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THREE-POINT METHOD IN THE THEORY OF HOMOGENEOUS TURBULENCE: APPENDICES B, C, D, E

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FOREWORD

This technical memorandum contains Appendices B, C, D, and E of the manuscript entitled "Three-point method in the theory of homogeneous turbulence" which is published in The Physics of Fluids.

These Appendices are published separately in order to conserve journal space.

CONTENTS

APPENDIX B (RENORMALIZED HIERARCHY FOR $Q^{(4)}$)

Equation (25) provides an expression for the third order velocity correlation function $\langle \underset{\approx}{T} \rangle$ in terms of fourth, and higher, order velocity correlation functions. Similarly, expressions can be derived for the fourth order velocity cumulant $Q^{(4)}$ in terms of <u>fifth</u> order velocity correlations; the <u>fifth</u> order velocity cumulant $Q^{(5)}$ in terms of sixth order velocity correlations; and so on. Such a series of expressions can be referred to as a hierarchy of equations, or, simply a hierarchy. The main difference between this hierarchy and the conventional hierarchy is that it is given in terms of the renormalized Green's functions, whereas the latter is not. For this reason, the present hierarchy is referred to as a <u>renormalized hierarchy</u>. The first equation of this hierarchy is given by (25). The second equation of this hierarchy, the equation for $Q^{(4)}$, will now be derived in this Appendix.

To derive the hierarchy equation for the fourth order cumulant $Q^{(4)}$, we first define the fourth order velocity function $T^{(4)}$ as at the end of Section III:

$$T^{(4)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \underset{\sim}{r}_4, t) \equiv \underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_2, t)\underset{\sim}{v}(\underset{\sim}{r}_3, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t) \quad , \qquad \text{(B1)}$$

which is seen to be a tensor of fourth rank. For convenience, let us use the condensed notation

$$\underset{\sim}{Y} \equiv (\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \underset{\sim}{r}_4)$$

so that

$$T^{(4)}(\underset{\sim}{Y}, t) \equiv T^{(4)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \underset{\sim}{r}_4, t) \quad .$$

The cumulant of $T^{(4)}$ is denoted by $Q^{(4)}$ so that we can write

$$Q^{(4)}(\underset{\sim}{Y}, t) = \langle T^{(4)}(\underset{\sim}{Y}, t)\rangle_c$$

$$= \langle \underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_2, t)\underset{\sim}{v}(\underset{\sim}{r}_3, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t)\rangle_c \quad , \qquad \text{(B2)}$$

where the subscript c is used to denote that the cumulant of the average is to be taken. Thus, the meaning of subscript c is that all correlations between two or more of the velocities in (B2) are to be subtracted out. [An explicit expression for the cumulant in (B2) is a little cumbersome to write out in the present vector notation because the vectors $\underset{\sim}{v}(\underset{\sim}{r}_2, t)$, $\underset{\sim}{v}(\underset{\sim}{r}_3, t)$ and $\underset{\sim}{v}(\underset{\sim}{r}_4, t)$ do not commute with each other. However, if we make use of the transpose symbol tr, then we can write out the cumulant in (B2) as follows:

$$\begin{aligned}\langle \underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_2, t)\underset{\sim}{v}(\underset{\sim}{r}_3, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t)\rangle_c &= \langle \underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_2, t)\underset{\sim}{v}(\underset{\sim}{r}_3, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t)\rangle \\ &- \langle \underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_2, t)\rangle\langle \underset{\sim}{v}(\underset{\sim}{r}_3, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t)\rangle \\ &- \mathrm{tr}\langle \underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_3, t)\rangle\langle \underset{\sim}{v}(\underset{\sim}{r}_2, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t)\rangle \\ &- \mathrm{tr}\langle \underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t)\rangle\langle \underset{\sim}{v}(\underset{\sim}{r}_2, t)\underset{\sim}{v}(\underset{\sim}{r}_3, t)\rangle \quad , \end{aligned} \tag{B3}$$

where the meaning of the transpose symbol in (B3) is that the vector <u>indices</u> be transposed so as to agree with the original ordering of vector indices in $\underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_2, t)\underset{\sim}{v}(\underset{\sim}{r}_3, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t)$. In obtaining this cumulant it is not necessary to subtract out terms like $\langle \underset{\sim}{v}\ \underset{\sim}{v}\ \underset{\sim}{v}\rangle\langle \underset{\sim}{v}\rangle$ since $\langle \underset{\sim}{v}\rangle = 0$ here.]

Continuing the derivation, we define the four point Green's function $\underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t')$ by

$$\begin{aligned}\underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t') \equiv\ & \underset{\approx}{G}(\underset{\sim}{r}_1, t; \underset{\sim}{r}_1', t') \cdot \underset{\approx}{G}(\underset{\sim}{r}_2, t; \underset{\sim}{r}_2', t') \\ & \cdot \underset{\approx}{G}(\underset{\sim}{r}_3, t; \underset{\sim}{r}_3', t') \cdot \underset{\approx}{G}(\underset{\sim}{r}_4, t; \underset{\sim}{r}_4', t') \quad , \end{aligned} \tag{B4}$$

so that

$$\begin{aligned}&\underset{\sim}{v}(\underset{\sim}{r}_1, t)\underset{\sim}{v}(\underset{\sim}{r}_2, t)\underset{\sim}{v}(\underset{\sim}{r}_3, t)\underset{\sim}{v}(\underset{\sim}{r}_4, t) \\ &\quad = \int d\underset{\sim}{Y}'\ \underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t') \cdot \underset{\sim}{v}(\underset{\sim}{r}_1', t')\underset{\sim}{v}(\underset{\sim}{r}_2', t')\underset{\sim}{v}(\underset{\sim}{r}_3', t')\underset{\sim}{v}(\underset{\sim}{r}_4', t') \quad . \end{aligned} \tag{B5}$$

where we have used the notation

$$\underset{\sim}{Y}' \equiv (\underset{\sim}{r}_1', \underset{\sim}{r}_2', \underset{\sim}{r}_3', \underset{\sim}{r}_4')$$

$$d\underset{\sim}{Y}' \equiv d\underset{\sim}{r}_1' d\underset{\sim}{r}_2' d\underset{\sim}{r}_3' d\underset{\sim}{r}_4' \quad .$$

