QC
801
.U65
no.14
c.2

NOAA Technical Memorandum NOS 14

DEPARTMENT OF COMMERCE ★ UNITED STATES OF AMERICA ★

SOME FEATURES OF THE DYNAMIC STRUCTURE OF A DEEP ESTUARY

Michael Devine

Rockville, Md.
April 1974

noaa NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION / NATIONAL OCEAN SURVEY

NOAA TECHNICAL MEMORANDA

National Ocean Survey Series

... (NOS) provides charts and related information for the safe navigation of ... The survey also furnishes other Earth science data--from geodetic, hydro- ...omagnetic, seismologic, gravimetric, and astronomic surveys or observations, ...rements--to protect life and property and to meet the needs of engineering, scientific, commercial, industrial, and defense interests.

NOAA Technical Memoranda NOS series facilitate rapid distribution of material that may be preliminary in nature and which may be published formally elsewhere at a later date. Publications 1 through 8 are in the former series, ESSA Technical Memoranda, Coast and Geodetic Survey Technical Memoranda (C&GSTM). Beginning with 9, publications are now part of the series, NOAA Technical Memoranda NOS.

Publications listed below are available from the National Technical Information Service (NTIS), U.S. Department of Commerce, Sills Bldg., 5285 Port Royal Road, Springfield, Va. 22151. Price varies for paper copy; $1.45 microfiche. Order by accession number (in parentheses) when given.

ESSA Technical Memoranda

C&GSTM 1 Preliminary Measurements With a Laser Geodimeter. S. E. Smathers, G. B. Lesley, R. Tomlinson, and H. W. Boyne, November 1966. (PB-174-649)

C&GSTM 2 Table of Meters to Fathoms for Selected Intervals. D. E. Westbrook, November 1966. (PB-174-655)

C&GSTM 3 Electronic Positioning Systems for Surveyors. Angelo A. Ferrara, May 1967. (PB-175-604)

C&GSTM 4 Specifications for Horizontal Control Marks. L. S. Baker, April 1968. (PB-179-343)

C&GSTM 5 Measurement of Ocean Currents by Photogrammetric Methods. Everett H. Ramey, May 1968. (PB-179-083)

C&GSTM 6 Preliminary Results of a Geophysical Study of Portions of the Juan de Fuca Ridge and Blanco Fracture Zone. William G. Melson, December 1969. (PB-189-226)

C&GSTM 7 Error Study for Determination of Center of Mass of the Earth From Pageos Observations. K. R. Koch and H. H. Schmid, January 1970. (PB-190-982)

C&GSTM 8 Performance Tests of Richardson-Type Current Meters: I. Tests 1 Through 7. R. L. Swanson and R. H. Kerley, January 1970. (PB-190-983)

NOAA Technical Memoranda

NOS 9 The Earth's Gravity Field Represented by a Simple Layer Potential From Doppler Tracking of Satellites. Karl-Rudolf Koch and Bertold U. Witte, April 1971. (COM-71-00668)

NOS 10 Evaluation of the Space Optic Monocomparator. Lawrence W. Fritz, June 1971. (COM-71-00768)

NOS 11 Errors of Quadrature Connected With the Simple Layer Model of the Geopotential. Karl-Rudolf Koch, December 1971. (COM-72-10135)

NOS 12 Trends and Variability of Yearly Mean Sea Level 1893-1971. Steacy D. Hicks, March 1973. (COM-73-10670)

NOS 13 Trends and Variability of Yearly Mean Sea Level 1893-1972. Steacy D. Hicks and James E. Crosby, March 1974.

QC
801
.U65
no.14
c.2

NOAA Technical Memorandum NOS 14

SOME FEATURES OF THE DYNAMIC STRUCTURE OF A DEEP ESTUARY

Michael Devine

Rockville, Md.
April 1974

MARINE AND EARTH SCIENCES LIBRARY
FEB 27 1975
U. S. Dept. of Commerce

UNITED STATES DEPARTMENT OF COMMERCE
Frederick B. Dent, Secretary
/ NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
Robert M. White, Administrator
/ NATIONAL OCEAN SURVEY
Allen L. Powell, Director

75 0251

CONTENTS

SOME FEATURES OF THE DYNAMIC STRUCTURE OF A DEEP ESTUARY

Michael Devine
Oceanographic Division, National Ocean Survey, NOAA, Rockville, Md.

ABSTRACT. A boundary-layer formulation for the dynamic structure of a deep estuary is developed. Cross-stream averages are used, but the boundary-layer structure is shown to depend on the cross-stream geostrophic constraint. A similarity transformation and a weighted residual method are used to derive an approximate solution for the velocity and salinity structure of the upper layer. This solution indicates that, in the central regime of the estuary, outflow extends through the entire halocline. Inflow takes place in a much less stratified lower layer, and mass exchange between the layers is by upwelling. This structure is modified in the outer regime of the estuary where mixing between the layers develops and in the inner regime where a sharp halocline develops and where mass exchange is by entrainment. The implications of the dynamics for the flushing process and for pollutant movement and dispersion are discussed.

I. INTRODUCTION

In the study of estuarine dynamics, a broad spectrum of hydrodynamic structures is encountered, including a wide range of horizontal and vertical salinity variations and tidal effects ranging from negligible to very large. Discussions of some of the physical and dynamical bases for the classification of estuaries are given by Pritchard (1967a) and Bowden (1967). No single analytical study can cover more than a narrow range of this spectrum. In this paper, we examine some factors of importance in understanding the circulation and flushing processes in partially mixed deep estuaries.

The most obvious class of deep estuary is the fjord; the results of this study will be most directly applicable to estuaries of this type. Formally, however, any estuarine region in which substantial stratification is confined to a near-surface layer and in which horizontal and vertical variations in salinity in the lower layer are small can be considered deep.

The understanding of circulation and flushing processes in these and in all types of estuaries is important to understand the related physical, chemical, and biological processes that occur and to deal with problems associated with the injection and dispersion of pollutants. In comparison with the attention directed toward the understanding of problems in ocean dynamics, however, relatively few studies have been made of the dynamics of estuarine circulation. This is due in part to the complexity of the problem that involves oscillating tidal motion superimposed on a net gravitational circulation in a three-dimensional rotating fluid with irregular and complex boundaries. For this reason, past studies of estuarine dynamics have treated them as idealized systems.

This study, also in the category of an idealized system, presents a simplified model to exhibit some of the physics of a complex hydrodynamic process as a necessary step toward a realistic representation of the complete physical system.

II. BACKGROUND

Estuarine gravitational circulation associated with density differences between outflowing river water and inflowing sea water generates an outflow of water near the surface that may be far in excess of the net outflow expected from river runoff alone. A mass balance is achieved by a strong inflow below the near-surface outflow. Examples are the Mersey River in England and the James River in Virginia where the net average rates of outflow are respectively about 40 times and 10 times the rate that should occur if river runoff alone were present (Pritchard 1967b, p. 41).

A series of studies of gravitational circulation in a partially mixed coastal plain estuary have been made by Rattray and Hansen (1962), Hansen (1967), and Hansen and Rattray (1965).

