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The Bruns Transformation and a Dual Setup of Geodetic Observational Equations

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Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1974 reprinted annually, 12 pp (PB265442). National specifications and tables show the closures required and tolerances permitted for first-, second-, and third-order geodetic control surveys. (A single free copy can be obtained, upon request, from the National Geodetic Survey, OA/C18x2, NOS/NOAA, Rockville, MD 20852.)

Specifications To Support Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1975, reprinted annually, 30 pp (PB261037). This publication provides the rationale behind the original publication, "Classification, Standards of Accuracy, ..." cited above. (A single free copy can be obtained, upon request, from the National Geodetic Survey, OA/C18x2, NOS/NOAA, Rockville, MD 20852.)

Proceedings of the Second International Symposium on Problems Related to the Redefinition of North American Geodetic Networks. Sponsored by U.S. Department of Commerce; Department of Energy, Mines and Resources (Canada); and Danish Geodetic Institute; Arlington, Va., 1978, 658 pp. (GPO #003-017-0426-1). Fifty-four papers present the progress of the new adjustment of the North American Datum at midpoint, including reports by participating nations, software descriptions, and theoretical considerations.

NOAA Technical Memorandums, NOS/NGS subseries

NOS NGS-1 Use of climatological and meteorological data in the planning and execution of National Geodetic Survey field operations. Robert J. Leffler, December 1975, 30 pp (PB249677). Availability, pertinence, uses, and procedures for using climatological and meteorological data are discussed as applicable to NGS field operations.

NOS NGS-2 Final report on responses to geodetic data questionnaire. John F. Spencer, Jr., March 1976, 39 pp (PB254641). Responses (20%) to a geodetic data questionnaire, mailed to 36,000 U.S. land surveyors, are analyzed for projecting future geodetic data needs.

(Continued at end of publication)

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Philip M. Klutznick, Secretary

National Oceanic and Atmospheric Administration
Richard A. Frank, Administrator

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THE BRUNS TRANSFORMATION AND A DUAL SETUP OF GEODETIC OBSERVATIONAL EQUATIONS

Erik W. Grafarend[1]
National Geodetic Survey
National Ocean Survey, NOAA
Rockville, Md. 20852

ABSTRACT. The Bruns formula, which equates the disturbing gravity potential modulo the length of the normal gravity vector to the height anomaly, is generalized into three dimensions and into horizontal, equatorial, and inertial reference frames. It is applied to formulate the space-like geodetic boundary value problem in geometry and gravity space. The Bruns transform allows a dual setup of geodetic observational equations in a network of mass points, the finite element approximation of the space-like geodetic boundary value problem, in the following sense: The observational equations can be expressed rigorously either as a function of geometric coordinate corrections alone without any gravity dependent quantity, or alone as a function of the gravity disturbing potential and its gradients alone without any geometric coordinate correction. For operational purposes, estimable quantities from reference-free observables are studied in geometry, gravity, and vorticity spaces. They correspond to invariants with respect to a linear similarity transformation typified by positional angles and length ratios in various vector spaces. A Cartesian series representation of the gravity potential and its gradients is given--the Cartesian coordinate system is known to be singularity-free--and is used for a unified Cartesian setup of observational equations.

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[1]Prepared during a 3-month period in 1978 when the author served as a Senior Scientist in Geodesy, National Research Council, National Academy of Sciences, Washington, D.C., while on leave from the University FAF at Munich, Federal Republic of Germany.

Science should be the friend of practice, but not its slave.

C.F. Gauss

INTRODUCTION

Geodesy is conventionally divided into two branches: geometric and physical. This separation has resulted in various geodetic schools, or research groups, concentrating on one or the other aspect with little intercommunication. We would like to show that geodesy is actually a unity. The two branches are only the sides of a single coin. In detail, we will prove that geodetic observational equations can be uniquely set up in either the geometric or the physical mode. For instance, a distance observation can be expressed in terms of either the coordinates of the end points of a line or the gravity disturbing potential and its gradient at these points. The proof is based on the classic Bruns formula which expresses the height anomaly in terms of the gravity disturbing potential modulo the magnitude of the normal gravity vector. The Bruns formula will be generalized into three dimensions and into various reference frames: horizontal, equatorial, and inertial.

To make the Bruns formula operational, we have to inject observable quantities. Therefore, the first section is devoted to geodetic observables. There are two perspectives from which to look upon geodetic observables. If we do not introduce an a priori reference system into the vector space of geodesy, only positional angles and length ratios are observable. They are invariant with respect to a linear similarity transformation, characterized by degrees of freedom of type translation, rotation, and scale. Referring to adjustment procedures, positional angles and length ratios are estimable quantities. This concept is applied to both geometric and physical space. For instance, we construct positional angles and length ratios in gravity space from a network based on gravity vectors. The geometric quantities are a function of the length of the gravity vector and astronomical longitude and latitude at three points.

In the second section we will derive the generalized three-dimensional Bruns equation from observables that are one-point functions. These can be computed from observations once we have established a reference system for origin, orientation, and scale in any geodetic vector space. The first step will be a transformation of one-point observables into Cartesian coordinates of points on the approximate surface of the Earth, the telluroid. We will use isoparametric mappings for astronomical longitude and latitude, gravity potential, and first- and second-order gradients. The mappings are one-to-one if we use the isotropic- or zero-order approximation of the gravity field. Uniqueness is lost if we use another order of approximation. The second step is formulation of the transform of disturbances of gravity into Cartesian coordinate corrections.

The third section deals with a dual setup of geodetic observational equations of one- and three-point type, either in the geometric or in the gravitational space. They refer to different formulations of the Bruns transformation based on astronomical longitudes and latitudes and gravity potential (or gravity or gravity gradient).

The appendices are a Cartesian form of series representing the gravity potential and its first- and second-order gradients.

The report reflects current research in space-time geodesy, especially with respect to the geodetic initial-boundary value problem and its finite element approximation, the setup of geodetic observational equations in networks of mass points.

Section 1 is influenced by the concept of geodetic invariants introduced by Baarda (1973) and estimable quantities introduced by Bossler (1973) and Grafarend and Schaffrin (1974, 1976). The isoparametric mappings of section 2 which led to the formulation of the three dimensional Bruns transform have been partly studied by Bocchio (1976 a,b,c), Bruns (1876), Hirvonen (1960, 1961), Krarup (1969, 1973 a,b), Livieratos (1976, 1978), Marussi (1973, 1974 a,b), Moritz (1965, 1977), Niemeier (1972) and Grafarend (1972, 1975, 1978 a,b,c). The first setup of geodetic observational equations to be expressed rigorously in the gravimetric mode was by Sanso (1978 a,b) by making use of his adjoint potential. Here, we will reverse his argument exactly by employing the inverse Bruns transformation and expressing the geodetic observational potential rigorously in the geometric mode.

Geodesists have hesitated to accept the new three-dimensional mapping. Therefore we would like to make the following comments. For two-dimensional cartographic mappings the isoparametric mapping is well known, e.g. Chovitz (1952, 1954, 1956), O'Keefe (1953), Lane (1939, p. 189), Levi-Civita (1926, p. 220). Let us quote from O'Keefe (1953): "It is evident that the deformations produced by the isoparametric method are of the same order as those produced by other methods."

Another comment is on the definition of a geodetic network. Much research in geodesy has been performed in two-dimensional network analysis. Such networks are better termed mathematical networks because they do not take the gravity field into account. Here, a geodetic network consists of mass points; thus there is gravitational interaction which we cannot switch off.

1. OBSERVABLES

Having decided foundational questions, we next introduce related observations which make geodesy operational. A majority of geodesists believe that geodesy is Euclidean geometry referred to linear space with finite dimensions and Hilbertian geometry referred to linear space with infinite dimensions. What then are the basic observables?

In Euclidean geodesy position is given by vectors, for instance,

the position vector

the gravity vector

the vorticity vector

moving in space-time. Let us give an illustration of these vectors, as shown in figure 1 (p.70). Choose an origin of reference, e.g., the geocenter. The position vector extends from the reference origin to a mass point in space, e.g., the topocenter. At this point we draw the gravity vector, the rotation vector, or any other vector of reference. The corresponding vector spaces are called the geometry space,

the gravity space, and

the vorticity space.

The set of all position vectors drawn from the reference origin is called the geometry space. Gravity space is constructed by a translation of the gravity vector along the position vector to the reference origin under Euclidean parallelism. In the same way the vorticity space or any other space of reference vectors is defined. Coordinates v^n of a vector $\underline{v}$ are provided after we select a frame of reference $\underline{e}_n$, e.g., the inertial frame, such that

$$\underline{v} = \sum_{n=o}^{N} v^n \underline{e}_n = v^o \underline{e}_o + v^1 \underline{e}_1 + \ldots + v^N \underline{e}_N. \qquad 1(1)$$

(Notation: vectors in Euclidean space are denoted by capital letters, or underlined small letters.)

If the base vectors are orthonormal,

$$(\underline{e}_i, \underline{e}_j) = \delta_{ij} \qquad 1(2)$$

where δ_{ij} is the Kronecker symbol for an element of the unit matrix and (.,.) is the sign for the scalar product. Coordinates are recovered by

$$(\underline{v}, \underline{e}_n) = v^n. \qquad 1(3)$$

$$||\underline{v}||^2 = \sum_{n=o}^{N} (v^n)^2 \qquad 1(4)$$

(where $||\cdot||$ is the norm sign) is the relation of completeness.

It is assumed that in space-time geodesy the number of independent base vectors, which is identical to the dimension of the vector space, is (3,1).

In Hilbertian geodesy, position is given by vectors, for instance,

the position potential,

the gravity potential, and

the vorticity potential.

(In Hilbert space, potential is a vector.)

Figure 2 (p. 70) illustrates these vectors. Coordinates v^{nm} of a vector $\bar{v}$ (vectors in Hilbert space carry an overbar) are provided once we have select a frame of reference, e.g.,

$$\bar{e}_{nm} = \begin{cases} \sqrt{2n+1}\; r^{-n-1}\; P_n(\sin\phi) & \text{for } m=0 \\ \sqrt{2(2n+1)\,\frac{(n-m)!}{(n+m)!}}\; r^{-n-1}\; P_{nm}(\sin\phi)\,\cos m\lambda & \text{for } m>0 \\ \sqrt{2(2n+1)\,\frac{(n-|m|)!}{(n+|m|)!}}\; r^{-n-1}\; P_{nm}(\sin\phi)\,\sin m\lambda & \text{for } m<0 \end{cases} \quad 1(5)$$

where P_n are Legendre and P_{nm} associated Legendre functions of the first kind, and λ, ϕ, r spherical coordinates, such that

$$\bar{v} = \sum_{n=o}^{\infty} \sum_{m=-n}^{+n} v^{nm}\, \bar{e}_{nm}. \quad 1(6)$$

Here, $\bar{v}$ is a "harmonic" function satisfying $\Delta\bar{v}=0$, where Δ is the three-dimensional Laplace operator.

If the base vectors are orthonormal, as in our example,

$$(\bar{e}_{ij}, \bar{e}_{k\ell}) = \delta_{ik}\, \delta_{j\ell} \quad 1(7)$$

e.g., the integral over the unit sphere divided by 4π. Coordinates are reproduced by

$$(\bar{v}, \bar{e}_{nm}) = v^{nm}. \qquad 1(8)$$

$$||\bar{v}|| = \sum_{n=o}^{\infty} \sum_{m=-n}^{+n} (v^{nm})^2 \qquad 1(9)$$

(where $||\bar{v}||^2$, the square of the norm, is the integral over the unit sphere of $(\bar{v},\bar{v})/4\pi$) is the relation of completeness.

The earlier question about basic observables can now be answered. Assume a network, e.g., a triangle, being constructed in a vector space. For the depiction of any vector by an arrow, as in figure 3 (p. 71), we require an origin, direction, and a length. To remove these artificial references for translation, rotation, and scale, we need quantities which are invariant with respect to changes of these parameters. In other words, we are looking for invariants under a similarity transformation

$$\underline{v} \rightarrow \underline{v}' = T + \lambda R\underline{v} \qquad 1(10)$$

where T is a translation vector, R an orthogonal matrix, and λ a scale factor. It is well known from analytical geometry that length ratios and angles are dual elements of the basis of invariants under the linear similarity transformation given above, e.g.,

$$\frac{||\underline{v}_2 - \underline{v}_1||}{||\underline{v}_3 - \underline{v}_1||} \qquad 1(11)$$

or

$$\frac{(\underline{v}_2 - \underline{v}_1, \underline{v}_3 - \underline{v}_1)}{||\underline{v}_2 - \underline{v}_1|| \; ||\underline{v}_3 - \underline{v}_1||} . \qquad 1(12)$$

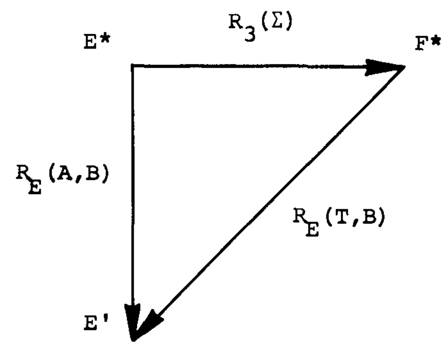
Finally, we will present three examples for basic observables in the geometry, gravity, and vorticity spaces.

EXAMPLE 1.1 (geometry space)

Let us introduce a triangle in the geometry space constructed from position vectors X_1, X_2, X_3 directed from the geocenter to three mass points in space.

At first the network is observed by a theodolite through horizontal directions and horizon distances (or zenith distances). Related reference frames are E', E*, and F*, defined as follows:

The orthonormal observational triad E' is based on the vector $E_{3'}$ directed from the station point at the topocenter to the target point. The base vector $E_{2'}$ is the normalized vector of the exterior product of the local, instantaneous gravity vector $-\Gamma$ and $E_{3'}$. $E_{1'}$ completes the orthonormal base. The orthonormal horizontal triad E* is based on the normalized local, instantaneous gravity vector E_{3*} at the topocenter. The base vector E_{2*} is the normalized vector of the exterior product of the local instantaneous rotation (vorticity) vector Ω and the local instantaneous gravity vector $-\Gamma$. E_{1*} completes the orthonormal base. The "carrousel" triad F* is related to the horizontal triad E* by $F^* = R_3(\Sigma)E^*$, where Σ is the horizontal orientation unknown such that F_{1*} is in the zero direction of the horizontal circle of the theodolite. To summarize, the frames are related as shown in the diagram



where $R_E(A,B) = R_2(\frac{\pi}{2} - B)R_3(A)$, and A the south azimuth, B the horizon distance, T the horizontal direction.