Equation (B5) is an application of the definition of $\underset{\approx}{G}$ given by (15). Substituting (B5) in (B2), and using (B1), we have

$$Q^{(4)}(\underset{\sim}{Y}, t) = \int d\underset{\sim}{Y}' \left\langle \underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t') \cdot T^{(4)}(\underset{\sim}{Y}', t') \right\rangle_c \quad , \tag{B6}$$

where the subscript c denotes that the cumulant refers to points $\underset{\sim}{Y} \equiv (\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \underset{\sim}{r}_4)$. But, as can be verified from (16), $\underset{\approx}{G}^{(4)}$ satisfies the four-point Navier-Stokes equation

$$\left[\frac{\partial}{\partial t} + L_o^{(4)} + L^{(4)}(\underset{\sim}{Y}, t)\right] \underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t') = 0 \tag{B7}$$

$$L_o^{(4)} \equiv L_o(\underset{\sim}{r}_1) + L_o(\underset{\sim}{r}_2) + L_o(\underset{\sim}{r}_3) + L_o(\underset{\sim}{r}_4)$$

$$L^{(4)}(\underset{\sim}{Y}, t) \equiv C\left[L(\underset{\sim}{r}_1, t) + L(\underset{\sim}{r}_2, t) + L(\underset{\sim}{r}_3, t) + L(\underset{\sim}{r}_4, t)\right] \quad , \tag{B8}$$

where C is the commutator operator defined after (18). (B7) can be solved for $\underset{\approx}{G}^{(4)}$ as an expansion involving $\langle \underset{\approx}{G}^{(4)} \rangle$. This solution for $\underset{\approx}{G}^{(4)}$ in terms of $\langle \underset{\approx}{G}^{(4)} \rangle$ is derived in the same way as $\underset{\approx}{G}^{(3)}$ was derived in terms of $\langle \underset{\approx}{G}^{(3)} \rangle$ in Appendix A. The only difference in the derivation is that $L^{(3)}(\underset{\sim}{X}, t)$ is replaced by $L^{(4)}(\underset{\sim}{Y}, t)$. The result is

$$\underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t') = \left\langle \underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t') \right\rangle$$

$$- \int_{t'}^{t} dt'' \int d\underset{\sim}{Y}'' \left\langle \underset{\approx}{G}^{(4)}(t; t'') \right\rangle \cdot L^{(4)}(\underset{\sim}{Y}'', t'') \cdot \underset{\approx}{G}^{(4)}(t''; t')$$

$$+ h^{(4)}(t; t') \quad , \tag{B9}$$

where $\underset{\approx}{G}^{(4)}(t; t'') \equiv \underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}'', t'')$, $\underset{\approx}{G}^{(4)}(t''; t') \equiv \underset{\approx}{G}^{(4)}(\underset{\sim}{Y}'', t''; \underset{\sim}{Y}', t')$ and $h^{(4)}(t; t')$ is given by $h^{(3)}(t; t')$ in (23b) with $L^{(3)}(\underset{\sim}{X}'', t'')$ replaced by $L^{(4)}(\underset{\sim}{Y}'', t)$. Substituting (B9) into (B6) we finally have $Q^{(4)}$ given by

$$Q^{(4)}(\underset{\sim}{Y}, t) = \int d\underset{\sim}{Y}' \left\langle \underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t') \right\rangle \cdot \left\langle T^{(4)}(\underset{\sim}{Y}', t') \right\rangle_c$$

$$- \int_{t'}^{t} dt'' \int d\underset{\sim}{Y}'' \left\langle \underset{\approx}{G}^{(4)}(t; t'') \right\rangle \cdot \left\langle L^{(4)}(\underset{\sim}{Y}'', t'') T^{(4)}(\underset{\sim}{Y}'', t'') \right\rangle_c$$

$$+ \int d\underset{\sim}{Y}' \left\langle \underset{\approx}{h}^{(4)}(t; t') \cdot T^{(4)}(\underset{\sim}{Y}', t') \right\rangle_c \quad . \qquad \text{(B10)}$$

This equation is the renormalized hierarchy expression for $Q^{(4)}$. It is seen that $Q^{(4)}(\underset{\sim}{Y}, t)$ is related by (B10) to the initial value of $\left\langle T^{(4)} \right\rangle$, and to the fifth order velocity correlation function $\left\langle L^{(4)}(\underset{\sim}{Y}'', t'') T^{(4)}(\underset{\sim}{Y}'', t'') \right\rangle_c$. There is also a sixth order correlation coming from the $\underset{\approx}{h}^{(4)}$ term in (B10). This hierarchy expression for $Q^{(4)}(\underset{\sim}{Y}, t)$ differs from the conventional hierarchy because it contains the renormalized Green's function $\underset{\approx}{G}^{(4)}$. If $\underset{\approx}{G}^{(4)}$ were replaced by the linearized (unrenormalized) Green's function, then $\underset{\approx}{h}^{(4)}$ would be zero and (B10) would reduce to the conventional hierarchy equation for $Q^{(4)}$.