These are studies of estuaries in which substantial vertical mixing occurs from the surface to the bottom between the inflowing and outflowing water; further, the assumption is made that the estuary is sufficiently narrow that lateral homogeniety can be assumed. It also is assumed that the gravitational circulation mechanism can be represented by equations that provide for averaging over a tidal cycle. The tide, presumed to be the main mixing mechanism in the estuary, has its dynamic effect represented by coefficients of mixing in the equations. On the basis of the observational work of Pritchard (1956), nonlinear field accelerations in the equations of motion are neglected. Attention is directed to shallow estuaries in which substantial horizontal and vertical salinity gradients are found at all depths. Even with these simplifications, a highly complex nonlinear hydrodynamic problem remains. Again, partly on the basis of Pritchard's observations and partly on dynamic considerations, the shallow estuary is broken up into outer, central, and inner regimes.

In the outer regime, vertical advection and horizontal diffusion of salt and the direct effect of river runoff are neglected. The dynamics of this regime are discussed in Rattray and Hansen (1962). The salinity is high in the outer regime, and fractional changes in salinity with horizontal distance are small. In the central regime, salinity is assumed to vary linearly with distance along the estuary. The vertical velocity is zero; therefore, vertical transfer of salt is entirely by diffusion. With an appropriate assumption about the dependence of the horizontal coefficient of diffusion of salt on distance along the estuary, analytical representations for the distribution of velocity and salinity can be derived. In the inner regime, salinity is low, and fractional changes with distance along the estuary are large. With certain assumptions about the mixing coefficients, a perturbation solution can be derived for an inner regime with strong tidal currents. The solutions for the central and inner regimes are discussed in Hansen and Rattray (1965).

Because of the complexity of this estuarine problem, all of the solutions given are similarity solutions (i.e., the relationship between horizontal and vertical variations is prescribed, and only a restricted set of boundary conditions can be satisfied). This is the usual situation for nonlinear hydrodynamic systems for which the goal of solving a classical boundary value problem in mathematical physics can be achieved only in isolated, very special cases. Nevertheless, considerable insight into the controlling dynamics in estuaries can be gained from similarity solutions, and important aspects of the operative physical processes can be deduced.

Among the important deductions of the work of Hansen and Rattray is that the nature of the estuarine flushing process is critically dependent on the state of turbulent mixing in the estuary. In a system in which little mixing takes place between outflowing and inflowing water, advective salt-balance concepts [such as that of Knudsen (see, e.g., Defant, 1961, Vol. I, p. 379 f.)] can be used to deduce the rate of movement of inflowing and outflowing water. Where, for example, the outflowing water has a mean salinity of 27°/₀₀ and the inflowing water has a salinity of 30°/₀₀, the Knudsen relations give $V_1 = 10R_0$ and $V_2 = 9R_0$ where R_0 is the river runoff rate, V_1 is the rate of outflow in the upper layer, and V_2 is the rate of inflow in the lower layer. Where mixing between the layers occurs, these deductions are no longer valid. Hansen (1967, p. 49) notes that, for reasonable values of the mixing coefficients in the central regime of a well-mixed estuary, the above salinity observations yield $V_1 = 2R_0$ and $V_2 = R_0$. This implies very different mechanisms for flushing and mass exchange in shallow estuaries than those inferred from the advective theory.

One can expect that similar considerations hold for flushing and mass exchange in a deep estuary. As with shallow estuaries, we anticipate that, for deep estuaries, one must consider several different dynamic regimes appropriate to the existing varying circulation patterns and salinity structures. Most of our present information about deep estuaries is only on the distributions of salinity and other chemical properties. Comprehensive velocity observations generally are not available. The various types of vertical salinity profiles found in British Columbia inlets have been used by Pickard (1961) to classify fjord-type estuaries on the basis of stratification. Observed salinity structures range from those in which the salinity increases more or less continuously with depth to an asymptotic value to two-layer cases with almost homogeneous upper and lower layers separated by a very sharp halocline. This latter profile is seen in the inner part of estuaries with strong river runoff. In the central and outer regimes of all types of estuaries, more gradual salinity changes with depth are observed. The classification of Pickard is discussed further in topic V.

Rattray (1967) has considered some aspects of the dynamics of a fjord-type estuary in a study of two-dimensional vertical circulation in an estuary with an asymptotic salinity value. Nonlinear momentum advection terms are included in the formulation but vanish in cases of negligible wind stress and are disregarded in the quantitative solutions given. A similarity approach is used, with specific assumptions about the vertical dependence of velocity, salinity, and turbulent mixing coefficients. Using this formulation, Rattray deduces velocity and salinity patterns in reasonable agreement with observations of horizontal dependence in Alberni Inlet, British Columbia, and with observations of vertical dependence in Silver Bay, Alaska. A more recent development of the theory by Winter (1972) yields qualitative agreement with observations in Knight Inlet, British Columbia. These analyses do not include a boundary-layer formulation but deal directly with nondimensional variables.

The boundary-layer approach successfully explains aspects of the deep ocean circulation (Johnson 1971, Devine 1972); this approach is well suited to the types of circulation occurring in deep estuaries. The near-surface stratified regime (by hypothesis, thin in comparison to the total depth of the estuary) can be used to define a vertical boundary-layer structure for the estuary. This structure is complex; in topic III, we shall see that its definition requires development of the relevant two-dimensional system in terms of cross-stream averages and departures therefrom. The approach taken in the analytical studies just referred to was to define the two-dimensional system directly and to regard cross-stream variations as negligible. In contrast to this, Cameron (1951) notes that geostrophy holds quite well for the transverse equation of motion in a British Columbia inlet. In topic III, we shall show that this geostrophic constraint should be considered to properly analyze the boundary-layer structure of a deep estuary.

III. GENERAL BOUNDARY-LAYER STRUCTURE

A. Basic Formulation

We begin with equations in three dimensions for an incompressible, hydrostatic Boussinesq fluid on a uniformly rotating Earth. Density changes are presumed to be caused only by salinity variations. The equations are to be applicable to steady, tidally averaged motion in the interior of a fjord-type estuary (deep and narrow in its interior with a well-defined horizontal direction of flow). Horizontal diffusion of momentum and salt and advection of momentum are neglected.

The steady-state equations of motion [with density eliminated by means of the linear equation of state $\rho = \rho_0(1 + kS)$ and the hydrostatic equation] are

$$kg\,\frac{\partial S}{\partial x} - \frac{\partial^2}{\partial z^2}\left(A_v \frac{\partial u}{\partial z}\right) = -f\,\frac{\partial v}{\partial z} \tag{1}$$

and

$$Kg\frac{\partial S}{\partial y} - \frac{\partial^2}{\partial z^2}\left(A_v\frac{\partial v}{\partial z}\right) = f\frac{\partial u}{\partial z}\ ; \tag{2}$$

and the equations for continuity of mass and salt are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3}$$

and

$$u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = \frac{\partial}{\partial z}\left(K_v\frac{\partial S}{\partial z}\right)\ . \tag{4}$$

Boundary conditions to be analytically specified require that there be continuity of stress and zero salt flux at the sea surface, appropriate conditions at horizontal boundaries, and proper asymptotic conditions at the base of horizontal boundary layers.

