NOAA Technical Memorandum NOS 11

U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
National Ocean Survey

Errors of Quadrature Connected With the Simple Layer Model of the Geopotential

KARL-RUDOLF KOCH

ROCKVILLE, MD.

December 1971

NOAA TECHNICAL MEMORANDA

National Ocean Survey Series

Ocean Survey (NOS) provides charts and related information for
on of marine and air commerce. The Survey also furnishes other
a--from geodetic, hydrographic, oceanographic, geomagnetic,
imetric, and astronomic surveys or observations, investigations,
-to protect life and property and to meet the needs of engineering,
rcial, defense, and industrial interests.

NOAA Technical Memoranda NOS series facilitate rapid distribution of material which may be preliminary in nature and which may be published formally elsewhere at a later date. Publications 1 to 8 are in the former series, ESSA Technical Memoranda, Coast and Geodetic Survey Technical Memoranda (C&GSTM). Beginning with 9, publications are now part of the series, NOAA Technical Memoranda NOS.

Publications listed below are available from the National Technical Information Service, U.S. Department of Commerce, Sills Bldg., 5285 Port Royal Road, Springfield, Va. 22151. Price: $3.00 paper copy; $0.95 microfiche. Order by accession number shown in parentheses at end of each entry.

ESSA Technical Memoranda

C&GSTM 1 Preliminary Measurements with a Laser Geodimeter. S. E. Smathers, G. B. Lesley, R. Tomlinson, and H. W. Boyne, November 1966. (PB-174 649)

C&GSTM 2 Table of Meters to Fathoms for Selected Intervals. D. E. Westbrook, November 1966. (PB-174 655)

C&GSTM 3 Electronic Positioning Systems for Surveyors. A. A. Ferrara, May 1967. (PB-175 604)

C&GSTM 4 Specifications for Horizontal Control Marks. L. S. Baker, April 1968. (PB-179 343)

C&GSTM 5 Measurement of Ocean Currents by Photogrammetric Methods. Everett H. Ramey, May 1968. (PB-179 083)

C&GSTM 6 Preliminary Results of a Geophysical Study of Portions of the Juan de Fuca Ridge and Blanco Fracture Zone. Dr. William G. Melson, December 1969. (PB-189 226)

C&GSTM 7 Error Study for the Determination of the Center of Mass of the Earth From Pageos Observations. K. R. Koch and H. H. Schmid, January 1970. (PB-190 982)

C&GSTM 8 Performance Tests of Richardson-Type Current Meters, I. Tests 1 through 7. Lt. Cdr. R. L. Swanson and Lt. R. H. Kerley, January 1970. (PB-190 983)

NOAA Technical Memoranda

NOS 9 The Earth's Gravity Field Represented by a Simple Layer Potential From Doppler Tracking of Satellites. Karl-Rudolf Koch and Bertold U. Witte, April 1971.

NOS 10 Evaluation of the Space Optic Monocomparator. Lawrence W. Fritz, June 1971.

U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
National Ocean Survey

NOAA Technical Memorandum NOS 11

ERRORS OF QUADRATURE CONNECTED WITH THE SIMPLE LAYER MODEL OF THE GEOPOTENTIAL

Karl-Rudolf Koch

Geodetic Research and Development Laboratory

ROCKVILLE, MD.
December 1971

UDC 528.11:629.783

528	Geodesy
.11	Errors and their adjustment
629.7	Aeronautics
.783	Earth satellites (Koch)

CONTENTS

ERRORS OF QUADRATURE CONNECTED WITH THE SIMPLE LAYER MODEL OF THE GEOPOTENTIAL

Karl-Rudolf Koch[1]
Geodetic Research and Development Laboratory,
National Ocean Survey, NOAA, Rockville, Md.

ABSTRACT. When using the simple layer model of the geopotential in satellite geodesy, one has to integrate over surface elements of the earth on which the density of the layer is assumed constant. The integration is solved numerically by subdividing the elements and by assuming constant kernels of the integrals for the subdivisions. This quadrature causes errors that are investigated for a sphere with the mean earth radius. By assuming unit density for the surface layer, exact values of the gravitational potential of the sphere and its gradient can be computed and compared with the values obtained by quadrature. Different sizes of surface elements and different methods of subdivisions are investigated. The error of quadrature, of course, increases with size of the surface elements and decreases with the number of subdivisions. An efficient and accurate way of subdividing is the method presently applied in the analysis of satellite data.