APPENDIX C

We wish to prove that $\underset{\approx}{S}_h$, given by (32), is a higher order term than is $\underset{\approx}{S}_c$, given by (31). In fact, we wish to prove that

$$\underset{\approx}{S}_h = \underset{\approx}{S}_c \; O(R^*) \tag{C1}$$

In order to estimate $\underset{\approx}{S}_h$ we must first be able to estimate $\underset{\approx}{h}(t; t')$ in (32). It is seen in (31) and (32) that (C1) could be proven to be true, if it could be proven that $\underset{\approx}{h}(t; t')$ is a higher order term than is

$$\int_{t'}^{t} dt' \int d\underset{\approx}{X}'' \left\langle \underset{\approx}{G}^{(3)}(t; t'') \right\rangle L^{(3)}(\underset{\sim}{X}'', t'').$$

To estimate $\underset{\approx}{h}(t; t')$, substitute (iterate) (23a) for $\underset{\approx}{G}^{(3)}$ into each of the three terms in the right side of (23b) as follows:

$$\begin{aligned}
\underset{\approx}{h}^{(3)}(t; t') = \int_{t'}^{t} dt'' \int d\underset{\sim}{X}'' \Bigg\{ & \underset{\approx}{G}^{(3)}(t; t'') \cdot \Bigg\langle L^{(3)}(t'') \\
& \left[\int_{t'}^{t''} dt^* \int d\underset{\sim}{X}^* \left\langle \underset{\approx}{G}^{(3)}(t''; t^*) \right\rangle \cdot L^{(3)}(t^*) \underset{\approx}{G}^{(3)}(t^*; t') + \underset{\approx}{h}^{(3)}(t''; t') \right] \Bigg\rangle \\
& + \left[\left\langle \underset{\approx}{G}^{(3)}(t; t'') \right\rangle \cdot L^{(3)}(t'') \right] \Bigg[\int_{t'}^{t''} dt^* \int d\underset{\sim}{X}^* \left\langle \underset{\approx}{G}^{(3)}(t''; t^*) \right\rangle L^{(3)}(t^*) \cdot \underset{\approx}{G}^{(3)}(t^*; t') \\
& \qquad + \underset{\approx}{h}^{(3)}(t''; t') \Bigg] \\
& - \left[\int_{t''}^{t} dt^* \int d\underset{\sim}{X}^* \left\langle \underset{\approx}{G}^{(3)}(t; t^*) \right\rangle \cdot L^{(3)}(t^*) \underset{\approx}{G}^{(3)}(t^*; t'') + \underset{\approx}{h}^{(3)}(t; t'') \right] \\
& \qquad \cdot \left[L^{(3)}(t'') \left\langle \underset{\approx}{G}^{(3)}(t''; t') \right\rangle \right] \Bigg\} ,
\end{aligned} \tag{C2}$$

where we use the notation

$$\underset{\approx}{G}^{(3)}(t;\, t^*) \equiv \underset{\approx}{G}^{(3)}(\underset{\sim}{X},\, t;\, \underset{\sim}{X}^*,\, t^*)$$

$$\underset{\approx}{G}^{(3)}(t^*;\, t\,) \equiv \underset{\approx}{G}^{(3)}(\underset{\sim}{X}^*,\, t^*;\, \underset{\sim}{X}',\, t\,)$$

$$L^{(3)}(t'') \equiv L^{(3)}(\underset{\sim}{X}'',\, t')$$

It is clear from (C2) that $\underset{\approx}{h}^{(3)}$ is second order in the velocity fluctuation (because $L^{(3)}$ is proportional to the velocity fluctuation). In fact, $\underset{\approx}{h}^{(3)} \propto O(L^{(3)}L^{(3)})\underset{\approx}{G}^{(3)}$. Hence, to second order, it is permissible to drop $\underset{\approx}{h}^{(3)}$ in the right side of (C2). Furthermore it can be seen that the second and third integral terms in the curly brackets on the right side of (C2) cancel each other out to second order in $L^{(3)}$. This can be seen by a second iteration (that is, by replacing $\underset{\approx}{G}^{(3)}$ with $\langle \underset{\approx}{G}^{(3)} \rangle$ in these two terms) and interchanging the integration variables $t^* \leftrightarrow t''$ in one of the terms. The result is that, to lowest order, (C2) becomes

$$\underset{\approx}{h}^{(3)}(t;\, t'\,) = \int_{t'}^{t} dt'' \int_{t'}^{t''} dt^* \int d\underset{\sim}{X}''d\underset{\sim}{X}^*\ \underset{\approx}{G}^{(3)}(t;\, t'')$$

$$\cdot \left\langle L^{(3)}(t'')\langle \underset{\approx}{G}^{(3)}(t'';\, t^*)\rangle \cdot L^{(3)}(t^*)\underset{\approx}{G}^{(3)}(t^*;\, t')\right\rangle \quad . \qquad \text{(C3)}$$

An estimate for $\underset{\approx}{S}_h$ can be obtained by substituting (28) and (C3) into (32) to yield

$$\underset{\approx}{S}_h = -\ (1 + tr) \int_{t'}^{t} dt'' \int_{t'}^{t''} dt^* \int d\underset{\sim}{X}''d\underset{\sim}{X}^* \int d\underset{\sim}{X}'d\underset{\sim}{r}_a\ \underset{\approx}{A}(\underset{\sim}{r}_o,\, \underset{\sim}{r}_a) : \underset{\approx}{G}^{(3)}(t;\, t'')$$

$$\cdot \left[\left\langle L^{(3)}(t'')\langle \underset{\approx}{G}^{(3)}(t'';\, t^*)\rangle \cdot L^{(3)}(t^*)\underset{\approx}{G}^{(3)}(t^*;\, t')\right\rangle\right] \cdot \underset{\approx}{T}(\underset{\sim}{X}',\, t') \quad . \qquad \text{(C4)}$$

But

$$\int d\underset{\sim}{X}'\ \underset{\approx}{G}^{(3)}(t^*;\, t') \cdot T(\underset{\sim}{X}',\, t') = \underset{\approx}{T}(\underset{\sim}{X}^*,\, t^*)$$

so that (C4) reduces a little to

$$\underset{\approx}{S}_h = -(1 + \mathrm{tr}) \int_{t'}^{t} dt'' \int_{t'}^{t''} dt^* \int d\underset{\sim}{X}'' d\underset{\sim}{X}^* d\underset{\sim}{r}_a \, \underset{\approx}{A}(\underset{\sim}{r}_o, \underset{\sim}{r}_a) : \left\langle \underset{\approx}{G}^{(3)}(t; t'') \right\rangle$$

$$\cdot \left\langle L^{(3)}(t'') \left\langle \underset{\approx}{G}^{(3)}(t''; t^*) \right\rangle \cdot L^{(3)}(t^*) \right\rangle \left\langle \underset{\approx}{T}(\underset{\sim}{X}^*, t^*) \right\rangle \quad . \qquad \text{(C5)}$$

In obtaining (C5) we have also replaced $\underset{\approx}{G}^{(3)}$ with $\langle \underset{\approx}{G}^{(3)} \rangle$, which is valid to lowest order.