In eq (1) through (4), x is distance along the estuary (positive toward the mouth); y is the distance across the estuary; z is the vertical distance downward from the sea surface; and u, v, and w are corresponding velocity components. In the other notation, f is the constant coriolis parameter at the latitude of the estuary; ρ, density; ρ_0, reference density of fresh water; S, salinity; $K=(1/\rho_0)(\partial\rho/\partial S)$ taken as constant; g, acceleration of gravity; A_v, vertical eddy coefficient of viscosity; and K_v, vertical eddy coefficient of diffusion.

As we anticipate that the salinity approaches a uniform asymptotic value with depth, we define a salinity defect E by the relation

$$S = S_0 - E\Delta S \tag{5}$$

where S_0 is the asymptotic value for salinity and ΔS is a scale factor for near-surface salinity variations. As we shall not consider cross-stream variations explicitly, we represent the dependent variables as the sum of means and departures from these means across the estuary (the m-subscripted variables are means, and the primed variables are departures):

$$\begin{aligned} u &= u_m + u'\ , \\ v &= v'\ , \\ w &= w_m + w'\ , \end{aligned}$$

and (6)

$$E = E_m + E'\ .$$

Using eq (5) and (6), we can integrate eq (1) through (4) across the estuary that is assumed to be of constant width B. We obtain

$$-Kg\Delta S\frac{\partial E_m}{\partial x}-\frac{\partial^2}{\partial z^2}\left(A_v\frac{\partial u_m}{\partial z}\right)=0\ , \tag{7}$$

$$-Kg\Delta S\left(\frac{E_r}{B}\right)=f\frac{\partial u_m}{\partial z}\ , \tag{8}$$

$$\frac{\partial u_m}{\partial x}+\frac{\partial w_m}{\partial z}=0\ , \tag{9}$$

and

$$u_m\frac{\partial E_m}{\partial x}+w_m\frac{\partial E_m}{\partial z}=\frac{\partial}{\partial z}\left(K_v\frac{\partial E_m}{\partial z}\right) \tag{10}$$

where

$$E_r=\int_0^B\frac{\partial E'}{\partial y}\,\partial y\ .$$

Horizontal dispersive terms generated by the advective terms in the salt equation are neglected, and A_v and K_v are presumed constant across the estuary.

Subsequently, we shall show this formulation leading to the deduction that there is zero net horizontal transport of mass in the estuary. The explicit effect of river flow, therefore, is not taken into account, although river effects appear implicitly in the parametric values used to describe a given estuarine situation. In shallow estuaries, the advection of salt by the gravitational circulation mechanism is primarily balanced by vertical diffusion while the lower order advection of salt by net river runoff is balanced primarily by horizontal diffusion, which is of lower order than vertical diffusion. This is noted in the central and outer regimes of coastal plain estuaries, both observationally (Pritchard 1956) and theoretically (Rattray and Hansen 1962, Hansen and Rattray 1965). We assume that this situation holds over a substantial part of a deep estuary as well. Both advection of salt by river runoff and horizontal diffusion of salt are then regarded as lower order effects as far as the primary gravitational circulation mechanism is concerned--so they are neglected.

To reduce eq (7) through (10) to nondimensional form, we define nondimensional variables (denoted by asterisks) as

$$u_m=\frac{Kg\Delta SD^3}{A_{vo}L}\,u^*,$$

$$w_m=\frac{Kg\Delta SD^4}{A_{vo}L^2}\,w^*,$$

$$z=Dz^*,$$

$$x = Lx^*,$$

$$A_r = A_{ro} a_r^*,$$

and (11)

$$K_r = K_{ro} \kappa_r^*$$

where D and L are respectively depth and scale length in the estuary and A_{ro} and K_{ro} are respectively scaling values for eddy viscosity and diffusion. Dropping asterisks, we obtain the nondimensional system

$$\frac{\partial E_m}{\partial x} + \frac{\partial^2}{\partial z^2}\left(a_r \frac{\partial u}{\partial z}\right) = 0, \tag{12}$$

$$\frac{r}{Ra} E_r + \frac{\partial u}{\partial z} = 0, \tag{13}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{14}$$

and

$$u\frac{\partial E_m}{\partial x} + w\frac{\partial E_m}{\partial z} = \frac{M}{Ra}\frac{\partial}{\partial z}\left(\kappa_r \frac{\partial E_m}{\partial z}\right) \tag{15}$$

where

$$Ra = \frac{Kg\Delta SD^3}{A_{ro} L U_r}, \text{ estuarine Rayleigh number;}$$

$$M = \frac{K_{ro} L}{U_r D^2}, \text{ tidal mixing parameter;}$$

and

$$r = \frac{Kg\Delta SD}{fBU_r}, \text{ rotational parameter.}$$

U_r is a reference velocity that may be taken as a mean upper layer river runoff rate.

Many fjords exhibit strong but continuous stratification near the surface over a considerable horizontal distance. Observations in the Kattegat (Proudman 1953, p. 101 f.) indicate that such flows take place at a relatively high Richardson number ($Ri > 1$), and hence the ratio A_r/K_r must be high as well. In terms of a boundary-layer scaling, we anticipate that the scale depth over which vertical eddy viscosity is an $O(1)$ process will be greater than the depth over which vertical eddy diffusion of salt is $O(1)$. Here, O stands for order.

For the case $O(M/Ra)^{2/5} < O(r/Ra) << 1$, which includes a wide range of physically relevant systems, we presume that the nondimensional variables can be expanded in terms of the small ordering parameter $\delta = (Ra/r)(M/Ra)^{2/5}$ as

$$u = u_0 + \delta u_1 + \dots ,$$

$$w = w_0 + \delta w_1 + \dots ,$$

$$E_m = E_{mo} + \delta E_{m1} + \dots ,$$

and (16)

$$E_r = \delta^{1/2} E_{ro} + \delta E_{r1} + \delta^{3/2} E_{r2} + \dots .$$

The development in eq (16) leads to a series of boundary layers, each with its own dynamic structure. The expansion in powers of $\delta^{1/2}$ of the cross-stream salinity variation E_r is made because of the nature of the geostrophic constraint on the flow, which we anticipate will hold at all depths. We also anticipate that cross-stream salinity variations are of lower order than cross-stream averages; hence there is no term in E_r corresponding to the order of E_{mo}.

B. Details of the Structure

1. Surface Boundary-Layer

Consideration of the lowest order system generated by E_{ro} leads to the definition of the boundary-layer variables

$$u_s = \left(\frac{Ra}{r}\right)^{1/4} \left(\frac{M}{Ra}\right)^{1/2} u_0 ,$$

$$w_s = 0 ,$$

and (17)

$$\zeta_s = \left(\frac{Ra}{r}\right)^{1/4} \left(\frac{M}{Ra}\right)^{3/10} z .$$

The use of 10th powers of M/Ra is considered reasonable since M/Ra is $O(10^{-8})$ or less for deep estuaries. Thus, equations for the upper layer are

$$\frac{\partial^2}{\partial \zeta_s^2}\left(a_r \frac{\partial u_s}{\partial \zeta_s}\right) = 0 , \tag{18}$$

$$E_{ro} + \frac{\partial u_s}{\partial \zeta_s} = 0 , \tag{19}$$

and

$$\frac{\partial}{\partial \zeta_s}\left(\kappa_r \frac{\partial E_{mo}}{\partial \zeta_s}\right) = 0 . \tag{20}$$

A vertically homogeneous surface layer can be accounted for with eq (18) through (20), and we expect that this system might apply to a vertically homogeneous layer generated by wind stress or to an upper layer under conditions of strong river runoff near the head of an estuary. In layers near a horizontal boundary, one often must assume particular forms for eddy coefficients from which a variety of boundary-layer structures can be derived. See, for example, the discussion of the surface atmospheric layers in Hess (1959, pp. 276-279). Such surface-layer structure may be neglected in shallow estuaries in the absence of wind stress (see Hansen and Rattray 1965, 1972) and will not be discussed further in this analysis. The dynamics of this layer and its relationship to the overall boundary-layer structure in a deep estuary will be examined in a future study.

