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U.S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
Environmental Research Laboratories

Outline of a Bayesian Approach to the EML Multiple Cloud Seeding Experiments

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of the Director
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OUTLINE OF A BAYESIAN APPROACH TO THE EML MULTIPLE CLOUD SEEDING EXPERIMENTS

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ABSTRACT

Decision analysis techniques, using Bayes equation in several forms, are evolved for use in analyzing NOAA's Florida cumulus seeding experiments, particularly the randomized multiple cumulus experiment begun in 1970 in a 2700 n mi^2 target area south of Lake Okeechobee.

In order to apply decision analysis to evaluate the seeding effect upon rainfall, it is necessary to know the distribution and its sufficient statistics for both seeded and natural rain. It is desirable to know these for both the total target and for a "floating target" which consists of the "seeded" cloud complexes and those neighboring clouds which merge with them.

Lacking this knowledge, several assumptions are made for use in the evolution of the Bayesian approach. Firstly, it is found that the fourth root of the single cloud rainfall (transformed data) fits a gamma distribution for both seeded and control cases. Seeding changes the mean but not the shape of the distribution. Using the same assumption for the area cases and assuming a natural mean from the one fair day control case available from 1970, simple and composite hypothesis testing are used to estimate the seeding effect on rainfall. All analyses would suggest a seeding effect of a factor of 2-3 for the floating targets. This figure cannot be relied on at present due to our poor knowledge of natural fluctuations. A procedure is outlined showing how decision analyses may be used with forthcoming experiments. These methods are likely to have rather wide application in the analysis of meteorological experiments.

OUTLINE OF A BAYESIAN APPROACH TO THE EML MULTIPLE CLOUD SEEDING EXPERIMENTS

Joanne Simpson and Jacques Pézier

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

The EML[1] single cumulus experiments have demonstrated that dynamic seeding increases the rainfall from supercooled Florida cumuli by a factor greater than three. The causal relationship was established by a series of conventional statistical tests on the data series collected in 1968 and 1970. The series contained a total of 52 GO clouds, 26 seeded and 26 controls (Simpson, Woodley and Miller, 1971). Numerous statistical tests were conducted on both the raw data and on the fourth root of the rainfall data, hereafter referred to as "transformed rainfall." Significance of the seeded-control differences was better than 5% with all tests on the combined 1968-1970 data and with most tests on the 1970 data alone. A significance of 5% or better is generally regarded by meteorologists as a satisfactory demonstration of causality.

The seeding effect was found to be larger and more significant on "fair" than on "rainy" days, when it appears to be negative. Fair days are defined as those with less than 13% coverage (4000 n mi^2 within radar range) by radar echoes, while rainy days are defined as those with more than 13% echo coverage. In the tropics, total rainfall splits about evenly between fair and rainy days, although the latter are only about 10% of the total number of days.

In 1970 for the first time EML conducted a multiple dynamic seeding experiment in a 2700 n mi^2 target area just south of Lake

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[1] Experimental Meteorology Laboratory, NOAA

Okeechobee. On each day of this experiment 10-15 clouds were seeded in rapid succession. Suitable days were selected objectively by a "meteorological suitability factor" (Simpson and Woodley, 1971; Woodley and Williamson, 1970) intended to screen out cases of poor seedability and excessive natural precipitation. Within the suitable cases, randomization was achieved by opening the sealed envelope instruction in flight on board the seeder aircraft but with outcome unknown to the scientists.

In 1970, only 21 operating days were available for the experiment for reasons (to be described) which make its future a serious decision problem. Only 6 GO cases were obtained, 4 seeded and two controls; one of the control cases fell into the "rainy" category and hence will not be too useful. One good case was missed because of malfunction of the seeder aircraft. Since the weather during the 1970 operational period was only slightly more unfavorable than normal it is reasonable to expect in the future that about one-third of the available operating days will result in useful GO cases. An experimental period of 45 days is scheduled for 1971; from this we can expect at best about 15 GO days. With 3:2 randomization in favor of the seed instruction one could hope for a total of about 9 seed cases and 6 control cases.

The problem limiting the operation of this important experiment is more serious and more irremediable than a mere financial shortage. Briefly, the experiment is squeezed between the Florida agriculturalists and the demands of the NOAA - Navy hurricane seeding experiments. The target area for the experiment is highly agricultural and supports a multi-million dollar tomato crop. Unfortunately, the spring tomato harvest often runs to as late as 15 June although in 1971 it will probably be finished by June 1. Since tomatoes are ruined from even small rainfalls during harvesting, the Florida farmers have made it quite clear that they will not tolerate any seeding until the harvest is over. The experimental area cannot be moved since the present target is the only large enough land area within the range of the University of Miami calibrated radar, which is the tool of rainfall analysis. Priorities within NOAA require the project aircraft to be turned over to the hurricane program after July 15. Unfortunately there are no periods of adequate numbers of seedable clouds in the area outside of those months of conflict with agriculturalists and/or hurricanes.

The purpose of this study is to apply the tools of decision analysis and Bayesian statistics not only to the existing data of this experiment but to guide future experimental strategy and data collection. This first report is preliminary and mainly outlines the approach. It will be shown that existing data are inadequate both to draw any scientific conclusions or to make any beyond tentative decisions concerning strategy.

It is, however, possible to conclude at the outset that conventional statistical methods will be unable to reach firm inferences regarding the seeding effect for the multiple seeding experiment based on the anticipated 20 or so GO cases available after completion of the 1971 operation. Due to the large natural variability of cumulus rainfall, 52 GO cases were required for adequate significance in the single cloud experiment and the situation is almost sure to be worse rather than better in the cloud group case. As will be seen in Appendix II, the rainfall distribution for single clouds is highly skewed, with most clouds raining little and a few very much. The standard deviation of the rainfall is roughly 50% greater than the mean value. With the area experiment, the factor of cloud mergers (Simpson and Woodley, 1971), avoided for single cloud analysis, makes it likely that the distributions may be more skewed and the standard deviations possibly larger.

Two other vital factors enter the decision process. Firstly, the NOAA RFF[2] aircraft may not be available for this research in 1972 and 1974. Secondly, increasing pressure is being brought upon NOAA and upon us to conduct practical rain enhancement operations. It is therefore highly likely that many if not most of the critical decisions on whether and how to go operational will have to be made on the basis of the 20 or so GO cases from 1970-1971. Decision analysis offers the best hope to maximize the learning benefit from these cases.

2. THE DATA AND ASSUMPTIONS

The rainfall summary of the 1970 multiple cloud seeding experiment is given in Table 1. The first row shows the

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[2] Research Flight Facility

TABLE 1. RAINFALL SUMMARY OF 1970 MULTIPLE CLOUD SEEDING EXPERIMENT FIVE HOURS AFTER INITIAL SEEDING

	JUNE 29	RANK	JUNE 30	RANK	JULY 2	RANK	JULY 7	RANK	JULY 8	RANK	JULY 17	RANK	JULY 18	RANK
$S-N_e$	2.95		1.10		5.00		2.85		3.70		1.90		2.90	
ECHO COVERAGE[a] $(n.mi.)^2$	1000		4145[b]		710		550		1275		775		42	
TIME: FIRST & LAST SEEDING (GMT)	1743 2058		1714 1950		2036 2302		1835 2113		1756 2005		1800		1851 2136	
TOTAL AgI (gm)	6800		(6050)		4800		(6300)		7550		0	—	6750	
TOTAL TARGET RAINFALL (acre-ft. $\times 10^4$)	1.41	6	6.06	4	1.71	5	6.13	3	8.08	1	3.75	—	7.53	2
FLOATING TARGET RAINFALL (acre-ft. $\times 10^4$)	0.16	6	3.08	3	1.12	4	0.73	5	6.41	1	—	—	4.37	2
W_{FT}/W_{TT}	0.11	6	0.51	4	0.65	2	0.12	5	0.79	1	—	—	0.58	3
WATER DEPTH FLOATING TARGET (in.)	1.29	4	2.72	2	1.20	5	1.06	6	2.38	3	—	—	3.91	1
dR/dt[c] (acre-ft./min.)	7.4	3	22.4	2	64.3	1	3.1	5	4.2	4	—	—	0.9	6
SUM OF RANKS		25		15		17		24		10		—		14

a. WITHIN 100 n.mi. OF MIAMI
b. MOST DISTURBED DAY
c. IN FIRST HOUR AFTER INITIAL SEEDING

"meteorological suitability factor" ($S-N_e$) which is higher the more suitable the day. The second row gives echo coverage within the whole radar range; the day is defined as "rainy" if this figure exceeds 4000 n mi^2. We will focus our attention most heavily on the data rows entitled "floating target rainfall" and, secondarily, "total target rainfall."

By "floating target rainfall" is meant the rainfall following the "seeded" radar echoes and those that merge with them. Comparison of seeded and control floating target rainfalls is likely to give a better measure of seeding effect than is comparison of seeded versus control total target rainfalls due to high natural variation in total target rainfalls introduced by the frequent moving in of heavily precipitating echoes which are not penetrated by the seeder aircraft. A qualitative measure of the effectiveness of the seeding may be inferred from the row W_{FT}/W_{TT}, the ratio of floating target to total target rainfall. Intuitively, we would expect this ratio to be greater on the days where seeding was having a greater effect. The water depth in the floating target (rainfall divided by echo area) turns out to be quite large, exceeding two inches on two seeded days and on the rainy control day. The row labeled dR/dt gives the rate of rainfall increase in the first hour after the initial seeding.

The only previous attempt to analyze this table was to rank each rainfall parameter with a rank 1 to 6 denoting where the observation stood (by size) among the 6 GO days. Thus July 8 (seeded) ranks first in total target and floating target rainfall as well as in the ratio W_{FT}/W_{TT}, the ratio of floating to total target rainfall. The sum of ranks for each day appears in the bottom row. It was encouraging to note that (if the rainy control day is excepted) the three lowest ranks are held by seed days. However, the highest rank is also held by a seed day (June 29) on which cloud lifetimes were very short and mergers were apparently not successfully promoted.

The goal here will be to apply Bayes equation to the multiple cloud seeding experiment to attempt to discriminate between two or more hypotheses regarding the effect of seeding. We first apply a two hypothesis and three hypothesis test to the floating target rainfall where:

H_1 = multiple seeding increases floating target rainfall by a factor of 3.3

H_2 = multiple seeding has no effect on floating target rainfall

and, in the 3-way test where:

H_{3A} = multiple seeding increases floating target rainfall by a factor of 5

H_{3B} = multiple seeding decreases floating target rainfall by a factor of 2

with H_1 and H_2 remaining as above. With the small data sample and large standard deviation it is not possible to make closer resolution with this type of hypothesis testing.

The Bayesian expert will recognize that the weakness in the above formulation of hypotheses is that we do not have a proper denial of H_1 in either H_2 or H_2 and H_3 together. So finally an attempt will be made to test improved hypotheses in the form

H_1 = seeding increases precipitation by a factor of n or more.

H_2 = seeding increases precipitation by less than a factor of n.

The hypotheses now cover all possibilities but greater difficulty ensues in trying to assess the probability of seeing the data with each hypothesis, as we shall see.

To apply Bayes equation to this problem it is necessary to know the distribution function for natural precipitation (and its sufficient statistics) and to be able to hypothesize sensibly how seeding alters the distribution function and its sufficient statistics. Concerning the floating targets, virtually no such information is presently available. Concerning the total target, radar data exist for roughly 100 cases in 1968, 1969 and 1970 but they have not yet been analyzed for use in this problem.

Hence for this preliminary study we shall have to draw upon the single cloud data and other rainfall information to characterize the distributions. Then we use these deduced distributions with the data for the one fair control day (July 7) for the natural case. The first is probably a safe procedure but the second is clearly untenable, as will soon be clear.

As will be seen from Appendix II it has so far been possible only to find tractable distribution functions for transformed (fourth root) rainfall. Although the inverted Rayleigh distribution makes a good fit to the raw data it does not have finite moments and hence is unusable for this type of study. Research is underway in the Dartmouth group to find a function to fit the raw data that does have finite moments since we clearly lose information by being forced to deal with fourth roots.

There are two important results from the single cloud data study in Appendix II. First, both seeded and control transformed data are best fitted by a gamma probability density function of the form

$$p(R) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} R^{\alpha-1} e^{-\beta R} \qquad (1)$$

where p(R) is the probability density of a rainfall amount R (here measured in acre-ft) from a single cloud. The scale of the distribution is determined by the parameter β and the shape by the parameter α, Γ is the gamma function (cf. Pearson et al.,1951). In meteorology the gamma function has been used extensively to fit untransformed rainfall data on much larger space and time scales, ranging from single storms to monthly and yearly distributions (see Thom, 1958; 1968 and references therein). That the fit is good in both seeded and control cases is attested to by values of χ^2 much lower than the number of degrees of freedom (see Appendix II, especially figs. A II-1 and A II-2).

The second important result is that seeding does not alter the shape of the distribution nor its coefficient of variation, but merely advances the mean (of the transformed data) by 35%. The coefficient of variation in both populations is about 0.4. This is the first of the two main assumptions made here -- namely that the gamma distribution with coefficient of variation of 0.4 carries over to both the natural and seeded cases of the transformed area rainfalls. A program is presently underway to test the validity of this carryover.

