aC_y$.

(6) (see section 3.3).

4.5 Advantages

(1) The log transform makes the equation more nearly linear and has the added benefit of removing heteroscedasticity in the error variance of the catch;

(2) The arbitrary choice of the terminal fishing mortality is removed;

(3) A measure of the variation explained by the model is available;

(4) Representing aFy as a product of availability and effective effort results in a substantial reduction in the number of parameters that need to be estimated;

(5) Variance estimates of the parameters are available so a determination as to their reliability can be made. This is especially true of the variance of the stock size estimates since it will show the amount of information contained in the catch data about the stock size;

(6) Because ultimately standard linear regression procedures are used, the residuals and many other diagnostic methods are available to evaluate the assumptions of the model. These include determining: systematic departures from the model (i.e. is there regularity left in the residuals); isolated departures from the model (i.e. do some points fit the model while others (outliers) don't); normality of errors; and nonconstant variance.

(7) The variance/covariance matrix is available to examine correlation between the independent and dependent variables. When present this tends to underestimate the error structure and overestimate the parameter variance; and

(8) The method has the correct stochastic orientation (Fournier and Archibald 1982), that is, the model addresses the fact that information submitted to the model and the underlying processes which the model attempts to describe (an exploited fishery) are subject to error.

4.6 Disadvantages

The disadvantages stem mainly from the linearization fitting procedure.

(1) The method does not monotonically converge, that is it has the potential to oscillate widely or reverse direction causing increases and decreases in the residual sum of squares;

(2) The method may not converge at all, so that the residual sum of squares increases iteration after iteration without bound;

(3) There is no guarantee of a global minimum and different starting values may produce different solutions;

(4) Long series of well sampled catches are required;

(5) Even when catches are well explained (as measured by a low residual sum of squares or a high R-squared value), parameter estimates have large variances and wide confidence intervals;

(6) Availability at age will often change with time;

(7) Natural mortality is assumed independent of age and year effects; and

(8) The variance of the random variable, predicted catch, is assumed approximately independent of the actual magnitude of catch. It seems reasonable to conclude that the variance of predicted log catch for age a should go to infinity as the percentage of age a in the catch goes to zero. Clearly some relationship exists between these two variables.

5.0 COHORT ANALYSIS - POPE AND SHEPHERD'S METHOD

The model of Pope and Shepherd (1982) is based on equations [1] and [2] which are nonlinear in F.

5.1 The Model

In Pope and Shepherd's model instantaneous fishing mortality is a product of two terms, exploitation pattern which only changes with age and exploited fishing mortality which only changes with year. The model consists of the ratio of catches in succeeding years. Substituting the product ${}_aVE_y$ for ${}_aF_y$ and taking logarithms produces

$$\ln\left(\frac{{}_{a+1}C_{y+1}}{{}_{a}C_{y}}\right) =$$

$$\ln({}_{a+1}V) + \ln(E_{y+1}) + \ln({}_aVE_y + M) - {}_aVE_y - M +$$

$$\ln(1 - \exp(-({}_{a+1}VE_{y+1}) - M)) - \ln({}_aV) - \ln(E_y) -$$

$$\ln({}_{a+1}VE_{y+1} + M) - \ln(1 - \exp(-({}_aVE_y) - M)) \qquad [9]$$

where ${}_aV$ - exploitation pattern for age a,
E_y - fully exploited fishing mortality for year y,
and other parameters as explained before.

Equation [9] is augmented by one equation that assures that all ${}_aV \le$ 1.0.

5.2 Parameter Estimation Procedure

Pope and Shepherd use a two stage least squares algorithm as their parameter estimation procedure. The first stage estimates exploitation pattern (${}_aV$) and fishing mortality (E_y) and the second stage estimates population numbers at age. Each stage utilizes a separate objective function.

5.2.1 Stage One

5.2.1.1 Stage One Objective Function

The objective function is the sum of squared differences between the observed log catch ratio and the predicted log catch ratio (given by [9]).

5.2.1.2 Stage One Parameter Estimation Procedure

Stage one proceeds as follows. All values of aV and Ey are set to their initial value aV and Ey (for a=A and y=Y) respectively. Next equation [9] is calculated, the objective function evaluated and residuals aR and Ry calculated, where aR is the residual for age summed over all years and Ry is the residual for year summed over all ages. New parameter estimates are calculated by multiplying the old estimates by empirical weighting factors aQ and Qy, where aQ = exp(aR/Y) and Qy = exp(Ry/A). The term aR/Y can be considered an average (over Y years) residual for age a. Similarly, Ry/A can be considered the average (over A ages) residual for year y. Exploitation patterns is renormalized and the procedure repeated until the solution converges.