2. Upper Layer Flow

This layer, associated with whatever large vertical salinity changes are observed in the estuary, is derived from the balance generated by E_{r1}. The boundary-layer variables are defined as

$$u_B = \left(\frac{M}{Ra}\right)^{-3/5} u_o ,$$

$$w_B = \left(\frac{M}{Ra}\right)^{-4/5} w_o ,$$

and

$$\zeta = \left(\frac{M}{Ra}\right)^{-1/5} z \tag{21}$$

and give the system

$$\frac{\partial E_{mo}}{\partial x} + \frac{\partial^2}{\partial \zeta^2}\left(a_v \frac{\partial u_B}{\partial \zeta}\right) = 0, \tag{22}$$

$$E_{v1} + \frac{\partial u_B}{\partial \zeta} = 0, \tag{23}$$

$$\frac{\partial u_B}{\partial x} + \frac{\partial w_B}{\partial z} = 0, \tag{24}$$

and

$$u_B \frac{\partial E_{mo}}{\partial x} + w_B \frac{\partial E_{mo}}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left(K_v \frac{\partial E_{mo}}{\partial \zeta}\right). \tag{25}$$

Stratification is of $O(1)$ importance in this layer, with the $O(1)$ nonlinear terms in the salt equation making the system intractable in the general case. The $O(1)$ diffusion of momentum and salt can be expected to be associated with the strong near-surface outflow observed in real estuaries. It is not necessary that any inflow occur in this layer, as mass balance can be achieved by upwelling from a lower layer in which inflow occurs.

3. Lower Layer Flow

This is a layer observed in real estuaries in which salinity variations are much less than in the upper layer but still are not negligible. This layer is generated by E_{r2} and is associated with the lower order salinity term E_{m1} so we expect that salt diffusion will not be an $O(1)$ process. Diffusion of $O(1)$ momentum is, however, still required to balance the horizontal salinity gradient term. We formally define the boundary-layer variables

$$u_g = \left(\frac{Ra}{r}\right)^{-1/4}\left(\frac{M}{Ra}\right)^{-7/10} u_o ,$$

$$w_g = \left(\frac{M}{Ra}\right)^{-4/5} w_o ,$$

and (26)

$$\zeta_g = \left(\frac{Ra}{r}\right)^{-1/4}\left(\frac{M}{Ra}\right)^{-1/10} z$$

that lead to the system

$$\frac{\partial E_{m1}}{\partial x} + \frac{\partial^2}{\partial \zeta_g^2}\left(a_v \frac{\partial u_g}{\partial \zeta_g}\right) = 0 , \tag{27}$$

$$E_{r2} + \frac{\partial u_g}{\partial \zeta_g} = 0, \tag{28}$$

$$\frac{\partial u_g}{\partial x} + \frac{\partial w_g}{\partial \zeta_g} = 0, \tag{29}$$

and

$$u_g \frac{\partial E_{m1}}{\partial x} + w_g \frac{\partial E_{m1}}{\partial \zeta_g} = 0 . \tag{30}$$

We anticipate that inflow will occur in this layer, with upwelling into the upper layer to achieve a mass balance. We note that, because of the asymptotic matching condition anticipated between upwelling out of the lower layer and into the upper layer, the vertical velocity must be of the same order in both layers.

IV. APPROXIMATE SOLUTION FOR THE FLOW REGIME

A. General Solution Form

We now consider an approximate solution for the upper stratified layer and also consider the matching to the lower layer through the asymptotic vertical velocity. In the following development, a possible surface layer is not considered, and a condition of zero stress is imposed on the top boundary of the upper layer. The mixing coefficients are taken as constant so that $a_v = \kappa_v = 1$. We feel that this assumption imposes the minimum constraint on the physical system. The boundary-layer stream function is defined by

$$\frac{\partial \psi}{\partial \zeta} = -u_B$$

and (31)

$$\frac{\partial \psi}{\partial x} = w_B \, .$$

The boundary-layer system, eq (22) through (25), then is

$$\frac{\partial E_{mo}}{\partial x} - \frac{\partial^4 \psi}{\partial \zeta^4} = 0, \tag{32}$$

$$E_{v1} - \frac{\partial^2 \psi}{\partial \zeta^2} = 0, \tag{33}$$

and

$$-\frac{\partial \psi}{\partial \zeta}\frac{\partial E_{mo}}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial E_{mo}}{\partial \zeta} = \frac{\partial^2 E_{mo}}{\partial \zeta^2} \, . \tag{34}$$

The boundary conditions are

$$\psi = \frac{\partial^2 \psi}{\partial \zeta^2} = \frac{\partial E_{mo}}{\partial \zeta} = 0 \text{ at } \zeta = 0 \tag{35}$$

and

$$\frac{\partial \psi}{\partial \zeta} \rightarrow 0, \quad \frac{\partial^2 \psi}{\partial \zeta^2} \rightarrow 0, \quad \text{and} \quad E_{mo} \rightarrow 0 \quad \text{as } \zeta \rightarrow \infty \, . \tag{36}$$

We anticipate a finite vertical velocity at the base of the boundary layer; hence the asymptotic value of ψ is to be determined as part of the solution. The cross-stream salinity variation E_v is to be determined parametically, after the solution of eq (32) and eq (34) through (36), which constitute a complete system.

The problem is reduced to a system of ordinary differential equations by a similarity transformation employed by Rattray (1967):

$$E = x^{\alpha}\theta(\chi), \quad \psi = x^{\beta}\phi(\chi), \quad \text{and} \quad \chi = x^{\gamma}\zeta \, . \tag{37}$$

The differential equations in the new similarity variables are

$$\phi^{\text{IV}} - (\alpha\phi + \gamma\chi\phi') = 0 \tag{38}$$

and

$$\theta'' + (\alpha\theta\phi' - \beta\phi\theta') = 0 \tag{39}$$

with the similarity conditions

$$\alpha - 1 = \beta + 4\gamma \quad \text{and} \quad \beta - \gamma = 1 . \tag{40}$$

The boundary conditions are

$$\phi = \phi'' = \theta' = 0 \text{ at } x = 0 \tag{41}$$

and

$$\phi' \to 0, \quad \phi'' \to 0, \quad \text{and} \quad \theta \to 0 \text{ as } x \to \infty . \tag{42}$$

This nonlinear system is still too complex for any general type of analytic solution to be derived. A technique for attacking such nonlinear problems is to linearize them through the use of a perturbation parameter. This is the approach taken in Rattray and Hansen (1962), Hansen and Rattray (1965), and (in a modified form) by Rattray (1967). The technique is extended by Hansen and Rattray (1972) to a case where a natural perturbation parameter is not available. In all cases, an exact solution to the perturbation equations is derived. The relationship of these equations to the original system is, however, neither analytically representable nor necessarily clear.