The next best fit, among tractable distributions, to the transformed single cloud data, was the truncated Gaussian, and this is also used with the primary purpose of testing the sensitivity of results to the exact form of the distribution.

The second and weakest assumption in this report is that the rainfall on the one fair control day of the experiment gives us the expected value of the fair day natural precipitation. We thus assume, for this exercise, that the natural floating target precipitation is a gamma or Gaussian distribution with a coefficient of variation of 0.40 and a mean value of 2.92 (fourth root of 73, units 10^2 acre-ft, see Table 1). Then the modified distributions for the various hypotheses are found by taking gamma or Gaussian distributions with the same coefficient of variation and the mean multiplied by the factor corresponding to the hypothesis. To test this assumption we compute in Appendix I the probability that with a gamma distribution with mean and coefficient of variation in this size range a single observation falls within one standard deviation of the true expected value. This probability only comes out about 70%; the probability that the one observation represents the expected value within a factor of two is only 35% (when untransformed rainfalls are considered). This result unfortunately renders the Bayesian calculations here only an interesting exercise to illustrate what can and should be done with the method; no physical inferences can be drawn. It does, however, bring home the importance of obtaining measurements on the natural distributions which can fortunately be achieved without expensive seeding programs.

3. BAYES EQUATION APPLIED TO THE 1970 MULTIPLE CUMULUS SEEDING DATA - SIMPLE HYPOTHESIS TESTING

The two hypothesis case will be treated first, as follows:

H_1 = the mean of the transformed rainfall distribution is increased 35% by seeding (a seeding factor of 3.3)

H_2 = transformed rainfall is unaffected by seeding; the distribution remains unaltered

D = floating target transformed rainfall for the 4 seeded cases, as given in Table 2.

X = other conditions and assumptions of the problem as outlined in Section 2

Table 2. Transformed Rainfall Data - Area Experiment 1970

Date	Case No.		R
June 29	1		2
July 2	2		3.25315
July 8	3		5.0317
July 18	4		4.57215
		mean =	3.71425
		σ =	1.36905
		$V = \sigma/m$ =	0.368595

Bayes equation is first written in the form

$$P(H_1/DX) = P(H_1) \frac{P(D/H_1 X)}{P(D/X)} \tag{2}$$

Extending the conversation in the denominator, we get

$$P(H_1/DX) = P(H_1) \frac{P(D/H_1X)}{P(DH_1/X) + P(DH_2/X)}$$

$$= P(H_1) \frac{P(D/H_1X)}{P(D/H_1X)\ P(H_1/X) + P(D/H_2X)\ P(H_2/X)} \tag{3}$$

A test will be made using two different prior probabilities:

Case 1 $P(H_1) = 0.50$

Case 2 $P(H_1) = 0.65$

Case 1 corresponds to that of no prior evidence for or against either hypothesis while with Case 2 we are assuming 2.69 dB of prior evidence for the hypothesis H_1 which might be justified on the grounds of the conclusiveness of the single cloud experiment.

Now it only remains to find the probabilities $P(D_i/H_iX)$. These are first evaluated using the gamma distribution which has the probability density (1).

The first two moments of the gamma distribution are well known to be (see, for example, Tribus and Pézier, 1970)

$$\mu_1 = \langle R \rangle = \alpha/\beta \tag{4}$$

and

$$\mu_2 = \sigma^2 = \alpha/\beta^2$$

where $\langle R \rangle$ is the expected value and σ^2 is the variance. Therefore the coefficient of variation V is

$$V \equiv \frac{\sigma}{\langle R \rangle} = \frac{1}{\sqrt{\alpha}} \tag{5}$$

Fortunately, with the coefficient of variation constant between seeded and control populations, α will not vary. With $V = 0.377$, $\alpha = 7.0$. In evaluating the denominator of (1) and in later work it is also highly fortunate that α is closely an integer. Hence we need only obtain β from (4) and the given value of $\langle R \rangle$ to specify $P(R)$.

For H_2 we have $\langle R \rangle = 2.92$

so that $\beta = 2.4$

and

For H_2 $$P(R) = 0.637R^6 e^{-2.4R} \tag{6}$$

and

For H_1 with $\langle R \rangle = 3.94$ (increased by 35%),

$\beta = 1.77575$

and

$$P(R) = 0.0773286R^6 e^{-1.77575R} \tag{7}$$

so that using Table 2 with (6) and (7) we obtain Table 3:

Table 3. Gamma Distribution - Probabilities of Seeing the Data Given the Two Simplest Hypotheses

Case	R	$P(D_i/H_1X)$	$P(D_i/H_2X)$
1	2	0.141946	0.33551
2	3.25315	0.284015	0.307032
3	5.0317	0.165268	0.0588644
4	4.57215	0.210391	0.0998342

The results are now straightforward:

Case 1: Prior probabilities $P(H_1) = 0.50; P(H_2) = 0.50$

Posterior probability for H_1

$$P(H_1/DX) = 0.70$$

Case 2: Prior probabilities $P(H_1) = 0.65; P(H_2) = 0.35$

Posterior probability for H_1

$$P(H_1/DX) = 0.8125$$

In each case the data set adds +3.67 db of evidence in favor of H_1. This might be interpreted as 0.9 dB of positive evidence per seeded case, although when we get to the evidence form of Bayes equation, reservations will be evident.

In order to go from a prior probability of 0.50, or zero prior evidence in favor of the hypothesis, to 0.95, required as the threshold by meteorologists, we need 12.78 dB of positive evidence. If it were true that we gain 0.9 dB per seeded case, or better 3.67 dB per set of 4 cases, we would need a total of 14.2 seeded cases. With 4 already in existence, this means 10-11 more, provided our knowledge about natural fluctuations becomes adequate.

More insight into this problem is gained by writing Bayes equation in evidence form. This is done first with two hypotheses, viz:

$$\mathrm{Ev}\ (H_1/DX) = \mathrm{Ev}\ (H_1) + 10 \log_{10} \frac{P(D/H_1X)}{P(D/H_2X)} \tag{8}$$

Here the units of evidence are decibels. For the definition of evidence, its relation to probability and the derivation of equation (8), the reader is referred to Tribus (1969). With the two hypothesis case, this formulation has the advantage that we can start with zero evidence for and against both hypotheses, namely the non-committal prior probabilities of 0.50, 0.50. With this we have

$$\mathrm{Ev}\ (H_1/DX) = 10 \log_{10} \frac{P(D/H_1X)}{P(D/H_2X)} \tag{9}$$

Applying (9) to the case we have just studied, we obtain Table 4 and figure 1.

From Table 4 we see the important result that the first two cases contributed negative evidence for H_1 while the last two cases contributed a positive amount of about 4 and 3 dB respectively. Due to the wide variation in apparent seeding effect found with single clouds, ranging from negative to an order of magnitude positive, it would appear wiser to consider the cumulative evidence provided by a given set of data rather than to take seriously the apparent evidence contributed by one seeded case.

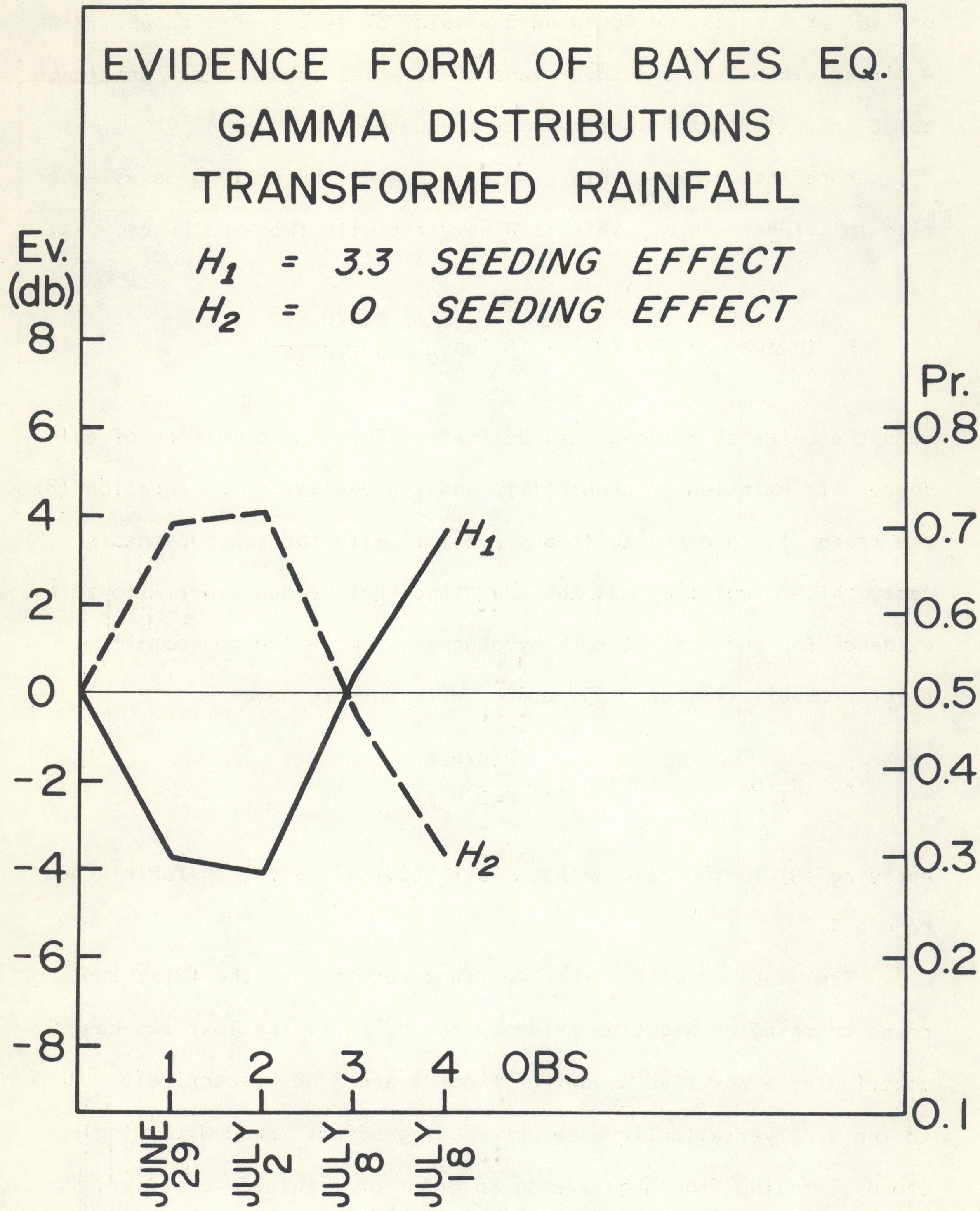

Figure 1. Plot of evidence for H_1 and H_2 in simple hypothesis test with Bayes equation. Evidence in decibels is the vertical scale in the left. Probability is the verical scale on the right.

Table 4. Evidence for H_1, H_2

Case	Gamma Distribution Ev H_1/DX	Transformed Rainfall Ev H_2/DX
0	0	0
1	-3.73	3.73
2	-4.07	4.07
3	0.41	-0.41
4	3.66	-3.66

Before proceeding to other hypotheses, it is important to test the sensitivity of results to the functional form of the rainfall distribution in the simplest case, namely with (9). We use the truncated Gaussian. Although this was the second best fit (among tractable functions) to the single cloud data and had a fairly good X^2, figures AII-1 and AII-2 in Appendix II show that the Gaussian function is a much poorer fit to the single cloud data than is the gamma distribution due to its smaller skewness. This relative merit may surely be expected to apply to the area rainfall, which is likely to be even more skewed. However, as Thom (1958 loc. cit.) pointed out, a gamma distribution approaches normality as α increases, for $\alpha > 100$ it is approximately normal for climatological applications.

The probability density for the truncated Gaussian distribution is given by

$$P(R) = A\, e^{BR - CR^2} \tag{10}$$

where B and C bear a complex relation to mean and standard deviation which is best found by means of a computer program (see Tribus, loc. cit., pp. 131 and ff). Using the Pézier program DAMAXS 2 (discussed and listed in Appendix II, Table AII-5) we find A, B and C for the cases

$$\langle R \rangle = 2.92 \qquad \sigma = 1.17$$

and

$$\langle R \rangle = 3.94 \qquad \sigma = 1.58$$

discussed previously and obtain

for H_1 $$P(R) = .00999181\, e^{1.64195R - 0.207249R^2} \tag{11}$$

and

for H_2 $$P(R) = .0133888\, e^{2.22078R - 0.378194R^2} \tag{12}$$

Using (11) and (12), Table 5 is the result.

Table 5. Evidence for H_1, H_2

Truncated Gaussian Distribution Case	 Ev (H_1/DX)	Transformed Rainfall Ev (H_2/DX)
0	0	0
1	-3.33	3.33
2	-4.92	4.92
3	-0.04	0.04
4	2.71	-2.71

Table 5 shows that while the general result is not too dissimilar from that presented in Table 4 using the gamma distribution, the values obtained are sufficiently different to make it clear that a real effort must be made to get adequate data to specify the distribution function fairly well.

