

# The Nonhydrostatic Solver of the GFDL Finite-Volume Cubed-Sphere Dynamical Core

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# 1 Foreword

The nonhydrostatic solver in FV3 was wholly designed by Dr. Shian-Jiann Lin, now retired from the Geophysical Fluid Dynamics Laboratory. What follows is our interpretation of the nonhydrostatic solver and sufficient background of the general FV3 algorithm to understand the nonhydrostatic implementation. This technical note acts as a means to document the nonhydrostatic FV3 and to ensure that credit is properly given to Dr. Lin.

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# 2 Introduction

This document describes the nonhydrostatic solver of the GFDL Finite-Volume Cubed-Sphere Dynamical Core, FV3. The nonhydrostatic solver works identically to the hydrostatic solver except for the need to solve for two new prognostic variables, the vertical velocity  $w$  and geometric layer depth  $\delta z$ ; and to use the full nonhydrostatic pressure in computing the pressure gradient force. In particular the Lagrangian dynamics described within L04 remains valid and all vertical processes (advection, wave propagation) remain implicit while all horizontal processes are explicit. This document assumes working knowledge of the hydrostatic discretization of FV3 described in LR96, LR97, L97, L04, PL07, and HL13. *It is strongly recommended that anyone who wishes to understand the nonhydrostatic FV3 solver read and understand these documents first.* Additional relevant material may be found in LPH17 and LH18. All of these documents may be found at [www.gfdl.noaa.gov/fv3/fv3-documentation-and-references/](http://www.gfdl.noaa.gov/fv3/fv3-documentation-and-references/).

The nonhydrostatic solver of FV3 has three main components: an explicit horizontal solver for all the modes, including the three-dimensional pressure-gradient force, described in section 4; a semi-implicit solver for the vertically-propagating sound-wave modes, described in section 5; and the Lagrangian vertical discretization described in section 3.

# 3 Lagrangian vertical coordinates

A *Lagrangian* vertical coordinate is used in FV3. This coordinate uses the depth of each layer (in terms of mass or as geometric height) as a prognostic variable, allowing the layer interfaces to deform freely as the flow evolves. Further, the flow is constrained within the Lagrangian layers, with no flow across the layer interfaces (even for non-adiabatic flows). Instead, the flow deforms the layers themselves by advecting the layer thickness and by straining the layers by the vertical gradient of explicit vertical motion. This form is automatically consistent with the LR96 scheme, avoids the need for explicit calculation and dimensional splitting of vertical advection, greatly reduces implicit vertical diffusion, and has no vertical Courant number restriction.

FV3 uses a hybrid-pressure coordinate based on the hydrostatic surface pressure  $p_s^*$ :

$$p_k^* = a_k + b_k p_s^*, \quad (1)$$

where  $k$  is the vertical index of the layer interface, counting from the top down, and  $a_k, b_k$  are pre-defined coefficients. Typically, the top interface is at a constant pressure  $p_T$ , so  $a_0 = p_T$  and  $b_0 = 0$ . The spacing of the levels depends on the particular application, and is chosen depending on how high of a model top is desired, where additional vertical resolution is applied (typically in the boundary layer, but sometimes also near the tropical tropopause), and where to transition from hybrid  $b_k > 0$  to pure pressure  $b_k = 0$  coordinates.

## 4 Prognostic variables and governing equations

The mass per unit area  $\delta m$  can be expressed in terms of the difference in hydrostatic pressure  $\delta p^*$  between the top and bottom of the layers; and, using the hydrostatic equation, can also be written in terms of the layer depth  $\delta z^1$ :

$$\delta m = \frac{\delta p^*}{g} = -\rho \delta z. \quad (2)$$

The continuous Lagrangian equations of motion, in a layer of finite depth  $\delta z$  and mass  $\delta p^*$ , are then given as

$$\begin{aligned} D_L \delta p^* + \nabla \cdot (\mathbf{V} \delta p^*) &= 0 \\ D_L \delta p^* \Theta_v + \nabla \cdot (\mathbf{V} \delta p^* \Theta_v) &= 0 \\ D_L \delta p^* w + \nabla \cdot (\mathbf{V} \delta p^* w) &= -g \delta z \frac{\partial p'}{\partial z} \\ D_L u &= \Omega v - \frac{\partial}{\partial x} K - \frac{1}{\rho} \frac{\partial p}{\partial x} \Big|_z \\ D_L v &= -\Omega u - \frac{\partial}{\partial y} K - \frac{1}{\rho} \frac{\partial p}{\partial y} \Big|_z \end{aligned} \quad (3)$$

Note that these equations are *exact*: no discretization has been made yet, and the only change from the original differential form of Euler's equations is to integrate over an arbitrary depth  $\delta p^*$ . The operator  $D_L$  is the "vertically-Lagrangian" derivative, formally equal to  $\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial z} (w \varphi)$  for an arbitrary scalar  $\varphi$ . The flow is entirely along the Lagrangian surfaces, including the vertical motion (which deforms the surfaces as appropriate, an effect included in the semi-implicit solver).