The approach taken in this study is to develop an approximate analytic solution to the exact system of equations, with boundary conditions satisfied exactly and with the relationship between the result and the exact solution to the original system in some sense measurable. With the asymptotic dependence of the variables and the polynomial dependence of the shallow water solution (Hansen and Rattray 1965) in mind, an approximate solution of the form

$$\phi = \phi_\infty + e^{-cx} \sum_{n=0}^{N} a_n x^n \quad \text{and} \quad \theta = e^{-cx} \sum_{n=0}^{N} b_n x^n \tag{43}$$

is developed.

Because of the boundary-layer nature of the flow, we can expect that a low-order solution will give a good approximation. We derive a solution that satisfies the boundary conditions exactly and which gives a "best fit" to the original equations by a method of weighted residuals. The details of the solution method are given in the appendix.

Only one set of similarity variables that corresponds to the parametric values $\alpha = -1/2$, $\beta = 1/2$, and $\gamma = -1/2$ is considered. These parametric values lead to a realistic physical situation with decreasing salinity defect, increasing transport, and increasing boundary-layer depth as the mouth of the estuary is approached.

The solutions for the similarity variables are

$$\phi = -1.35 + (1.35 + 0.76\, x + 0.09\, x^2) e^{-x} \tag{44}$$

and

$$\theta = (1 + x + 0.36\, x^2) e^{-x} . \tag{45}$$

The velocity components and the salinity defect in terms of the boundary-layer variables then are

$$u_B = \left(0.58 + 0.58\frac{\zeta}{\sqrt{x}} + 0.09\,\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}}, \tag{46}$$

$$w_B = \frac{1}{2\sqrt{x}}\left\{-1.35 + \left(1.35 + 0.76\frac{\zeta}{\sqrt{x}} + 0.09\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}}\right\} - \frac{\zeta}{2\sqrt{x}}\left(0.58 + 0.58\frac{\zeta}{\sqrt{x}} + 0.09\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}}, \tag{47}$$

$$E_{mo} = \frac{1}{\sqrt{x}}\left(1 + \frac{\zeta}{\sqrt{x}} + 0.36\,\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}}, \tag{48}$$

and

$$E_{r1} = \frac{1}{\sqrt{x}}\left(0.40\frac{\zeta}{\sqrt{x}} + 0.09\,\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}}. \tag{49}$$

The asymptotic vertical velocity in eq (47) provides an upper boundary condition for the lower layer through the asymptotic condition

$$\lim_{\zeta\to\infty} w_B = w_g(0). \tag{50}$$

The same similarity structure that holds in the upper layer is presumed to hold in the lower layer. Without considering the details of the solution, we can observe from the formulation in topic IIIB3 that lower-layer salinity changes are of lower order than those in the upper layer through the perturbation parameter δ while the horizontal velocity is of lower order through the boundary-layer scaling. For near-surface salinity variations of $O(10°/_{oo})$, salinity variation in the lower layer is $O(1°/_{oo})$ or less. This is compatible with the salinity structure observed in the lower region of deep estuaries. From the condition that all velocity components vanish at the base of the lower layer,

$$\int_0^{\infty} u_g\, dz = -1.35\, x^{1/2} \tag{51}$$

follows so that inflow in the lower layer exactly compensates outflow in the upper layer and gives a net zero horizontal transport.

Some general features of the velocity and salinity structure of the solution are now apparent. The upper layer, in which the flow is entirely seaward, contains all of the substantial salinity gradients. The return inward flow takes place entirely in the lower layer where the stratification is less marked. The inward and outward transports balance, each increasing in magnitude as $\sqrt{x}$ toward the mouth of the estuary as does the depth of the boundary layer. As would be expected, none of the characteristics of the boundary layers depend on the depth of the estuary. The horizontal dependence (see topic V) provides a mechanism for modification of the deduced boundary-layer structure in both the landward and seaward directions.

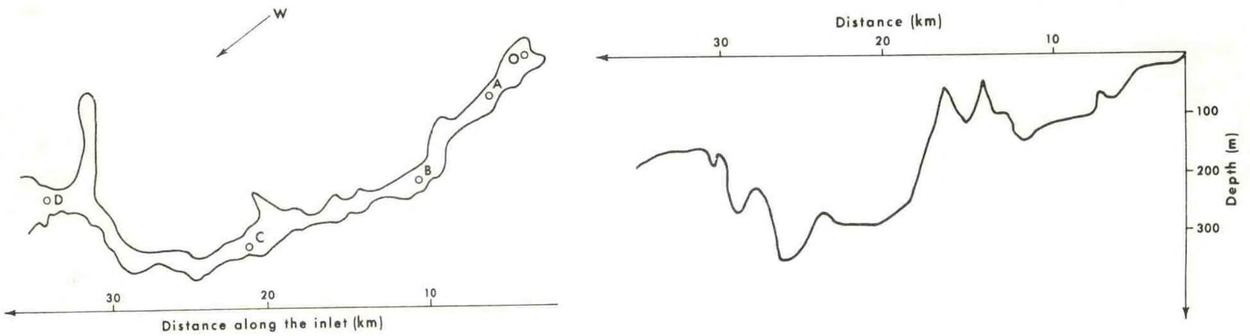


Figure 1.--(Left) the plan of Alberni Inlet with locations of long-period stations and (right) the longitudinal cross-section of the inlet

B. Illustrative Example

As an illustration of the solution, we consider Alberni Inlet, a British Columbia fjord, that has been studied extensively by Tully (1949). Horizontal and vertical sections of the inlet are shown in figure 1. We take the parametric values as

$$\mathcal{L} = 24 \text{ km}, \qquad \mathcal{D} = 200 \text{ m}, \qquad \mathcal{S}_o = 31.5°/_{oo},$$

$$\mathcal{S} = 15.8°/_{oo}, \quad \mathcal{A}_v = 1 \text{ cm}^2/\text{s}, \quad \mathcal{K}_v = 0.1 \text{ cm}^2/\text{s},$$

$$f = 10^{-4} \text{ s}^{-1}, \qquad \text{and} \qquad \mathcal{K}g = 1 \text{ cm/s}.$$

The dimensional values of the dependent variables in the upper layer in terms of nondimensional ζ, x then are

$$u = 19.8 \left(0.58 + 0.58\frac{\zeta}{\sqrt{x}} + 0.09\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}} \text{ cm} \cdot \text{s}^{-1}, \tag{52}$$

$$w = \frac{3.9 \times 10^{-3}}{2\sqrt{x}}\left\{-1.35 + \left(1.35 + 0.76\frac{\zeta}{\sqrt{x}} + 0.09\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}} - \zeta\left(0.58 + 0.58\frac{\zeta}{\sqrt{x}} + 0.09\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}}\right\} \text{ cm} \cdot \text{s}^{-1}, \tag{53}$$