Next, three hypotheses are tested jointly. The first set of three is:

H_1 = the seeding effect is +3.3

H_2 = the seeding effect is 0

H_{3A} = the seeding effect is +5

for H_3 we have

$\langle R \rangle = 4.38 \qquad \sigma = 1.65$, so that for H_3

$$P(R) = 0.0369852\ R^6\ e^{-1.59817R} \tag{13}$$

For this test Bayes equation is written

$$\text{ev}\ (H_1/DX) = 10 \log_{10} \frac{P(H_1/DX)\ P(H_1/X)}{P(H_2/DX)\ P(H_2/H) + P(H_3/DX)\ P(H_3/X)} \tag{14}$$

Unfortunately in this three-way case we are unable to assign zero prior evidence to all three hypotheses since the probabilities of the three must add up to one. Hence for the first test we assign the prior probabilities as follows:

$P(H_1) = 0.40$

$P(H_2) = 0.30$

$P(H_{3A}) = 0.30$

The slight prior preference for H_1 is justified by the single cloud seeding effect.

The results for this set up are shown in Tables 6 and 7 and figure 2.

Table 6. Probabilities of the Data Given the Hypotheses

		Gamma Distribution	Floating Target	Transformed Rainfall
Date	Case	$P(D_n/H_1X)$	$P(D_n/H_2X)$	$P(D_n/H_3X)$
June 29	1	0.141946	0.33551	0.0968401
July 2	2	0.284015	0.307032	0.242057
July 8	3	0.165268	0.0588644	0.193167
July 18	4	0.210391	0.0998342	0.226636

Table 7. Evidence for H_1, H_2, H_{3A}

$P(H_1) = 0.40$, $P(H_2) = 0.30$, $P(H_{3A}) = 0.30$

	Gamma Distribution	Transformed Floating Target Rainfall	
Case	$Ev\ (H_1/DX)$	$Ev\ (H_2/DX)$	$Ev\ (H_{3A}/DX)$
0	-1.76	-3.68	-3.68
1	-3.59	0.69	-7.34
2	-3.71	1.25	-8.25
3	-0.76	-3.45	-5.19
4	0.59	-6.80	-3.82

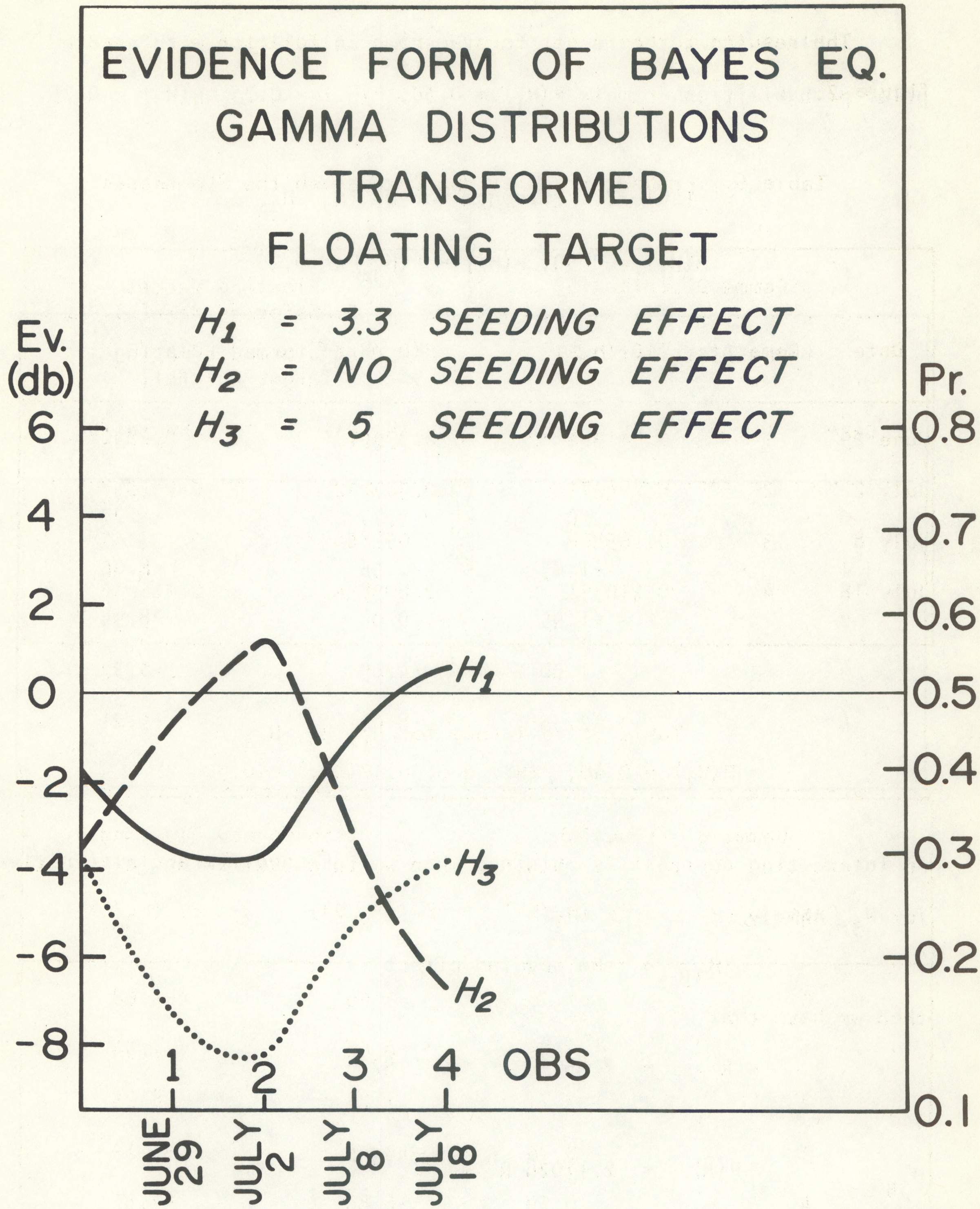

Figure 2. Plot of evidence for H_1, H_2 and H_{3A} in simple hypothesis test with Bayes equation. Evidence in decibels is the vertical scale on the left. Probability is the vertical scale on the right.

Table 8 shows the result for the same calculation with altered prior probabilities, namely $P(H_1) = 0.50$, $P(H_2) = 0.25$, $P(H_{3A}) = 0.25$.

Table 8. Evidence for H_1, H_2, H_{3A}

$P(H_1) = 0.50$; $P(H_2) = P(H_{3A}) = 0.25$

	Gamma Distribution		Transformed Floating Target Rainfall
Case	Ev (H_1/DX)	Ev (H_2/DX)	Ev (H_{3A}/DX)
0	0	-4.77	-4.77
1	-1.83	-0.55	-8.06
2	-1.95	-0.04	-8.94
3	1.00	-4.69	-6.32
4	2.35	-8.01	-5.21

An interesting contrast is obtained when we take a different alternative for H_3, namely

$$H_{3B} = \text{the seeding effect is } -2$$

then we have that

$$< R > = 2.45 \qquad \sigma = 0.926$$

and

$$\underset{H_{3B}}{P(R)} = 2.11028\ R^6\ e^{-2.8479R}$$

In this test, we assign nearly even prior probabilities as $P(H_1) = 0.34$, $P(H_2) = 0.33$, $P(H_{3B}) = 0.33$

With H_{3B} the negative seeding effect hypothesis, results are shown in Table 9.

Table 9. Comparison of H_1, H_2 and H_{3B}

Gamma Distribution - Transformed Floating Target Rainfall						
Case	$P(D/H_1)$	$P(D/H_2)$	$P(D/H_3)$	$Ev(H_1/D)$	$Ev(H_2/D)$	$Ev(H_{3B}/D)$
0	0.34	0.33	0.33	-2.88	-3.08	-3.08
1	0.141946	0.33551	0.453801	-7.32	-2.52	-0.26
2	0.284015	0.307032	0.236908	-7.04	-1.60	-1.29
3	0.165268	0.0588644	0.0204779	-0.81	-1.75	-7.69
4	0.210391	0.0998342	0.0426681	3.15	-4.05	-13.39

An important result of these tests is that they show the evidence for H_1 varies considerably depending upon what are the other hypotheses that it is tested against. When H_3 is a large positive seeding effect, H_1 shows up much less well than when H_3 is a sizeable (-2) negative seeding effect. If we could be confident that the natural transformed rainfall did indeed have a gamma distribution with expected value 2.92, these results would be very encouraging, in that H_1 consistently shows up well relative to other hypotheses under a wide variation of tests and assumptions. Unfortunately, a quite opposite result arises when we treat total target transformed rainfall. The 1970 data for the total target rainfall are given in Table 10.

Table 10. Seeded and Control Total Target Rainfall 1970

Action	Date	Rainfall $\times 10^2$ acre-ft	Transformed Rainfall
Control	July 7	613	4.97583
Seeded	July 29	141	3.44592
Seeded	July 2	171	3.61617
Seeded	July 8	808	5.33154
Seeded	July 18	753	5.2384

So for the "no effect" hypothesis we have

$$\langle R \rangle = 4.97583 \qquad \sigma = 1.87$$

and

H_2 $$P(R) = .0151459\, R^6\, e^{-1.4068R}$$

and for H_1 $$\langle R \rangle = 6.71737 \qquad \sigma = 2.53$$

H_1 $$P(R) = .0018533\, R^6\, e^{-1.04207R}$$

Taking prior probabilities of 0.5 for each hypothesis, Table 11 gives the probabilities of the data given each hypothesis.

Putting the ingredients of Table 11 in Bayes equation in form (3) we find that

$$P(H_1/DX) = 0.12!$$

$$P(H_2/DX) = 0.88$$

Table 11. Probabilities of H_1, H_2

Gamma Distribution - Total Target Rainfall		
Case	$P(D/H_1)$	$P(D/H_2)$
1	0.08556	0.19897
2	0.095694	0.209138
3	0.16451	0.19232
4	0.163087	0.197245

After seeing the data, there would be 8.65 dB of evidence against hypothesis H_1 and for H_2 if the control case value in Table 10 were to be taken seriously.

We plan a radar evaluation of the numerous (about 100) natural cases for the total target rainfall in our files and will repeat this analysis thereafter. However, with an area this large (2700 n mi^2) it may be that a meaningful "expected value" cannot be obtained. It would appear much more meteorologically sound to take the trouble and expense to obtain an adequate number of unmodified cases to specify the "floating target" rainfall distribution and expected value which is much more likely to be a tractable problem.

4. COMPOSITE HYPOTHESIS TESTING WITH BAYES EQUATION

In Section 3 we have tested several simple hypotheses using Bayes equation. The shortcoming of this procedure is that the hypotheses do not cover all possibilities and hence a proper denial for H_1 has not been given, either by H_2 alone or by H_2 and H_3 together. Since extending the conversation in the denominator depends upon the fact that the probabilities for all the summed hypotheses add up to one, strictly speaking the extension of conversation in the denominator in Section 3 was not correct.

In a meteorological problem it makes much more sense to hypothesize as follows:

H_1 = the seeding effect is a factor of n or greater

H_2 = the seeding effect is less than a factor of n.

Now H_2 is a proper denial of H_1 and all possibilities are covered. The difficulty is that it is harder to assess the probability of the data given this type of composite hypothesis.

However, with the gamma distribution for a population whose coefficient of variation does not change, we are in a very fortunate position. The coefficient α does not change but remains fixed at 7 and the seeding effect (alteration of expected R) depends on the parameter β alone. Table 12 and figure 3 show the dependence of the seeding effect on β. Note that we have defined the seeding effect as a ratio of untransformed seeded rain to natural rain, so that a seeding effect of 1 is no change, a seeding effect of 0.5 is a factor of two reduction while a seeding effect of 2 is doubling and so forth.

A difficulty of course is that beyond a seeding effect of five, β changes very little. However, this is not important in that it is almost inconceivable that the seeding effect is this large or larger and if it were, establishing it anywhere in this range would be quite adequate for most purposes.