Weighting factors are determined by considering the change in the parameter required to eliminate the residual. The actual functional form is a result of three approximations and one very strong assumption. For example, when Ey is estimated the resulting residual Ry is a function of four variables, Ey, Ey+1, aV, and a+1V. To derive the functional form of the appropriate empirical weighting factor, the dependence of Ry on all other variables except Ey is ignored.

5.2.2 Stage Two

The least squares algorithm of stage one does not estimate population at age. However if the population numbers of the youngest age of each cohort in the catch-at age data matrix (1Ny and aN1) can be estimated, then any aNy can be estimated from the recurrence relationship [2].

5.2.2.1 Stage Two Objective Function

Two separate objective functions are used in the stage two parameter estimation procedure. The first is

$$\sum_{k=1}^{kmax} \left[\ln(kCy+k-1) - \ln(1Ny) \frac{kFy+k-1}{kZy+k-1} (1-\exp(-kZy+k-1))\right]^2 \qquad [10]$$

where k is the k th age and kmax is the oldest age of a cohort in the catch-at-age data matrix. The second is

$$\sum_{t=1}^{tmax} \left[\ln(a+k-1Ct) - \ln(aN1) \frac{a+k-1Ft}{a+k-1Zt} (1-\exp(-a+k-1Zt))\right]^2 \qquad [11]$$

where t is the t th year and tmax is the last year that the cohort is in the catch-at-age data matrix. Equations [10] and [11] have closed analytical solutions.

5.2.2.2 Stage Two Parameter Estimation Procedure

Estimates of the population numbers of the youngest age of each cohort (1Ny and aN1) are determined from the analytical solutions to the objective functions [10] and [11]. Estimates of aNy for all succeeding ages and years are obtained with the recurrence relationship [2].

5.3 Assumptions

See (1) through (4) and (6) in section 4.4.

5.4 Advantages

See (1) through (4) in section 4.5.

5.5 Disadvantages

(1) Variance estimates of the parameters aV and Ey are difficult because of the parameter estimation algorithm. Variance estimates of population at age are conditional upon estimates of aV and Ey; and

(2) The method does not do a good job of predicting the current situation. Parameter estimates converge towards the correct values on earlier ages as they do in VPA.

(3) See (6) in section 4.6

6.0 DISCUSSION

Discussion will concentrate on techniques that do not require effort data since estimates of effort can be unreliable (see 2.0). Specifically these will be the methods of Fry, Doubleday and Pope and Shepherd. In comparing these methods, particular attention will be placed on data requirements, what a priori parameters are required, what parameters are estimated and their variances, the mathematical models, assumptions about errors, observation-to-parameter ratio, and parameter estimation methods.

Before proceeding with the comparisons, perhaps it would be a good idea to construct a list of attributes one would find desirable in the "ideal cohort analysis technique". In this way, the characteristics of the cohort analysis techniques described below can be evaluated against the goal or ideal. The method should (1) provide parameter estimates that are unbiased, accompanied by variance estimates so that confidence intervals can be constructed, and uncorrelated with other parameter estimates; (2) provide some measure of how well the model under consideration explaines the observed data; (3) be able to assess the assumptions underlying the model; and (4) be easy to use. Finally the estimation procedure, when nonlinear, should be well behaved. This means it should converge quickly to the true unique solution (i.e. the global minimum on the residual sums of squares response surface).

6.1 Comparison of Methods

6.1.1 Data Requirements

Raw data for cohort analysis techniques consist of catch-at-age data. This information can be put into a matrix consisting of Y rows (years) and A columns (ages). Thus the catch-at-age matrix includes YA catch observations. Fry's method uses the matrix directly, Doubleday's method uses the natural log of the catch matrix, and Pope and Shepherd use the natural log of a catch ratio matrix. In Pope and Shepherd's method two catch observations ($_aC_y$ and $_{a+1}C_{y+1}$) are required to make one catch ratio observation, thus the data matrix consists of (Y-1) rows and (A-1) columns, and has (Y-1)(A-1) observations (see Table 2).