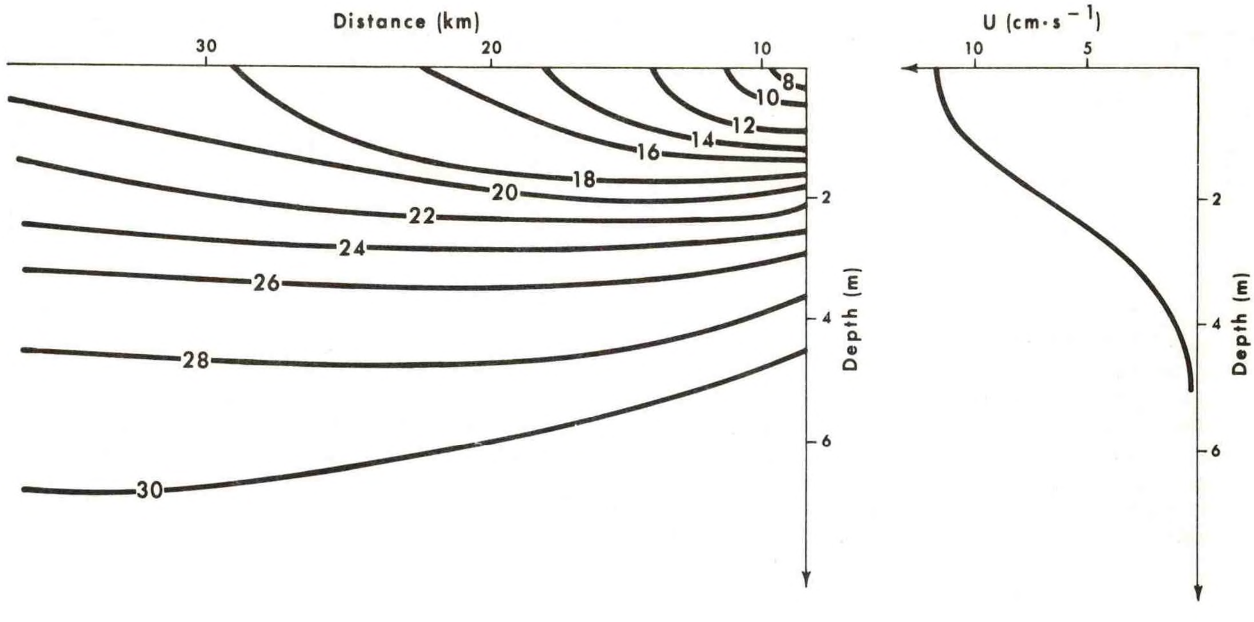
and

$$\mathcal{S}_m = \left\{31.5 - 15.8\left(1 + \frac{\zeta}{\sqrt{x}} + 0.36\frac{\zeta^2}{x}\right)e^{-\zeta/\sqrt{x}}\right\} °/_{oo} \tag{54}$$

where $\mathcal{S}_m = \mathcal{S}_o - \mathcal{E}_m \Delta\mathcal{S}$ and where $\zeta = 1$ corresponds to a depth of about 1.3 m at $x = 1$ (dimensional $x = 24$ km).

Tully's report gives many series of salinity observations that vary considerably depending on tidal stage, river discharge, and other factors. Velocity observations are not given, although it is noted that the upper layer transport increases to seaward.

Two series of observations, those of June 4-7 and those of July 4-5 are felt by Tully to yield salinity values representable of the mean, tidally averaged state. A longitudinal cross-section of the mean of the observations is given in figure 2 (bottom). The salinity



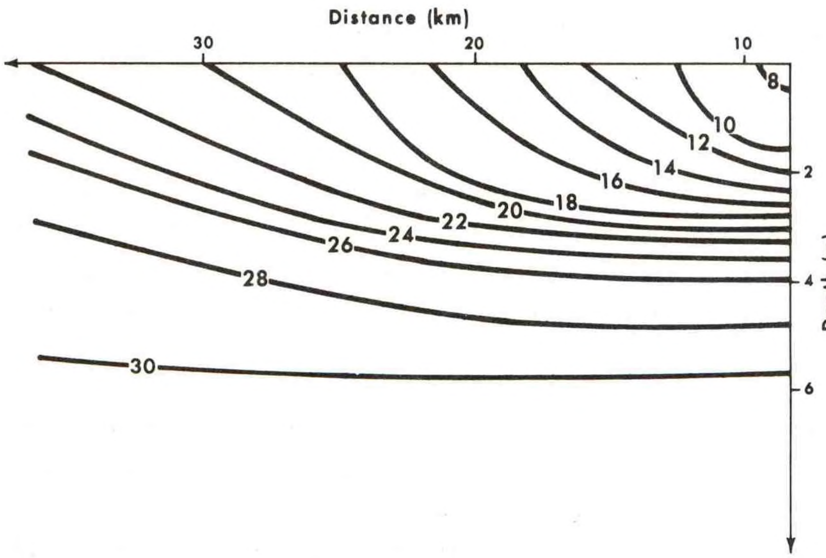


Figure 2.--(Bottom) the observed salinity (°/oo) in a longitudinal section of Alberni Inlet; (top left) the salinity profile for the bottom longitudinal section, calculated from eq (54); and (top right) the vertical velocity profile in the upper layer at $x = 24$ km, calculated from eq (52)

structure, with velocity vectors and a representative velocity profile deduced from eq (52) and (54) are given in figure 2 (top). Tully does not give the cross-inlet salinity structure so that comparison with deductions from eq (49) is not possible. Agreement with observed salinity is good in the central part of the inlet, but the predicted salinity in the outermost region is too low. We infer from this that mixing with the lower layer takes place in this region and that the theoretically predicted velocities are too high. We do not expect agreement in the innermost part of the inlet, as the solution becomes singular as $x \to 0$. These points are discussed further in topic V. We note that the theoretical upper layer includes the entire halocline, whereas Tully considers the level of maximum salinity gradient as marking a dividing point between upper and lower layers. In this connection, observations in Silver Bay, Alaska (McAlister et al. 1959), tend to support the deduction that outflow extends below the level of maximum salinity gradient.

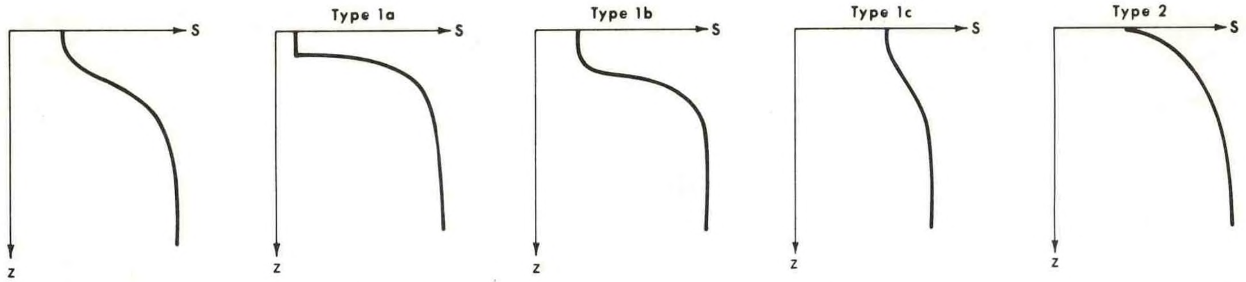


Figure 3.--(Left) the salinity profile calculated from eq (54); and (remainder) the salinity profiles typical of each type in the Pickard Classification: type la, homogeneous layer near the surface, separated by a sharp halocline from a high-salinity deep layer; characteristic of the heads of large-runoff estuaries; types lb and lc, modifications of type la; the upper layer is only weakly stratified, but the halocline becomes progressively much less sharp than in la; characteristic of the mouth and much of the interior region of large and moderate runoff estuaries; type 2, no sharp halocline, but a continuous increase of salinity from the surface downward; characteristic of low runoff estuaries and of regions near the mouth of some moderate and high runoff estuaries

V. DISCUSSION

The dynamic structure of a deep estuary is complex and, like that of a shallow estuary, can be expected to vary with horizontal distance along the estuary and with changes in controlling factors such as tidal forcing and river runoff.