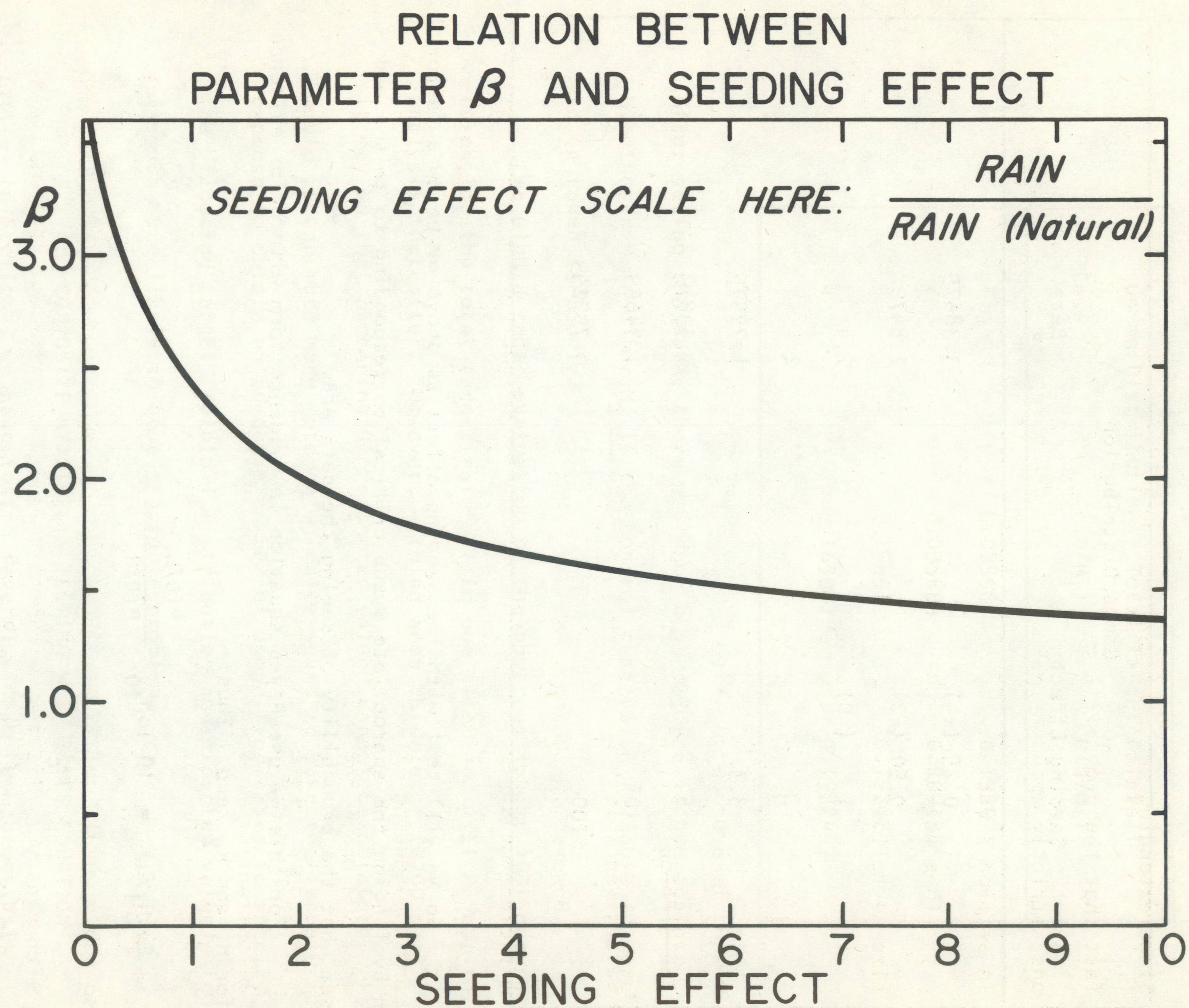

Figure 3. Relation between the gamma distribution parameter β and seeding effect. Seeding effect here is defined as the ratio of the (untransformed) rain to the natural rain. That is, a value of 0.5 of the abscissa means a decrease of a factor of two, a value of 1 means no increase and a value of two means a factor of two increase.

Table 12. Parameter β Related to Seeding Effect $\frac{R}{R(\text{Natural})}$ (Untransformed)

Gamma Distribution	
Seeding Effect	Beta
0.25 (-4)	3.38675
0.50 (-2)	2.8479
1 (0)	2.4
2	2
3.3	1.77575
5	1.59817
10	1.34669
100	0.757299

So we will test with the seeding effect as above or below a factor n by finding the appropriate β sub critical corresponding to this n and see what the probability of seeing the data are.

For this we use Bayes equation in evidence form with prior evidence for H_1 and H_2 zero. Thus:

$$Ev(H_1/D) = 10 \log_{10} \frac{P(D/H_1)}{P(D/H_2)} \tag{15}$$

so now

$$P(D_i/H_1) = \int_0^{\beta_c} \frac{\beta^7}{\Gamma(\alpha)} R_i^{\,6} e^{-\beta R_i} P(\beta/H_i)\, d\beta = I_{1i} \tag{16}$$

and

$$P(D_i/H_2) = \int_{\beta_c}^{\infty} \frac{\beta^7}{\Gamma(\alpha)} R_i^6 \; e^{-\beta R_i} \; P(\beta/H_2) \; d\beta = I_{2i}$$

and

$$Ev(H_1/D) = 10 \log_{10} \frac{I_{11} \cdot I_{12} \cdot I_{13} \cdot I_{14}}{I_{21} \cdot I_{22} \cdot I_{23} \cdot I_{24}} \qquad (17)$$

where $R_i = R_1, R_2, R_3, R_4$ are the rainfall data (transformed) for the floating target cases as presented in Table 2.

To illustrate this procedure we will first take uniform prior probabilities on β in various ranges.

The first case considered for β_c will be 1.77575, the value corresponding to a seeding effect of 3.3. Hence the hypotheses become:

H_1 = the seeding effect is +3.3 or more

H_2 = the seeding effect is less than +3.3

The first assignment of prior probability on β will be uniform in the range corresponding to a seeding effect of 0.5 (-2) to +10, or β in the range 2.8479 to 1.34669. Then for the observation R_i, the ratio of integrals becomes:

$$\frac{I_{1i}}{I_{2i}} = \frac{\dfrac{\int_{1.34669}^{1.77575} \beta^7 \, e^{-\beta R_i} \, d\beta}{1.77575 - 1.34669}}{\dfrac{\int_{1.77575}^{2.8479} \beta^7 \, e^{-\beta R_i} \, d\beta}{2.8479 - 1.77575}} \qquad (18)$$

with the other factors depending on α and R_i in both numerator and denominator cancelling.

The integral tables given for the integrals

$$\int \beta^7 e^{-\beta R} d\beta = \frac{-e^{-\beta R}}{R^8} [(\beta R)^7 + 7\cdot(\beta R)^6 + 7\cdot 6(\beta R)^5$$
$$+ 7\cdot 6\cdot 5 (\beta R)^4 + 7\cdot 6\cdot 5\cdot 4(\beta R)^3$$
$$+ 7\cdot 6\cdot 5\cdot 4\cdot 3(\beta R)^2 + 7\cdot 6\cdot 5\cdot 4\cdot 3(\beta R)^2$$
$$+ 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2(\beta R) + 7!] \qquad (19)$$

A computer program to evaluate this integral with various value of β_c and variations in the range of the uniform prior β is shown in Table 13 for the particular example just cited. Table 14 summarizes results for β_c corresponding to a seeding effect of 3.3, with various prior ranges of β.

The results of Table 14 suggest that so far the evidence is about evenly balanced between H_1 and H_2. This result suggests testing a higher value of β_c, namely a lower seeding effect. We now try the following two versions of H_1, namely:

H_{1A} = the seeding effect is 2 or more

H_{1B} = the seeding effect is 0 or more

Retaining the most reasonable prior on β, namely uniform for seeding effects between 0.5 (-2) and +10, we obtain Table 15.

Table 13. Computer Program

```
100 REM THIS PROGRAM CALCULATES THE EVIDENCE IN FAVOR OF THE
110 REM HYPOTHESIS THAT THE AREA SEEDING EFFECT IS 3.3 OR MORE
115 REM THE PRIOR PROBABILITY ON BETA IS UNIFORM FROM A SEEDING
116 REM EFFECT OF -2 TO +10
120 PRINT "JOANNE"
130 PRINT
140 PRINT "R", "I1", "I2", "RATIO I1/I2", "EVIDENCE"
145 PRINT
150 DEF FNR (B,R)
160 LET X=1*2*3*4*5*6*7
170 LET X=X+7*6*5*4*3*2*(B*R)
180 LET X=X+7*6*5*4*3*((B*R)↑ 2)
190 LET X=X+7*6*5*4* ((B*R) ↑ 3)
200 LET X=X+7*6*5*((B*R)↑ 4)
210 LET X=X+7*6* ((B*R)↑ 5)
220 LET X=X+7* ((B*R)↑ 6)
230 LET X=X+ ((B*R)↑ 7)
240 LET X=X*(-EXP (-B*R) /(R ↑ 8))
250 LET FNR=X
260 FNEND
270 FOR K=1 TO 4
280 READ R
290 DATA 2,3.25315,5.0317,4.57215
300 LET I1= (FNR(1.77575,R)-FNR(1.344669,R))/(1.77575-1.34669)
310 LET I2= (FNR(2.8479,R)-FNR(1.77575,R))/(2.8479-1.77575)
320 PRINT R, I1, I2, I1/I2,10*(LOG(I1/I2)/LOG(10))
330 NEXT K
340 END
```

R	I1	I2	RATIO I1/I2	EVIDENCE
2	1.02333	3.4252	0.298765	-5.2467
3.25315	0.139808	0.179982	0.776789	-1.09697
5.0317	8.6332 E-3	3.49201 E-3	2.47227	3.93096
4.57215	1.76463 E-2	9.42874 E-3	1.87154	2.722

Thus if the assumptions regarding the floating target natural rainfall could be regarded as valid, we could already consider that we had fairly good evidence in favor of a positive seeding effect, perhaps as large as a factor of 2 or more but probably not larger than three.

Table 14. Evidence for H_1 = Seeding Effect More Than 3.3; Uniform Prior on β

Range of β	Case (R)	Evidence H_1
Seeding Effect		
0.5(-2) to 10	2.0	-5.2467
	3.25315	-1.09697
	5.0317	3.93096
	4.57215	2.722
		Acc. +0.31
Seeding Effect		
0.25(-4) to 32	2.0	-7.70423
	3.25315	-1.66252
	5.0317	5.29503
	4.57215	3.64041
		Acc. -0.43
Seeding Effect		
1 (0) to 10	2.0	-4.18013
	3.25315	-1.33448
	5.0317	2.51033
	4.57215	1.53799
		Acc. -1.47

Table 15. Uniform Prior on β from Seeding Effect 0.5 (-2) to 10

Case (R)	Evid H_{1A}	Evid H_{1B}
2	-4.47067	-3.51044
3.25315	-0.585455	0.056718
5.0317	4.6582	5.89442
4.57215	3.32961	4.29482
	Acc. +2.93	Acc. +6.73

The final method of hypothesis testing is perhaps the most preferable. It consists of using Bayes equation to find a posterior probability density for β given the data and a prior probability assignment on β. We write

$$P(\beta/D) = P(\beta)\,\frac{P(D/\beta)}{P(D)} \tag{20}$$

where the denominator may be regarded as just a normalizing constant which will be determined later.

This time let us first assume a gamma function for the prior probability distribution of β, namely

$$P(\beta) = \frac{K_2^{K_1}}{\Gamma(K_1)}\,\beta^{K_1 - 1}\,e^{-K_2\beta} \tag{21}$$

where the selection of the distribution parameters K_1 and K_2 will be described presently.

Now we know that

$$P(D/\beta) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} R_i^{\alpha-1} e^{-\beta R_i} \tag{22}$$

and substituting (22) and (21) into (20)

we find

$$P(\beta/D) = \frac{\left(\sum_{i=1}^{n} R_i + K_2\right)^{n\alpha+K_1}}{\Gamma(n\alpha + K_1)} \beta^{n\alpha+K_1 - 1} e^{-\left(\sum_{i=1}^{n} R_i + K_2\right)\beta} \tag{23}$$

where the normalizing constant is found from the exponents, since we know the resulting distribution is also a gamma distribution. Thus we further know that

$$\langle\beta/D\rangle = \frac{n\alpha + K_1}{\sum_{i=1}^{n} R_i + K_2} = \frac{\alpha + K_1/n}{\bar{R}_i + K_2/n} \tag{24}$$

so that

$$\lim_{n \to \text{large}} \langle\beta/D\rangle = \frac{\alpha}{\bar{R}_i} \tag{24a}$$

and

$$V^2(\beta/D) = \frac{1}{n\alpha + K_1} \tag{25}$$

Before we consider any specific prior probabilities for β_1 let us examine the limiting case, thus assuming our data is repeated enough times for (24a) to hold. In this case we find the limit of $\langle\beta/D\rangle$ to be

$$\lim \langle\beta/D\rangle = \frac{\alpha}{\bar{R}_i} = \frac{4 \times 7}{14.85} = 1.88$$

which corresponds to a seeding effect of about 2.5.

Now we consider two classes of priors on β, both classes gamma distributions. The first class all have $K_1 = 12$ and are shown in figure 4. In these, β is confined to the physically reasonable range but unfortunately the prior probability is highly peaked at the expected value. Table 16 shows results of considering three priors of this type.

Table 16. Prior β is a Gamma Distribution with $K_1 = 12$

Prior Expected β	Posterior Expected β	Coeff. of Variation
2.4	2.02	0.16
2.0	1.92	0.16
1.8	1.86	0.16

The next trials consider the prior β distributions shown in figure 5, with $K_1 = 2$ and a much wider spread of values. Using the same values of prior expected β, we get the results of Table 17.