1. INTRODUCTION

If the simple layer model of the geopotential is applied in satellite geodesy, the earth's gravitational potential is divided into a known part, represented by an expansion in spherical harmonics of finite degree, and into an unknown part, T, to be determined by satellite observations. The potential T is represented by the potential of a simple layer distributed over the surface of the earth. The unknown density of the layer is a

---

[1]Now with the University of Bonn, Germany

function of the position at the earth's surface. For simple numerical evaluation, constant density is assumed for the surface elements into which the surface of the earth is divided. Hence, we obtain

$$T = \sum_{i=1}^{k} \chi_i \iint_{\Delta E_i} \frac{dE}{\ell} \qquad (1)$$

(Koch and Morrison 1970, Morrison 1971b) where k is the number of surface elements ΔE_i; χ_i, the density of ΔE_i; and ℓ, the distance between the fixed point and the moving point.

The integral over the surface element ΔE_i is solved numerically, so errors of quadrature arise. These errors are investigated here; Morrison (1971a) investigated their influence on satellite orbits.

2. DIVISION INTO SURFACE ELEMENTS

To define the surface of the earth, one may use an ellipsoid of revolution whose shape approximates that of the earth. The surface elements for the ellipsoid are formed by means of the meridians L = const and the parallels B = const, B and L being the geographic latitude and longitude. To obtain surface elements of nearly equal size, one applies the following method. If $s°$ (e.g., $s° = 20°$) is the chosen side length in latitude for the surface elements, the surface of the ellipsoid is devided into strips bordered by the parallels $B = 90°$ and $B = 90° - s°$, $B = 90° - s°$ and $B = 90° - 2s°$, and so on, provided $180°/s°$ is an integer. The area of the strip, which includes the Equator or is bordered by the Equator, is computed and divided by $360°/s°$. Thus, the area of a block of $5° \times 5°$ at the Equator is obtained. The areas of the strips are then divided by the area of the block at the Equator; the result is rounded to the nearest integer denoted by j_B. Each strip is then divided by the meridians $L = 0°$, $L = 360°/j_B$, $L = 2(360°/j_B)$, and so on, to obtain the surface elements $\Delta E_i'$ at the surface of the ellipsoid. The elements $\Delta E_i'$ are triangular shaped at the poles and rectangular elsewhere.

The area of a surface element $\Delta E_i'$ on the ellipsoid is computed by

$$\Delta E_i' = \int_{B_i}^{B_{i+1}} \int_{L_i}^{L_{i+1}} MN \cos B \, dB \, dL$$

$$= (L_{i+1} - L_i) \, b^2 \int_{B_i}^{B_{i+1}} \frac{\cos B \, dB}{(1-e^2 \sin^2 B)^2}$$

where M and N are the radii of curvature of the meridian and the parallel; B_i, B_{i+1}, L_i, and L_{i+1}, the latitudes and longitudes of the two parallels and meridians that border $\Delta E_i'$; and where b is the semiminor axis of the ellipsoid, and e is its eccentricity. The integral over B can be solved by substituting $e \sin B = \sin \psi$. We obtain

$$\Delta E_i' = (L_{i+1} - L_i) \frac{b^2}{2} \left[\frac{\sin B}{1-e^2 \sin^2 B} + \frac{1}{2e} \ln \frac{1+e \sin B}{1-e \sin B} \right]_{B_i}^{B_{i+1}} . \qquad (2)$$

The surface of the reference ellipsoid does not coincide with the surface of the earth; thus area $\Delta E_i'$ must be corrected. If H denotes the height of the earth's surface (consisting of the geoid undulation plus the topographic height) above the reference ellipsoid, the area of the earth's surface element ΔE_i is computed from $\Delta E_i'$ by

$$\Delta E_i = \int_{B_i}^{B_{i+1}} \int_{L_i}^{L_{i+1}} (M+H)(N+H) \cos B \, dB \, dL$$

$$\approx \left(1 + \frac{2H}{R} + \frac{H^2}{R^2}\right) \Delta E_i' \qquad (3)$$

where H is assumed constant over $\Delta E_i'$ and R denotes the mean radius of the earth.

If a spherical surface is used instead of an ellipsoidal reference surface, we obtain

$$\Delta E_i' = (L_{i+1} - L_i)\, R^2\, [\sin B]_{B_i}^{B_{i+1}} \tag{4}$$

instead of eq (2).