This result for $\underset{\approx}{S}_h$ is to be compared with $\underset{\approx}{S}_c$ given by

$$\underset{\approx}{S}_c = (1 + \mathrm{tr}) \int d\underset{\sim}{r}_a \int_{t'}^{t} dt'' \int d\underset{\sim}{X}'' \, \underset{\approx}{A}(\underset{\sim}{r}_o, \underset{\sim}{r}_a) : \left\langle \underset{\approx}{G}^{(3)}(t; t'') \right\rangle$$

$$\cdot \left\langle L^{(3)}(\underset{\sim}{X}'', t'') \underset{\approx}{T}(\underset{\sim}{X}'', t'') \right\rangle \quad . \qquad \text{(C6)}$$

To compare the orders of magnitude of $\underset{\approx}{S}_h$ and $\underset{\approx}{S}_c$, for $\underset{\approx}{S}_h$ has an extra factor of

$$\int_{t'}^{t''} dt^* \int d\underset{\sim}{X}^* \left\langle \underset{\approx}{G}^{(3)}(t''; t^*) \right\rangle$$

than does the expression for $\underset{\approx}{S}_c$. But, the magnitude of this factor is less than or on the order of τ_c because, $\langle \underset{\approx}{G}^{(3)}(t''; t^*) \rangle$ cuts off the t^* integral when $t'' - t^* > \tau_c$. Hence, using (C5) and (C6), the relative magnitudes of $\underset{\approx}{S}_h$ and $\underset{\approx}{S}_c$ can be roughly stated as

$$\frac{\underset{\approx}{S}_h}{\underset{\approx}{S}_c} = O\left(\frac{\tau_c \left| \langle L^{(3)} L^{(3)} \rangle \langle \underset{\approx}{T} \rangle \right|}{\left| \langle L^{(3)} \underset{\approx}{T} \rangle \right|} \right)$$

$$= O\left(\frac{\tau_c \left|\langle \underset{\sim}{v} L^{(3)}\rangle\langle \underset{\approx}{T}\rangle\right|}{\left|\langle \underset{\sim\approx}{vT}\rangle\right|}\right) \tag{C7}$$

Therefore we have

$$\frac{\underset{\approx}{S}_h}{\underset{\approx}{S}_c} = O(R^*) \quad , \tag{C8}$$

where

$$R^* = \frac{\tau_c \left|\langle \underset{\sim}{v} L^{(3)}\rangle\langle \underset{\approx}{T}\rangle\right|}{\left|\langle \underset{\sim\approx}{vT}\rangle\right|} \tag{C9}$$

as we set out to prove.

ESTIMATE OF R^*

Here, we wish to estimate an upper bound for the magnitude of R^*. One way to do this is to first note from dimensional considerations, that the magnitude of $L^{(3)}$ can be described by

$$L^{(3)} < O(\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \approx O(k_o v) \quad , \tag{C10}$$

where $\underset{\sim}{v}$ is the velocity at a point.

(k_o is a characteristic wavenumber of the energy containing region of the spectrum.) Notice that (C10) is written as an upper bound of $L^{(3)}$. This is a potentially important point. The reason we view (C10) as an (approximate) upper bound of $L^{(3)}$ is simply because $L^{(3)}$ is a sum of three fluctuating terms at three different points; that is, $L^{(3)}(\underset{\sim}{X}) = L(\underset{\sim}{r}_1) + L(\underset{\sim}{r}_2) + L(\underset{\sim}{r}_3)$. Since L is a fluctuation, it follows that in some realizations we can have $L(\underset{\sim}{r}_1) \approx -\left[L(\underset{\sim}{r}_2) + L(\underset{\sim}{r}_3)\right]$. In such cases $L^{(3)}(\underset{\sim}{X}) \approx 0$ even though $L(\underset{\sim}{r}_1)$ may be large. Consequently, we expect that in computing averages, such as (C7), cancellations between $L(\underset{\sim}{r}_1)$, $L(\underset{\sim}{r}_2)$ and $L(\underset{\sim}{r}_3)$ may take place, with the effect of "reducing" the value of $L^{(3)}$. This cancellation or "reduction" is not unrelated to the cancellations found by Orzag and Kruskal for the inertial range forms of third and higher order cumulants. Combining (C10) with (C9), for simplicity, we have

$$R^* \lesssim O\left(\tau_c \frac{|\langle \underset{\approx}{vv} \rangle \cdot \langle \underset{\sim}{\nabla} T \rangle|}{|\langle \underset{\sim}{v} T \rangle|}\right) \tag{C11}$$

From the structure of the $\underset{\approx}{S}_c$ and $\underset{\approx}{S}_h$ terms it can be seen that the coordinate variables in (C11) can be typically expressed as follows:

$$R^* \lesssim O\left(\tau_c \frac{|\langle \underset{\sim}{v}(\underset{\sim}{r}_b)\underset{\sim}{v}(\underset{\sim}{r}_a)\rangle \cdot \langle (\partial/\partial \underset{\sim}{r}_b) T(\underset{\sim}{r}_b, \underset{\sim}{r}_a, \underset{\sim}{r})\rangle|}{|\langle \underset{\sim}{v}(\underset{\sim}{r}_b) T(\underset{\sim}{r}_b, \underset{\sim}{r}_a, \underset{\sim}{r})\rangle|}\right) . \tag{C12}$$