Table 17. Prior β is a Gamma Distribution with $K_1 = 2$

Prior Expected β	Posterior Expected β	Coeff. of Variation
2.4	1.91	0.18
2.0	1.88	0.18
1.8	1.89	0.18

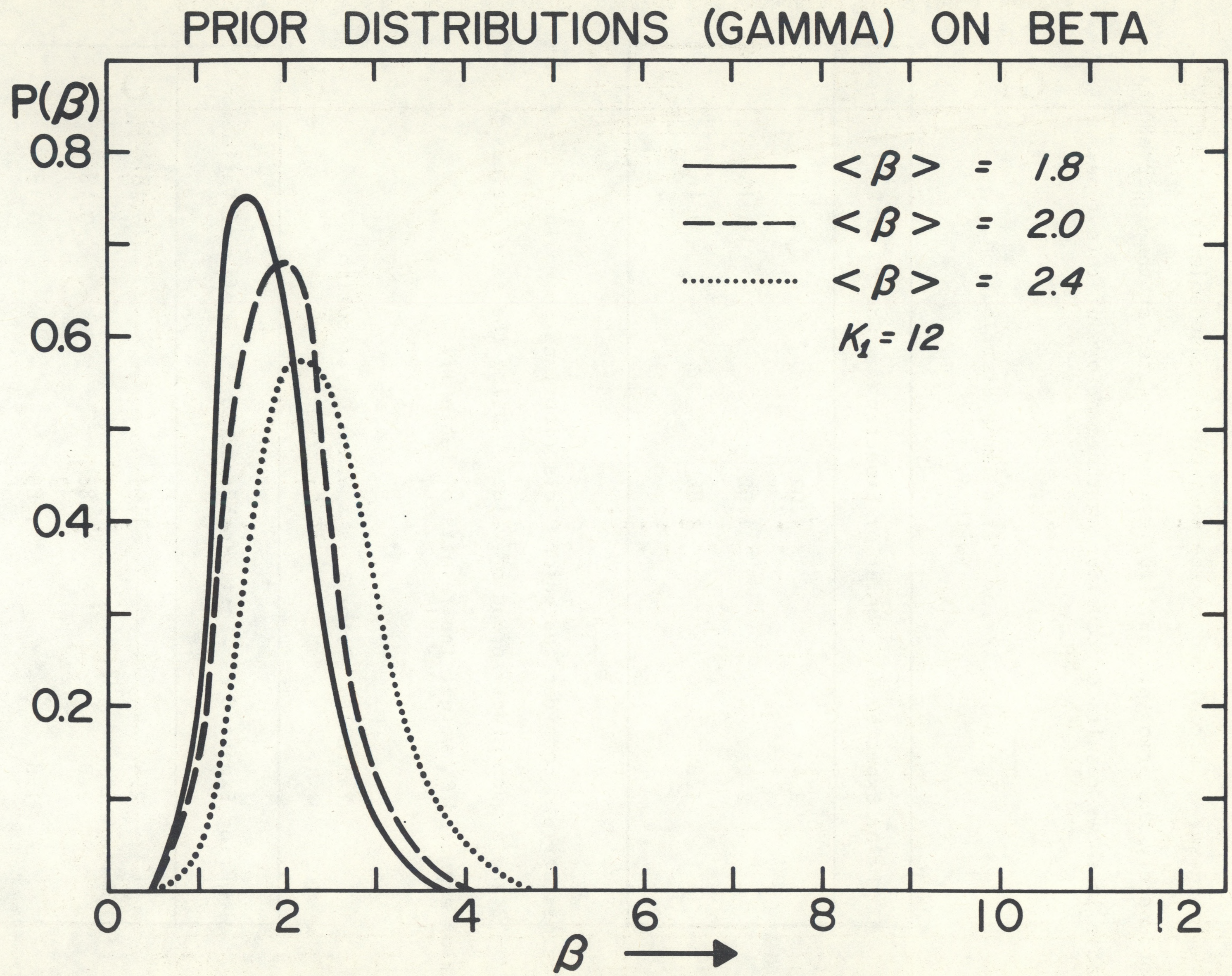

Figure 4. Prior distribution on β given gamma function. Here K_1 + 12.

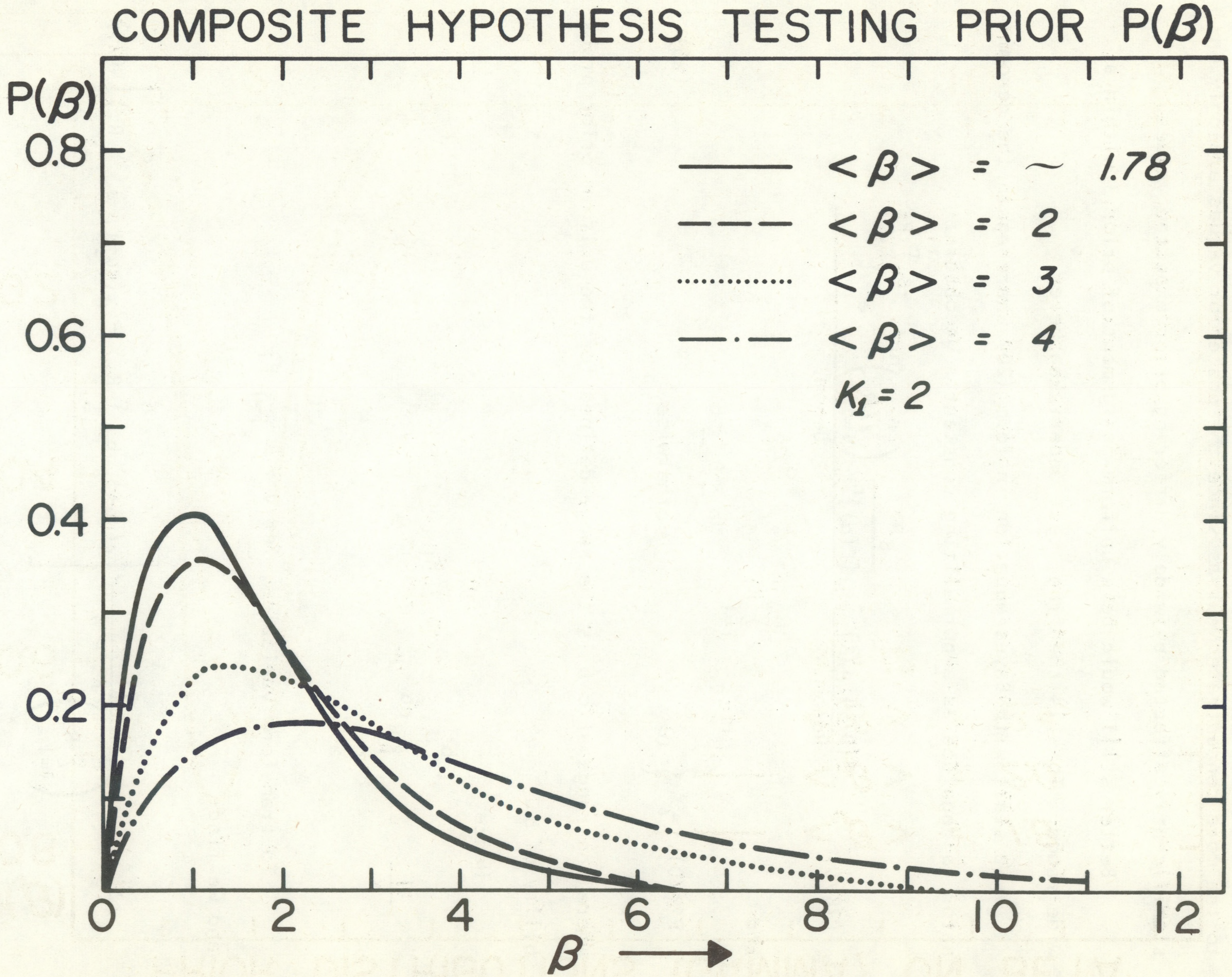

Figure 5. Prior distribution on β given by the gamma function. Here $K_1 = 2$.

This type of spread-out prior distribution for β is clearly preferable to the more concentrated curves since the posterior values are only insignificantly affected by widely differing prior expected values.

Better still would be a uniform assignment of prior β within a wide range. This is the final test undertaken here.

We start with Bayes equation in form (20) where again the denominator is regarded as a normalizing constant. We obtain:

$$P(\beta/D) = \frac{P(D/\beta)\ p(\beta)}{\text{Denom.}} = \frac{\dfrac{\beta^{n\alpha}}{[\Gamma(\alpha)]^n} \left(\prod_{i=1}^{n} R_i\right)^{\alpha-1} e^{-\beta \sum_{i=1}^{n} R_i}}{\text{Denom.}}$$

and

$$P(\beta/D) = \begin{cases} \kappa\beta^{n\alpha}\ e^{-\beta \sum_{i=1}^{n} R_i} & a \leq \beta \leq b \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

where the range a to b is the range defined for the uniform prior probability of β.

Hence

$$\frac{1}{\kappa} = \int_a^b \beta^{n\alpha}\ e^{-\beta \sum_{i=1}^{n} R_i}\ d\beta$$

To transform variables, let $y = \beta \sum_{i=1}^{n} R_i$

and we find

$$\frac{1}{\kappa} = \frac{1}{\left(\sum_{i=1}^{n} R_i\right)^{n\alpha+1}} \left[\gamma(n\alpha+1,\ b \sum_{i=1}^{n} R_i) - \gamma(n\alpha+1,\ a \sum_{i=1}^{n} R_i)\right] \tag{27}$$

where γ is the incomplete gamma function.

It is noteworthy that $\sum_{i=1}^{n} R_i$ is a sufficient statistic for the distribution.

We want the moments of the posterior probability distribution on β. Firstly

$$<\beta/D> = \kappa \int_a^b \beta^{n\alpha+1} e^{-\beta \sum_{i=1}^{\infty} R_i} d\beta$$

$$= \frac{\kappa}{\left(\sum R_i\right)^{n\alpha+2}} \int_{a\sum R_i}^{b\sum R_i} y^{n\alpha+1} e^{-y} dy$$

$$= \frac{\kappa}{\left(\sum_{i=1}^{n} R_i\right)^{n\alpha+2}} \left[\gamma(n\alpha+2,\ b\sum R_i) - \gamma(n\alpha+2,\ a\sum R_i) \right] \quad (28)$$

multiplying (28) by κ from (27) we get

$$<\beta/D> = \frac{1}{\sum_{i=1}^{n} R_i} \quad \frac{\gamma(n\alpha+2,\ b\sum_{i=1}^{n} R_i) - \gamma(n\alpha+2,\ a\sum_{i=1}^{n} R_i)}{\gamma(n\alpha+1),b\sum_{i=1}^{n} R_i) - \gamma(n\alpha+1,\ a\sum_{i=1}^{n} R_i)} \quad (29)$$

or for the m^{th} moment of the distribution

$$<\beta^m/D> = \frac{1}{\left(\sum R_i\right)^m} \quad \frac{\gamma(n\alpha+m+1,\ b\sum R_i) - \gamma(n\alpha+m+1,\ a\sum R_i)}{\gamma(n\alpha+1,\ b\sum R_i) - \gamma(n\alpha+1,\ a\sum R_i)} \quad (30)$$

A computer program is presented in Table 18 which computes the posterior expected value of β and also its variance and standard deviation. A summary of results of this program are shown in Table 19.

Table 18. A Computer Program for the Expected Value of Beta

```
100 REM THIS PROGRAM COMPUTES THE EXPECTED VALUE OF BETA
110 REM AFTER SEEING THE AREA DATA, WITH A UNIFORM PRIOR BETA
120 REM IN THIS CASE SEEDING EFFECT IN RANGE -2 TO +10
130 PRINT "JOANNE"
140 PRINT
150 PRINT "POSTERIOR EXPECTED BETA", "VARIANCE", "STANDARD DEVIATION"
155 PRINT
160 DEF FNI(X,Y)
170 IF Y<X/2+4 THEN260
180 LET G7=G8=1
190 FOR G6=1 TO INT(X)
200   LET G7=G7*(X-G6)/Y
210   IF G7<1E-4 THEN 240
220   LET G8=G8+G7
230 NEXT G6
240 LET FNI=EXP(FNG(X))-G8*EXP((X-1)*LOG(Y)-Y)
250 GO TO 330
260 LET G7=G8=1/X
270 FOR G6=1 TO 50
280   LET G7=G7*Y/(X+G6)
290   IF G7*X <1E-4 THEN 320
300   LET G8=G8+G7
310 NEXT G6
320 LET FNI=G8*EXP(X*LOG(Y)-Y)
330 FNEND
340 DEF FNG(X)
350 IF X<4 THEN 390
360 LET G=X*(LOG(X)-1)+.5*LOG(6.2831853/X)
370 LET FNG=G+(1-1/(30*X*X))/(12*X)
380 GO TO 480
390 LET G1=X-INT(X)
400 LET G2=INT(X)-1
410 LET G=1-.57710166*G1+.98585399*G1*G1-.87642182*G1*G1*G1
420 LET G=G+(.8328212-.5684729*G1+.25482049*G1*G1-.0514993*G1*G1*G1)*G1↑4
430 IF G2=0 THEN 470
440 FOR G9=1 TO G2
450 LET G=G*(G1+G9)
460 NEXT G9
470 LET FNG-LOG(G)
480 FNEND
490 LET R=2+3.25315+5.0317+4.57215
500 LET B=2.8479
510 LET A=1.34669
520 LET Q=FNI(30,B*R)-FNI(30,A*R)
530 LET T=FNI(29,B*R)-FNI(29,A*R)
540 LET U=(1/R)*(Q/T)
560 REM THIS PART OF THE PROGRAM WILL COMPUTE THE VARIANCE AND
570 REM STANDARD DEVIATION OF THE POSTERIOR BETA DISTRIBUTION
580 LET W=FNI(31,B*R)-FNI(31,A*R)
590 LET J=(1/(R*R))*(W/T)-U*U
600 LET S=SQR(J)
610 PRINT U; TAB(30); J; TAB(50); S
620 END
```

Table 19. Posterior β given the Data with Uniform Prior

Range of Prior Seeding Effect	< β/D >	σ^2	σ
-2 to +10	1.96	0.1036	0.32
-4 to +100	1.95	0.1303	0.36
-2 to +5	2.03	0.0790	0.28
-4 to +5	2.05	0.0915	0.30

In Table 19, probably the most reasonable prior distribution is the uniform prior from seeding effect of -2 to 10; it could hardly fall outside this range. Even extending the uniform prior through the ridiculous range of seeding effects from -4 to 100 changes the posterior only negligibly. The last two computations try out the most unfavorable possible prior distributions; the last one is unreasonably unfavorable but after seeing the data we still come out with a seeding effect close to two.