The extent of the validity of the boundary-layer analysis developed in topics III and IV can be interpreted in terms of the classification of fjord-type inlets according to type of stratification given by Pickard (1961). The stratification types and the classes of estuary to which each belongs together with a salinity profile taken from eq (54) are illustrated in figure 3.

Note that what Pickard considers to be an upper homogeneous layer may be a layer which is much less strongly stratified than the halocline region but in which the salinity at a depth of 1 m is distinctly higher (usually 1°/₀₀ or more) than the salinity at the surface. This structure is found in Alberni Inlet and Silver Bay and would be expected wherever substantial but not complete vertical mixing takes place in the outflowing water. Continuous but greatly varying stratification is a predicted consequence of the structure of the dynamic model; this stratification is illustrated by the sample profile in figure 3 (left). With this in mind, one can expect that the deduced boundary-layer structure applies over a considerable horizontal distance for medium and large runoff estuaries. The dynamics of the various types of estuaries cannot be assessed, as Pickard's report does not include velocity.

Quite detailed and sophisticated data collection and analysis techniques are needed to study the dynamic structure of an estuary. A combination of highly accurate, long-period current data with complete horizontal and vertical salinity profiles is required. The task of data gathering and analysis is made particularly complex by the large oscillations about mean values produced by tidal motions. Maximum tidal velocities are often an order of magnitude or more larger than net nontidal velocities, and the magnitude of salinity fluctuations may be greater than the mean value of the salinity in the inner parts of estuaries. Tidal motion, however, gives zero net advection of salt into the estuary and, in particular, zero net advection into the upper layer. The net nontidal mechanism, on the other hand, may advect salt into the upper layer by means of upwelling. This is compensated for by horizontal advection in the upper layer. Thus the movement of salt and, by inference, of near-surface pollutants may be controlled in many deep estuarine situations by gravitational circulation rather than directly by the numerically greater tidal oscillation.

The oscillatory tide itself contributes to estuarine renewal only through horizontal mixing. It is this renewal mechanism that is the basis for the modified tidal prism method of Ketchum (1951). Ketchum uses a knowledge of river discharge and horizontal tidal movement to estimate flushing rates for a wide variety of estuarine situations. Good flushing rate estimates are obtained in some cases, but it appears that, in many deep estuarine situations, the method may not properly represent the mechanism by which near-surface water, salt, and constituents are removed from the estuary.

In general, vertical processes such as those associated with gravitational circulation can be represented in one-dimensional models only through an artificially enhanced coefficient of horizontal diffusion such as that employed by Arons and Stommel (1951) in an attempt to represent Ketchum's theory in differential equation form. Although horizontal diffusion effects may be small, they provide the only mechanism for the inward movement of salt in a one-dimensional system.

The deduced seaward movement of water in the upper layer of a deep estuary points up the different dynamical considerations that may be valid in deep and shallow estuaries. In a region where upper layer water is renewed by upwelling, advectively deduced water budgets are valid even though substantial mixing may take place in the upper layer itself. This is in contrast to the situation in the central regime of a partially mixed shallow estuary, discussed in topic II. In this regime, vertical transfer of salt from inflowing to outflowing water is entirely by vertical mixing, and advective concepts cannot be applied. The rate of near-surface outflow in a deep estuary may then be considerably greater than would be the case if a shallow bottom generated mixing between inflowing and outflowing water.

The dynamic structure deduced for what can be referred to as the central regime of a deep estuary is modified in both the inner and outer regimes. As the innermost part of the estuary is approached, the explicit effect of river runoff becomes dominant, and a two-layer flow tends to develop. The salinity profile tends toward type 1a of Pickard, with a very sharp halocline separating two layers that are only slightly stratified. Salt exchange between the layers is presumably by entrainment from an almost motionless lower layer to a moving upper layer (Bowden 1967, Stommel 1953). As distance from the head of the estuary increases, the gravitational circulation mechanism just analyzed develops, with substantial movement in both upper and lower layers.

The central regime flow is modified in a different manner as the mouth of the estuary is approached. The thickness of the upper layer and the strength of the horizontal circulation both increase with distance from the head of the estuary while the vertical stratification decreases. These factors can be expected to contribute to a breakdown of the separation between upper and lower layers and to mixing between the two. In the outer regions of an estuary, then, where the salinity difference between the upper and lower layers is small, the large transports deduced from an advective mass balance are not expected to hold. Vertical diffusion contributes to the salt balance at all levels and results in the surface-abyssal salinity difference being less than would be expected from advective balance considerations. This is noted in Alberni Inlet where the observed near-surface salinity in the outer region is higher than that deduced from the advective balance model.

From these considerations, we see that inner, central, and outer regimes can be defined for a deep estuary on the basis of the mechanism of salt transfer from inflowing to outflowing water. This basis differs from that of Hansen and Rattray (1965) who utilize observed horizontal salinity gradients, although it is expected that similar observational criteria can be developed for deep estuaries as well. In the inner regime, characterized by a very low salinity upper layer and a high salinity lower layer separated by a very sharp halocline, salt transfer from the lower layer to the upper layer is by entrainment. Gravitational circulation is poorly developed in this regime. In the central regime, stratification is continuous and substantial in the upper layer but slight in the lower layer, and salt transfer between the layers is by upwelling. In the outer regime, stratification is continuous but smaller, and salt transfer between layers is substantially by vertical mixing. Gravitational circulation is well developed in both the central and the outer regimes. The boundaries between the individual regimes and the overall process of salt intrusion depend ultimately on the external mechanisms of the problem such as river runoff and tidal excursion.

Concerning the estimation of flushing rates for a deep estuary, it appears that different approaches to different parts of the estuary are necessary. In the inner and central regimes, advective exchange estimates derived from salinity profiles are valid as substantial mixing between inflowing and outflowing water does not occur. In the outer regime, however, horizontal transports and flushing rates are less than would be expected from advective considerations. The mixing between inflowing and outflowing water that occurs in the outer regime complicates the entire flushing process because part of the constituents in the outflowing water are returned to the inflowing layer. Knowledge of the details of the flushing process is of undoubted importance in the study of pollutant movement and dispersion in all aspects; this aspect of estuarine dynamics requires further observational and theoretical study.

The type of stratified flow that occurs in deep estuaries presents considerable analytical difficulties; many factors that are important in real estuaries have not been directly dealt with here. Most obvious is the explicit tidal effect. The direct effect of river runoff, horizontal diffusion, and changing cross-section are also subjects for future study, as is the detailed analysis of the surface and lower boundary layers. Among the important effects considered in this study, that of the geostrophic constraint is particularly noteworthy. It is apparent that this factor should be considered in future estuarine studies even if they, as does this study, concentrate on the laterally averaged state.