But, in (C12), and in $\underset{\approx}{S}_c$,

$$\left|\langle \underset{\sim}{v}(\underset{\sim}{r}_b) T(\underset{\sim}{r}_b, \underset{\sim}{r}_a, r)\rangle\right| \approx 2\left|\langle \underset{\sim}{v}(\underset{\sim}{r}_b)\underset{\sim}{v}(\underset{\sim}{r}_a)\rangle \langle \underset{\sim}{v}(\underset{\sim}{r}_b)\underset{\sim}{v}(\underset{\sim}{r})\rangle\right| . \tag{C13}$$

Also, we have

$$\left|\langle (\partial/\partial \underset{\sim}{r}_b) T(\underset{\sim}{r}_b, \underset{\sim}{r}_a, \underset{\sim}{r})\rangle\right| \lesssim \Big|\langle \underset{\sim}{v}(\underset{\sim}{r}_b) v(\underset{\sim}{r}_b) \partial \underset{\sim}{v}(\underset{\sim}{r}_b)/\partial \underset{\sim}{r}_b\rangle$$

$$\times \langle \underset{\sim}{v}(\underset{\sim}{r}_b)\underset{\sim}{v}(\underset{\sim}{r})\rangle \Big/ \langle \underset{\sim}{v}(\underset{\sim}{r}_b)\underset{\sim}{v}(\underset{\sim}{r}_b)\rangle\Big| . \tag{C14}$$

Finally, we substitute (C13) and (C14) into (C12) to obtain the following approximate upper bound:

$$R^* \lesssim O\left(\tau_c \left|\langle \underset{\sim}{v}^2 \; \partial \underset{\sim}{v}/\partial \underset{\sim}{r}_b\rangle\right| \Big/ \left|\langle \underset{\sim}{v}^2\rangle\right|\right) . \tag{C15}$$

Here we have used the notation

$$\langle \underset{\sim}{v}(\underset{\sim}{r}_b)\underset{\sim}{v}(\underset{\sim}{r}_b) \quad \partial \underset{\sim}{v}(\underset{\sim}{r}_b)/\partial \underset{\sim}{r}_b\rangle = \langle \underset{\sim}{v}^2 \quad \partial \underset{\sim}{v}/\partial \underset{\sim}{r}_b\rangle$$

$$\langle \underset{\sim}{v}(\underset{\sim}{r}_b)\underset{\sim}{v}(\underset{\sim}{r}_b)\rangle = \langle \underset{\sim}{v}^2\rangle \; ,$$

which are independent of $\underset{\sim}{r}_b$ for a homogeneous system.

APPENDIX D

We wish to derive (41a) and (41b). Let us consider (41b) for $Q^{(4)}$ first. An hierarchy equation which expresses $Q^{(4)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \underset{\sim}{r}_4, t)$ in terms of fifth and higher order velocity correlations is given in Appendix by (B10). There, the condensed notation $\underset{\sim}{Y} \equiv (\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \underset{\sim}{r}_4, t)$ is used so that

$$Q^{(4)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \underset{\sim}{r}_4, t) \equiv Q^{(4)}(\underset{\sim}{Y}, t)$$

There are seen to be three terms on the right side of (B10). The third term, containing $\underset{\approx}{h}^{(4)}(t; t')$, is a higher order term in $\underset{\sim}{v}$ as is proven for $\underset{\approx}{h}^{(3)}(t; t')$ at some length in Appendix C. (The results for $\underset{\approx}{h}^{(4)}$ are precisely the same as for $\underset{\approx}{h}^{(3)}$ when $L^{(3)}$ is replaced by $L^{(4)}$ and $\underset{\approx}{G}^{(3)}$ is replaced by $\underset{\approx}{G}^{(4)}$. This replacement is a formality.) This term can thus be neglected to lowest order so that (B10) can be approximately written as

$$Q^{(4)}(\underset{\sim}{Y}, t) \approx \int d\underset{\sim}{Y}' \langle \underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t') \rangle \cdot \langle T^{(4)}(\underset{\sim}{Y}', t') \rangle_c$$

$$- \int_{t'}^{t} dt'' \int d\underset{\sim}{Y}'' \langle \underset{\approx}{G}^{(4)}(t; t'') \rangle \cdot \langle L^{(4)}(\underset{\sim}{Y}'', t'') T^{(4)}(\underset{\sim}{Y}'', t'') \rangle_c \quad . \tag{D1}$$

The first term on the right side of (D1) is seen to be a decaying initial value term, since $\underset{\approx}{G}^{(4)}(\underset{\sim}{Y}, t; \underset{\sim}{Y}', t')$ decays towards zero as $t - t'$ increases. This (first) term is equal to $Q^{(4)}(\underset{\sim}{Y}, t)$ at $t = t'$ and is less than $Q^{(4)}(\underset{\sim}{Y}, t)$ at later times ($t > t'$). In sum, the first term on the right side of (D1) is equal to or less than $Q^{(4)}(\underset{\sim}{Y}, t)$. If τ_c denotes the characteristic decay time, e-folding time, of $\underset{\approx}{G}^{(4)}$, then the initial value term can be neglected for $t - t' \gtrsim \tau_c$, and (D1) reduces to

$$Q^{(4)}(\underset{\sim}{Y}, t) \approx - \int_{t'}^{t} dt'' \int d\underset{\sim}{Y}'' \left\langle \underset{\approx}{G}^{(4)}(t; t'') \right\rangle \cdot \left\langle L^{(4)}(\underset{\sim}{Y}'', t'') T^{(4)}(\underset{\sim}{Y}'', t'') \right\rangle \quad . \tag{D2}$$

Equation (D2) is valid for $t - t' \gtrsim \tau_c$. The value of $Q^{(4)}(\underset{\sim}{Y}, t)$ for small t, i.e., for $t - t' < \tau_c$, is not important since it is seen in (39) that $\underset{\approx}{E}^{(4)}(t)$ approaches zero as $t - t'$ approaches zero. To complete the estimation of $Q^{(4)}$, we note, again, that $\left\langle \underset{\approx}{G}^{(4)}(t; t'') \right\rangle$ decreases towards zero as $t - t''$ increases, and effectively cuts-off the t'' integration at $t - t'' \lesssim \tau_c$. Hence, from (D2) we have the following order of magnitude result:

$$Q^{(4)}(\underset{\sim}{Y}, t) = O\left[\tau_c \left\langle L^{(4)}(\underset{\sim}{Y}, t) T^{(4)}(\underset{\sim}{Y}, t) \right\rangle_c\right] \quad . \tag{D3}$$