It is clear that this last method is the best one evolved since it introduces the least possible prejudice on the posterior distribution of β, thus giving the data the most weight. _If_ we could rely on the assumption about the natural rainfall, we could say that with all tests our data so far indicate an area seeding effect of about two for the floating target rainfall.

5. CONCLUDING REMARKS

The most important result of this work is that it has shown that the Bayesian approach is extremely promising for the EML area experiment, both in drawing causal inferences from the data and in planning experimental and operational strategy. The work herein shows that there is real hope of obtaining conclusive or near conclusive results from the 1970-1971 experiments. The most urgent requisite then is to specify the distribution and expected value for natural floating target rainfall using something like 25-50 case studies.

Even if the data distributions do not prove to be as tractable as those described here, computer methods can still be evolved to apply the same analyses. The composite hypothesis testing methods have been shown much more suitable for this and probably for most meteorological problems. No physical inferences at all can be drawn at this time regarding the seeding effect in the area experiment, not just because there are too few seeded cases but because we have inadequate knowledge of the distribution and particularly the expected value of the natural rainfall, either for the floating or total target.

6. ACKNOWLEDGMENTS

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APPENDIX I

CRITIQUE OF THE ASSUMPTION DEFINING THE EXPECTED VALUE OF THE NATURAL RAINFALL IN TERMS OF THE ONE AVAILABLE CONTROL DAY OBSERVATION

Since the weakest assumption in this report is that the natural rainfall's expected value is represented by the observation on the one fair control day, it is proper to examine honestly just how bad this assumption is.

The floating target rainfall for this case was 7300 acre-ft. Using our unit of 100 acre-ft, the transformed value is $(73)^{1/4} = 2.92$. Let us suppose that the floating target transformed rainfall is indeed well described by a gamma distribution with a coefficient of variation of 0.377 and thus an $\alpha = 7.0$. If the expected value of R, $\langle R \rangle = 2.92$ then the gamma probability distribution would be

$$P(R) = 0.637\, R^6\, e^{-2.4R} \qquad \text{(A1-1)}$$

and since $\sigma = 1.10$, $R + \sigma = 4.02$ and $R - \sigma = 1.82$. Hence to obtain the probability that one randomly made observation will fall within one σ of $\langle R \rangle$ we take the following integral:

$$0.637 \int_{1.82}^{4.02} R^6\, e^{-2.4R}\, dR =$$

$$0.637 \left[\frac{-e^{-2.4R}}{(2.4)^7} \left\{ (2.4R)^6 + 6(2.4R)^5 + 6\cdot5\,(2.4R^4) \right. \right.$$

$$\left. \left. 6\cdot5\cdot4\,(2.4R)^3 + 6\cdot5\cdot4\cdot3\cdot\,(2.4R)^2 + 6\cdot5\cdot4\cdot3\cdot2\,(2.4R) + 6! \right\} \right] \qquad \text{(A1-2)}$$

The accompanying computer program computes the result from equation (Al-2). As we see, the probability is only about 0.7 that the observation will lie within this very wide range.

To bring this sad result home more palpably, we now compute with the same program the probability that a single observation will show an untransformed rainfall that lies between one-half and twice the expected value.

Twice the (untransformed) rainfall is

14600 acre-ft, transformed = 3.47607

Half the (untransformed) rainfall is

3650 acre-ft, transformed = 2.45795

We only change the limits of integration in (AI-2) to range between 2.45795 and 3.47607 and rerun the program. The discouraging result is that the integrated probability is 0.35. In the real situation, the standard deviation will surely not be smaller than we have assumed.

Hence the urgency of obtaining a more valid expected value of floating target natural rainfall is re-emphasized.

Table AI-1. Computer Subprogram

```
10 REM THIS SUBPROGRAM COMPUTES THE INTEGRATED PROBABILITY THAT
20 REM A SINGLE OBSERVATION WILL LIE WITHIN ONE STANDARD DEVIATION
30 REM OF THE EXPECTED VALUE FOR A GAMMA DISTRIBUTION WITH A
40 REM COEFFICIENT OF VARIATION OF 0.377
50 PRINT "JOANNE"
60 PRINT
70 PRINT "INTEGRATED PROBABILITY"
80 PRINT
90 DEF FNR(R)
100 LET X=1*2*3*4*5*6
110 LET X=X+6*5*4*3*2*(2.4*R)
120 LET X=X+6*5*4*3*((2.4*R)↑ 2)
130 LET X=X+6*5*4*((2.4*R) ↑ 3)
140 LET X=X+6*5*((2.4*R) ↑ 4)
150 LET X=X+6*((2.4*R) ↑ 5)
160 LET X=X+(2.4*R) ↑ 6)
170 LET X=X*0.637*((-EXP(-2.4*R) /((2.4)↑ 7)))
175 LET FNR=X
180 FNEND
190 PRINT FNR(4.02)-FNR(1.82)
200 END



INTEGRATED PROBABILITY

0.693626
```

APPENDIX II

THE BEST FIT DISTRIBUTION FUNCTIONS FOR THE EML SINGLE CLOUD RAINFALL DATA

FROM

1968 AND 1970 EXPERIMENTS COMBINED

The techniques outlined here for studying the distribution of the single cloud rainfall data will later be applied to floating and total target distributions for the area when these become available. The primary analysis tool of this part of the study is Pézier's computer program, DAMAXS2, which is listed at the end of this Appendix. Taking the data, this program finds (with the principle of maximum entropy) the parameters of the best fit curves to the data for seven different distributions. It also finds the values of ψ and χ^2 for each of these distributions.

The 52 single cloud rainfalls (26 seeded and 26 control) presented here were calculated from the cloud base radar echoes as described elsewhere by Woodley (1970). For this analysis we use the rainfall measured for the total lifetime of each cloud. Limitations of the data are numerous. Measurements were terminated when a cloud echo merged with a neighboring echo, which happened with 9 seeded and 5 control clouds. Also four seeded clouds moved into radar blind cones and thus required adjustment in their radar-measured rainfall. We have shown that if corrections could be made for the several data limitations, the difference between seeded and control populations would be larger than we have shown.

Table AII-1. 1968 and 1970 Single Cloud Data

Total Cloud Lifetime			
Seeded Rain		Control Rain	
Acre-ft	Fourth Root	Acre-ft	Fourth Root
129.6	3.37405	26.1	2.26027
31.4	2.36719	26.3	2.26459
2745.6	7.23868	87.0	3.05408
489.1	4.70272	95.0	3.12199
430.0	4.55373	372.4	4.39291
302.8	4.17147	0	0 (1)
119.0	3.30283	17.3	2.03944
4.1	1.42297	24.4	2.22253
92.4	3.1004	11.5	1.84151
17.5	2.04531	321.2	4.23344
200.7	3.76389	68.5	2.87689
274.7	4.07113	81.2	3.00185
274.7	4.07113	47.3	2.6225
7.7	1.6658	28.6	2.31255
1656.0	6.37918	830.1	5.36763
978.0	5.59223	345.5	4.31134
198.6	3.754	1202.6	5.88885
703.4	5.14992	36.6	2.45963
1697.8	6.41906	4.9	1.48782
334.1	4.27532	4.9	1.48782
118.3	3.29797	41.1	2.53198
255.0	3.99609	29.0	2.3206
115.3	3.27686	163.0	3.57311
242.5	3.94619	244.3	3.95349
32.7	2.39132	147.8	3.48673
40.6	2.52424	21.7	2.15832

Table AII-2. Important Parameters of Single Cloud Rainfall Distributions - 1968 and 1970

	Untransformed Data Seeded	Transformed Data Seeded
Mean	441.985	3.87899
σ	650.787	1.44994
V	1.47242	0.373794
	Control	Control
Mean	164.55	2.89507
σ	278.45	1.26776
V	1.69219	0.437903

Many of the conventional statistical tests made with the single cloud data were done on the fourth root transformed rainfalls because the transformation made the distribution more nearly normal and the validity of quite a few of the standard tests depends on having a normal distribution of the data. Here we work with the transformed data for a slightly different reason, namely for a Bayesian analysis we need a tractable distribution with first and second moments fairly simply related to the key parameters in the probability density equation.

Hence the remainder of this Appendix deals with single cloud transformed rainfall. Table AII-3 shows the important results of applying the program DAMAXS2 to these data.

Table AII-3. Results of Program DAMAXS2-Single Cloud Transformed Rainfalls

I. Unseeded Cases

	Parameters			
Distribution	-Log A	B	C	Probability
Tr. Normal	4.38621	2.26211	0.383128	5.15×10^{-2}
Gamma	0.491139	5.523	2.2236	0.271176
Weibull	2.34166	1.801	3.43×10^{-2}	9.43×10^{-2}
Log-Normal	3.08051	5.15809	3.08645	0.256951
Rayleigh	2.31008	2.5878	0.180015	0.169842
Inv. Gamma	-11.7724	-7.16194	15.3737	0.124435
Inv. Rayleigh	-4.22864	-4.21716	8.40303	3.16×10^{-2}

Distribution	X^2	
Tr. Normal	12.78	
Gamma	9.46	16 Degrees
Weibull	11.57	of
Log-Normal	9.57	Freedom
Rayleigh	10.39	
Inv. Gamma	11.02	
Inv. Rayleigh	13.75	

Table AII-3 (Continued). Results of Program DAMAXS2-Single Cloud Transformed Rainfalls

II. Seeded Cases

Distribution	Parameters -Log A	B	C	Probability
Tr. Normal	5.43878	2.14824	0.273406	0.121835
Gamma	2.47642	6.10433	1.83149	0.244059
Weibull	3.34546	2.017	1.168×10^{-2}	0.183406
Log-Normal	5.34034	7.34813	3.25198	0.142415
Rayleigh	3.59768	3.00345	0.117279	0.251999
Inv. Gamma	-13.8461	-7.29359	20.9373	4.67×10^{-2}
Inv. Rayleigh	-5.17874	-4.23249	14.9841	9.58×10^{-2}

Distribution	χ^2	
Tr. Normal	13.02	
Gamma	11.63	16 Degrees
Weibull	12.21	of
Log-Normal	12.71	Freedom
Rayleigh	11.57	
Inv. Gamma	14.94	
Inv. Rayleigh	18.11	

For each distribution in Table AII-3, the appropriate probability density equation is given in Table AII-4.

Table AII-4. Distribution of Probability Equations

Distribution	Equation for P(R)
Tr. Normal	$A \, EXP \, (BX-CX^2)$
Gamma	$A \, X^B \, EXP(-CX)$
Weibull	$A \, X^B \, EXP(-CX^{B+1})$
Log-Normal	$A \, X^B \, EXP(-C \, LOG \, X^2)$
Rayleigh	$A \, X^B \, EXP(-CX^2)$
Inv. Gamma	$A \, X^B \, EXP(-C/X)$
Inv. Rayleigh	$A \, X^B \, EXP(-C/X^2)$

Figures AII-1 and AII-2 show graphically the important results of this study. The gamma distribution is a rather good distribution for the data, catching the skewness and mode at low rainfalls. The truncated normal, on the other hand, although not apparently much worse from the χ^2, looks much worse when plotted as it does not catch the skewness or mode of the data.

Finally a histogram of the raw data for seeded and control cases combined is shown in figure AII-3. We see that there are actually two populations, one with small values and another with large, almost separate from the first. An attempt is being made to treat the raw data as the sum of two gamma distributions.