Now we can compute the positional angle

$$\cos \Psi_x = \frac{(X_2 - X_1, X_3 - X_1)}{||X_2 - X_1|| \; ||X_3 - X_1||} = \left(E_{3'}^{12}, E_{3'}^{13}\right). \qquad 1(13)$$

From the diagram we read

$$E' = R_E(A,B)E^* = R_E(A,B)R_3^T(\Sigma)F^* = R_E(T,B)F^* \qquad 1(14)$$

so that

$$E_{3'} = \cos T \cos B\, F_{1*} + \sin T \cos B\, F_{2*} + \sin B\, F_{3*} \qquad 1(15)$$

$$\cos \Psi_x = ||X_2 - X_1||^{-1}||X_3 - X_1||$$

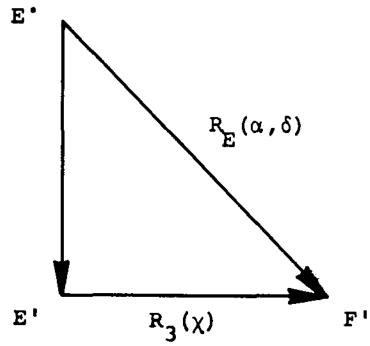
$$\left\{(X_{2*} - X_{1*})(X_{3*} - X_{1*}) + (Y_{2*} - Y_{1*})(Y_{3*} - Y_{1*}) + (Z_{2*} - Z_{1*})(Z_{3*} - Z_{1*})\right\}$$

$$= \cos T_{12} \cos B_{12} \cos T_{13} \cos B_{13} + \sin T_{12} \cos B_{12} \sin T_{13} \cos B_{13} + \sin B_{12} \sin B_{13}. \qquad 1(16)$$

Here, the positional angle is represented by horizontal directions and horizon distances at the three network points and is independent of the origin, orientation, and scale of the reference systems.

Next, the network is observed by a camera through right ascension and declination. Related reference frames are E', F', and E˙ which are defined as follows:

The orthonormal equatorial triad E˙ is based on the normalized local instantaneous rotation (vorticity) vector $E_{3˙}$ at the topocenter. The base vector $E_{2˙}$ is the normalized vector of the exterior product of the instantaneous ecliptic normal vector and the local instantaneous rotation (vorticity) vector. $E_{1˙}$ completes the orthonormal base. The "carrousel" triad F' is related to the observational triad E' by $F' = R_3(\chi)E'$, where χ is the observational orientation unknown. To summarize, the frames are related as shown in the diagram



where $R_E(\alpha,\delta) = R_2(\frac{\pi}{2} - \delta)\ R_3(\alpha)$, and α the right ascension, δ the declination.

From the diagram we see

$$F' = R_E(\alpha,\delta)E^{\cdot} \qquad 1(17)$$

so that

$$E_{3'} = \cos\alpha\cos\delta\ E_{1^{\cdot}} + \sin\alpha\cos\delta\ E_{2^{\cdot}} + \sin\delta\ E_{3^{\cdot}} \qquad 1(18)$$

$$\cos\Psi_X = ||X_2 - X_1||^{-1}\ ||X_3 - X_1||^{-1} \qquad 1(19)$$

$$\Big\{(X_{2^{\cdot}} - X_{1^{\cdot}})(X_{3^{\cdot}} - X_{1^{\cdot}}) + (Y_{2^{\cdot}} - Y_{1^{\cdot}})(Y_{3^{\cdot}} - Y_{1^{\cdot}}) + (Z_{2^{\cdot}} - Z_{1^{\cdot}})(Z_{3^{\cdot}} - Z_{1^{\cdot}})\Big\}$$

$$= \cos\alpha_{12}\cos\delta_{12}\cos\alpha_{13}\cos\delta_{13} + \sin\alpha_{12}\cos\delta_{12}\sin\alpha_{13}\cos\delta_{13} + \sin\delta_{12}\sin\delta_{13}$$

(where δ_{ij}, α_{ij} represent differences in δ, α between points i and j)

holds because $E_{3'} = F_{3'}$ by definition.

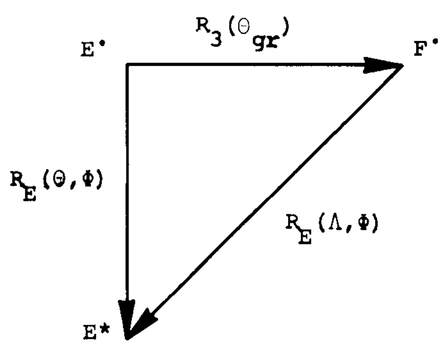
The positional angle above is represented by differences in right ascension and declination at the three network points and is independent of the origin, orientation, and scale of the reference systems.

EXAMPLE 1.2 (gravity space)

Let us introduce a triangle in the gravity space constructed from gravity vectors Γ_1, Γ_2, Γ_3.

The network is observed by an astronomic instrument and a gravimeter. Related reference frames are E^*, $E^{\cdot}$, and $F^{\cdot}$, defined as follows

The frames E^* and $E^{\cdot}$ have been introduced in the first example. The "carrousel" triad $F^{\cdot}$ is related to the equatorial triad $E^{\cdot}$ by $F^{\cdot} = R_3\ (\Theta_{gr})\ E^{\cdot}$, where Θ_{gr} is the equatorial orientation angle, (also called Greenwich sidereal time), such that $F_{1^{\cdot}}$ is in the Greenwich direction, the projection of the local instantaneous gravity vector at Greenwich onto the equatorial plane. The frames are related in the following manner:



where $R_E(\Theta,\Phi) = R_2(\frac{\pi}{2} - \Phi)R_3(\Theta)$, and Λ astronomic longitude, Φ astronomic latitude, Θ sidereal time angle.

We can again compute the positional angle

$$\cos \Psi_\Gamma = \frac{(\Gamma_2 - \Gamma_1, \Gamma_3 - \Gamma_1)}{||\Gamma_2 - \Gamma_1||\ ||\Gamma_3 - \Gamma_1||} . \qquad 1(20)$$

From the diagram,

$$E^* = R_E(\theta,\Phi)E^{\cdot} = R_E(\theta,\Phi) \quad R_3^T(\theta_{gr}) \; F^{\cdot} = R_E(\Lambda,\Phi)F^{\cdot} \tag{1(21)}$$

so that

$$E_{3*} = \cos\Lambda \cos\Phi \; F_{1^{\cdot}} + \sin\Lambda \cos\Phi \; F_{2^{\cdot}} + \sin\Phi \; F_{3^{\cdot}} \tag{1(22)}$$

$$\Gamma = - \,||\Gamma||E_{3*} \tag{1(23)}$$

$$\cos\Psi_\Gamma = ||\Gamma_2 - \Gamma_1||^{-1} \; ||\Gamma_3 - \Gamma_1||^{-1} \tag{1(24)}$$

$$\Big\{[||\Gamma_2|| \cos\Lambda_2 \cos\Phi_2 - ||\Gamma_1|| \cos_1\Lambda \;\; \cos\Phi_1] \; [||\Gamma_3|| \cos\Lambda_3 \cos\Phi_3$$
$$- ||\Gamma_1|| \cos\Lambda_1 \cos\Phi_1] + [||\Gamma_2|| \sin\Lambda_2 \cos\Phi_2 - ||\Gamma_1|| \sin\Lambda_1 \cos\Phi_1]$$
$$[||\Gamma_3|| \sin\Lambda_3 \cos\Phi_3 - ||\Gamma_1|| \sin\Lambda_1 \cos\Phi_1] + [||\Gamma_2|| \sin\Phi_2$$
$$- ||\Gamma_1|| \sin\Phi_1] \; [||\Gamma_3|| \sin\Phi_3 - ||\Gamma_1|| \sin\Phi_1]\Big\}.$$

Thus, the positional angle is represented by the magnitudes of the gravity vectors and astronomic longitude and latitude at three network points, and is independent of the origin, orientation, and scale of the reference systems.

EXAMPLE 1.3 (vorticity space)

Let us introduce a triangle in the rotation space constructed from rotation (vorticity) vectors Ω_1, Ω_2, Ω_3.

The network is observed with respect to a frame $f^{\cdot}$ uniformly rotating with rotational speed ω. $F^{\cdot}$, defined in the second example, and $f^{\cdot}$ are related by $F^{\cdot} = R_C(\gamma,\beta,\alpha)\; f^{\cdot}$, where $R_C(\gamma,\beta,\alpha) = R_1(\alpha)\; R_2(\beta)\; R_3(\gamma)$ are rotation matrices, and α,β,γ Cardan angles (Grafarend et al., 1979: p. 208).

The positional angle

$$\cos \Psi_\omega = \frac{(\Omega_2 - \Omega_1,\ \Omega_3 - \Omega_1)}{||\Omega_2 - \Omega_1||\ ||\Omega_3 - \Omega_1||} \qquad 1(25)$$

can now be computed, taking into account

$$\begin{aligned} F_{3\cdot} = {} & (\sin\alpha \sin\gamma + \cos\alpha \sin\beta \cos\gamma)\, f_{1\cdot} \\ & - (\sin\alpha \cos\gamma + \cos\alpha \sin\beta \cos\gamma)\, f_{2\cdot} \\ & + \cos\alpha \cos\beta\, f_{3\cdot} \end{aligned} \qquad 1(26)$$

$$\Omega = ||\Omega||F_{3\cdot} \qquad 1(27)$$

$$\begin{aligned} \cos \Psi_\omega = {} & ||\Omega_2 - \Omega_1||^{-1}\ ||\Omega_3 - \Omega_1||^{-1} \\ & \Big\{ \Big[||\Omega_2|| (\sin\alpha_2 \sin\gamma_2 + \cos\alpha_2 \sin\beta_2 \cos\gamma_2) \\ & - ||\Omega_1|| (\sin\alpha_1 \sin\gamma_1 + \cos\alpha_1 \sin\beta_1 \cos\gamma_1)] \\ & [||\Omega_3|| (\sin\alpha_3 \sin\gamma_3 + \cos\alpha_3 \sin\beta_3 \cos\gamma_3) \\ & - ||\Omega_1|| (\sin\alpha_1 \sin\gamma_1 + \cos\alpha_1 \sin\beta_1 \cos\gamma_1)] \\ & + [||\Omega_2|| (\sin\alpha_2 \cos\gamma_2 - \cos\alpha_2 \sin\beta_2 \cos\gamma_2) \\ & - ||\Omega_1|| (\sin\alpha_1 \cos\gamma_1 - \cos\alpha_1 \sin\beta_1 \cos\gamma_1)] \\ & [||\Omega_3|| (\sin\alpha_3 \cos\gamma_3 - \cos\alpha_3 \sin\beta_3 \cos\gamma_3) \\ & - ||\Omega_1|| (\sin\alpha_1 \cos\gamma_1 - \cos\alpha_1 \sin\beta_1 \cos\gamma_1)] \\ & + [||\Omega_2|| \cos\alpha_2 \cos\beta_2 - ||\Omega_1|| \cos\alpha_1 \cos\beta_1] \\ & [||\Omega_3|| \cos\alpha_3 \cos\beta_3 - ||\Omega_1|| \cos\alpha_1 \cos\beta_1 \Big] \Big\} \end{aligned} \qquad 1(28)$$

is represented as a function of the magnitude of the rotation (vorticity) vectors and the three Cardan angles at three network points and is independent of the origin, orientation, and scale of the reference systems.

2. THE BRUNS TRANSFORMATION

Operational geodesy uses observables as input data and coordinates of the position vector as output data. This input-output relation is called the Bruns transformation, originally presented in its linear form by Bruns (1878) when referring to the horizontal frame. The Bruns transformation classically yields the height anomaly from the disturbing potential divided by normal gravity. Thus it transforms the "observable" disturbing potential into the vertical coordinate called "height." Now, we will present a three-dimensional generalization of the Bruns transformation which can be used in both terrestrial and satellite geodesy.

The idea of the Bruns transformation is the following: Let a vector field $\bar{V}$ be observed, for instance, at a point P on the Earth's surface. Decompose the vector field into a normal part, whose structure is known (which approximates the real vector field), and into a disturbing part:

$$\bar{V}_P = \bar{v}_P + \delta\bar{v}_P. \qquad 2(1)$$

The normal part $\bar{v}_P$ at P can be linearized by a Taylor series with origin at a point p of the telluroid:

$$\bar{v}_P = \bar{v}_p + (\mathrm{grad}\ \bar{v}_P)_p\ (P-\underline{p}) + o_2 \qquad 2(2)$$

where o_2 indicates second- and higher-order terms. If we know the approximate position vector $\bar{p}$, we can determine the "displacement vector" $P-\underline{p}$ from

$$\bar{V}_P - \bar{v}_p = (\mathrm{grad}\ \bar{v}_P)_p\ (P-\underline{p}) + \delta\bar{v}_P. \qquad 2(3)$$

Figure 4 (p.71) illustrates the vector field in geometric space. Let us call the two-point functions, $\bar{V}_P - \bar{v}_p = \Delta\bar{v}$ and $P-\underline{p} = \Delta\underline{x}$, anomalies of the vector field and the position vector, respectively, so that

$$\Delta\bar{v} = (\mathrm{grad}\ \bar{v}_P)_p\ \Delta\underline{x} + \delta\bar{v}_P. \qquad 2(4)$$

We can choose $\Delta\bar{v} = 0$ which we refer to as the isoparametric mapping of $\delta\bar{v}_p \to \Delta\underline{x}$

$$\Delta\underline{x} = B\delta\bar{v}, \quad \delta\bar{v} = C\Delta\underline{x}, \tag{2(5)}$$

where

$$B = C^{-1} = -(\text{grad } \bar{v}_P)_p^{-1}, \tag{2(6)}$$

the Brun's matrix, which is the inverse of the gradient of the vector field at point p. Here we have assumed that grad $\bar{v}P$ is a regular matrix excluding rank deficiencies with respect to injectivity. In practice, singularities appear and have to be treated separately.