We next use (C10), the definition of $T^{(4)}$ in (B1), and expand the right side of (D3) in terms of cumulants to obtain

$$\begin{aligned} Q^{(4)}(\underset{\sim}{Y}, t) &\approx O(\tau_c \langle L^{(4)} \underset{\sim}{v} \rangle \langle T \rangle) \\ &+ O(\tau_c \underset{\sim}{\nabla} Q^{(5)}) \quad , \end{aligned} \tag{D4}$$

where $Q^{(5)}$ is the fifth order cumulant:

$$Q^{(5)} = \langle \underset{\sim}{vvvvv} \rangle_c \quad .$$

But $Q^{(5)}$ is higher order in $\underset{\sim}{v}$ than is the $\langle L^{(3)} \underset{\sim}{v} \rangle \langle T \rangle$ term in (D4). [In fact, it can be shown, in precisely the same way as done here and in Appendix B, that $Q^{(5)} \propto \langle vvv \rangle \langle vvv \rangle$ + sixth order cumulants. This is done by replacing respectively $L^{(4)}$, $\underset{\approx}{G}^{(4)}$, and $T^{(4)}$ by $L^{(5)}$, $\underset{\approx}{G}^{(5)}$, and $T^{(5)}$ everywhere in this Appendix and in Appendix B. Here, $T^{(5)}$ is the fifth order velocity function as $T^{(5)} = \underset{\sim}{v}(\underset{\sim}{r}_1)\underset{\sim}{v}(\underset{\sim}{r}_2)\underset{\sim}{v}(\underset{\sim}{r}_3)\underset{\sim}{v}(\underset{\sim}{r}_4)\underset{\sim}{v}(\underset{\sim}{r}_5)$, $L^{(5)} = L^{(4)}(\underset{\sim}{Y}, t) + L(\underset{\sim}{r}_5, t)$, and $\underset{\approx}{G}^{(5)}$ is the five-point generalization of the four-point Green's function $\underset{\approx}{G}^{(4)}$. In this way we obtain

$$Q^{(5)} = O\left[\tau_c \langle L^{(5)} T^{(5)} \rangle_c\right]$$

$$= O\left[\tau_c \langle \underset{\sim}{\nabla} \cdot (\underset{\sim\sim\sim}{vvv}) \rangle \langle \underset{\sim\sim\sim}{vvv} \rangle + Q^{(6)}\right] \qquad \text{(D5)}$$

where $Q^{(6)}$ is the sixth order velocity cumulant. Hence, it is seen that $Q^{(5)}$ is higher order than the $\langle \underset{\sim\sim}{vv} \rangle \langle \underset{\sim\sim\sim}{vvv} \rangle$ term, or, with (D4), that $\tau_c \underset{\sim}{\nabla} Q^{(5)}$ is higher order than $Q^{(4)}$ and can consequently be neglected in (D4). It is interesting to note that (D4) comes from the renormalized hierarchy equation for $Q^{(4)}$, and that (D5) comes from the renormalized hierarchy equation for $Q^{(5)}$.] Since $Q^{(5)}$ is a higher order term, it can be neglected in (D4) to obtain

$$Q^{(4)}(\underset{\sim}{Y}, t) = O(\tau_c \langle L^{(4)} \underset{\sim}{v} \rangle \langle T \rangle) \qquad . \qquad \text{(D6)}$$

To estimate the order of magnitude of the right side of (D6) in terms of R^*, we use the following:

$$\langle L^{(4)} \underset{\sim}{v} \rangle \langle T \rangle = O(\langle \underset{\sim}{v} L^{(4)} \rangle \langle T \rangle)$$

$$= O(\langle \underset{\sim}{v} L^{(3)} \rangle \langle T \rangle)$$

$$= O(R^* |\langle \underset{\sim}{v} T \rangle|)$$

$$= O(R^* |\langle \underset{\approx}{U} \rangle \langle \underset{\approx}{U} \rangle|) \qquad , \qquad \text{(D7)}$$

where we have used (C 9) for R^*. Substituting (D7) in (D6) we have the sought result.

$$Q^{(4)} = O(R^* |\langle \underset{\approx}{U} \rangle \langle \underset{\approx}{U} \rangle|) \qquad . \qquad \text{(D8)}$$

This result is (41a) as we set out to derive.

Next, we wish to show that (41b) is true. Actually, (41b) follows almost immediately from (41a). To understand this, it is observed in (38) and (39) that

$$\frac{(1 + tr)\int d\underset{\sim}{r}_a d\underset{\sim}{r}_b \; \underset{\approx}{A}(\underset{\sim}{r}_o, \underset{\sim}{r}_a) : \underset{\approx}{E}^{(4)}(t)}{\underset{\approx}{S}_c(\underset{\sim}{r}, \underset{\sim}{r}_o, t)} = O\left(\frac{Q^{(4)}}{\langle \underset{\approx}{U} \rangle \langle \underset{\approx}{U} \rangle}\right) \quad . \tag{D9}$$

But, we have just seen in (D8), or (41a), that

$$Q^{(4)} = O(R^* \langle \underset{\approx}{U} \rangle \langle \underset{\approx}{U} \rangle)$$

Hence, (41b) follows by combining (D8) with (D9).