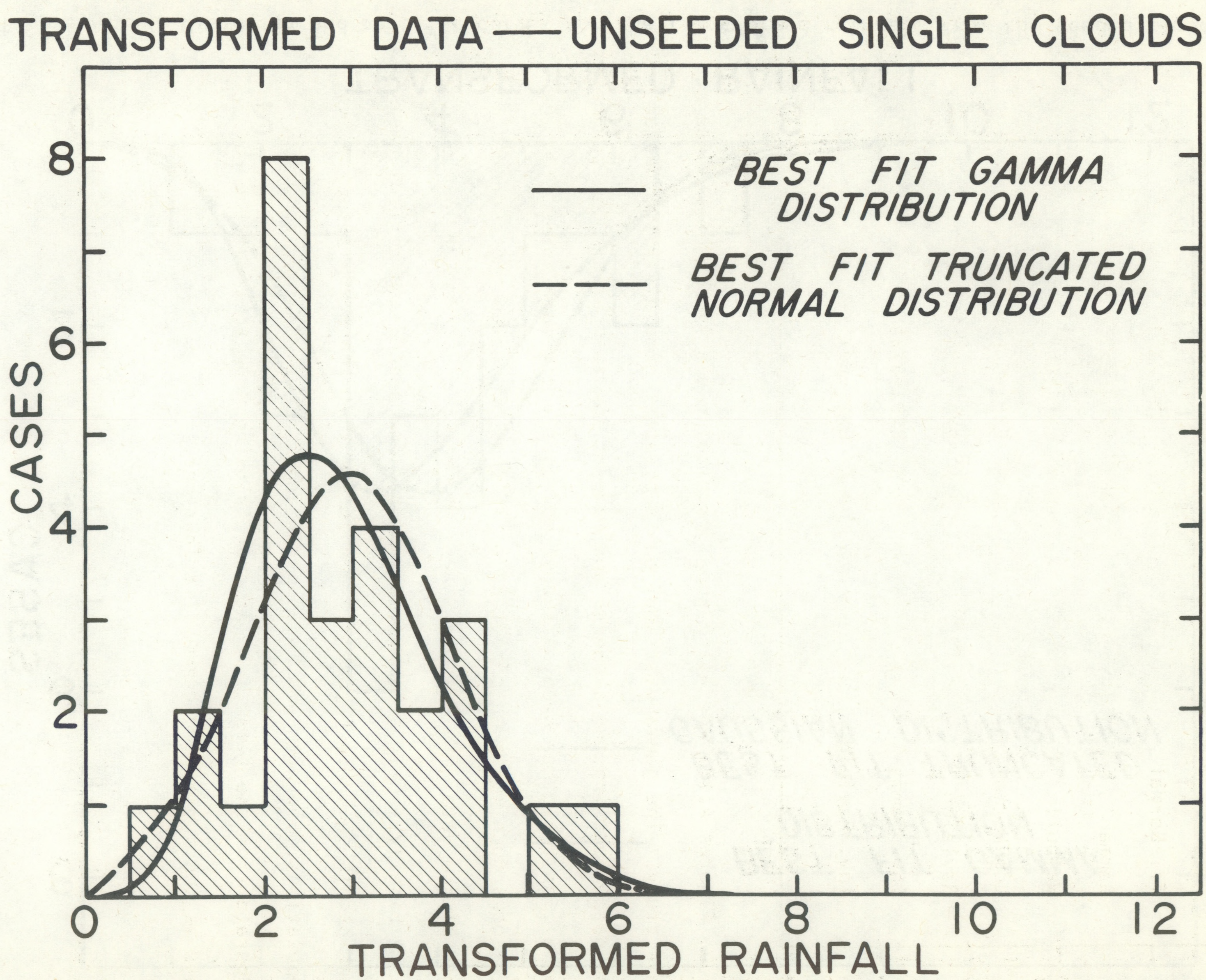

Figure A11-1. Histogram and best fit curves for the transformed rainfall from the unseeded clouds. The grouping of the data is by intervals of 0.5. In the DAMAXS2 program the group interval is $\sigma/3$.

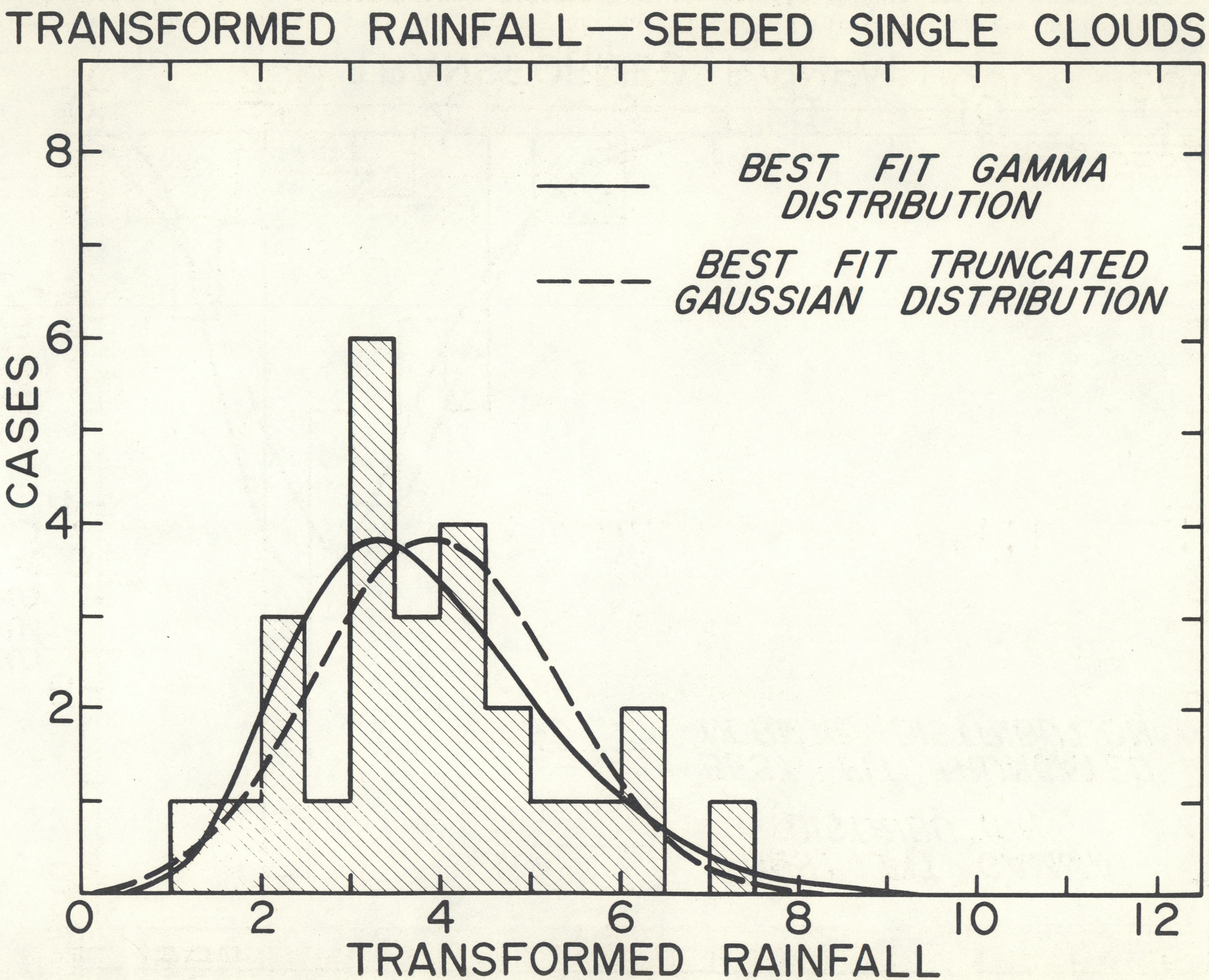

Figure A11-2. Histogram and best fit curves for the transformed rainfall from the seeded clouds.

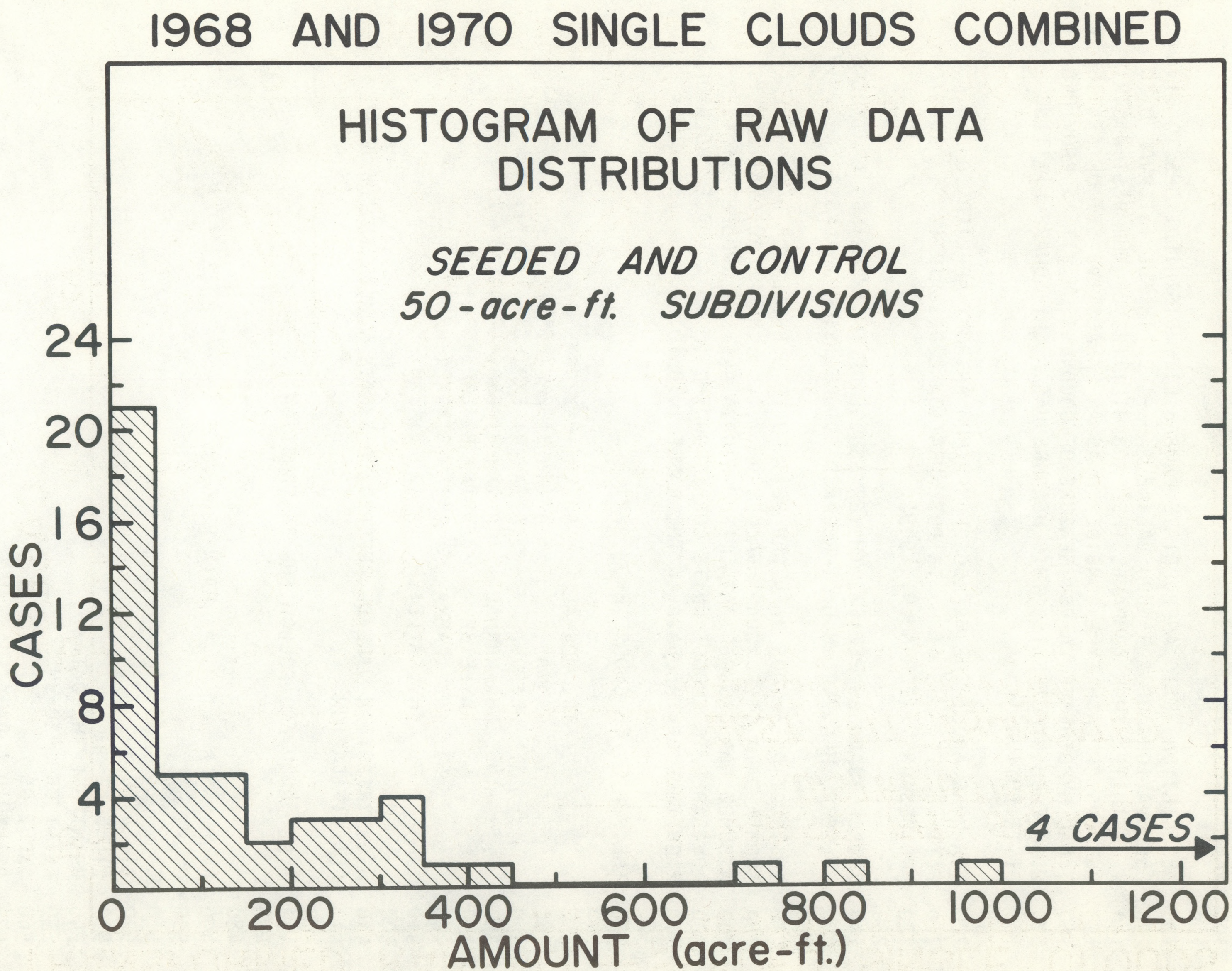

Figure A11-3. Histogram of raw data distributions.

Table AII-5. Listing of Program DAMAXS2

```
100  NAME--DAMAXS2
110
120  DESCRIPTION--FITS APPROPRIATE PROBABILITY DISTRIBUTIONS TO GIVEN
130  OBSERVATIONS OF A RANDOM VARIABLE DEFINED IN THE INTERVAL 0 TO
140  + INFINITY. THEN COMPARES THE PROBABILITIES OF THE BEST DISTRI-
150  BUTIONS OF EACH TYPE CONSIDERED AS AN EXHAUSTIVE SET OF STATIS-
160  TICAL HYPOTHESES. A PSI MEASURE OF GOODNESS OF FIT IS PROVIDED
170  WITH EACH HYPOTHESIS. THIS MEASURE HAS A CHI-SQUARE LIMITING
180  DISTRIBUTION.
190
200  FOR MORE ABOUT THE METHOD OF MAXIMUM ENTROPY APPLIED TO P.D.F.
210  FITTING SEE CH.5 of M. TRIBUS, "RATIONAL DESCRIPTIONS, DECISIONS
220  AND DESIGNS" (PERGAMON, 1969).
230
240  SOURCE--JACQUES PEZIER, THAYER SCHOOL OF ENGINEERING, DARTMOUTH
250  COLLEGE, HANOVER, N.H. (PREPARED FOR DECISION ANALYSIS ENGG. 175)
260
270  INSTRUCTIONS--YOUR DATA CAN BE INPUTED DIRECTLY OR CAN BE READ
280  FROM A DATA FILE. CLASS VALUES OF 1,2,...,M ARE AUTOMATICALLY
290  ASSUMED FOR GROUPED DATA. IF THE DATA ARE NOT GROUPED IT IS
300  ADVISABLE TO CHOOSE UNITS CORRESPONDING TO AN AVERAGE ORDER OF
310  MAGNITUDE OF THE DATA IN THE RANGE 1 TO 10.
320
330  THE P.D.F. TO CHOOSE FROM ARE:
340
350            1.  TR. NORMAL    : A*EXP(+B*X-C*X↑2)
360            2.  GAMMA         : A*(X↑B)*EXP(-C*X)
370            3.  WEIBULL       : A*(X↑B)*EXP(-C*X↑(B+1))
380            4.  LOG-NORMAL    : A*(X↑B)*EXP(-C*LOG(X)↑2)
390            5.  RAYLEIGH      : A*(X↑B)*EXP(-C*X↑2)
400            6.  INV. GAMMA    : A*(X↑B)*EXP(-C/X)
410            7.  INV.RAYLEIGH  : A*(X↑B)*EXP(-C/X↑2)
420
430  A DISTRIBUTION WILL BE REFERED TO BY ITS NUMBER IN THE ABOVE
440  CLASSIFICATION.
450
460  TYPE "RUN" AND FOLLOW THE INSTRUCTIONS.
470
480
490  * * * * * * * MAIN PROGRAM * * * * * * * * * *
500
510
520  DIM D(200)
530  FILES *%
540  PRINT"ARE THE DATA:"
550  PRINT"GROUPED--";
560  INPUT G$
570  PRINT"IN A DATA FILE--";
580  INPUT D$
590  IF D$="YES" THEN 680
```