3. SUBDIVISION OF THE SURFACE ELEMENTS

The integral over the surface element ΔE_i in eq (1) is solved numerically by subdividing ΔE_i into n^2 elements ΔE_{im} ($m = 1, 2, \ldots, n^2$) and replacing the distance ℓ by ℓ_{im} computed between the fixed point and the midpoint M_{im} of ΔE_{im}. We thus obtain

$$\iint_{\Delta E_i} \frac{dE}{\ell} \approx \sum_{m=1}^{n^2} \frac{\Delta E_{im}}{\ell_{im}} . \tag{5}$$

This quadrature causes errors dependent upon the size of the surface elements ΔE_i, the number n^2 of subdivisions ΔE_{im}, and the definition of the midpoint M_{im} of ΔE_{im}. To decrease the influence of the errors of quadrature in the analysis of satellite data, one sets the preliminary density values equal to zero so that the errors of quadrature enter only the variational equations for the parameter-sensitivity matrix and not the trajectory equations (Koch and Witte 1971).

In the application of the surface-layer model of the geopotential by Koch and Morrison (1970) and Koch and Witte (1971), the following method of subdividing the surface elements and defining the midpoints of the subdivisions was used. This method, which also forms the subdivisions ΔE_{im} by parallels and meridians, shall be called method A. If ΔB_i and ΔL_i are the differences in latitude and longitude between the parallels and meridians that border the surface element ΔE_i, the differences ΔB_{im} and ΔL_{im} in latitude and longitude between the parallels and meridians that border the subdivisions ΔE_{im} are found simply by

$$\Delta B_{im} = \Delta B_i / n$$

and (6)

$$\Delta L_{im} = \Delta L_i / n \ .$$

The differences ΔB_M and ΔL_M between the latitude and longitude of the midpoint M_{im} of ΔE_{im} and the borders are given by

$$\Delta B_M = \Delta B_{im} / 2$$

and (7)

$$\Delta L_M = \Delta L_{im} / 2 \ .$$

Although the division of the surface of the earth into the elements ΔE_i provides approximately equal areas for ΔE_i, the subdivision according to method A does not lead to equal areas for the elements ΔE_{im}. Given an element ΔE_i subdivided into ΔE_{im}, the elements ΔE_{im} closer to the poles are smaller than the ones farther away. The differences between the areas of ΔE_{im} reach a maximum for the triangular elements ΔE_i containing one of the poles. Also, the meridian and the parallel through the midpoint M_{im} of ΔE_{im} do not divide ΔE_{im} into equal areas. Thus, the areas closer to the pole are smaller than the ones farther away.

Since a uniform division of the surface of the earth promises to give the best results for the quadrature, a subdivision called method B is tried; this method uses equal areas for the subdivisions ΔE_{im} and places the midpoint M_{im} of ΔE_{im} in such a way that the meridian and parallel through M_{im} divides ΔE_{im} into four equal areas. The difference ΔL_{im} in longitude between two meridians bordering ΔE_{im} and the difference ΔL_M in longitude between the midpoint M_{im} of ΔE_{im} and the meridians bordering

ΔE_{im} again are found as in eq (6) and (7) by

$$\Delta L_{im} = \Delta L_i / n$$

and (8)

$$\Delta L_M = \Delta L_{im}/2 \ .$$

The difference ΔB_{im} in latitude between two parallels bordering ΔE_{im}, however, is determined in such a way that

$$\Delta E_{im} = \frac{\Delta E_i}{n^2} \ . \qquad (9)$$

Likewise, the difference ΔB_M in latitude between the midpoint M_{im} of ΔE_{im} and the parallels bordering ΔE_{im} is found by the parallel through M_{im}, dividing ΔE_{im} into equal areas. If B_i and B_{i+1} now denote latitude of the parallels bordering ΔE_{im}, then (from eq 4, Rapp 1971) for a spherical surface we obtain

$$\Delta E_{im} = \frac{\Delta L_i R^2}{n} (\sin B_{i+1} - \sin B_i) \qquad (10)$$

where

$$\sin B_{i+1} = \sin B_i + \frac{\Delta E_i}{n \Delta L_i \ R^2}$$

from which we find

$$\Delta B_{im} = B_{i+1} - B_i \qquad (11)$$

and, correspondingly, ΔB_M. For an ellipsoidal surface, eq (2) must be applied. However, a closed formula like (10) cannot be derived for an ellipsoidal surface; $\sin B_{i+1}$ must be computed by successive approximation.