APPENDIX E

The purpose of this appendix is to derive (66c) and, in the process, elucidate the propagation of $L^{(3)}(t'')$ by $\underset{\approx}{G}^{(3)}(t;t'')$. We begin with $\underset{\sim}{X}''$ integral in (65), which we denote by $\underset{\approx}{J}$ for convenience:

$$\underset{\approx}{J} \equiv \int d\underset{\sim}{X}\, \underset{\approx}{G}^{(3)}(t;t'')\cdot L^{(3)}(t'') \left\langle \underset{\approx}{G}^{(3)}(t'';t') \right\rangle \quad . \tag{E1}$$

With (69c), or with the definition of $L^{(3)}$ given by (21') and (10), (E1) becomes

$$\underset{\approx}{J} \equiv \int d\underset{\sim}{X}''\, \underset{\approx}{G}^{(3)}(t;t')\cdot \sum_{i}^{1,2,3} \int d\underset{\sim}{r}_a\, \underset{\approx}{A}(\underset{\sim}{r}_i'',\underset{\sim}{r}_a): \underset{\sim}{v}(\underset{\sim}{r}_a,t'')\, S(\underset{\sim}{r}_a,\underset{\sim}{r}_i'')\, \underset{\approx}{G}^{(3)}(t'';t'') \tag{E2}$$

where the substitution operator $S(\underset{\sim}{r}_a,\underset{\sim}{r}_i'')$ simply means that any function of $\underset{\sim}{r}_i''$ to its right side has $\underset{\sim}{r}_i''$ replaced by $\underset{\sim}{r}_a$; i.e.,

$$S(\underset{\sim}{r}_a,\underset{\sim}{r}_i'')\, f(\underset{\sim}{r}_i'') \equiv f(\underset{\sim}{r}_a)$$

for any function f.

Stated simply, the goal now is to expand $L^{(3)}(t'') \equiv L^{(3)}(\underset{\sim}{X}'',t'')$ about $L^{(3)}(t) \equiv L^{(3)}(\underset{\sim}{X},t)$ within the integral in (E1), [or, equivalently, to expand $\underset{\approx}{A}(\underset{\sim}{r}_i'',\underset{\sim}{r}_a): \underset{\sim}{v}(\underset{\sim}{r}_a,t'')$ about $\underset{\approx}{A}(\underset{\sim}{r}_i,\underset{\sim}{r}_a): \underset{\sim}{v}(\underset{\sim}{r}_a,t)$]. Furthermore, this expansion need only be correct to lowest order in $\tau_c L$ in order to satisfy the requirements in Sec. V [see discussion following (65)]; that is, the error can be second order in $\tau_c L$. The difficulty with a straightforward power series expansion of $L^{(3)}(t'')$ about $t'' = t$ is that it could lead to time secularities due to powers of $(t''- t)$. Such secularities, however, can be avoided by using the propagator properties of $\underset{\approx}{G}$ -- which, by the way, is equivalent to a renormalized expansion.

Hence, to derive (66c) from (E2), we will use such a propagator. This propagator, denoted by W, is defined by

$$\underset{\sim}{v}(\underset{\sim}{r},t) = W(\underset{\sim}{r},t,t')\, \underset{\sim}{v}(\underset{\sim}{r}',t') \tag{E3}$$

Clearly, $W(\underset{\sim}{r},t,t')$ must satisfy the Navier-Stokes equation (3) in the variable t, just as $\underset{\sim}{v}(\underset{\sim}{r},t)$ does:

$$\left[\frac{\partial}{\partial t} + L_o(\underset{\sim}{r}) + L(\underset{\sim}{r},t)\right] W(\underset{\sim}{r},t,t') = 0 \quad , \tag{E4}$$

with the initial value equal to unity

$$W(\underset{\sim}{r},t',t') = 1 \tag{E5}$$

In fact, the formal solution of (E4), or (3), is given by

$$W(\underset{\sim}{r},t,t') = \text{Ti}\exp\left\{-\int_{t'}^{t}[L_o(\underset{\sim}{r}) + L(\underset{\sim}{r},t'')]\right\} \quad , \tag{E6}$$

where Ti denotes the time ordered product of the expansion of the exponential. It is not difficult to verify that (E6) satisfies (E4), by differentiating both sides with respect to (3). [The term <u>propagator</u> derives from the fact that $W(t,t')$ <u>propagates</u> a function ($\underset{\sim}{v}$ in this case) from its value at one time to its value at another time. More descriptively, $W(t,t')$ is sometimes referred to as a "time-displacement operator".] The Green's function $\underset{\approx}{G}(t,t') \equiv \underset{\approx}{G}(\underset{\sim}{r},t;\underset{\sim}{r}',t')$ is also very closely related to $W(\underset{\sim}{r},t;t'')$. This relation is made evident by the following formal solution of (16)

$$\underset{\approx}{G}(\underset{\sim}{r},t;\underset{\sim}{r}',t') = W(\underset{\sim}{r},t,t')\, \underset{\approx}{G}(\underset{\sim}{r},t';\underset{\sim}{r}',t') \quad . \tag{E7}$$

This solution is verified by differentiating both sides of (E7) with respect to t, and noting that $\underset{\approx}{G}(\underset{\sim}{r},t;\underset{\sim}{r}',t')$ and $W(\underset{\sim}{r},t;t')$ both satisfy the same equation

$$(\partial/\partial t + L_o + L)\, \underset{\approx}{G},\ W = 0.$$

Substituting (17) for the initial value $\underset{\approx}{G}(\underset{\sim}{r},t';\underset{\sim}{r}',t')$ into (E7) we thus have the propagator form of $\underset{\approx}{G}$ given by

$$\underset{\approx}{G}(\underset{\sim}{r},t;\underset{\sim}{r}',t') = W(\underset{\sim}{r},t,t')\ \underset{\approx}{P}^{o}(\underset{\sim}{\nabla})\ \delta(r-r') \quad , \tag{E8}$$

which shows the close relation between $\underset{\approx}{G}$ and W.