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<sup>1</sup> In this document, to avoid confusion we write  $\delta z$  as if it is a positive-definite quantity. In the solver itself,  $\delta z$  is defined to be negative-definite, incorporating the negative sign into the definition of  $\delta z$ ; this definition has the additional advantage of being consistent with how  $\delta p$  is defined, being measured as the difference in hydrostatic pressure between the bottom and top of a layer.

Variable	Description
$\delta p^*$	Vertical difference in hydrostatic pressure, proportional to mass
$u$	D-grid face-mean horizontal x-direction wind
$v$	D-grid face-mean horizontal y-direction wind
$\Theta_v$	Cell-mean virtual potential temperature
$w$	Cell-mean vertical velocity
$\delta z$	Geometric layer height

**Table 1:** Prognostic variables in FV3

The vertical component of absolute vorticity is given as  $\Omega$  and  $p$  is the full nonhydrostatic pressure. The kinetic energy is given as  $K = \frac{1}{2}(\tilde{u}u + \tilde{v}v)$ : since FV3 does not assume that the horizontal coordinate system is orthogonal, we use the covariant ( $u$  and  $v$ ) components of the wind vector as prognostic variables and the contravariant ( $\tilde{u}$  and  $\tilde{v}$ ) components for advection, avoiding the need to explicitly include metric terms. See PL07 and HL13 for more information about covariant and contravariant components.

The nonhydrostatic pressure gradient term in the  $w$  equation is computed by the semi-implicit solver described section 5, which also computes the prognostic equation for  $\delta z$ . There is no projection of the vertical pressure gradient force into the horizontal; similarly, there is no projection of the horizontal winds  $u$ ,  $v$  into the vertical, despite the slopes of the Lagrangian surfaces.

Finally, the ideal gas law:

$$p = p^* + p' = \rho R_d T_v = -\frac{\delta p^*}{g \delta z} R_d T_v, \quad (4)$$

where

$$T_v = T(1 + \epsilon q_v)(1 - q_{cond}), \quad (5)$$

is the “condensate modified” virtual temperature, or density temperature, is used to close the system of equations. Here,  $q_{cond}$  is the (moist) mixing ratio of all of the liquid and solid-phase microphysical species, if present. When the gas law is used, the mass  $p^*$  in this computation must be the mass of only the dry air and water vapor, and not including the mass of the condensate (non-gas) species. A rigorous derivation of the virtual and density temperatures is given in K. Emanuel, *Atmospheric Convection* (1994, Oxford), Sec. 4.3.

These equations are also applicable to hydrostatic flow, in which  $w$  is not prognosed and  $p = p^*$  is entirely hydrostatic.

## 5 Dynamics along Lagrangian surfaces

The layer-integrated equations (3) are discretized along the Lagrangian surfaces and integrated on the “acoustic” or “dynamical” time step  $\delta t$  using forward-backward time-stepping as in LR97. The vertical velocity  $w$  is a three-dimensional cell-mean value and partially advanced using the advection scheme. The geometric layer depth  $\delta z$  is simply the difference of the heights of the successive layer interfaces, which with  $\delta p^*$  defines the layer-mean density and the location of the Lagrangian surfaces. The air mass is the total air mass, including water vapor and condensate species.

FV3 places the wind components using the D-grid (following Arakawa’s terminology), which defines the winds as face-tangential quantities. The D-grid permits us to compute the cell-mean absolute vorticity  $\Omega$  *exactly* using Stokes’ theorem and a cell-mean value of the local Coriolis parameter  $f$ , without performing any averages or interpolations. The wind components themselves are face-mean values “along the cell edges” (not cell-mean values).