4. SPHERICAL MODEL

For obtaining a simple computation of the errors of quadrature, a spherical surface of the earth with R = 6368 km is assumed. The sphere is covered by a simple layer of unit density so that the potential T is obtained from eq (1) and (5) by

$$T = \sum_{i=1}^{k} \sum_{m=1}^{n^2} \frac{\Delta E_{im}}{\ell_{im}} \tag{12}$$

with

$$\ell_{im} = [(x_M - x_F)^2 + (y_M - y_F)^2 + (z_M - z_F)^2]^{1/2} \tag{13}$$

where x, y, and z are the coordinates of an earth-centered coordinate system in which the z axis points toward the North Pole and the x axis toward the intersection of the Greenwich Meridian with the Equator. The index F denotes the fixed point, and M denotes the midpoint of ΔE_{im}.

The mass of a sphere covered with a simple layer of constant density may be concentrated at the center of the sphere so that, in case of unit density, the potential T can be computed from

$$T = \frac{4\pi R^2}{(x_F^2 + y_F^2 + z_F^2)^{1/2}} \, . \tag{14}$$

Methods A and B are now applied to form the subdivisions ΔE_{im}; thus, potential T and its gradient can be obtained from eq (12). By comparing the results with eq (14), the errors of the quadrature are obtained.

5. RESULTS

The way of forming the surface elements ΔE_i provides symmetry with respect to the Equator and to the meridian L = 0° and L = 180°. Hence, only fixed points with $90° \geq B_F \geq 0°$ and $180° \geq L_F \geq 0°$ have to be considered. The coordinates B_F and L_F of 26 fixed points are selected at random with heights of H_F = 800 km and H_F = 1000 km above the surface of the sphere with the mean earth radius R = 6368 km. For these fixed points, one computes the derivatives $\partial T/\partial z$ from eq (12) and compares them with eq (14) to compute the relative errors given in units of

10^{-3} in tables 1 through 9. The z component of grad T has been selected since it is defined above the North Pole or South Pole where the maximum errors of quadrature are found. The results are computed with s° = 20° and s° = 15° for ΔE_i. A side length of 20° for ΔE_i has been used by Koch and Witte (1971); a side length of 15° will be applied in a forthcoming analysis of satellite data.

Tables 1 to 3 show the relative errors of $\partial T/\partial z$ with H_F = 1000 km and s° = 20° for ΔE_i subdivided by method A with $n^2 = 1$ in table 1, $n^2 = 4$ in table 2, and $n^2 = 9$ in table 3. The values demonstrate the increase of accuracy with the increase of the number of subdivisions ΔE_{im}. Although there is considerable gain with the change from one to four subdivisions, the gain is less from four to nine subdivisions. Tables 4 and 5 confirm this fact. Hence, good accuracy with little computational effort is obtained with four subdivisions ΔE_{im}.

Tables 4 and 5 show the relative errors of quadrature with H_F = 1000 km and s° = 20° for ΔE_i subdivided by method B with $n^2 = 4$ and $n^2 = 9$. When comparing these results with tables 2 and 3, one cannot readily judge whether the subdivision by method A or B gives better results. The same is true for the comparison between tables 6 and 8 and between 7 and 9. However, the relative error of method A for the fixed points above the pole is almost half the error of method B; therefore, method A is preferred. These smaller errors result because subdivisions ΔE_{im} at the pole are smaller with method A than with B.

In tables 6 through 9, relative errors are given with s° = 15° for ΔE_i subdivided by methods A and B. Comparisons of tables 2 and 6 and 4 and 8 show the increase of accuracy gained by use of smaller surface elements ΔE_i. Tables 6 and 7 and 8 and 9 give the relative errors when heights of 1000 and 800 km are used for the fixed point H_F above the surface of the sphere. Although the height is only reduced by one-fifth when 800 km is used instead of 1000, the errors more than double at some fixed points.

Since the error for a fixed point above the pole considerably exceeds that of any other fixed point, especially when method B is used, method A was applied to subdivide ΔE_i with $s° = 15°$ into nine subdivisions at the poles and into four subdivisions elsewhere. The results are given in table 10. While the accuracy increased at the pole, the results did not change for fixed points toward the Equator (cf. table 8). Therefore, the subdivisions of the surface element below the fixed point contribute mainly to the error of quadrature. This suggests using a scheme in which the subdivisions are varied for different positions of the fixed point (e.g., $n^2 = 9$ for the surface element below the fixed point, $n^2 = 4$ for the surrounding elements, and $n^2 = 1$ for the remainder).