REFERENCES

Ames, W.F., Nonlinear Ordinary Differential Equations in Transport Processes, Academic Press, New York, N.Y., 1968, 265 pp.

Arons, Arnold B., and Stommel, Henry, "A Mixing-Length Theory of Tidal Flushing," Transactions American Geophysical Union, Vol. 32, No. 3, June 1951, pp. 419-421.

Bowden, K.F., "Circulation and Diffusion," Estuaries, American Association for the Advancement of Science, Washington, D.C., 1967, pp. 15-36.

Cameron, W.M., "On the Transverse Forces in a British Columbia Inlet," Proceedings and Transactions of the Royal Society of Canada, Vol. 45, No. 5, Ottowa, Ontario, Canada, May 1951, pp. 1-8.

Defant, Albert, Physical Oceanography, Vol. 1, Pergamon Press, New York, N.Y., 1961, 729 pp.

Devine, Michael, "Some Aspects of the Dynamics of the Antarctic Circumpolar Current," Journal of Geophysical Research, Vol. 77, No. 30, Oct. 20, 1972, pp. 5987-5992.

Hansen, Donald V., "Salt Balance and Circulation in Partially Mixed Estuaries," Estuaries, American Association for the Advancement of Science, Washington, D.C., 1967, pp. 45-51.

Hansen, Donald V., and Rattray, Maurice, Jr., "Gravitational Circulation in Straits and Estuaries," Journal of Marine Research, Vol. 23, No. 2, May 1965, pp. 104-122.

Hansen, Donald V., and Rattray, Maurice, Jr., "Estuarine Circulation Induced by Diffusion," Journal of Marine Research, Vol. 30, No. 3, Sept. 1972, pp. 281-294.

Hess, Seymour L., Introduction to Theoretical Meteorology, Henry Holt and Co., New York, N.Y., 1959, 362 pp.

Jain, M.K., "On External Point Collocation Method for Fluid Flow Problems," Applied Scientific Research, Sec. A, Vol. 11, B.V. Martinus Nijhoff, The Hague, Netherlands, 1963, pp. 177-188.

Johnson, J.A., "On the Wind-Driven Circulation of a Stratified Ocean," Journal of Marine Research, Vol. 12, No. 2, May 1971, pp. 197-213.

Ketchum, Bostwick H., "The Exchange of Fresh and Salt Waters in Tidal Estuaries," Journal of Marine Research, Vol. 10, No. 1, Jan. 1951, pp. 18-38.

McAlister, W.B., Rattray, Maurice, Jr., and Barnes, Clifford A., "The Dynamics of a Fjord Estuary: Silver Bay, Alaska," Technical Report No. 62, Department of Oceanography, University of Washington, Seattle, Nov. 1959, 70 pp.

Pickard, G.L., "Oceanographic Features of Inlets in the British Columbia Mainland Coast," Journal of the Fisheries Research Board of Canada, Vol. 18, No. 6, Ottowa, Ontario, Canada, June 1961, pp. 907-999.

Pritchard, D.W., "The Dynamic Structure of a Coastal Plain Estuary," Journal of Marine Research, Vol. 15, No. 1, Jan. 1956, pp. 33-42.

Pritchard, D.W., "What Is an Estuary: Physical Viewpoint," Estuaries, American Association for the Advancement of Science, Washington, D.C., 1967a, pp. 3-5.

Pritchard, D.W.,"Observations of Circulation in Coastal Plain Estuaries," Estuaries, American Association for the Advancement of Science, Washington, D.C., 1967b, pp. 37-44.

Proudman, Joseph, Dynamical Oceanography, Methuen, London, England, 1953, 409 pp.

Rattray, Maurice, Jr., "Some Aspects of the Dynamics of the Circulation in Fjords," Estuaries, American Association for the Advancement of Science, Washington, D.C., 1967, pp. 52-62.

Rattray, Maurice, Jr., and Hansen, Donald V., "A Similarity Solution for Circulation in an Estuary," Journal of Marine Research, Vol. 20, No. 2, May 1962, pp. 121-133.

Stommel, Henry, "The Role of Density Currents in Estuaries," Proceedings of the Minnesota International Hydraulics Convention, September 1-4, 1953, Minneapolis, Minnesota, St. Anthony Falls Hydraulic Laboratory, Minneapolis, Minn., 1953, pp. 305-312.

Tully, John P., "Oceanography and Prediction of Pulp Mill Pollution in Alberni Inlet," Bulletin of the Fisheries Research Board of Canada, No. 83, Ottowa, Ontario, Canada, 1949, pp. 1-169.

Winter, D., "A Similarity Solution for Circulation in Stratified Fjords," Technical Report WSG-TA-72-12, Department of Oceanography, University of Washington, Seattle, Dec. 1972, 9 pp.

APPENDIX I. DETAILS OF THE APPROXIMATE SOLUTION METHODS

We seek an approximate solution to the system (38) through (42) of the general form of eq (43). The approximating functions are written as

$$\phi_A = c_1 \phi_1 + c_2 \phi_2 \tag{55}$$

and

$$\theta_A = \theta_0 + c_3 \theta_3 + c_4 \theta_4 \tag{56}$$

where we choose

$$\phi_1 = -1 + \left(1 + \frac{az}{2}\right) e^{-az},$$

$$\phi_2 = -1 + \left(1 + \frac{(az)^2}{2}\right) e^{-az},$$

$$\theta_0 = (1 + az) e^{-az},$$

$$\theta_1 = (az)^2 e^{-az},$$

and (57)

$$\theta_2 = (az)^3 e^{-az}$$

as a set of lineary independent functions that exactly satisfy the boundary conditions. The approximations to eq (38) and (39) are

$$\phi_A + (\alpha \theta_A + \gamma x \theta_A') = R_1 [c_j, x] \tag{58}$$

and

$$\theta_A'' + (\alpha \theta_A \phi_A' - \beta \phi_A \theta_A') = R_2 [c_j, x] \tag{59}$$

where R_1 and R_2 are residuals. The c_j are to be chosen to minimize the residuals in some sense. The particular method chosen is the point collocation method of Jain (1963) whereby the residuals are made to vanish at appropriate points on the half-open interval $[0,\infty]$. This gives the relations

$$R_1 [c_j, x_1] + R_1 [c_j, 0] = 0 ,$$

$$R_1 [c_j, x_2] - R_1 [c_j, 0] = 0 ,$$

$$R_2 [c_j, x_3] + R_2 [c_j, 0] = 0 ,$$

and (60)

$$R_2 [c_j, x_4] - R_2 [c_j, 0] = 0$$

that constitute four nonlinear algebraic equations for the c_j. These equations are solved by a computerized application of a modified Newton-Raphson scheme, which also optimizes the x_j. For further details, see Jain or Ames (1968, p. 190 f.). The choice $a=1$ is not necessary, but it gives the best results. With $a=1$, we obtain

$$c_1 = 1.528 , \quad c_2 = -0.179 , \quad c_3 = 0.358 , \quad \text{and} \quad c_4 = 0.001 \tag{61}$$

that gives solutions (44) and (45) when rounded to two decimal places, which we consider appropriate for a boundary-layer analysis.

N.O.A.A. CENTRAL LIBRARY
3 8398 1000 1918 5