Thus far the Bruns transformation 2(5) of vector field disturbances into position vector anomalies is coordinate-free. Its form with respect to geodetic reference frames is the following:

$$\Delta\underline{x}^{\cdot} = B^{\cdot}\delta\bar{v} \tag{2(7)}$$

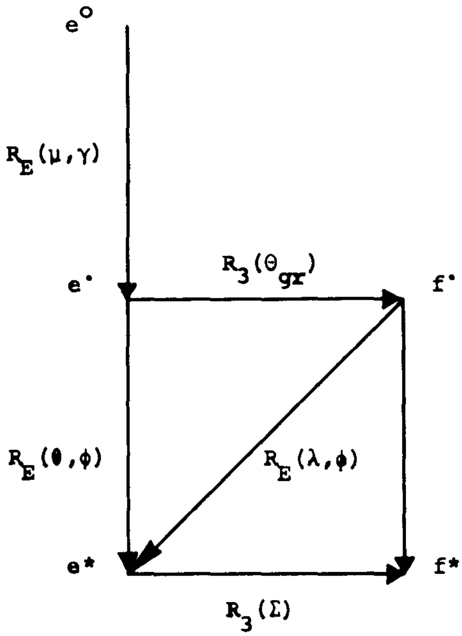
$$\Delta\underline{x}^{*} = B^{*}\delta\bar{v} \tag{2(8)}$$

$$\Delta\underline{x}^{o} = B^{o}\delta\bar{v} \tag{2(9)}$$

$$B^{*} = R_E(\lambda,\phi)B^{\cdot} \tag{2(10)}$$

$$B^{o} = R_E^T(\mu,\nu)\ R_E^T(\theta_{gr})B^{\cdot}\ . \tag{2(11)}$$

$\Delta\underline{x}^{\cdot} = [\Delta x^{\cdot}, \Delta y^{\cdot}, \Delta z^{\cdot}]^T$ displays the coordinates of the "displacement vector" in the Earth-fixed equatorial triad $f^{\cdot}$, $\Delta\underline{x}^{*} = [\Delta x^{*}, \Delta y^{*}, \Delta z^{*}]^T$, and the corresponding coordinates in the horizon triad e^{*}, $\Delta\underline{x}^{o} = [\Delta x^{o}, \Delta y^{o}, \Delta z^{o}]^T$ in the "fixed" or inertial (ecliptic) triad e^{o}, where μ,ν are Eulerian angles, as shown in the following diagram:



Instead we could represent the "displacement vector" in the "network" frames (written by capital letters). In the theory of mappings the two frames are called Eulerian and Lagrangian; thus we have here the Eulerian description.

Before we show examples of the thus far abstract Bruns transformation, we first note another remarkable property.

In many applications, the vector field of observables is "conservative," i.e.,

$$\text{div } \delta\bar{v} = 0, \text{ rot } \delta\bar{v} = 0 \tag{2(12)}$$

at least insofar as we are outside of the masses. A consequence of the Bruns transformation 2(5) is then

$$\text{div } \Delta\underline{x} = 0, \text{ rot } \Delta\underline{x} = 0; \tag{2(13)}$$

thus $\delta\bar{v}$ and $\Delta\underline{x}$ can be expressed as the gradient of a scalar potential. If we introduce

$$\delta\bar{v} = \text{grad } \delta w, \tag{2(14)}$$

$$\Delta\underline{x} = \text{grad } \delta x = B \text{ grad } \delta w \tag{2(15)}$$

follows. δx will be called the adjoint potential. In the first instance, it might be surprising that the "displacement vector" which leads to the position vector is a "harmonic" function, div $\Delta\underline{x}$ = div grad $\delta x = 0$, but the result is quite "natural."

Another basic assumption we have made is that we know the approximate position vector $\underline{p}$, but from where? If we have settled a convention about origin, orientation, and scale of a geodetic network to, for example, the geocenter, directions to extragalactic objects and a unit length in geometry space, how can we find the approximate position vector $\underline{p}$ in a geodetic reference system? The factor of uncertainty is introduced by the fact that nearly all geodetic observables depend on the gravity field whose coordinates in Hilbert space (e.g., coefficients in a spherical harmonic representation of the gravitational field) are unknown. Fortunately, there are geodetic observables that are gravity free, like positional angles and distance ratios. Only because of this can geodesy be made operational: coordinates in Euclidean and Hilbert spaces can be determined. This general statement will be verified in example 2.1.

<u>EXAMPLE 2.1 (longitude, latitude, potential)</u>

Let us introduce the isoparametric mapping

$$\Lambda_P = \lambda_p, \ \Phi_P = \phi_p, \ W_P = w_p \tag{2(16)}$$

where Λ and Φ are astronomic longitude and latitude, W the scalar gravity potential, λ and ϕ geodetic longitude and latitude, and w the scalar normal gravity potential (better known in geodesy by the letter U). Longitude and latitude are spherical coordinates in gravity space defined by

$$\Lambda = \text{arc tan } \Gamma_y/\Gamma_x \tag{2(17)}$$

$$\lambda = \text{arc tan } \gamma_y/\gamma_x \tag{2(18)}$$

$$\Phi = \text{arc tan } \Gamma_z/\sqrt{\Gamma_x^{\ 2} + \Gamma_y^{\ 2}} \tag{2(19)}$$

and

$$\phi = \text{arc tan } \gamma_z / \sqrt{\gamma_x^2 + \gamma_y^2} \qquad 2(20)$$

where $(\Gamma_x, \Gamma_y, \Gamma_z)$, $(\gamma_x, \gamma_y, \gamma_z)$ are Cartesian coordinates of the gravity vector Γ, $\underline{\gamma}$, respectively, in the "Eulerian" frame $\underline{f}^{\cdot}$. With respect to a chosen reference system, (Λ, Φ, W) are "observable," whereas (λ, ϕ, w) are "computable," as in the representation of the potential given in the first section.

A zero-order approximation of the actual gravity potential is

$$w = - \text{gm } ||\underline{x}||^{-1} \qquad 2(21)$$

where gm is the product of the gravitational constant and the mass of the model terrestrial body, $||\underline{x}|| = \sqrt{x^2 + y^2 + z^2}$, the length of the vector $\underline{x}$.

$$\partial w / \partial x = \text{gm } ||\underline{x}||^{-3} \, x \qquad 2(22)$$

(and similarly for y and z).

$$\Lambda = \text{arc tan } y/x \qquad 2(23)$$

$$\Phi = \text{arc tan } z/\sqrt{x^2 + y^2} \qquad 2(24)$$

and

$$W = \text{gm}/\sqrt{x^2 + y^2 + z^2} \qquad 2(25)$$

are the corresponding zero-order mapping equations. They can be inverted into

$$x = \frac{\text{gm}}{W} \cos \Lambda \cos \Phi \qquad 2(26)$$

$$y = \frac{\text{gm}}{W} \sin \Lambda \cos \Phi \qquad 2(27)$$

$$z = \frac{gm}{W} \sin \Phi \qquad 2(28)$$

(excluding, of course, $\phi = \pm \pi/2$). If we know (Λ, Φ, W) and (gm) from observed quantities, Cartesian coordinates of the approximate position vector can be computed. But how can we call the quantities (Λ, Φ, W, gm) observable?

Astronomical longitude and latitude are quantities derived from observations related to geodetic astronomy. Fundamental catalogs and a variety of reductions (precession, nutation, polar motion, aberration, parallax, etc.) are involved. The setup of an observational equation in geodetic astronomy is not routine and assumes, strictly speaking, approximate a priori information about the position of an observer in geometry space. From gravimetric leveling we obtain only potential differences. In order to be able to derive absolute potential, the reference system should contain sufficient information in its definition. In addition, if we introduce length observations and we extend the isoparametric mapping to the isometric case in geometry space by

$$\overline{P_1P_2} = \overline{p_1p_2} , \qquad 2(29)$$

for example, by imposing equal length of an observational line between two points on the Earth's surface and the corresponding points on the telluroid, we will be able to determine a value for gm. More details will be given in section 3.

The next step is a computation of the Bruns matrix.

$$\underline{v} = \begin{bmatrix} \lambda \\ \phi \\ w \end{bmatrix}, \quad \delta\underline{v} = \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w \end{bmatrix} \qquad 2(30)$$

$$C^{\bullet} = - \begin{bmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{bmatrix} \qquad 2(31)$$

$$= \begin{bmatrix} -\dfrac{y}{x^2+y^2} & +\dfrac{x}{x^2+y^2} & 0 \\ -\dfrac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & -\dfrac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & +\dfrac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \\ -\dfrac{gmx}{(x^2+y^2+z^2)^{3/2}} & -\dfrac{gmy}{(x^2+y^2+z^2)^{3/2}} & -\dfrac{gmz}{(x^2+y^2+z^2)^{3/2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{W}{gm}\dfrac{\sin\Lambda}{\cos\Phi} & +\dfrac{W}{gm}\dfrac{\cos\Lambda}{\cos\Phi} & 0 \\ -\dfrac{W}{gm}\cos\Lambda\sin\Phi & -\dfrac{W}{gm}\sin\Lambda\sin\Phi & \dfrac{W}{gm}\cos\Phi \\ -\dfrac{W^2}{(gm)^2}\cos\Lambda\cos\Phi & -\dfrac{W^2}{(gm)^2}\sin\Lambda\cos\Phi & \dfrac{W^2}{(gm)^2}\sin\Phi \end{bmatrix}$$

$$B^{\cdot} = C^{\cdot\,-1}$$

$$A^* = C^{\cdot}\, R_E^T(\lambda,\phi)$$

$$B^* = C^{*^{-1}}$$

$$A^o = C^o \, R_E(\theta_{gr}) \, R_E(\mu,\gamma)$$

$$B^o = C^{o^{-1}}$$

where the dot, asterisk, and circle denote equatorial, horizontal, and ecliptic coordinates, respectively.

A zero-order approximation of the vector $\Delta\underline{x}^*$ is

$$\Delta x^* = 0 \tag{2(32)}$$

$$\Delta y^* = 0 \tag{2(33)}$$

$$\Delta z^* = \frac{gm}{W^2} \, \delta w \tag{2(34)}$$

which corresponds to the original Bruns formula, because within the zero-order approximation

$$\gamma = \frac{gm}{||\underline{x}||^2} = \frac{W^2}{gm} \quad . \tag{2(35)}$$

Finally, figure 5 (p. 71) illustrates the isoparametric mapping in the curvi-linear gravity and Cartesian geometry space.

EXAMPLE 2.2 (longitude, latitude, gravity)

Let us introduce the isoparametric mapping

$$\Lambda_P = \lambda_p, \quad \Phi_P = \phi_p, \quad \Gamma_P = \gamma_p \tag{2(36)}$$

where Λ and Φ are astronomic longitude and latitude, Γ the length of the gravity vector, λ and ϕ geodetic longitude and latitude, and γ the length of the normal gravity vector. We refer also to (Λ, Φ, Γ) as the spherical coordinates of the gravity vector. Alternatively, a Cartesian representation of 2(36) is the vector identity

$$\underline{\Gamma}_P = \underline{\gamma}_p \qquad 2(37)$$

or

$$\Gamma_x = \gamma_x \qquad 2(38)$$

$$\Gamma_y = \gamma_y \qquad 2(39)$$

$$\Gamma_z = \gamma_z \qquad 2(40)$$

in an "Eulerian" frame f'. Because

$$\Gamma_x = \Gamma \cos \Lambda \cos \Phi \qquad 2(41)$$

$$\Gamma_y = \Gamma \sin \Lambda \cos \Phi \qquad 2(42)$$

$$\Gamma_z = \Gamma \sin \Phi \qquad 2(43)$$

the spherical and the Cartesian mappings are equivalent (excluding again $\phi = \pm \pi/2$).

Corresponding to the first example, astronomical longitude and latitude are observed by astronomical instruments, and the length of the gravity vector by gravimeters. In addition, the scale of gravity space has to be included in the reference system.

A zero-order approximation of the actual gravity potential is

$$w = - gm ||\underline{x}||^{-1} \qquad 2(44)$$

$$\partial w/\partial x = - gm ||\underline{x}||^{-3} x = \gamma \qquad 2(45)$$

and similarly for y and z.

$$\Lambda = \text{arc tan } y/x \qquad 2(46)$$

$$\Phi = \text{arc tan } z/\sqrt{x^2+y^2} \qquad 2(47)$$

$$\Gamma = gm ||\underline{x}||^{-2} \qquad 2(48)$$

are the global mapping equations. They can be inverted into

$$x = \sqrt{\frac{gm}{\Gamma}} \cos \Lambda \cos \Phi \qquad 2(49)$$

$$y = \sqrt{\frac{gm}{\Gamma}} \sin \Lambda \cos \Phi \qquad 2(50)$$

$$z = \sqrt{\frac{gm}{\Gamma}} \sin \Phi \qquad 2(51)$$

(excluding $\phi = \pm \pi/2$). Compare 2(26), 2(27), 2(28) to 2(49), 2(50), 2(51) to see that only the "radial" component has changed from gm W^{-1} to $\sqrt{gm\ \Gamma^{-1}}$.

A "Cartesian" proof of 2(49), 2(50), 2(51) follows. Starting with global mapping equations

$$\Gamma_x = \gamma_x = - gm \, ||\underline{x}||^{-3} \, x \qquad 2(52)$$

$$\Gamma_y = \gamma_y = - gm \, ||\underline{x}||^{-3} \, y \qquad 2(53)$$

and

$$\Gamma_z = \gamma_z = - gm \, ||\underline{x}||^{-3} \, z \, . \qquad 2(54)$$

We write these in the general form:

$$A = \frac{a}{(x^2+y^2+z^2)^{3/2}} \, x \qquad 2(55)$$

$$B = \frac{a}{(x^2+y^2+z^2)^{3/2}} \, y \qquad 2(56)$$

$$C = \frac{a}{(x^2+y^2+z^2)^{3/2}} \, z \quad . \qquad 2(57)$$

Insert 2(57) into 2(55) and 2(56) to derive

$$A = \frac{C}{z} \, x \qquad 2(58)$$

$$B = \frac{C}{z} \, y \qquad 2(59)$$

or

$$x = AC^{-1} \, z \qquad 2(60)$$

$$y = BC^{-1} \, z \quad . \qquad 2(61)$$

Writing 2(57) as

$$(x^2+y^2+z^2)^{3/2} = C^{-1}\ az \qquad 2(62)$$

leads to

$$(A^2C^{-2} + B^2C^{-2} + 1)^{3/2}\ z^3 = C^{-1}\ az \qquad 2(63)$$

$$(A^2 + B^2 + C^2)^{3/2}\ C^{-3}\ z^2 = C^{-1}\ a \qquad 2(64)$$

$$z^2 = a\ C^2\ (A^2 + B^2 + C^2)^{-3/2} \qquad 2(65)$$

$$x = \Gamma_x\ \Gamma_z^{-1}\ z = \sqrt{\frac{gm}{\Gamma}}\cos\Lambda\ \cos\Phi \qquad 2(66)$$

$$y = \Gamma_y\ \Gamma_z^{-1}\ z = \sqrt{\frac{gm}{\Gamma}}\sin\Lambda\ \cos\Phi \qquad 2(67)$$

$$z = (\pm)\ \sqrt{gm}\ \Gamma_z \left(\Gamma_x^{\ 2} + \Gamma_y^{\ 2} + \Gamma_z^{\ 2}\right)^{-3/4} = \sqrt{\frac{gm}{\Gamma}}\sin\Phi\ . \qquad 2(68)$$