6.1.2 A priori/Initial Parameter Estimates

In Fry's method the number of a priori parameters required for each year is one terminal fishing mortality $_aF_y$ (a=A) and an estimate of the natural mortality. To analyze the entire catch-at-age matrix Y $_aF_y$'s (a=A) and one M are required. Doubleday's method analyzes the catch-at-age matrix all at once, so the number of initial parameter estimates are A $_aV$'s (availability), Y E_y's (effective effort), and one M for a total of A+Y+1. Pope and Shepherd's method analyzes the catch ratio matrix similar to Doubleday, yet it reduces the number of a priori parameters required from (Y-1) E_y's, (A-1) $_aV$'s, and one M (= Y+A-1) to only three, one E_y (y=Y), one $_aV$ (a=A), and one M (see Table 2).

6.1.3 Parameter Estimates

6.1.3.1 Number of Parameters Estimated

In Fry's method Y(A-1) $_aF_y$'s and Y $_aN_y$'s are estimated, thus Y(A-1) + Y (= YA) parameters are estimated. In Doubleday's method A $_aV$'s, Y E_y's, and Y+A-1 fundamental cohorts (the $_aN_y$'s) are the parameters estimated. In Pope Shepherd's method the parameters estimated are A-1 $_aV$'s, Y-1 E_y's, and (Y-1)+(A-1)-1 (= Y+A-3) fundamental cohorts (see Table 2).

In Doubleday and Pope and Shepherd's method, a more restrictive assumption about fishing mortality (i.e that it is a product of availability and effective effort), reduce the number of parameters that need to be estimated from YA+Y (in Fry's method) to 2(Y+A)-1 and 2(Y+A)-5 respectively (see Table 2). This results in a substantial increase in the observation-to-parameter ratio (see Table 3) and the ability to calculate a meaningful goodness-of-fit measure.

6.1.3.2 Bias

To obtain unbiased estimates of parameters from a given mathematical relationship, a knowledge of the variability and observation errors inherent in the data is essential. It is difficult to evaluate if parameter estimates from various methods are biased since the degree of bias will be dependent on the model, the data, and the degree to which the underlying assumptions are violated. Given adequate data (i.e. a long time series of catch-at-age data) it is safe to say that if parameters are assumed to be constant and free from measurement error when in fact they are random variables subject to measurement error, then resulting estimates will be biased. It would seem that methods based on assumptions of constant parameter values over year and/or age are more likely to produce biased estimates. Fry's method has received the most thorough treatment in this regard (see 3.4 and 3.5).

One final point regarding bias is that models fitted to logarithmically transformed variables are fitted to the geometric rather than the arithmetic mean and are biased towards low expected values (i.e. $\exp(E(\ln(X))) < E(X)$, where E( ) represents the expected value). In the Doubleday model the estimate of population at age is really an estimate of the natural log of the population. Since estimates of population size could range over an order of magnitude, this source of bias could be significant in some applications. This is especially true when computing variables such as total biomass which involve summming a series of age-specific exponential transformations. Corrections for this type of bias are described in Beauchamp and Olson (1973).

6.1.3.3 Variances

Interpretation of parameter estimates not accompanied by variances is extremely difficult. Without variances there is no way of knowing the reliability of the parameters. For example, if the confidence limits of a parameter are plus or minus 100% of the actual estimate, then I would accept the parameter estimates with a great deal of caution. I might be more willing to accept a parameter estimate known to be biased but accompanied by a reasonable variance estimate than an unbiased estimate with a coefficient of variation greater than one.

Fry's method (especially Pope's approximation) does allow computation of variances in N and F resulting from sampling errors in the catch, but this is primarily a result of the analytical nature of equations [1] and [2]. This is the basis for many of the analyses mentioned in sections 3.4 and 3.5. The resulting variances are highly sensitive to the estimates of terminal fishing mortality. Siddeek (1982) has recently identified an error in several variance formulas that appeared in Pope (1972). Doubleday's method allows variance estimates since the variance/covariance matrix is available from the least squares approximation to the Jacobian matrix in the estimation procedure. In Pope and Shepherd's method no variances can be calculated from the procedure directly. However, the ratio of catches will have a higher sampling variance than the catches themselves. Thus even if variance estimates were available, they would probably be larger when compared with variance estimates from Doubleday's method.