Table AII-5 (continued). Listing of Program DAMAXS2.

```
600  PRINT"INPUT YOUR DATA--";
610  MAT INPUT D
620  LET M=NUM
630  FOR I=1 TO M
640  WRITE #1:D(I)
650  NEXT I
660  RESET #1
670  GO TO 710
680  PRINT "DATA FILE NAME--";
690  INPUT N$
700  FILE #1:N$
710  PRINT"WHAT DISTRIBUTIONS DO YOU WANT TO TRY; GIVE THE #";
720  MAT INPUT R
730  LET R=NUM
740  FOR I=1 TO 7
750   READ L$(I)
760  NEXT I
770  DATA "TR. NORMAL", "GAMMA", "WEIBULL", "LOG-NORMAL, "RAYLEIGH"
780  DATA "INV. GAMMA", "INV. RAYLEIGH"
790
800                  *** COMPUTATION OF THE MEANS ***
810
820  IF G$="YES" THEN 980
830
840  LET N=LOF(1)
850  FOR I=1+U TO N+U
860    READ #1:X
870    LET X=X+U
880    LET L=LOG(X)
890    LET L1=L1+L
900    LET L2=L2+L*L
910    LET M1=M1+X
920    LET M2=M2+X*X
930    LET N1=N1+1/X
940    LET N2=N2+1/(X*X)
950  NEXT I
960  GO TO 1110
970
980  LET M=LOF(1)
990  FOR I=1+U TO M+U
1000    READ#1:X
1010    LET L=LOG(I)
1020    LET L1=L1+X*L
1030    LET L2=L2+X*L*L
1040    LET M1=M1+X*I
1050    LET M2=M2+X*I*I
1060    LET N1=N1+X/I
1070    LET N2=N2+X/(I*I)
1080    LET N=N+X
1090 NEXT I
```

Table AII-5 (continued). Listing of Program DAMAXS2.

```
1100
1110 LET L1=L1/N                    L1=<LOG(X)>
1120 LET L2=L2/N                    L2=<LOG(X)↑ 2>
1130 LET M1=M1/N                    M1=<X>
1140 LET M2=M2/N                    M2=<X ↑ 2>
1150 LET N1=N1/N                    N1=<1/X>
1160 LET N2=N2/N                    N2=<1/X↑ 2>
1170 LET L0=L2=L1*L1
1180 LET M0=M2-M1*M1
1190 LET N0=N2-N1*N1
1200 LET V0=M0/(M1*M1)
1210
1220                        *** COMPUTATION OF THE PARAMETERS ***
1230
1240 FOR I=1 TO R
1250   LET J=R(I)
1260   ON J GO TO 1300,1530,1640,1990,2070,2180,2290
1270
1280   *** TR. NORMAL ***
1290
1300    LET S1=1/SQR(3.1415927)
1310    IF V0<.04 THEN 1450
1320    LET Y=M1/SQR(M0)-.7
1330    LET T=.5/Y-(V0+1)*Y
1340    IF ABS(Y1)<1E-5 THEN 1410
1350    LET F7=S1*EXP(-T*T)/(1-FNE(T))
1360    LET F9=Y+T-F7
1370    LET F8=1-(1/(2*Y*Y)+V0+1)*(1+2*F7*(T-F7))
1380    LET Y1=F9/F8
1390    LET Y=Y-Y1
1400    GO TO 1330
1410    LET C(J)=Y*Y/(M1*M1)
1420    LET B(J)=(2*Y*Y*(V0+1)-1)/M1
1430    LET A(J)=T*T-.5*LOG(C(J))+LOG((1-FNE(T))/(2*S1))
1440    GO TO 1480
1450    LET C(J)=.5/M0
1460    LET B(J)=M1/M0
1470    LET A(J)=.5/V0+.5*LOG(6.2831853*M0)
1480    LET S(J)=A(J)-M1*B(J)+M2*C(J)
1490    GO TO 2370
1500
1510   *** GAMMA ***
1520
1530    LET Y=L1-LOG(M1)
1540    LET X=FNX(Y)
1550    LET C(J)=X/M1
1560    LET B(J)=X-1
1570    LET G=FNG(X)
1580    LET A(J)=G-X*LOG(C(J))
1590    LET S(J)+A(J)-L1*B(J)+M1*C(J)
```

Table AII-5 (continued). Listing of Program DAMAXS2.

```
1600   GO TO 2370
1610
1620  *** WEIBULL ***
1630
1640   LET W9=.729
1650   LET W0=2
1660   GO SUB 1810
1670   LET W8=SGN(W9)*(W0*L1+.57721566-LOG(D1))
1680   IF W8<0 THEN 1710
1690   LET W0=W0+W9
1700   GO TO 1660
1710   IF ABS(W9)<.002 THEN 1740
1720   LET W9=-W9/3
1730   GO TO 1690
1740   LET C(J)=1/D1
1750   LET B(J)=W0-1
1760   LET A(J)=LOG(D1/W0)
1770   LET S(J)=A(J)-L1*B(J)+1
1780   GO TO 2370
1790
1800   COMPUTATION OF D1=<X↑ W0>
1810   RESET #1
1820   IF G$="YES" THEN 1890
1830   FOR K=1 TO N
1840     READ#1:X
1850     LET X=X+U
1860     LET W7=W7+X↑ W0
1870   NEXT K
1880   GO TO 1930
1890   FOR K=1+U TO M+U
1900     READ#1:X
1910     LET W7=W7+X*K↑ W0
1920   NEXT K
1930   LET D1=W7/N
1940   LET W7=0
1950   RETURN
1960
1970  *** LOG-NORMAL ***
1980
1990   LET C(J)=.5/L0
2000   LET B(J)=L1/L0-1
2010   LET A(J)=.5*L1*L1/L0+.5*LOG(6.2831853*L0)
2020   LET S(J)=A(J)-L1*B(J)+L2*C(J)
2030   GO TO  2370
2040
2050  *** GEN. RAYLEIGH ***
2060
2070   LET Y=2*L1-LOG(M2)
2080   LET X=FNX(Y)
2090   LET G=FNG(X)
```

```
2100       LET C(J)=X/M2
2110       LET B(J)=2*X-1
2120       LET A(J)=G-X*LOG(C(J))-LOG(2)
2130       LET S(J)=A(J)-L1*B(J)+M2*C(J)
2140       GO TO 2370
2150
2160      *** INV. GAMMA ***
2170
2180       LET Y=-L1-LOG(N1)
2190       LET X=FNX(Y)
2200       LET C(J)=X/N1
2210       LET B(J)=-X-1
2220       LET G=FNG(X)
2230       LET A(J)=G-X*LOG(C(J))
2240       LET S(J)=A(J)-L1*B(J)+N1*C(J)
2250       GO TO 2370
2260
2270      *** INV. RAYLEIGH ***
2280
2290       LET Y=-2*L1-LOG(N2)
2300       LET X=FNX(Y)
2310       LET G=FNG(X)
2320       LET C(J)=X/N2
2330       LET B(J)=-2*X-1
2340       LET A(J)=G-X*LOG(C(J))-LOG(2)
2350       LET S(J)=A(J)-L1*B(J)+N2*C(J)
2360
2370 NEXT I
2380
2390                *** EVALUATION OF FIT AND PRINT OUT ***
2400
2410 IF U<>0 THEN 2650
2420 RESET #1
2430 IF G$="YES" THEN 2590
2440   MAT D=ZER(100)
2450   LET S2=SQR(M0)/3
2460   FOR I=1 TO N
2470     READ#1:X
2480     LET J=INT(X/S2+1)
2490     LET D(J)=D(J)+1
2500     IF X<M3 THEN 2520
2510     LET M3=X
2520     IF X>N3 THEN 2540
2530     LET N3=X
2540   NEXT I
2550   LET M=INT(M3/S2+1)
2560   LET N3=INT(N3/S2)
2570   LET P1=N*LOG(S2)
2580   GO TO 2620
2590   FOR I=1 TO M
```

Table AII-5 (continued). Listing of Program DAMAXS2.

```
2600        READ #1:D(I)
2610      NEXT I
2620 FOR I=1+N3 TO M
2625   IF D(I)=0 THEN 2640
2630   LET Q1=Q1+D(I)*LOG(D(I))
2640 NEXT I
2650 PRINT
2660 PRINT"DISTRIBUTION","2*PSI (D.F.=";M-N3;")"
2670 PRINT
2680 LET S0=1E8
2690 FOR I=1 TO R
2700   LET J=R(I)
2710   PRINT L$(J),2*(N*(S(J)-LOG(N))+Q1-P1)
2720   IF S(J)>S0 THEN 2740
2730   LET S0=S(J)
2740 NEXT I
2750 FOR I=1 TO R
2760   LET J=R(I)
2770   LET S(J)=N*(S0-S(J))
2780   IF S(J)<-60 THEN 2810
2790   LET P(J)=EXP(S(J))
2800   GO TO 2820
2810   LET P(J)=0
2820   LET P0=P0+P(J)
2830 NEXT I
2840 PRINT
2850 PRINT TAB(30);"PARAMETERS"
2860 PRINT"DISTRIBUTION","-LOG(A)","B","C","PROBABILITY"
2870 PRINT
2880 FOR I=1 TO R
2890   LET J=R(I)
2900   PRINT L$(J),A(J),B(J),C(J); TAB(59);P(J)/P0
2910 NEXT I
2920 PRINT
2930 PRINT
2940 PRINT"TRANSLATE THE ORIGIN OF THE DATA BY T= (TYPE '0' TO STOP)";
2950 INPUT U
2960 RESET #1
2970 LET L1=L2=M1=M2=N1=N2=M=N=P0=S0=0
2980 ON ABS(SGN(U))+1 GO TO 3580,820
2990
3000                *** FUNCTIONS USED ABOVE ***
3010
3020 DEF FNE(T)                    ERROR FUNCTION
3030 LET T0=ABS(T)
3040 IF T0>1 THEN 3130
3050 LET T2=T1=T0
3060 FOR T8=1 TO 50
3070   LET T2=T2*T0*T0/(T8+.5)
3080   IF T2<1E-8 THEN 3110
```


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Table AII-5 (continued). Listing of Program DAMAXS2.

```
3090       LET T1=T1+T2
3100    NEXT T8
3110    LET FNE=SGN(T)*EXP(-T0*T0)*2*S1*T1
3120    GO TO 3190
3130    LET T9=INT(12*T0+15)
3140    LET T1=T9/(2*T0)
3150    FOR T8=T9-1 TO 1 STEP -1
3160      LET T1=(T8/2)/(T0+T1)
3170    NEXT T8
3180    LET FNE=SGN(T)*(1-EXP(-T0*T0)*S1/(T0+T1))
3190    FNEND
3200
3210    DEF FNG(X)                        LOG(GAMMA(X))
3220    IF X<4 THEN 3260
3230    LET G=X*(LOG(X)-1)+.5*LOG(6.2831853/X)
3240    LET FNG=G+(1-1/(30*X*X))/(12*X)
3250    GO TO 3350
3260    LET G1=X-INT(X)
3270    LET G2=INT(X)-1
3280    LET G=1-.57710166*G1+98585399*G1*G1-.87642182*G1*G1*G1
3290    LET G=G*(.8328212-.5684729*G1+.25482049*G1*G1-.0514993*G1*G1*G1)*G1
3300    IF G2=0 THEN 3340
3310    FOR G9=1 TO G2
3320      LET G=G*(G1+G9)
3330    NEXT G9
3340    LET FNG=LOG(G)
3350    FNEND
3360
3370    DEF FNX(Y)                        INVERSE PHI FUNCTION
3380    LET X=-1/(2*Y)
3390    LET X1=INT(X)-4
3400    IF X1=>0 THEN 3420
3410    LET X=X=X1
3420    LET X2=X*X
3430    LET Y9=-6/X-(1-.1/X2+1/(21*X2*X2)-.05/(X2*X2*X2))/X2-12*Y
3440    LET Y8=6/X2+(2-.4/X2+2/(7*X2*X2)-.4/(X2*X2*X2))/(X*X2)
3450    IF X1=>0 THEN 3520
3460    LET X=X+X1
3470    FOR X3=0 TO -1-X1
3480      LET Y9=Y9-12/(X+X3)
3490      LET Y8=Y8+12/((X+X3)*(X+X3)*(X+X3+1))
3500    NEXT X3
3510    LET Y9=Y9+12*LOG(1=X1/X)
3520    LET X4=Y9/Y8
3530    LET X=X-X4
3540    IF ABS(X4)<1E-4 THEN 3560
3550    GO TO 3390
3560    LET FNX=X
3570    FNEND
3580    END
```