A first-order approximation of the actual gravity potential is

$$w = -\ gm\ ||\underline{x}||^{-1} - \omega^2\ (x^2 + y^2) \qquad 2(69)$$

where ω is the length of the rotation (vorticity) vector.

$$\partial w/\partial x = \gamma_x = (+\ gm\ ||\underline{x}||^{-3} - 2\omega^2)\ x \qquad 2(70)$$

$$\partial w/\partial y = \gamma_y = (+\ gm\ ||\underline{\underline{x}}||^{-3} - 2\omega^2)\ y \tag{2(71)}$$

$$\partial w/\partial z = \gamma_z = +\ gm\ ||\underline{\underline{x}}||^{-3}\ z\quad . \tag{2(72)}$$

Equations 2(70), 2(71), 2(72) can be written in the general form:

$$A = \left[\frac{a}{(x^2+y^2+z^2)^{3/2}} + b\right]\ x \tag{2(73)}$$

$$B = \left[\frac{a}{(x^2+y^2+z^2)^{3/2}} + b\right]\ y \tag{2(74)}$$

$$C = \frac{a}{(x^2+y^2+z^2)^{3/2}}\ z\quad . \tag{2(75)}$$

Insert 2(75) into 2(73) and 2(74) to obtain

$$A = (\frac{C}{z} + b)\ x \tag{2(76)}$$

$$B = (\frac{C}{z} + b)\ y \tag{2(77)}$$

or

$$x = Az\ (B + bz)^{-1} \tag{2(78)}$$

$$y = Bz\ (C + bz)^{-1} \tag{2(79)}$$

which, together with 2(57), is written as

$$(x^2 + y^2 + z^2)^3 = C^{-2} a^2 z^2 \quad , \qquad 2(80)$$

leads to

$$z^4[(A^2 + B^2) + (C + bz)^2]^3 = C^{-2} a^2 (C + bz)^6 \quad . \qquad 2(81)$$

This is an equation of the tenth order, i.e., of the form

$$z^{10} + \alpha z^9 + \ldots + \beta = 0 \quad . \qquad 2(82)$$

Equation 2(82) gives a set of solutions for z, then 2(78) for x, and 2(79) for y. Thus we have inverted 2(38), 2(39), 2(40) in gravity space into equations in geometry space. Of course, the inversion is not single-valued, but the solution space can be easily obtained.

The next setup will be a computation of the Bruns matrix.

$$\underline{\gamma} = \begin{bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{bmatrix} , \quad \delta\underline{\gamma} = \begin{bmatrix} \delta\gamma_x \\ \delta\gamma_y \\ \delta\gamma_z \end{bmatrix} \qquad 2(83)$$

$$C^{\bullet} = -\partial\underline{\gamma}/\partial\underline{x} \qquad 2(84)$$

$$C^{\bullet} = 3\ gm\ ||\underline{x}||^{-5} \begin{bmatrix} x - \frac{1}{3}||\underline{x}||^2 + 2\omega^2 & xy & xz \\ xy & y^2 - \frac{1}{3}||\underline{x}||^2 + 2\omega^2 & yz \\ xz & yz & z^2 - \frac{1}{3}||\underline{x}||^2 \end{bmatrix} \qquad 2(85)$$

to a first-order approximation.

Because the gravity vector field to the first order is conservative, div $\delta \underline{v} = 0$, rot $\delta \underline{v} = 0$ leads via the Bruns transformation to div $\Delta \underline{x} = 0$, rot $\Delta \underline{x} = 0$. Thus if $\delta \underline{v}$ is taken from the space of spherical harmonics, so is $\Delta \underline{x}$.

Figure 6 (p. 72) illustrates the isoparametric mapping in the Cartesian gravity and geometric space.

EXAMPLE 2.3 (longitude, latitude, gravity gradient)

Let us introduce the isoparametric mapping

$$\Lambda_P = \lambda_p, \ \Phi_P = \phi_p, \ W_{ij}{}^P = w_{ij}{}^p \qquad 2(86)$$

where Λ and Φ are astronomic longitude and latitude, W_{ij} second-order gradients of the actual gravity potential (i,j ranging over x,y,z), λ and ϕ geodetic longitude and latitude, and w_{ij} second-order gradients of the normal gravity potential in the horizontal triad. Specific gravity gradients W_{xz}, W_{yz}, W_{xy} and $W_\Delta = W_{yy} - W_{xx}$ are assumed to be measured by a torsion balance, or any W_{ij} by a Gradiometer.

The first problem is to find a representation of model gravity gradients in the equatorial triad. Because of the transformation $f^{\cdot} \to e^* = R_E(\lambda,\phi) f^{\cdot}$, the first-order gradient tensor (grad $\underline{\gamma}$) can be determined by

$$(\text{grad } \underline{\gamma})^* = R_E(\lambda,\phi) \ (\text{grad } \underline{\gamma})^{\cdot} \ R_E^T(\lambda,\phi). \qquad 2(87)$$

For instance ,

$$\begin{aligned} w_{x^*z^*} = & - 3 \text{ gm } ||\underline{x}||^{-5} (x^2 - z^2) \cos\lambda \sin\phi \cos\phi \qquad 2(88) \\ & + 3 \text{ gm } ||\underline{x}||^{-5} xz \cos\lambda \cos 2\phi \\ & + 3 \text{ gm } ||\underline{x}||^{-5} xy \sin\lambda \cos\phi \\ & + 3 \text{ gm } ||\underline{x}||^{-5} yz \sin\lambda \sin\phi \end{aligned}$$

$$w_{y*z*} = +3\ gm\ ||\underline{x}||^{-5}\ (x^2 - z^2)\ \sin\lambda\ \sin\phi\ \cos\phi \qquad 2(89)$$
$$-3\ gm\ ||\underline{x}||^{-5}\ xz\ \sin\phi\ \cos 2\phi$$
$$+3\ gm\ ||\underline{x}||^{-5}\ xy\ \cos\lambda\ \cos\phi$$
$$+3\ gm\ ||\underline{x}||^{-5}\ yz\ \cos\lambda\ \sin\phi$$

$$w_{x*y*} = +3\ gm\ ||\underline{x}||^{-5}\ x^2\ \sin\lambda\ \cos\lambda\ \sin^2\phi \qquad 2(90)$$
$$-3\ gm\ ||\underline{x}||^{-5}\ y^2\ \sin\lambda\ \cos\lambda$$
$$+3\ gm\ ||\underline{x}||^{-5}\ z^2\ \sin\lambda\ \cos\lambda\ \cos^2\phi$$
$$+3\ gm\ ||\underline{x}||^{-5}\ xy\ \cos 2\lambda\ \sin\phi$$
$$-3\ gm\ ||\underline{x}||^{-5}\ xz\ \sin 2\lambda\ \sin\phi\ \cos\phi$$
$$-3\ gm\ ||\underline{x}||^{-5}\ yz\ \cos 2\lambda\ \cos\phi\ .$$

If we choose the zero-order approximation of the normal gravity potential

$$w = +\ gm\ ||\underline{x}||^{-1} \qquad 2(91)$$

the gravity gradients are given by

$$w_{ij} = -3\ gm\ ||\underline{x}||^{-5}\ \left[(\underline{e}_i^T\underline{x})(\underline{e}_j^T\underline{x}) - \frac{1}{3}||\underline{x}||^2\ \delta_{ij}\right]\ . \qquad 2(92)$$

Examples of 2(92) are

$$w_{x^{\cdot}x^{\cdot}} = -3\ gm\ ||\underline{x}||^{-5}\left[\frac{2}{3}x^2 - \frac{1}{3}y^2 - \frac{1}{3}z^2\right] \qquad 2(93)$$

$$w_{x^{\cdot}y^{\cdot}} = -3\ gm\ ||\underline{x}||^{-5}\ xy\ . \qquad 2(94)$$

If we are interested in the zero-order approximation 2(91), the isoparametric mapping equations 2(86) can be summed up to be

$$\Lambda = \text{arc tan } y/x \qquad 2(95)$$

$$\Phi = \text{arc tan } z/\sqrt{x^2+y^2} \qquad 2(96)$$

$$W_{ij} = w_{ij}\ (x, y, z) \quad . \qquad 2(97)$$

The general solution can be represented by

$$x = ||\underline{x}|| \cos \Lambda \cos \Phi \qquad 2(98)$$

$$y = ||\underline{x}|| \sin \Lambda \cos \Phi \qquad 2(99)$$

$$z = ||\underline{x}|| \sin \Phi \ . \qquad 2(100)$$

Scale is taken from

$$W_{ij} = gm\ ||\underline{x}||^{-3}\ f_{ij}\ (\Lambda,\Phi) \qquad 2(101)$$

$$||\underline{x}|| = \sqrt[3]{f_{ij}\ (\Lambda,\Phi)} \quad \sqrt[3]{\frac{gm}{W_{ij}}} \quad \text{(no summation)} \qquad 2(102)$$

where f_{ij} (Λ,ϕ) is a specific expression, an example of which, for i=x and j=z, is shown below. Substituting equations 2(98) to 2(100) into 2(88) yields

$$\begin{aligned} w_{x*z*} &= +\ 3\ gm\ ||\underline{x}||^{-3} \left\{(\sin^2\Phi - \cos^2\Lambda \cos^2\Phi) \cos \Lambda \sin \Phi \cos \Phi \right. \\ &\quad \left. + \cos^2\Lambda \sin \Phi \cos \Phi \cos 2\Phi + \sin^2\Lambda \cos \Lambda \cos^3\Phi + \sin^2\Lambda \sin^2\Phi \cos \Phi\right\} \\ &= gm\ ||\underline{x}||^{-3}\ f_{xz}(\Lambda,\Phi) . \end{aligned} \qquad 2(103)$$

Compare 2(26) - 2(28), 2(49) - 2(51) and 2(98) - 2(103) in order to see that only the "radial" component has changed in the following way:

$$\frac{gm}{W} \qquad 2(104)$$

$$\sqrt{\frac{gm}{\Gamma}} \qquad 2(105)$$

$$\sqrt[3]{\frac{gm}{W_{ij}}}\ f_{ij}\ (\Lambda,\Phi) \qquad \text{(no summation).} \qquad 2(106)$$

Thus any coordinate of the Cartesian tensor of gravity gradients is as easily chosen as another.

In addition to the isoparametric mapping of 2(86), I tried one with only the gravity gradients mapped isoparametrically, but the mapping equations turned out extremely nonlinear and I have been unable to invert them.

The next step is the computation of the Bruns matrix.

$$v = \begin{bmatrix} \lambda \\ \phi \\ w_{ij} \end{bmatrix}, \quad \delta v = \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} \qquad 2(107)$$

$$C^{\cdot} = - \begin{bmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ \partial w_{ij}/\partial x & \partial w_{ij}/\partial y & \partial w_{ij}/\partial z \end{bmatrix}. \qquad 2(108)$$

The first two rows of the matrix $C^{\cdot}$ were computed within 2(31); in addition

$$\partial w_{ij}/\partial x,\ \partial w_{ij}/\partial y,\ \partial w_{ij}/\partial z \qquad 2(109)$$

have to be computed. Using 2(92) we arrive at

$$\partial w_{ij}/\partial \underline{x} = \qquad 2(110)$$

$$-3\ gm\ ||\underline{x}||^{-7} \left\{-5\ \underline{e}_i^T\underline{x}\ \underline{e}_j^T\underline{x}\ \underline{x}^T + ||\underline{x}||^2 \left(\delta_{ij}\underline{x}^T + \underline{e}_i^T\underline{x}\ \underline{e}_j^T + \underline{e}_j^T\underline{x}\ \underline{e}_i^T\right)\right\}$$

$$B^{\bullet} = C^{\bullet -1} \ .$$

Finally, figure 7 (p. 72) illustrates the isoparametric mapping in the generalized gravity and geometric space.

Let us summarize what the examples tell us. If we refer to the isotropic (zero-order) approximation of the normal gravity field

$$w = -\ gm\ ||\underline{x}||^{-1}$$

we can represent zero-order coordinates of telluroid points by

$$x = ||\underline{x}||\ \cos\Lambda\ \cos\Phi$$

$$y = ||\underline{x}||\ \sin\Lambda\ \cos\Phi$$

$$z = ||\underline{x}||\ \sin\Phi$$

where scale is taken from a quantity referring to the gravity field, like $||\underline{x}||$, as given by 2(104), 2(105), 2(106). For a higher order normal gravity field the (x, y, z) representation is more complicated as can be inferred from 2(78), 2(79) and 2(82). In addition, the examples emphasize the nearly arbitrary choice of the isoparametric mapping $p \rightarrow P$. As we will see in the next chapter, the isoparametric mapping

$$\Lambda_P = \lambda_p,\ \Phi_P = \phi_p,\ W_P = w_p$$

leads to the Stokes approximation of the geodetic boundary value problem and its finite element form.

3. THE DUAL SETUP OF GEODETIC OBSERVATIONAL EQUATIONS

Once we have decided upon the reference system in either the geometry, gravity, or vorticity space, we are able to set up geodetic observational equations. In general, these depend on coordinates in these spaces. Let us assume for a moment we know the approximate coordinates such that we can linearize observational equations. The quantities "observed minus computed" $\underline{Y}_P - \underline{y}_P$ can be represented by the gradients with respect to these coordinates, such as

$$\underline{Y}_P - \underline{y}_p = (\text{grad}_x\ \underline{y}_P)_p\ \Delta\underline{x} + \delta\underline{y}_p \qquad 3(1)$$

where δy is the disturbance vector. There are geodetic observational equations which depend on coordinates only of the geometry space, but, in general, they are a function of gravity space coordinates of Hilbert type, e.g., spherical harmonic coefficients.

To present the idea of dual setup of geodetic observational equations in a simple way, we will start with a priori parameters which describe the normal gravity field, e.g., (gm).

If we use the dual Bruns transformation

$$\Delta\underline{x} = B\ \delta\underline{v} \qquad 3(2)$$

$$\delta\underline{v} = C\ \Delta\underline{x} \qquad 3(3)$$

which expresses coordinate corrections in the geometry space in terms of disturbances in the gravity space (with $BC = I$, $\det B \neq 0$, $\det C \neq 0$) to replace geometric coordinate corrections by gravimetric coordinate disturbances and vice versa, we arrive at the observational equations

$$\underline{Y}_P - \underline{y}_p = A_1\ \Delta\underline{x} \qquad 3(4)$$

$$\underline{Y}_P - \underline{y}_p = A_2\ \delta\underline{y}\ . \qquad 3(5)$$

These depend either on geometric or on gravimetric unknowns. Thus we have two alternatives in adjusting a geodetic network, a geometric mode or a gravimetric mode, as shown by the following examples.