6.1.3.4 Correlation

Correlation causes variances of parameter estimates to be large since the variance expressions contain significant covariance terms (note: $corr(X,Y) = cov(X,Y)/sqrt(var(X)var(Y))$). Doubleday (1976) found that when he analyzed the log catch-at-age matrix fishing mortality and population estimates were negatively correlated (i.e. as F increased N decreased). The problem of correlation between parameters is especially pronounced in Pope and Shepherd's method. In this case the raw data matrix is a matrix of log catch ratios, thus successive catch ratios of the same year-class are correlated by year and age in addition to the correlation mentioned earlier. In Doubleday and Pope and Shepherd's methods all population numbers at age (aNy) can be expressed in terms of the fundamental cohorts (i.e. those cohorts that occupy the first row and first column of the catch-at-age matrix). Also as the number of parameters in a model increases so does the probability of spurious correlations. For example, if there are 5 ages and 10 years of data, Doubleday's method would estimate 29 parameters. If the probability is 5% that two variables are correlated due to chance alone, then we could expect one random spurious correlation to occur from this data set.

6.1.4. Goodness-of-fit Measures

When the number of parameters are fewer than the number of observations a useful measure of the goodness-of-fit can be calculated. This measure is available in the methods of Doubleday and Pope and Shepherd and can be used to describe the amount of the variation in the data explained by the model. In Fry's method each parameter estimate is supported one observation so the observed catches are predicted exactly. The goodness- of-fit measure could be calculated for Fry's method, but it would be equal to 1.0 (i.e. 100% of the variation in the data is explained by the model) and not very meaningful.

6.1.5 Mathematical Models

The mathematical model for all three methods being compared are based on the catch equation and exponential survival model given in section 3.1.

The major difference between Fry's method and the methods of Doubleday and Pope and Shepherd is that the latter two use a more restictive assumption regarding fishing mortality. Both Doubleday and Pope and Shepherd assume that fishing mortality is a product of two factors, availability by age (aV',an age effect) and effective effort by year (Ey',a year effect) however they express this assumption differently (note that I am using a slight modification of my variable naming convention). Doubleday expresses fishing mortality as aFy = exp(aV + Ey) where aV is the natural log of availability and Ey is the natural log of effective effort. This can be shown to be equivalent to aFy = aV'Ey', which is identical to Pope and Shepherd's expression. In either case the expressions can be made linear by taking logs. The model in Doubleday's method is the natural log of the catch equation. The model in Pope and Shepherd's method is the natural log of the ratio of two catch equations. This is similar to the equation Doubleday used to get initial parameter estimates except that Doubleday used log(aCy/a+1Cy+1) and Pope and Shepherd used the inverse, log(a+1Cy+1/aCy).

6.1.6 Consideration of Errors

Of the three methods, Doubleday's method is the only one that considers that some of the parameters and/or variables in the model might be subject to error. His approach is somewhat unsatisfactory for reasons explained in section 4.6 (7). Fournier and Archibald (1982) have suggested a general theory for analyzing catch- at-age data based on constructing maximum likelihood functions. This appears to be the most flexible method available designed to address various sources of error. Their method estimates total catch in numbers and the percentage of the catch at each age.

6.1.7 Parameter Estimation Methods

Fry and Pope and Shepherd's methods use a sequential method of estimating the parameters of the nonlinear equations. Doubleday's gradient method estimates the parameters simultaneously, however it has problems with convergence and uniqueness. Indeed, Fournier and Archibald (1982) attempted to fit Doubleday's model but failed because they could not get stable parameter estimates. Pope and Shepherd developed their method in response to computational problems which they encountered when trying to use Doubleday's method.

The main question to answer is this; What is the best method of estimating parameters of a nonlinear equation? Draper and Smith (1966) mention three methods. The first, linearization, has been described in section 4.2. The next is the steepest decent method. Problems with this method are that it is often very inefficient, requiring a large number of iterations which tend to zigzag in a so-called hemstiching pattern. Steepest decent is slightly favorable over linearization. The third method is the Marquardt method. It can be shown that Marquardt's method is identical to the linearization method and the steepest decent method under the correct limiting conditions. Except for extremely ill-conditioned problems, Marquardt's method almost always converges to the global minimum of the objective function and does so in few iterations. Draper and Smith (1966; p272) report, "Marquardt's method represents a compromise between the linearization method and the steepest descent method and appears to combine the best features of both while avoiding their most serious limitations". A modified Levenberg-Marquardt method is commonly available from the International Mathematical and Statistical Software Library. An additional advantage of using the modified Levenberg- Marquardt method is that variances for parameter estimates can be directly approximated by multiplying the residual sums of squares by the diagonal elements of the inverse of an approximation to the Hessian matrix divided by the appropriate degrees of freedom (Bard 1974). The approximation is very good when the residuals are small. If the linear aproximation in the neighborhood of the true parameter vector is appropriate, then some idea of the joint variability of the parameters can be obtained by evaluating the ellipsoidal confidence region (Draper and Smith 1966).