Following the notation from L04, PL07, and HL13, we can write the discretized forms of (3), excluding the vertical components, as:

$$\delta p^{*(n+1)} = \delta p^{*n} + F[\widetilde{u}^*, \delta p_y] + G[\widetilde{v}^*, \delta p_x] \quad (6)$$

$$\Theta^{n+1} = \frac{1}{\delta p^{*(n+1)}} \{ \Theta^n \delta p^{*n} + F[X^*, \Theta_y] + G[Y^*, \Theta_x] \} \quad (7)$$

$$w^* = \frac{1}{\delta p^{*(n+1)}} \{ w^n \delta p^{*n} + F[X^*, w_y] + G[Y^*, w_x] \} \quad (8)$$

$$u^{n+1} = u^n + \Delta \tau [Y(\widetilde{v}^*, \Omega_x) - \delta_x (K^* - v \nabla^2 D) + \widehat{P}_x] \quad (9)$$

$$v^{n+1} = v^n + \Delta \tau [X(\widetilde{u}^*, \Omega_y) - \delta_y (K^* - v \nabla^2 D) + \widehat{P}_y]. \quad (10)$$

The quantities  $\widehat{P}_x, \widehat{P}_y$  are the horizontal pressure-gradient force terms computed as in L97, the primary difference being that the forces due to hydrostatic and nonhydrostatic pressures are computed separately, and that the hydrostatic pressure gradient computation uses the log of the pressure to improve accuracy. The vertical nonhydrostatic pressure-gradient force is evaluated by the semi-implicit solver described in section 5; only the forward advection of  $w$  is performed during the Lagrangian dynamics, producing a partially-updated  $w^*$ .

For stability, the pressure gradient force is evaluated backwards-in-time: the flux terms in the momentum, mass, and entropy equations are evaluated forward by the advection scheme, and the resulting updated fields are used to compute the pressure gradient force. This forward-backward time-stepping is stable without needing to use predictor-corrector or Runge-Kutta methods.

In nonhydrostatic simulations it is recommended that the time off-centering for the horizontal pressure-gradient force be consistent with that used in the semi-implicit solver, which includes the vertical nonhydrostatic pressure-gradient force computation, to ensure consistency between the two. If the semi-implicit solver is run fully-implicit ( $\alpha = 1$ ) then the pressure-gradient force should be evaluated fully backward ( $\beta = 0$ ); otherwise use  $\beta = 1 - \alpha$ .

## 6 Nonhydrostatic semi-implicit solver

An equation for  $z$  can be derived from the definition of  $w$ :

$$w = \frac{Dz}{Dt} = D_L z + \vec{U} \cdot \nabla z \quad (11)$$

The time-tendency of geopotential height is then the sum of the advective height flux along the Lagrangian interfaces and the vertical distortion of the surfaces by the gradient of  $z$ .

Discretizing:

$$z^{n+1} = z^n + F[\vec{u}^*, \delta z_y] + G[\vec{v}^*, \delta z_x] + w^{n+1} \Delta t. \quad (12)$$

Since  $z$  is solved for on the interfaces, we can then simply take the vertical difference to get  $\delta z$ .

Recalling that the Lagrangian dynamics in (7) only performs the forward advection of the vertical velocity, yielding  $w^*$ , we then need to evaluate the vertical pressure-gradient force:

$$w^{n+1} = w^* - g \delta z^{n+1} \delta_z p'^{n+1}. \quad (13)$$

The pressure perturbation  $p'$  can be evaluated from the ideal gas law,

$$p' = p - p^* = \frac{\delta p}{g \delta z} R_d T_v - p^*, \quad (14)$$

requiring simultaneous solution of  $w$ ,  $p'$ , and  $\delta z$  using a tridiagonal solver.

There is an option to off-center the semi-implicit solver to reduce implicit diffusion. The parameter  $\alpha$  can be varied between 0.5 and 1 to control the amount of off-centering, with  $\alpha = 1$  being fully-implicit. As discussed in Section 4 this off-centering parameter should be set to  $\alpha = \beta - 1$ , consistent with that used for the horizontal pressure-gradient force.

The boundary conditions used are  $p' = 0$  at the model top, and  $\vec{U} \cdot \hat{n}_s = 0$  at the lower boundary of  $z = z_s$ . This is the “free-slip” boundary condition, that the lower boundary is a streamline. The surface vertical velocity  $w_s$  can be computed from (11) by advecting the surface height  $z_s$ :

$$w_s = \frac{z_s^* - z_s}{\Delta t}, \quad (15)$$

where  $z_s^*$  is the advected value and  $z_s$  the height of the topography.

## 7 References

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## 8 Revision History

**v1.0 $\alpha$  20 Oct 2020** First draft of nonhydrostatic technical note

**v1.0 $\beta$  10 Nov 2020**