EXAMPLE 3.1 (geometry space)

As shown in example 1.1, geometric positional angles or length ratios are independent of a reference system with degrees of freedom for translation, rotation, and scale. The linearized observational equations read

$$\underline{Y}^x_{312} - \underline{y}^x_{312} = \Delta\underline{y}^x_{312} = \left[A_{11} \mid A_{12} \mid A_{13}\right] \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \\ \hline \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \\ \hline \Delta x_3 \\ \Delta y_3 \\ \Delta z_3 \end{bmatrix} = A_1 \, \Delta\underline{x} \qquad 3(6)$$

where the matrices A_{1i} are functions of the coordinates (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) at the points (p_1, p_2, p_3), and $\underline{Y}^x_{312}$ represents either positional angles or length ratios. If horizontal directions and horizon distances have been observed, they are correlated observations, in general. (See formula 1(16).)

The vector $\Delta\underline{x}$ of geometric coordinate corrections can now be transformed into gravimetric coordinate disturbances by the three-dimensional Bruns formula, e.g., 2(31), 2(84) or 2(108).

(a) Isoparametric mapping of type longitude, latitude, geopotential:

$$\begin{bmatrix} \Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot} \end{bmatrix} = B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w \end{bmatrix} \qquad 3(7)$$

$$\delta\lambda = \left(\gamma_x^2 + \gamma_y^2\right)^{-1} \left(-\gamma_y \frac{\partial \delta w}{\partial x} + \gamma_x \frac{\partial \delta w}{\partial y}\right) \qquad 3(8)$$

$$\delta\phi = \gamma^{-2} \left(\gamma_x^2 + \gamma_y^2\right)^{-1/2} \left[-\gamma_x\gamma_z \frac{\partial \delta w}{\partial x} - \gamma_y\gamma_z \frac{\partial \delta w}{\partial y} + \left(\gamma_x^2 + \gamma_y^2\right) \frac{\partial \delta w}{\partial z}\right] \qquad 3(9)$$

(Grafarend (1978a): formulas 1(21), 1(22)).

$$\underline{Y}^{x}_{312} - \underline{y}^{x}_{312} = \Delta\underline{y}^{x}_{312} = \left[A_{11}\; B_1 \mid A_{12}\; B_2 \mid A_{13}\; B_3\right] \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta w_1 \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta w_2 \\ \delta\lambda_3 \\ \delta\phi_3 \\ \delta w_3 \end{bmatrix} = A_2\, \delta\underline{y} \quad . \qquad 3(10)$$

Thus, we have found that the observed geometric positional angles or distance ratios depend now only on the disturbing potential δw and the coordinates $\partial_i \delta w$ of its gradients. Equations 3(6) and 3(10) are dual.

If we know scale, distance observations can be approached in the same way.

(b) Isoparametric mapping of type longitude, latitude, gravity:

$$\begin{bmatrix} \Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot} \end{bmatrix} = B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta\gamma \end{bmatrix} \qquad 3(11)$$

$\delta\lambda$ 3(8)
$\delta\phi$ 3(9)

$$\delta\gamma = \gamma^{-1}\left(\gamma_x \frac{\partial\delta w}{\partial x} + \gamma_y \frac{\partial\delta w}{\partial y} + \gamma_z \frac{\partial\delta w}{\partial z}\right) \qquad 3(12)$$

(Grafarend (1978b): formulas (1.38)),

or

$$\begin{bmatrix} \Delta x^{\bullet} \\ \Delta y^{\bullet} \\ \Delta z^{\bullet} \end{bmatrix} = B^{\bullet} \begin{bmatrix} \delta\gamma_x \\ \delta\gamma_y \\ \delta\gamma_z \end{bmatrix} . \qquad 3(13)$$

($B^{\bullet}$ is, of course, a general notation, and not equal to $B^{\bullet}$ in 3(11).)

$$\delta\gamma_i = \partial_i \, \delta w \quad . \qquad 3(14)$$

$$\underline{Y}^{x}_{312} - \underline{y}^{x}_{312} = \Delta\underline{y}^{x}_{312} = \left[A_{11} \; B_1 \mid A_{12} \; B_2 \mid A_{13} \; B_3 \right] \begin{bmatrix} \delta\gamma_{x_1} \\ \delta\gamma_{y_1} \\ \delta\gamma_{z_1} \\ \delta\gamma_{x_2} \\ \delta\gamma_{y_2} \\ \delta\gamma_{z_2} \\ \delta\gamma_{x_3} \\ \delta\gamma_{y_3} \\ \delta\gamma_{z_3} \end{bmatrix} = A_2 \, \delta\underline{y} \quad . \qquad 3(15)$$

Thus, we have found that the observed geometric positional angles or distance ratios depend now only on the coordinates $\partial_i\delta w$ of the gradient of the disturbing potential. If we know scale, distance observations can be approached in the same way.

(c) Isoparametric mapping of type longitude, latitude, gravity gradient:

$$\begin{bmatrix} \Delta x^{\bullet} \\ \Delta y^{\bullet} \\ \Delta z^{\bullet} \end{bmatrix} = B^{\bullet} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} \qquad 3(16)$$

$\delta\lambda$ 3(8)
$\delta\phi$ 3(9)

$$\delta w_{ij} = \partial_i \partial_j \, \delta w \qquad 3(17)$$

$$\underline{Y}^x_{312} - \underline{y}^x_{312} = \Delta \underline{y}^x_{312} = \begin{bmatrix} A_{11} & B_1 & | & A_{12} & B_2 & | & A_{13} & B_3 \end{bmatrix} = \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta w^1_{ij} \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta w^2_{ij} \\ \delta\lambda_3 \\ \delta\phi_3 \\ \delta w^3_{ij} \end{bmatrix} = A_2 \, \delta \underline{y} \quad . \qquad 3(18)$$

Thus, we have found that the observed geometric positional angles or distance ratios depend now only on the coordinates $\partial_i \delta w$ and $\partial_i \partial_j \delta w$ of the first- and second-order gradients of the disturbing potential.

If we know scale, distance observations can be approached in the same way.

<u>EXAMPLE 3.2 (gravity space)</u>:

As we have seen in example 1.2 gravimetric positional angles or length ratios are independent of a reference system with degrees of freedom for translation, rotation, and scale. The linearized observational equations read

$$\underline{y}^{\Gamma}_{312} - \underline{y}^{\Gamma}_{312} = \Delta\underline{y}^{\Gamma}_{312} = \left[A_{21} \mid A_{22} \mid A_{23}\right] \begin{bmatrix} \delta\gamma_{x_1} \\ \delta\gamma_{y_1} \\ \delta\gamma_{z_1} \\ \delta\gamma_{x_2} \\ \delta\gamma_{y_2} \\ \delta\gamma_{z_2} \\ \delta\gamma_{x_3} \\ \delta\gamma_{y_3} \\ \delta\gamma_{z_3} \end{bmatrix} = A_2\ \delta\underline{y} \qquad 3(19)$$

where the matrices A_{2i} are functions of the coordinates (γ_{x_1}, γ_{y_1}, γ_{z_1}, γ_{x_2}, γ_{y_2}, γ_{z_2}, γ_{x_3}, γ_{y_3}, γ_{z_3}) at the points (γ_1, γ_2, γ_3) and y^{Γ}_{312} represents either positional angles or length ratios in gravity space. If astronomical longitude, latitude, and gravity have been observed, they are correlated observations, in general. (See formula 1(24).)

The vector δy of gravimetric coordinate corrections can now be transformed into geometric coordinate corrections by the inverse of the three-dimensional Bruns formula, e.g., 2(84).

$$\begin{bmatrix} \delta\gamma_x \\ \delta\gamma_y \\ \delta\gamma_z \end{bmatrix} = C^{\bullet} \begin{bmatrix} \Delta x^{\bullet} \\ \Delta y^{\bullet} \\ \Delta z^{\bullet} \end{bmatrix} \qquad 3(20)$$

$$\underline{y}^{\Gamma}_{312} - \underline{y}^{\Gamma}_{312} = \Delta\underline{y}^{\Gamma}_{312} = \left[A_{21}\ C_1 \mid A_{22}\ C_2 \mid A_{23}\ C_3\right] \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \\ \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \\ \Delta x_3 \\ \Delta y_3 \\ \Delta z_3 \end{bmatrix} = A_1 \Delta\underline{x} \quad . \qquad 3(21)$$

Thus, we have found that the observed gravimetric positional angles or length ratios depend now only on geometric coordinate corrections. Equations 3(15) and 3(21) are dual.

The next step is to assume that we know orientation and scale in gravity space. The linear observational equations for astronomical longitude and latitude, the length of the gravity vector, and potential differences read

$$\Lambda_P - \lambda_p = (\partial\lambda/\partial x)_p \, \Delta x + (\partial\lambda/\partial y)_p \, \Delta y + (\partial\lambda/\partial z)_p \, \Delta z + \delta\lambda \qquad 3(22)$$

$$\Phi_P - \phi_p = (\partial\phi/\partial x)_p \, \Delta x + (\partial\phi/\partial y)_p \, \Delta y + (\partial\phi/\partial z)_p \, \Delta z + \delta\phi \qquad 3(23)$$

$$\Gamma_P - \gamma_p = (\partial\gamma/\partial z)_p \, \Delta x + (\partial\gamma/\partial y)_p \, \Delta y + (\partial\gamma/\partial z)_p \, \Delta z + \delta\gamma \qquad 3(24)$$

$$\begin{aligned} W_2 - W_1 - (w_2 - w_1) = & - (\partial w/\partial x)_{P_1} \, \Delta x^1 - (\partial w/\partial y)_{P_1} \, \Delta y^1 - (\partial w/\partial z)_{P_1} \, \Delta z^1 \\ & + (\partial w/\partial z)_{P_2} \, \Delta x^2 + (\partial w/\partial y)_{P_2} \, \Delta y^2 + (\partial w/\partial z)_{P_2} \, \Delta z^2 \\ & + \delta w_2 - \delta w_1 . \end{aligned} \qquad 3(25)$$

(a) Isoparametric mapping of type longitude, latitude, geopotential:

$$\begin{bmatrix} \Delta x^{\bullet} \\ \Delta y^{\bullet} \\ \Delta z^{\bullet} \end{bmatrix} = B^{\bullet} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w \end{bmatrix} \qquad 3(26)$$

$\delta\lambda$ 3(8)
$\delta\phi$ 3(9)
$\delta\gamma$ 3(12)

$$\Lambda_P - \lambda_p = \Delta\lambda = 0$$
$$\Phi_P - \phi_p = \Delta\phi = 0$$
$$W_P - w_p = \Delta w = 0$$

$$2\ \delta w + \rho\ \frac{\partial \delta w}{\partial \rho} = \frac{\Delta\gamma}{\gamma_p}\ \rho\ \frac{\partial w}{\partial \rho} \qquad 3(27)$$

where

$$\rho = \gamma^{-1/2}$$

(Grafarend (1978a): formula 1(23)).

The observational equation for the length of the gravity vector is well known; it is the boundary condition for the "harmonic" potential δw. Even better known is its zero-order approximation based on 2(21)

$$\frac{2}{r}\ \delta w + \frac{\partial}{\partial r}\ (\delta w) = -\ \Delta\gamma \qquad 3(28)$$

(Grafarend (1978a): formula 1(26)).

Thus, we have found that the observed length of the gravity vector depends only on the disturbing potential δw and the coordinate $\partial\delta w/\partial\rho$ or $\partial\delta w/\partial r$ of its gradient vector, where $r = ||\underline{x}||$.

The dual formulation is obtained if we make use of the inverse three-dimensional Bruns transformation

$$\delta w = C_3\ \Delta\underline{\underline{x}}\quad . \qquad 3(29)$$

Employing the summation convention over i = x,y,z,

$$\delta\underline{\gamma} = \gamma^{-1} C_3 \gamma_i \partial_i \Delta\underline{x} \tag{3(30)}$$

$$\underline{\Gamma}_P - \underline{\gamma}_p = \Delta\underline{\gamma} = A_\gamma \begin{bmatrix}\Delta x \\ \Delta y \\ \Delta z\end{bmatrix} + \gamma^{-1} C_3 \gamma_i \partial_i \begin{bmatrix}\Delta x \\ \Delta y \\ \Delta z\end{bmatrix} \tag{3(31)}$$

where the vector of geometric coordinate corrections $\Delta\underline{x}$ is a "harmonic" function as long as we measure in empty space. Again we have found that the observed length of the gravity vector depends only on the geometry vector $\Delta\underline{x}$ and its gradient. Equations 3(27) (or 3(28)) and 3(31) are dual.

(b) Isoparametric mapping of type longitude, latitude, gravity

$$\begin{bmatrix}\Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot}\end{bmatrix} = B^{\cdot} \begin{bmatrix}\delta\lambda \\ \delta\phi \\ \delta\gamma\end{bmatrix} \tag{3(32)}$$

$\delta\lambda$ 3(8)

$\delta\phi$ 3(9)

$\delta\gamma$ 3(12)

$$\Lambda_P - \lambda_p = \Delta\lambda = 0$$

$$\Phi_P - \phi_p = \Delta\phi = 0$$

$$\Gamma_P - \gamma_p = \Delta\gamma = 0$$

$$W_P - w_p = \Delta w = \delta w + \frac{\rho}{2} \frac{\partial \delta w}{\partial \rho} \tag{3(33)}$$

where

$$\rho = \gamma^{-1/2}$$

(Grafarend (1978b): formula (1.39)).

The observational equation for the potential is well known; it serves alternatively as the boundary condition for the "harmonic" potential δw. For the zero-order approximation 2(21) we find

$$\delta w + \frac{r}{2}\frac{\partial}{\partial r}(\delta w) = \Delta w \quad . \qquad 3(34)$$

Thus, we have found that the absolute potential depends only on the disturbing potential δw and the coordinate $\partial \delta w/\partial \rho$ or $\partial \delta w/\partial r$ of its gradient vector where

$$r = ||\underline{x}|| \quad .$$

For the dual formulation we mention only that we have to integrate $\delta\gamma$ in order to arrive at δw.

(c) Isoparametric mapping of type longitude, latitude, and gravity gradient

$$\begin{bmatrix} \Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot} \end{bmatrix} = B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} \qquad 3(35)$$

$\delta\lambda$ 3(8)

$\delta\phi$ 3(9)

δw_{ij} 3(17)

$$\Lambda_P - \lambda_p = \Delta\lambda = 0$$

$$\Phi_P - \phi_p = \Delta\phi = 0$$

$$W_{ij}{}^P - w_{ij}{}^p = \Delta w_{ij} = 0$$

$$W_P - w_p = (\text{grad } w)_p \; B^{\bullet} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} + \delta w \qquad 3(36)$$

$$\Gamma_P - \gamma_p = (\text{grad } \gamma)_p \; B^{\bullet} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} + \delta\gamma \quad . \qquad 3(37)$$

Thus, we have found that the observed potential and length of the gravity vector depend only on the disturbing potential and coordinates of its first- and second-order gradient.