Finally, it should be emphasized that "... no single method has emerged which is the best for the solution of all nonlinear programming problems" (Bard 1974; p 84).

7.0 CONCLUSIONS

Based on the evaluation criteria described in section 6.0 the following conclusion can be drawn. First, the method of Fry (including Pope's approximation) is very robust, given a good estimate of terminal fishing mortality. The robust nature of this model is probably due to its simple assumptions. When sample size is small, this method is probably the best to use. Doubleday (1976) showed that to a first order approximation, the variance of the parameter estimates will have the same coefficient of variation as a single catch observations. Second, the method of Pope and

Shepherd should not be used because, (1) it lacks a statistical basis, and (2) the parameter estimation procedure is sequential instead of simultaneous (i.e. new parameter estimates are conditional upon the old estimates vs. jointly estimating all parameters at once). In fact, I feel Fry or Pope's method is a much better approach than Pope and Shepherd's method. The extra effort required by Pope and Shepherd's method is not compensated by better parameter estimates. The method of Doubleday is the best of the methods considered because it

(1) reduces the number of parameters to be estimated when compared to Fry's method and has a higher observation-to-parameter ratio when compared to Pope and Shepherd's method. The difference between Doubleday's method and the method of Pope and Shepherd decreases as sample size gets small. (see Table 3);

(2) permits multiple observations for one cohort to be analyzed, thus when long series of catch-at-age data are available variances on population estimates for the middle cohorts will be small while variances for cohort that occupy the lower left and upper right corner of the catch-at-age matrix will have larger variances. This can be seen from Doubleday's (1976) principal component analysis of the year class variable;

(3) uses a log catch-at-age matrix as the raw data instead of a log catch ratio matrix. This should result in a lower sampling variance when compared to the sampling variance of catch ratio; and

(4) uses an estimation procedure that (1) fits all the data points simultaneously instead of piecemeal, and (2) permits evaluation of the assumptions of the model. This is perhaps the most important advantage since a variance/covariance matrix is available for further scrutiny regarding residuals and correlation among the parameters. Also other univariate and multivariate statistical methods can be applied once the variance/covariance matrix is calculated (the principal component analysis carried out by Doubleday is a prime example). Estimation of the parameter vector from the Jacobian matrix by least squares theory provides approximate maximum likelihood parameter estimates. The adequacy of the approximation depends on the extent to which the elements of the Jacobian matrix conform to a linear hypothesis.

Doubleday's method appears to be the best cohort analysis method to use since it is more statistically and theoretically sound. In actual practice, however, the best method will depend on factors such as 1) population characteristics such as type of recruitment, stock productivity, and rate of increase or decrease in population numbers, 2) sample size (length of catch-at-age time series), 3) correlations between parameters, 4) sources of variability in aging, estimates of effort, estimates of natural mortality, and estimates of catchability/availability, and 4) violations of the underlying assumptions. Doubleday's method is not without its problems. A more sophisticated parameter estimation procedure, such as the modified Levenberg-Marquardt method, should be used instead of his linearization method. This would remove the disadvantages related to problems with convergence. Also his method of incorporating random errors is lacking, since it does not acknowledge the fact the catch observations are usually distributed lognormally.

A problem that persists no matter which cohort analysis procedure is used is the problem of large variances in the parameter estimates. Pope (1977) points out the inescapable fact that catch-at-age data alone does not contain enough information to estimate parameters with high precision. Independent supplementary information is required to tie down the estimates.

Even when the fit is extremely good (R-squared > 0.90), the variances of the parameter estimates are very high due to correlation problems. Alton and Deriso (1982) found this to be true from their analysis, even when the variances were determined with a Monte-Carlo simulation technique. The correlation problem is the major factor contributing to low precision of the parameter estimates.

The procedure used by Alton and Deriso (1982) to estimate Pollock abundance is based on the method of Doubleday only to the extent that they use his mathematical formulation (i.e. use of the catch equations with fishing mortality represented as a product of availability and effective effort) to structure the catch-at-age model. The nonlinear parameter estimation procedure employed is based on the Marquardt algorithm, so in this respect the two analyses differ (Rick Deriso, personal communication 1/18/83). To completely generalize, the problem comes down to two steps, (1) hypothesizing a structural model, and (2) choosing a least squares procedure to estimate the parameters. The only difference between Alton and Deriso's analysis and the Doubleday method is that the former used a more sophisticated nonlinear least squares parameter estimation procedure.