Method A of subdividing the elements ΔE_i gives negative errors of quadrature for fixed points above and close to the pole while method B leads to positive errors in that region. Therefore, an attempt was made to combine both methods by subdividing the elements ΔE_i according to method A but defining the midpoints as in B. The results given in table 11 indicate that the combination method is not successful. The relative errors for fixed points close to the pole considerably exceed those of tables 6 and 8.

6. CONCLUSIONS

The relative errors of quadrature for the z component of grad T presented in tables 1 through 11 are obtained after summing the contribution of each surface element ΔE_i. One may assume from the results of table 10 that the quadrature error for the surface element closest to the fixed point is the main contributor to the error given in the tables. Thus, the errors from the individual elements ΔE_i for the three components of grad T will not surpass the maximum error of the tables found for the fixed points above the pole where the z component only is defined. The contribution of each individual element ΔE_i to the three components of grad T, however, enters the variational equations (Witte 1971). The relative errors of these contributions will generally be less than 1 percent and always less than 4 percent for satellites with perigees between 800 and 1000 km and for surface elements ΔE_i (with s° = 15°) divided into four subdivisions by method A. The corresponding values for elements of 20° sidelength are 6 percent and 3 percent. This accuracy is sufficient for the variational equations.

Acknowledgments

The work was supported partially by the Deutsche Forschungsgemeinschaft. The author is indebted to Mr. H. Sckweickhard for computing the results on the IBM 7090 of the Gesellschaft für Mathematik und Datenverarbeitung, Bonn.

References

Koch, Karl-Rudolf, and Morrison, Foster, "A Simple Layer Model of the Geopotential From a Combination of Satellite and Gravity Data," _Journal of Geophysical Research_, Vol. 75, No. 8, Mar. 10, 1970, pp. 1483-1492.

Koch, Karl-Rudolf, and Witte, Bertold U., "The Earth's Gravity Field Represented by a Simple Layer Potential From Doppler Tracking of Satellites, "_NOAA Technical Memorandum_ NOS 9, National Ocean Survey, Rockville, Md., Apr. 1971, 19 pp.

Morrison, Foster (Geodetic Research and Development Laboratory, National Ocean Survey, NOAA, Rockville, Md.), "Propagation of Errors in Orbits Computed From Density-Layer Models," paper presented at the AGU/AIAA Symposium on Applications of Artificial Satellites to Geodesy, Washington, D.C., Apr. 15-17, 1971a, 49 pp. (proceedings to be published by the AGU in 1972?).

Morrison, Foster, "Density Layer Models for the Geopotential," _Bulletin Géodésique_, Vol. 101, Paris, France, Sept. 1971b (in press).

Rapp, Richard H., "Equal-Area Blocks," _Bulletin Géodésique_, Vol. 99, Paris, France, Mar. 1971, pp. 113-125.

Witte, Bertold U., "Computational Procedures for the Determination of a Simple Layer Model of the Geopotential From Doppler Observations," _NOAA Technical Report_ NOS 42, National Ocean Survey, Rockville, Md., Apr. 1971, 63 pp.

Table 1. -- Relative errors of quadrature (in units of 10^{-3}) for method A, $s° = 20°$, $n^2 = 1$, and $H_F = 1000$ km

B_F	L_F 0°	30°	100°	130°	180°
90°	-70.9				
80°	34.4	-87.5	-20.3	20.8	-280.0
70°	107.0	72.6	69.5	91.0	28.6
50°	100.0	85.8	87.6	95.6	94.2
40°	54.4	1.9	33.2	49.8	56.6
10°	102.0	28.7	203.0	-108.0	282.0

Table 2. -- Relative errors of quadrature (in units of 10^{-3}) for method A, $s° = 20°$, $n^2 = 4$, $H_F = 1000$ km

B_F	L_F 0°	30°	100°	130°	180°
90°	-44.9				
80°	10.4	-11.9	-6.1	5.0	10.4
70°	24.2	-15.3	-3.2	14.8	24.2
50°	11.3	2.5	10.2	4.3	11.3
40°	11.6	4.2	7.5	10.6	11.6
10°	10.1	13.0	21.6	17.0	10.1

Table 3. -- Relative errors of quadrature (in units of 10^{-3}) for method A, $s° = 20°$, $n^2 = 9$, $H_F = 1000$ km

	L_F				
B_F	0°	30°	100°	130°	180°
90°	-18.5				
80°	-2.2	-3.4	-4.6	-3.4	-4.6
70°	6.8	0.5	-6.2	-0.5	-6.2
50°	1.7	0.9	0.3	1.0	0.4
40°	0.7	-0.2	0.0	0.5	0.7
10°	1.3	0.4	1.4	0.7	1.9