Another set of geodetic observations depends on the gravity field, e.g., astronomical azimuth A and horizon distance B. Their observational equations are structured according to

$$\begin{aligned} A_{12} - \alpha_{12} = {} & (\partial\alpha_{12}/\partial x^1)\Delta x^1 + (\partial\alpha_{12}/\partial y^1)\Delta y^1 + (\partial\alpha_{12}/\partial z^1)\Delta z^1 \\ & + (\partial\alpha_{12}/\partial x^2)\Delta x^2 + (\partial\alpha_{12}/\partial y^2)\Delta y^2 + (\partial\alpha_{12}/\partial z^2)\Delta z^2 + \delta\alpha_{12} \end{aligned} \qquad 3(38)$$

$$\begin{aligned} B_{12} - \beta_{12} = {} & (\partial\beta_{12}/\partial x^1)\Delta x^1 + (\partial\beta_{12}/\partial y^1)\Delta y^1 + (\partial\beta_{12}/\partial z^1)\Delta z^1 \\ & + (\partial\beta_{12}/\partial x^2)\Delta x^2 + (\partial\beta_{12}/\partial y^2)\Delta y^2 + (\partial\beta_{12}/\partial z^2)\Delta z^2 + \delta\beta_{12} \end{aligned} \qquad 3(39)$$

where α, β are zero-order approximations of A, B and $\delta\alpha$. $\delta\beta$ are subject to the Laplace condition.

$$\begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix} = L \begin{bmatrix} \delta\lambda \\ \delta\phi \end{bmatrix} \qquad 3(40)$$

Here, we assumed that the base vectors $F^{\bullet}$ and $f^{\bullet}$ are parallel in the Euclidean sense. Hence, the observational equations will read

$$\underline{Y}_{12} - \underline{y}_{12} = \Delta\underline{y}_{12} = A_y \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \\ \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \end{bmatrix} + L \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \end{bmatrix} . \qquad 3(41)$$

The equations can be formulated in the geometric mode if we transform $\delta\lambda$, $\delta\phi$ according to the inverse three-dimensional Bruns formula of types (a), (b) or (c). In all cases we will obtain

$$\Delta\underline{y}_{12} = A_2 \, \Delta\underline{x} . \qquad 3(42)$$

The dual representation is found to be

$$\Delta\underline{y}_{12} = A_1 \, \delta\underline{y} \qquad 3(43)$$

where

$$\text{(i)} \quad \delta\underline{y} = \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta w_1 \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta w_2 \end{bmatrix} \qquad \text{(ii)} \quad \delta\underline{y} = \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta\gamma_1 \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta\gamma_2 \end{bmatrix} \qquad \text{(iii)} \quad \delta\underline{y} = \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta w_{ij}^{1} \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta w_{ij}^{2} \end{bmatrix} \qquad 3(44)$$

if we choose different isoparametric mappings.

At this point we introduce the unknowns describing the gravity field of the Earth, for instance, the mass density virials

$$I_o, \; I_i, \; I_{ij}, \; I_{ijk}, \; \ldots$$

given in the appendix. They appear in the form

$$I_o \,||x||^{-1},\; I_i \, x_i \,||x||^{-3},\; I_{ij} \, x_i \, x_j \,||x||^{-5},\; \ldots \qquad 3(45)$$

and must fulfill the Laplace differential equation. Because of this condition, the degree of freedom is 2n+1, where n is the order of approximation in a series for the gravitational potential, e.g., for n = 2 only 5 coefficients of the six

$$I_{11},\; I_{22},\; I_{12},\; I_{13},\; I_{23} \qquad 3(46)$$

are independent as a result of tr $I_{ij} = 0$.

The quantities "observed minus computed" $\underline{Y}_P - \underline{y}_p$ are represented by

$$\begin{aligned} \underline{Y}_P - \underline{y}_p = {} & (\partial\underline{y}/\partial x)_p \Delta x + (\partial\underline{y}/\partial y)_p \Delta y + (\partial\underline{y}/\partial z)_p \Delta z \qquad 3(47) \\ & + (\partial\underline{y}/\partial I_o)_p \, \delta I_o \\ & + (\partial\underline{y}/\partial I_1)_p \, \delta I_1 + (\partial\underline{y}/\partial I_2)_p \, \delta I_2 + (\partial\underline{y}/\partial I_3)_p \, \delta I_3 \\ & + (\partial\underline{y}/\partial I_{11})_p \, \delta I_{11} \\ & + (\partial\underline{y}/\partial I_{22})_p \, \delta I_{22} \\ & + (\partial\underline{y}/\partial I_{12})_p \, \delta I_{12} \\ & + (\partial\underline{y}/\partial I_{13})_p \, \delta I_{13} \\ & + (\partial\underline{y}/\partial I_{23})_p \, \delta I_{23} \\ & + \ldots + \delta\underline{y} \quad . \end{aligned}$$

By the symbol "p" we understand the origin of the Taylor series, a set of approximate coordinates of points in the geometry space and of approximate gravity field parameters. How can these be determined?

To answer this question we present an example.

Let us introduce the isoparametric mapping

$$\begin{bmatrix} \Delta\lambda = 0 \\ \Delta\phi = 0 \\ \Delta w = 0 \\ \Delta\gamma = 0 \end{bmatrix} \qquad 3(48)$$

applied to a zero-order approximation

$$w = - gm \, ||\mathbf{x}||^{-1}$$

of the gravity field.

$$\Lambda = \text{arc tan} \frac{y}{x}$$

$$\Phi = \text{arc tan} \frac{z}{\sqrt{x^2 + y^2}}$$

$$W = - \frac{gm}{\sqrt{x^2 + y^2 + z^2}}$$

$$\Gamma = \frac{gm}{x^2 + y^2 + z^2}$$

are the global mapping equations. The general solution can be represented by

$$x = ||\underline{x}|| \cos \Lambda \cos \Phi$$

$$y = ||\underline{x}|| \sin \Lambda \cos \Phi$$

$$z = ||\underline{x}|| \sin \Phi$$

$$||\underline{x}|| = -\frac{gm}{W} = \sqrt{\frac{gm}{\Gamma}} \qquad 3(49)$$

$$gm = \frac{W^2}{\Gamma} \quad . \qquad 3(50)$$

Thus we have derived approximate values for (x, y, z) and gm:

$$x = \frac{W}{\Gamma} \cos \Lambda \cos \Phi \qquad 3(51)$$

$$y = \frac{W}{\Gamma} \sin \Lambda \cos \Phi \qquad 3(52)$$

$$z = \frac{W}{\Gamma} \sin \Phi \qquad 3(53)$$

$$gm = \frac{W^2}{\Gamma} \quad . \qquad 3(54)$$

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APPENDIX A.--INVARIANT REPRESENTATION OF $||x - x'||^{-m}$

In terms of Hilbert invariants

$$i = ||\underline{x}|| \tag{A1}$$

$$i' = ||\underline{x}'|| \tag{A2}$$

$$i'' = \frac{(\underline{x}, \underline{x}')}{||\underline{x}|| \; ||\underline{x}'||} \tag{A3}$$

we will set up series for

$$||\underline{x} - \underline{x}'||^{-m} = i^{-m} \left[1 + \frac{i'^2}{i^2} - 2\,\frac{i'}{i}\,i''\right]^{\frac{m}{2}} \quad . \tag{A4}$$

Set

$$\varepsilon = \frac{i'^2}{i^2} - 2\,\frac{i'}{i}\,i'' \tag{A5}$$

$$(1+\varepsilon)^{-\frac{m}{2}} = 1 - \frac{m}{1!\;2}\,\varepsilon + \frac{m(m+2)}{2!\;4}\,\varepsilon^2 \tag{A6}$$

$$- \frac{m(m+2)(m+4)}{3!\;8}\,\varepsilon^3 + \frac{m(m+2)(m+4)(m+6)}{4!\;16}\,\varepsilon^4$$

$$- o(\varepsilon^5)$$

$$= \sum_{n=o}^{\infty} (-1)^n \frac{m(m+2)(m+4)\ldots(m+2n-2)}{n!\;2^n}\,\varepsilon^n \quad .$$

$i'' \leq 1$; thus the series are convergent within the sphere determined by $\frac{||\underline{x}||}{||\underline{x}||} < 1$. Under this assumption we can represent (A6) by

$$||\underline{x}-\underline{x}'||^{-m} = \sum_{n=o}^{\infty} i'^{\,n} \; i^{-n-m} \; P_n^m(i'') \qquad \text{(A7)}$$

where $P_n^m(i'')$ are Gegenbauer (ultraspherical) polynomials (Abramowitz and Stegun, 1964: p. 774)

$$P_o^m(x) = 1 \qquad \text{(A8)}$$

$$P_1^m(x) = mx \qquad \text{(A9)}$$

$$P_2^m(x) = \frac{m}{2}\,[(m+2)x^2 - 1] \qquad \text{(A10)}$$

$$P_3^m(x) = \frac{m(m+2)}{6}\, x\,[(m+4)x^2 - 3] \qquad \text{(A11)}$$

$$P_4^m(x) = \frac{m(m+2)}{24}\,[(m+4)(m+6)x^4 - 6\,(m+4)x^2 + 3] \qquad \text{(A12)}$$

...

$$P_n^m(x) = \frac{1}{n!}\left\{m(m+2)\ldots(m+2n-2)x^n\right.$$

$$- \frac{n(n-1)}{2}\, m(m+2)\ldots(m+2n-4)x^{n-2}$$

(A13)

$$+ \frac{n(n-1)(n-2)(n-3)}{2\cdot 4} m(m+2)\ldots(m+2n-6)x^{n-4}$$

$$- \ldots$$

$$\left. + (-1)^{\ell} \frac{n(n-1)\cdots(n-2\ell+1)}{2\cdot 4\cdots 2\ell} m(m+2)\cdots[m+2(n-\ell-1)]\; x^{n-2\ell} \right\} ,$$

$$\ell = 0,1,\cdots \leq \frac{n}{2} \quad .$$

Despite the identical notation, these should not be confused with associated Legendre polynomials.

APPENDIX B.--CARTESIAN REPRESENTATION OF THE GRAVITATIONAL POTENTIAL

In the appendices B-E, we change to index notation (x_1, x_2, x_3) in place of (x, y, z) and utilize the summation convention.

Let

$$u(\underline{x}) = - g \int dx' \rho(\underline{x}') \, ||\underline{x}-\underline{x}'||^{-1} \tag{B1}$$

be the Newtonian representation of the gravitational potential. In terms of Hilbert invariants

$$i = ||\underline{x}|| \tag{B2}$$

$$i' = ||\underline{x}'|| \tag{B3}$$

$$i'' = \frac{(\underline{x},\underline{x}')}{||\underline{x}|| \; ||\underline{x}'||} \tag{B4}$$

$$||\underline{x}-\underline{x}'||^{-1} = \sum_{n=o}^{\infty} i'^{n} \, i^{-n-1} \, P_n(i'') \tag{B5}$$

holds if $\frac{||\underline{x}'||}{||\underline{x}||} < 1$. Under this assumption the gravitational potential can be represented by

$$u(\underline{x}) = -\sum_{n=o}^{\infty} g \, i^{-n-1} \int dx' i'^{n} \, P_n(i'') \, \rho(\underline{x}') \tag{B6}$$

where P_n are the standard Legendre polynomials.

In cartesian coordinates the Hilbert invariants read

$$i = ||\underline{x}|| = \sqrt{x_i x_i} \tag{B7}$$

$$i' = ||\underline{x}'|| = \sqrt{x'_i x'_i} \tag{B8}$$

$$i'' = \frac{x_i x'_i}{\sqrt{x_i x_i}\sqrt{x'_i x'_i}} \tag{B9}$$

and the related Legendre polynomials $P_n(i'')$

$$P_o(i'') = 1 \tag{B10}$$

$$P_1(i'') = x_i x_i' \div \left(\sqrt{x_j x_j}\ \sqrt{x'_k x'_k}\right) \tag{B11}$$

$$P_2(i'') = \frac{3}{2}\ \frac{x_i x'_i\ x_j x'_j - \frac{1}{3}\ x_p x_p\ x'_q x'_q}{x_k x_k\ x'_\ell x'_\ell} \tag{B12}$$

$$P_3(i'') = \frac{5}{2}\ \frac{x_i x'_i}{\sqrt{x_j x_j}\ \sqrt{x'_k x'_k}}\ \frac{x_\ell x'_\ell\ x_m x'_m - \frac{3}{5}\ x_r x_r\ x'_s x'_s}{x_p x_p\ x'_q x'_q} \tag{B13}$$

$$P_4(i'') = \frac{35}{8}\ \frac{(x_i x'_i)^4 - \frac{6}{7}\ (x_i x'_i)(x_p x_p)(x'_q x'_q) + \frac{3}{35}\ (x_r x_r)^2 (x'_s x'_s)^2}{(x_j x_j)^2 (x'_k x'_k)^2} \tag{B14}$$

etc.

(B6) can be written

In cartesian coordinates the Hilbert invariants read

$$i = ||\underline{x}|| = \sqrt{x_i x_i} \tag{B7}$$

$$i' = ||\underline{x}'|| = \sqrt{x_i' x_i'} \tag{B8}$$

$$i'' = \frac{x_i x_i'}{\sqrt{x_i x_i}\sqrt{x_i' x_i'}} \tag{B9}$$

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$$P_2(i'') = \frac{3}{2}\,\frac{x_i x_i' \; x_j x_j' - \frac{1}{3}\, x_p x_p \; x_q' x_q'}{x_k x_k \; x_\ell' x_\ell'} \tag{B12}$$

$$P_3(i'') = \frac{5}{2}\,\frac{x_i x_i'}{\sqrt{x_j x_j}\sqrt{x_k' x_k'}}\;\frac{x_\ell x_\ell' \; x_m x_m' - \frac{3}{5}\, x_r x_r \; x_s' x_s'}{x_p x_p \; x_q' x_q'} \tag{B13}$$

$$P_4(i'') = \frac{35}{8}\,\frac{(x_i x_i')^4 - \frac{6}{7}\,(x_i x_i')(x_p x_p)(x_q' x_q') + \frac{3}{35}\,(x_r x_r)^2 (x_s' x_s')^2}{(x_j x_j)^2 (x_k' x_k')^2} \tag{B14}$$

etc.