8.0 RECOMMENDATIONS

If I were to recommend one cohort analysis technique to someone I would choose a method that

(1) uses a modified Levenberg-Marquardt nonlinear least squares regression parameter estimation procedure;

(2) uses log catch-at-age as the raw data;

(3) considers fishing mortality as a product of availability and effective effort; and

(4) permits calculation of variances of the parameter estimates. This can be done through approximation methods such as the delta method (Seber 1973), approximations derived from the Hessian matrix, or through Monte-Carlo simulation techniques.

In addition, I would strongly suggest that when using a stochastic technique, a residual sum of squares response surface be generated so as to better ascertain the shape of the parameter hypervolume. Further, Monte-Carlo simulations should be carried out in order to determine the sampling distribution of the parameter estimates.

An alternative approach might be to use a combination of methods so as to take maximum advantage of the strong aspects of each method. For example, Fry's method is very simple and robust yet extremely sensitive to the initial estimate of terminal fishing mortality and somewhat sensitive to errors in the estimation of M. One way to approach the problem might be to spend most of your effort getting the best estimate possible of fishing mortality in the current year (terminal fishing mortality) from a nonlinear procedure, and use that estimate as the input parameter in Fry or Pope's procedure. Paloheimo (1980) indicates that once the value of M is known, cohort analysis (Fry's method) results in more reliable estimates of year class abundance.

Further areas of research that appear to hold promise are

(1) use of the catch equation directly in the nonlinear parameter estimation procedures. This would tend to reduce bias associated with logarithmic transformations or approximations;

(2) use of weighted least squares fits in the parameter estimation procedures. Weights could be set inversely proportional to the residual or the residual standardized by the error mean square. In the application of (1) above, the objective function could be modified to include a weighting factor. This would account for the more realistic assumption that observed catches are distributed lognormally;

(3) reparameterization of the natural mortality and fishing mortality parameters to decrease the number of parameters that need to be estimated;

(4) use of alternative objective functions in the nonlinear parameter estimation procedure that take into account an unknown covariance matrix. Bard (1974) (see Table 5-1 on page 99) provides alternative objective functions for the assumption that errors are normally distributed;

(5) use of alternative objective functions that fit the observed catches to other distributions, such as multinomial, log normal, or negative binomial and looking at the sensitivity of the results to these assumptions; and

(6) incorporating supplemental information into the model, such as effort or CPUE, so that more precise parameter estimates can be derived.

Table 2. Comparison of three cohort analysis methods. Raw data consist of catch at age for ages a=1,...,A and years y=1,...,Y. A is the oldest age represented in the sample and Y is the number of years. Usually by convention Y represents the last years data.

	Fry's Method	Doubleday's Method	Pope & Shepherd's Method
Number of a priori/initial parameters required	Y aFy's (a=A) 1 M	A aV's Y Ey's 1 M	1 Ey (y=Y) 1 aV (a=A) 1 M
TOTALS	Y + 1	Y + A + 1	3
Number of parameters estimated	Y(A-1) aFy's Y aNy's (a=A)	A aV's Y Ey's Y+A-1 aNy's	A-1 aV's Y-1 Ey's Y+A-3 aNy's
TOTALS	YA	2(Y + A) - 1	2(Y + A) - 5
Number of observations	A ages for Y years	A ages for Y years	A-1 ages for Y-1 years
TOTALS	YA	YA	(Y - 1)(A - 1)
Observation-to-parameter ratio	YA / YA	YA / 2(Y + A) - 1	(Y - 1)(A - 1) / 2(Y + A) - 5
Number of equations in the model	YA	Y + A - 1	Y + A - 3

Table 3. Comparison of the number of a priori/initial parameter estimates required, number of parameters estimated, number of observations, and observation-to-parameter ratio for three common cohort analysis methods and different combinations of ages (A) and years (Y). Cohort analysis methods are Fry's method (FM), Doubleday's method (DM), and Pope and Shepherd's method (PSM).

	A = 5 Y = 10			A = 10 Y = 25			A = 20 Y = 100			Gulf of Alaska Pollock A = 8 Y = 6		
	FM	DM	PSM	FM	DM	PSM	FM	DM	PSM	FM	DM	PSM
# a priori parms.	11	16	3	26	36	3	101	121	3	7	15	3
# parms est.	50	29	25	250	69	65	2000	239	235	48	27	21
# obs.	50	50	36	250	250	216	2000	2000	1881	48	48	35
obs. per parm.	1.00	1.72	1.44	1.00	3.62	3.32	1.00	8.37	8.00	1.00	1.78	1.67

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