Table 4. -- Relative errors of quadrature (in units of 10^{-3}) for method B, $s° = 20°$, $n^2 = 4$, $H_F = 1000$ km

	L_F				
B_F	0°	30°	100°	130°	180°
90°	75.5				
80°	14.5	-11.5	-4.8	8.2	14.5
70°	34.8	-19.7	-2.3	23.4	34.8
50°	5.9	-5.2	5.0	-3.7	5.9
40°	13.3	6.7	9.6	12.5	13.3
10°	1.2	4.1	12.7	8.1	1.2

Table 5. -- Relative errors of quadrature (in units of 10^{-3}) for method B, $s° = 20°$, $n^2 = 9$, $H_F = 1000$ km

	L_F				
B_F	0°	30°	100°	130°	180°
90°	43.4				
80°	2.9	1.8	0.6	1.8	0.6
70°	8.8	1.6	-6.2	1.6	-6.2
50°	-1.6	-2.4	-3.1	-2.3	-3.0
40°	0.0	-1.0	-0.7	-0.3	0.5
10°	-3.0	-3.9	-2.8	-3.6	-2.4

Table 6. -- Relative errors of quadrature (in units of 10^{-3}) for method A, $s° = 15°$, $n^2 = 4$, $H_F = 1000$ km

	L_F				
B_F	0°	30°	100°	130°	180°
90°	-24.4				
80°	3.8	-13.5	-9.0	-0.4	3.8
70°	2.6	-2.6	-0.3	0.3	2.6
50°	0.9	-1.9	-1.2	0.2	0.9
40°	2.2	1.8	0.6	1.2	2.2
10°	5.8	6.1	4.8	4.6	5.8

Table 7. -- Relative errors of quadrature (in units of 10^{-3}) for method A, $s° = 15°$, $n^2 = 4$, $H_F = 800$ km

B_F	L_F 0°	30°	100°	130°	180°
90°	-37.9				
80°	15.0	-30.1	-17.8	4.7	15.0
70°	4.9	-4.8	1.4	-1.1	4.9
50°	1.7	-9.3	-6.6	-1.0	1.7
40°	6.1	4.7	0.1	2.4	6.1
10°	18.6	19.3	15.9	15.6	18.6

Table 8. -- Relative errors of quadrature (in units of 10^{-3}) for method B, $s° = 15°$, $n^2 = 4$, $H_F = 1000$ km

B_F	L_F 0°	30°	100°	130°	180°
90°	39.9				
80°	17.2	2.0	5.9	13.5	17.2
70°	-1.6	-9.6	-6.3	-4.6	-1.6
50°	2.0	-0.6	0.0	1.4	2.0
40°	-1.5	-1.9	-3.3	-2.6	-1.5
10°	3.2	3.5	2.1	2.0	3.2

Table 9. -- Relative errors of quadrature (in units of 10^{-3}) for method B, $s° = 15°$, $n^2 = 4$, $H_F = 800$ km

	L_F				
B_F	0°	30°	100°	130°	180°
90°	62.9				
80°	28.1	-8.7	1.1	19.5	28.1
70°	-5.1	-20.1	-11.3	-13.4	-5.1
50°	6.0	-4.0	-1.5	3.5	6.0
40°	0.7	-0.9	-5.9	-3.3	0.7
10°	15.8	16.6	13.1	12.8	15.8

Table 10. -- Relative errors of quadrature (in units of 10^{-3}) for method B, $s° = 15°$, $n = 4$, for polar triangles $n = 9$, $H_F = 1000$ km

	L_F				
B_F	0°	30°	100°	130°	180°
90°	25.9				
80°	13.2	12.4	11.6	12.4	11.6
70°	-4.4	-6.5	-5.5	-6.5	-5.5
50°	2.1	-0.6	0.0	1.4	2.1
40°	-1.5	-1.9	-3.3	-2.6	-1.5
10°	3.2	3.5	2.1	2.0	3.2

Table 11. -- Relative errors of quadrature (in units of 10^{-3}) for the combination of methods A and B, s° = 15°, n = 4, and H_F = 1000 km

	L_F				
B_F	0°	30°	100°	130°	180°
90°	108.0				
80°	44.8	26.2	31.0	40.4	44.8
70°	6.1	-4.8	-0.9	2.4	6.1
50°	-5.5	-8.3	-7.6	-6.2	-5.5
40°	-4.9	-5.3	-6.6	-5.9	-4.9
10°	-0.6	-0.3	-1.7	-1.8	-0.6