(B6) can be written

$$u(x) = \sum_{n=o}^{\infty} u_n \tag{B15}$$

where

$$u_o = -\frac{gm}{\sqrt{x_i x_i}} = -\frac{g}{(x_i x_i)^{1/2}} I_o \tag{B16}$$

$$u_1 = -\frac{g x_i}{(x_j x_j)^{3/2}} \int dm' x_i' = -\frac{g}{(x_j x_j)^{3/2}} x_i I_i \tag{B17}$$

$$u_2 = -\frac{3g}{2(x_k x_k)^{5/2}} x_i x_j \int dm' x_i' x_j' + \frac{g}{2(x_k x_k)^{3/2}} dm' x_i' x_i' \tag{B18}$$

$$= -\frac{3g}{2(x_k x_k)^{5/2}} x_i x_j I_{ij}$$

$$u_3 = -\frac{5g}{2(x_\ell x_\ell)^{7/2}} x_i x_j x_k \int dm' x_i' x_j' x_k' \tag{B19}$$

$$+ \frac{3g}{2(x_\ell x_\ell)^{3/2}} x_i \delta_{jk} =$$

$$- \frac{5g}{2(x_\ell x_\ell)^{7/2}} x_i x_j x_k I_{ijk}$$

$$u_4 = -\frac{35g}{8(x_n x_n)^{9/2}} x_i x_j x_k x_\ell \int dm' \left[x_i' x_j' x_k' x_\ell' - \frac{6}{7} x_i' x_j' \delta_{k\ell} (x_m' x_m') + \frac{3}{35} \delta_{ij} \delta_{k\ell} (x_m' x_m')^2\right] = -\frac{35g}{8(x_m x_m)^{9/2}} x_i x_j x_k x_\ell I_{ijk\ell} \quad \text{(B20)}$$

and

$$I_o = \int dm = m \quad \text{(B21)}$$

$$I_i = \int dm \; x_i \quad \text{(B22)}$$

$$I_{ij} = \int dm \left[x_i x_j - \frac{1}{3} (x_k x_k) \delta_{ij}\right] \quad \text{(B23)}$$

$$I_{ijk} = \int dm \left[x_i x_j x_k - \frac{3}{5} (x_\ell x_\ell) x_i \delta_{jk}\right] \quad \text{(B24)}$$

$$I_{ijk\ell} = \int dm \left[x_i x_j x_k x_\ell - \frac{6}{7} x_i x_j \delta_{k\ell} (x_m x_m) + \frac{3}{35} \delta_{ij} \delta_{k\ell} (x_m x_m)^2\right] \quad \text{(B25)}$$

etc.

In summarizing, we can represent the gravitational potential in Cartesian coordinates by

$$u(x) = -\frac{gm}{\sqrt{x_j x_j}} - \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)}{n! \; (x_j x_j)^{(2n+1)/2}} g x_{i_1} x_{i_2} \ldots x_{i_n} I_{i_1 i_2 \ldots i_n} \quad \text{(B26)}$$

where

$$I_{i_1 i_2 \ldots i_n} = \int dm \quad x_{i_1} x_{i_2} \ldots x_{i_n} - \frac{n(n-1)}{2(2n-1)} x_{i_1} x_{i_2} \ldots x_{i_{n-2}} \delta_{i_{n-1} i_n} x_k x_k \tag{B27}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x_{i_1} x_{i_2} \ldots x_{i_{n-4}} \delta_{i_{n-3} i_{n-2}} \delta_{i_{n-1} i_n} (x_k x_k)^2$$

$$- \ldots + (-1)^{\ell} \frac{n(n-1)\ldots(n-2\ell+1)}{2 \cdot 4 \ldots 2\ell (2n-1)(2n-3)\ldots(2n-2\ell+1)}$$

$$\left[\left(x_{i_1} x_{i_2} \ldots x_{i_{n-2\ell}} \delta_{i_{n-(2\ell+1)} i_{n-(2\ell+2)}} \ldots \delta_{i_{n-1} i_n} (x_k x_k)^{\ell}\right)\right] ,$$

$$\ell = 0, 1, \ldots, \leq \frac{n}{2} .$$

APPENDIX C.--CARTESIAN REPRESENTATION OF FIRST-ORDER GRADIENTS OF THE GRAVITATIONAL POTENTIAL

Let

$$\partial u/\partial x_i = + g \int dx' \rho(\underline{x}') \; (x_i - x_i') \; ||\underline{x} - \underline{x}'||^{-3} \tag{C1}$$

be the Newtonian representation of the first-order gradients of the gravitational potential. In terms of Hilbert invariants

$$i = ||\underline{x}|| \tag{C2}$$

$$i' = ||\underline{x}'|| \tag{C3}$$

$$i'' = \frac{(\underline{x},\underline{x}')}{||\underline{x}|| \; ||\underline{x}'||} \tag{C4}$$

$$||\underline{x} - \underline{x}'||^{-3} = \sum_{n=o}^{\infty} i'^{n} \; i^{-n-3} \; P_n^3 \; (i'') \tag{C5}$$

holds if $\frac{||x'||}{||x||} < 1$. Under this assumption the first-order gradients of the gravitational potential can be represented by

$$\partial u/\partial x_i = + \sum_{n=o}^{\infty} g \; i^{-n-3} \int dx' i'^{n} \; P_n^3(i'') \; (x_i - x_i') \; \rho(x') \tag{C6}$$

where P_n^3 are Gegenbauer polynomials (see (A8) - (A13))

(C7)

$$\left[\begin{array}{l} P_0^3(x) = 1 \\ P_1^3(x) = 3x \\ P_2^3(x) = \frac{3}{2}(5x^2-1) \\ P_3^3(x) = \frac{5}{2}x(7x^2-3) \\ P_4^3(x) = \frac{15}{8}(21x^4-14x^2+1) \end{array}\right.$$

...

(C8)

$$P_n^3(x) = \frac{1}{n!}\Bigg[3\cdot 5\ldots(3+2n-2)x^n$$

$$- \frac{n(n-1)}{2}\, 3\cdot 5\ldots(3+2n-4)x^{n-2}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2\cdot 4}\, 3\cdot 5\ldots(3+2n-6)x^{n-4}$$

$$- \ldots$$

$$+ (-1)^{\ell}\frac{n(n-1)\ldots(n-2\ell+1)}{2\cdot 4\ldots 2\ell}\, 3\cdot 5\ldots(2n-2\ell+1)x^{n-2\ell}\Bigg] ,$$

$$\ell = 0,1,\ldots < \frac{n}{2} \,.$$

If we refer to Equations (B15) to (B27) as the Cartesian representation of the gravitational potential, we will arrive at

$$\partial u_o/\partial x_i = + g\ ||\underline{x}||^{-3}\ x_i\ I_o \tag{C9}$$

$$\partial u_1/\partial x_i = - g\ ||\underline{x}||^{-3}\ I_i + 3g\ ||\underline{x}||^{-5}\ x_i x_j\ I_j \tag{C10}$$

$$\partial u_2/\partial x_i = + 3g\ ||\underline{x}||^{-5}\ x_j\ I_{ij} - \frac{15}{2}\ g\ ||\underline{x}||^{-7}\ x_i x_j x_k\ I_{jk} \tag{C11}$$

$$\partial u_3/\partial x_i = - \frac{15}{2}\ g\ ||\underline{x}||^{-7}\ x_j x_k\ I_{ijk} + \frac{35}{2}\ g\ ||\underline{x}||^{-9}\ x_i x_j x_k x_\ell\ I_{jk\ell} \tag{C12}$$

$$\partial u_4/\partial x_i = - \frac{35}{2}\ g\ ||\underline{x}||^{-9}\ x_j x_k x_\ell\ I_{ijk\ell} + \frac{315}{8}\ g\ ||\underline{x}||^{-11}\ x_i x_j x_k x_\ell x_m\ I_{jk\ell m} \tag{C13}$$

or, in general,

$$\begin{aligned}\partial u/\partial x_i = & + g\ ||\underline{x}||^{-3}\ x_i\ I_o \\ & - \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)n}{n!}\ g\ ||\underline{x}||^{-(2n+1)}\ x_{i_1} \ldots x_{i_{n-1}}\ I_{ii_1 \ldots i_{n-1}} \\ & + \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)(2n+1)}{n!}\ g\ ||\underline{x}||^{-(2n+3)}\ x_i x_{i_1} \ldots x_{i_n}\ I_{i_1 \ldots i_n}\end{aligned} \tag{C14}$$

where

$$||\underline{x}|| = \sqrt{x_k x_k}\ .$$

APPENDIX D.--CARTESIAN REPRESENTATION OF SECOND-ORDER GRADIENTS OF THE GRAVITATIONAL POTENTIAL

We refer to equations (C9) to (C14) as the Cartesian representation of the first-order gradients of the gravitational potential. Here we will compute second-order gradients.

$$\partial^2 u_o/\partial x_i \partial x_j = -\,3g\,||\underline{x}||^{-5}\, I_o \left[x_i x_j - \frac{1}{3}\,||\underline{x}||^2\, \delta_{ij}\right] \tag{D1}$$

$$\begin{aligned}\partial^2 u_1/\partial x_i \partial x_j = &+ 3g\,||\underline{x}||^{-5}\,(x_i\, I_j + x_j\, I_i)\\ &- 15g\,||\underline{x}||^{-7}\, x_k\, I_k \left[x_i x_j - \frac{1}{5}\,||\underline{x}||^2\, \delta_{ij}\right]\end{aligned} \tag{D2}$$

$$\begin{aligned}\partial^2 u_2/\partial x_i \partial x_j = &\ 3g\,||\underline{x}||^{-5}\, I_{ij}\\ &- 15g\,||\underline{x}||^{-7}\, x_k\,(x_j\, I_{ik} + x_i\, I_{jk})\\ &+ \frac{105}{2}\, g\,||\underline{x}||^{-9}\, x_k x_\ell\, I_{k\ell} \left[x_i x_j - \frac{1}{7}\,||\underline{x}||^2\, \delta_{ij}\right]\end{aligned} \tag{D3}$$

$$\begin{aligned}\partial^2 u_3/\partial x_i \partial x_j = &-\,15g\,||\underline{x}||^{-7}\, x_k\, I_{ikj}\\ &+ \frac{105}{2}\, g\,||\underline{x}||^{-9}\, x_k x_\ell\,(x_i\, I_{jk\ell} + x_j\, I_{ik\ell})\\ &- \frac{315}{2}\, g\,||\underline{x}||^{-11}\, x_k x_\ell x_m\, I_{k\ell m} \left[x_i x_j - \frac{1}{9}\,||\underline{x}||^2\, \delta_{ij}\right]\end{aligned} \tag{D4}$$

$$\partial^2 u_4/\partial x_i \partial x_j = - \frac{105}{2} g \, ||\underline{x}||^{-9} \, x_k x_\ell \, I_{ijk\ell}$$

$$+ \frac{315}{2} g \, ||\underline{x}||^{-11} \, x_k x_\ell x_m \, (x_i \, I_{jk\ell m} + x_j \, I_{ik\ell m}) \tag{D5}$$

$$- \frac{3465}{8} g \, ||\underline{x}||^{-13} \, x_k x_\ell x_m x_p \, I_{k\ell mp} \left[x_i x_j - \frac{1}{11} ||\underline{x}||^2 \, \delta_{ij}\right]$$

or, in general,

$$\partial^2 u/\partial x_i \partial x_j = - 3g \, ||\underline{x}||^{-5} \, I_o \left[x_i x_j - \frac{1}{3} ||\underline{x}||^2 \, \delta_{ij}\right] \tag{D6}$$

$$- \sum_{n=1}^{\infty} \frac{1 \cdot 3 \ldots (2n-1)(n-1)n}{n!} g \, ||\underline{x}||^{-(2n+1)} \, I_{iji_2 \ldots i_{n-1}}$$

$$+ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \ldots (2n-1)(2n+1)n}{n!} g \, ||\underline{x}||^{-(2n+3)}$$

$$\cdot \, (x_i \, I_{ji_2 \ldots i_n} + x_j \, I_{ii_2 \ldots i_n})$$

$$+ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \ldots (2n-1)(2n+1)(2n+3)}{n!} g \, ||\underline{x}||^{-(2n+5)}$$

$$\cdot \left(x_{i_1} \ldots x_{i_n} \, I_{i_1 \ldots i_{n-1}}\right)\left(x_i x_j - \frac{1}{2n+3} ||\underline{x}||^2 \, \delta_{ij}\right)$$

where $||\underline{x}|| = \sqrt{x_k x_k}$. It is a "nice" exercise to prove

$\mathrm{tr}\left(\partial^2 u_o/\partial x_i \partial x_j\right) = 0$, $\mathrm{tr}\left(\partial^2 u_1/\partial x_i \partial x_j\right) = 0$, $\mathrm{tr}\left(\partial^2 u_2/\partial x_i \partial x_j\right) = 0$,

$\mathrm{tr}\left(\partial^2 u_3/\partial x_i \partial x_j\right) = 0$, $\mathrm{tr}\left(\partial^2 u_4/\partial x_i \partial x_j\right) = 0$, or, in general, $\mathrm{tr}\left(\partial^2 u/\partial x_i \partial x_j\right) = 0$.

APPENDIX E.--CARTESIAN REPRESENTATION OF THE EUCLIDEAN NORM OF FIRST-ORDER GRADIENTS OF THE GRAVITY POTENTIAL

Let $w = u - \frac{1}{2}\omega^2(x_1{}^2 + x_2{}^2)$ be the scalar part of the gravity potential such that $\gamma^2 = (\partial w/\partial x_i)(\partial w/\partial x_i)$ is the square of the Euclidean norm of its first-order gradients. We refer to (C14) as the Cartesian representation of the gravitational potential. The formula

$$\frac{\partial w}{\partial x_i} = + g\,||\underline{x}||^{-3}\,x_i\,I_o$$
$$-\sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)n}{n!}\,g\,||\underline{x}||^{-(2n+1)}\,x_{i_1}\ldots x_{i_{n-1}}\,I_{ii_1\ldots i_{n-1}}$$
$$+\sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)(2n+1)}{n!}\,g\,||\underline{x}||^{-(2n+3)}\,x_i x_{i_1}\ldots x_{i_n}\,I_{i_1\ldots i_n}$$
$$-\,\omega^2\,x_\alpha\,\delta_{\alpha i} \tag{E1}$$

(where Greek indices range over 1, 2 only)

contains four terms and we designate it as a + b + c + d. The four-term scheme leads to

$$\gamma^2 = a^2 + b^2 + c^2 + d^2$$
$$+ 2ab + 2ac + 2ad + 2bc + 2bd + 2cd. \tag{E2}$$

Explicitly, it has the form

$$\gamma^2 = g^2 \, ||\underline{x}||^{-4} \, I_o^{\,2} \tag{E3}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)n}{n!} \, \frac{1\cdot 3 \ldots (2m-1)m}{m!} \, g^2$$

$$||\underline{x}||^{-(2n+1)} \, ||\underline{x}||^{-(2m+1)} \, x_{i_1} \ldots x_{i_{n-1}} \, x_{j_1} \ldots x_{j_{m-1}}$$

$$\cdot \Big(I_{1i_1 \ldots i_{n-1}} \, I_{1j_1 \ldots j_{m-1}} + I_{2i_1 \ldots i_{n-1}} \, I_{2j_1 \ldots j_{m-1}}$$

$$+ I_{3i_1 \ldots i_{n-1}} \, I_{3j_1 \ldots j_{m-1}} \Big)$$

$$+ g^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)(2n+1)}{n!} \, \frac{1\cdot 3 \ldots (2m-1)(2m+1)}{m!}$$

$$||\underline{x}||^{-(2n+2)} \, ||\underline{x}||^{-(2m+2)} \, x_{i_1} \ldots x_{i_n} \, x_{j_1} \ldots x_{j_m} \, I_{i_1 \ldots i_n} \, I_{j_1 \ldots j_m}$$

$$+ \omega^4 \, (x_\alpha x_\alpha)$$

$$- 2g^2 \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)n}{n!} \, ||\underline{x}||^{-(2n+4)} \, x_{i_1} \ldots x_{i_{n-1}} \, I_o$$

$$\cdot \Big(x_1 I_{1i_1 \ldots i_{n-1}} + x_2 I_{2i_1 \ldots i_{n-1}} + x_3 I_{3i_1 \ldots i_{n-1}} \Big)$$

$$+ 2g^2 \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)(2n+1)}{n!} \, ||\underline{x}||^{-(2n+4)} \, x_{i_1} \ldots x_{i_n} \, I_o \, I_{i_1 \ldots i_n}$$

$$- 2g\,\omega^2\,||\underline{x}||^{-3}\,(x_\alpha x_\alpha)\,I_o$$

$$- 2g^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1\cdot3\ldots(2n-1)n}{n!}\,\frac{1\cdot3\ldots(2m-1)(2m+1)}{m!}$$

$$\cdot\,||\underline{x}||^{-(2n+1)}\,||\underline{x}||^{-(2m+3)}\,x_{i_1}\ldots x_{i_{n-1}}\,x_{j_1}\ldots x_{j_m}\,I_{j_1\ldots j_m}$$

$$\left(x_1 I_{1i_1\ldots i_{n-1}} + x_2 I_{2i_1\ldots i_{n-1}} + x_3 I_{3i_1\ldots i_{n-1}}\right)$$

$$+ 2\omega^2 g \sum_{n=1}^{\infty} \frac{1\cdot3\ldots(2n-1)n}{n!}\,||\underline{x}||^{-(2n+1)}\,x_{i_1}\ldots x_{i_{n-1}}$$

$$\cdot\,x_1 I_{1i_1\ldots i_{n-1}} + x_2 I_{2i_1\ldots i_{n-1}}$$

$$- 2\omega^2 g \sum_{n=1}^{\infty} \frac{1\cdot3\ldots(2n-1)(2n+1)}{n!}\,||\underline{x}||^{-(2n+3)}\,(x_\alpha x_\alpha)\,x_{i_1}\ldots x_{i_n}\,I_{i_1\ldots i_n}$$

or

$$\gamma^2 = g^2\,||\underline{x}||^{-4}\,I_o^{\,2} \qquad \text{(E4)}$$

$$+ g^2\,||\underline{x}||^{-6}\,I_i\,I_i$$

$$+ 3g^2\,||\underline{x}||^{-8}\,x_i\,x_j\,I_i\,I_j$$

$$+ 6g^2\,||\underline{x}||^{-8}\,x_i\,I_j\,I_{ij}$$

$$+ 9g^2\,||\underline{x}||^{-10}\,x_i\,x_j\,I_{ki}\,I_{kj}$$

$$+ 15 \, ||\underline{x}||^{-10} \, x_i \, x_j \, x_k \, I_i \, I_{jk}$$

$$+ \frac{225}{4} \, ||\underline{x}||^{-12} \, x_i \, x_j \, x_k \, x_\ell \, I_{ij} \, I_{k\ell}$$

$$+ 4g^2 \, ||\underline{x}||^{-6} \, x_i \, I_o \, I_i$$

$$- 6g^2 \, ||\underline{x}||^{-8} \, x_i \, x_j \, I_i \, I_j$$

$$+ 9g^2 \, ||\underline{x}||^{-8} \, x_i \, x_j \, I_o \, I_{ij}$$

$$+ \omega^4 \, (x_\alpha x_\alpha)$$

$$- 2g\omega^2 \, ||\underline{x}||^{-3} \, (x_\alpha x_\alpha) \, I_o$$

$$+ 2g\omega^2 \, ||\underline{x}||^{-3} \, x_i \, I_i$$

$$+ 6g\omega^2 \, ||\underline{x}||^{-5} \, x_i \, x_\alpha \, I_{i\alpha}$$

$$- 6g\omega^2 \, ||\underline{x}||^{-5} \, x_i \, (x_\alpha x_\alpha) \, I_i$$

$$- 15g\omega^2 \, ||\underline{x}||^{-7} \, x_i \, x_j \, (x_\alpha x_\alpha) \, I_{ij}$$

$$+ o(I_{ijk})$$

up to third-order terms.

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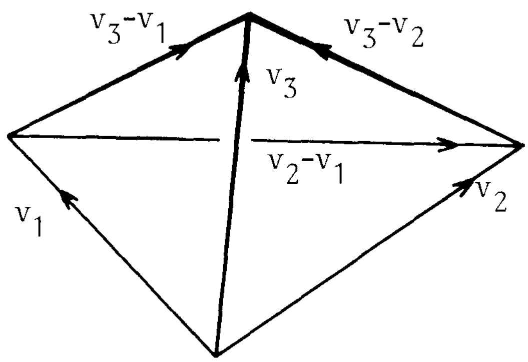

Figure 3.--Triangular network in a vector space.

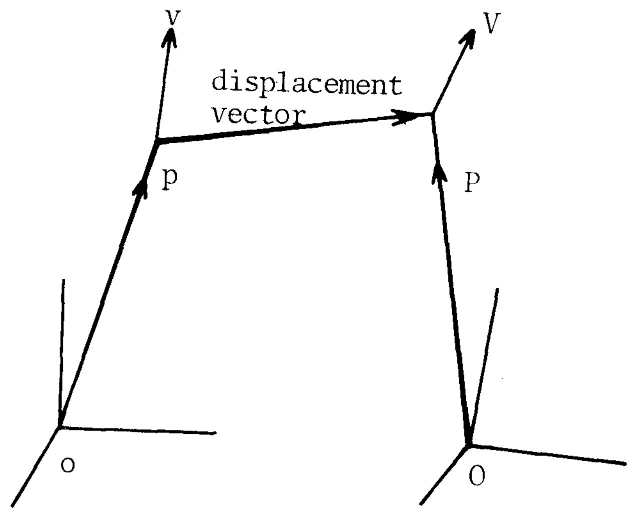

Figure 4.--The displacement vector.

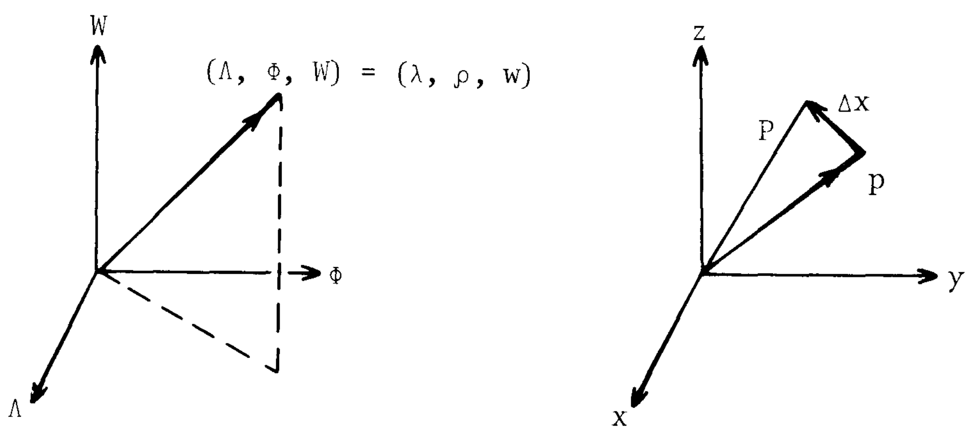

Figure 5.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and potential.

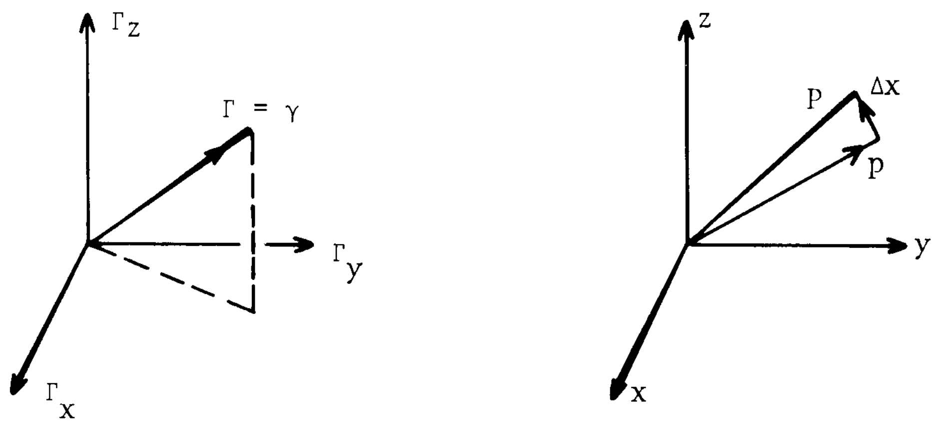

Figure 6.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and gravity.

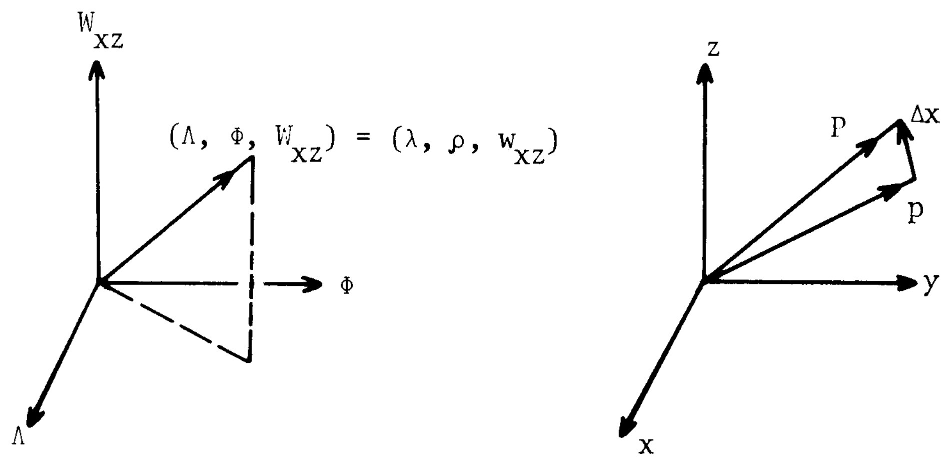

Figure 7.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and gravity gradient.

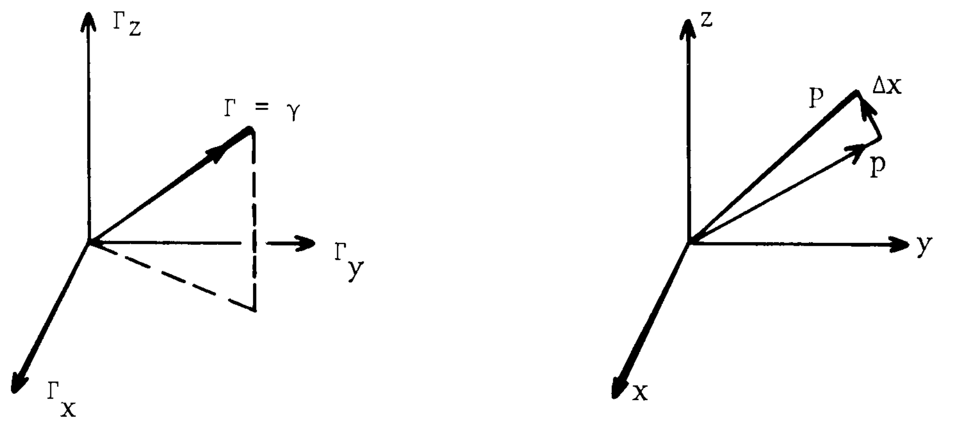

Figure 6.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and gravity.

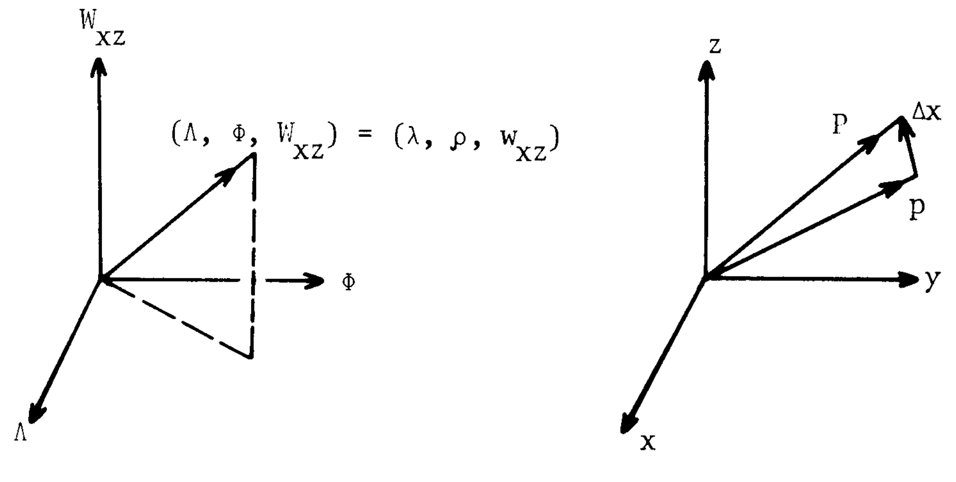

Figure 7.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and gravity gradient.

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NOS NGS-9 National Geodetic Survey publications on surveying and geodesy 1976. September 1977, 17 pp (PB275181). Compilation lists publications authored by NGS staff in 1976, source availability for out-of-print Coast and Geodetic Survey publications, and subscription information on the Geodetic Control Data Automatic Mailing List.

NOS NGS-10 Use of calibration base lines. Charles J. Fronczek, December 1977, 38 pp (PB279574). Detailed explanation allows the user to evaluate electromagnetic distance measuring instruments.

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