Our intention, now, is to use W, as in (E3), to derive (66c) from (E2) to lowest order in $\tau_c L^{(3)}$. There are several related properties of W that will be used for this purpose, and which only need be correct to zeroth order in $\tau_c L$. First there is the inverse property

$$W(\underset{\sim}{r},t,t')\ W(\underset{\sim}{r},t',t) = 1 \quad , \tag{E9}$$

which is exact, and can be verified by using (E6). Next is the "product" property

$$[W(\underset{\sim}{r},t,t')\ f(\underset{\sim}{r})\ g(\underset{\sim}{r})] = [W(\underset{\sim}{r},t,t')\ f(\underset{\sim}{r})][W(\underset{\sim}{r},t,t')\ g(\underset{\sim}{r})] + D \quad , \tag{E10}$$

$$D = O[\tau_c L(\underset{\sim}{r})]\ f(\underset{\sim}{r})\ g(\underset{\sim}{r})$$

where $f(\underset{\sim}{r})$ and $g(\underset{\sim}{r})$ are any two functions of r, D is a "correction" term that is first order in $\tau_c L$, and we use the convention that $W(\underset{\sim}{r},t,t')$ only acts on (propagates) functions within the square bracket; e.g., $[W(\underset{\sim}{r},t,t')]\ f(\underset{\sim}{r})]g(\underset{\sim}{r}) = g(\underset{\sim}{r})[W,\underset{\sim}{r},t,t')\ f(\underset{\sim}{r})]$. The "correction" D is necessarily at least first order in $\tau_c L$ because $W = 1$ to zeroth order. To demonstrate this, and to show how D could be calculated, let us consider the simplifying case of zero viscosity, $L_o = 0$. The result also applies to when $L_o \neq 0$, but requires more algebra. An integration of (E4), with $L_o = 0$, gives

$$W(\underset{\sim}{r},t,t') = 1 - \int_{t'}^{t} dt''\ L(\underset{\sim}{r},t'')\ W(\underset{\sim}{r},t'',t') \quad , \tag{E11}$$

Note that W cuts off the t" integration in (E11) at $t''-t' \approx \tau_c$ so that

$$W(\underset{\sim}{r},t;t'') = 1 + O[\tau_c L(\underset{\sim}{r})] \quad . \qquad t - t' > \tau_c \tag{E11'}$$

Substituting (E11) or (E11') in (E10) trivially verifies that $D = O(\tau_c L) f(\underset{\sim}{r})\, g(\underset{\sim}{r})$ and, therefore, that

$$[W(\underset{\sim}{r},t,t')\, f(\underset{\sim}{r})\, g(\underset{\sim}{r})] = [W(\underset{\sim}{r},t,t')\, f(\underset{\sim}{r})]\, [W(\underset{\sim}{r},t,t')\, g(\underset{\sim}{r})] \cdot \quad , \tag{E12}$$
$$+ O[\tau_c L(\underset{\sim}{r})]$$

Equation (E12) is actually more accurate than it might appear. For example, (E12) is exact (i.e., $D = 0$) for the scalar diffusion problem in which $L(\underset{\sim}{r},t)$ is given by $\underset{\sim}{v}(r,t) \cdot \partial/\partial\underset{\sim}{r}$ [that is, when the pressure contribution to $L(\underset{\sim}{r},t)$ is ignored]. An evaluation of D can be made to obtain higher order terms in $\tau_c L$ by substituting (E11) in (E10), but will not be done because we do not think it is important.

To proceed, we use (E3) to write

$$\underset{\sim}{v}(\underset{\sim}{r}_a,t'') = [W(\underset{\sim}{r}_a,t'',t)\; \underset{\sim}{v}(\underset{\sim}{r}_a,t)] \quad ,$$

which is then substituted into (E2) to obtain

$$\underset{\approx}{J} = \int d\underset{\sim}{X}''\; \underset{\approx}{G}^{(3)}(t;t'') \cdot \sum_i^{1,2,3} \int d\underset{\sim}{r}_a\; \underset{\approx}{A}(\underset{\sim}{r}_i'',\underset{\sim}{r}_a) : [W(\underset{\sim}{r}_a,t'',t)\; \underset{\sim}{v}(\underset{\sim}{r}_a,t)]$$
$$S(\underset{\sim}{r}_a,\underset{\sim}{r}_i'') \langle \underset{\approx}{G}^{(3)}(t'',t') \rangle \tag{E13}$$

Combining (E12) with (E9), we have $W(\underset{\sim}{r}_a,t'',t)\, \underset{\sim}{v}(\underset{\sim}{r}_a,t)\, W(\underset{\sim}{r}_a,t,t'')\, f(\underset{\sim}{r}_a) = [W(\underset{\sim}{r}_a,t'',t)\, \underset{\sim}{v}(\underset{\sim}{r}_a,t)]\, f(\underset{\sim}{r}_a)\, \{1 + O[\tau_c L(\underset{\sim}{r}_a)]\}$, for any f, which is substituted into (E13) to yield $\underset{\approx}{J}$ as

$$\underset{\approx}{J} = \int d\underset{\sim}{X}'' \, \underset{\approx}{G}^{(3)}(t;t'') \cdot \sum_{i}^{1,2,3} \int d\underset{\sim}{r}_a \, \underset{\approx}{A}(\underset{\sim}{r}_i'', \underset{\sim}{r}_a) : W(\underset{\sim}{r}_a, t'', t) \, \underset{\sim}{v}(\underset{\sim}{r}_a, t)$$

$$W(\underset{\sim}{r}_a, t, t'') \, S(\underset{\sim}{r}_a, \underset{\sim}{r}_1'') \langle \underset{\approx}{G}^{(3)}(t'';t') \rangle \{1 + O[\tau_c L(\underset{\sim}{r}_a)]\} \; . \qquad \text{(E14)}$$

To use the propagation property of $\underset{\approx}{G}(\underset{\sim}{r}_1, t; \underset{\sim}{r}_i'', t'')$ contained in $\underset{\approx}{G}^{(3)}(t;t'')$, we substitute the definition of $\underset{\approx}{G}^{(3)}(t;t'')$, (19a), in (E14), and, for definiteness we take $i = 1$ (i.e., $\underset{\sim}{r}_i = \underset{\sim}{r}_1$) so that (E14) becomes

$$\underset{\approx}{J} = {\sum_{1}^{1,2,3}}' \int d\underset{\sim}{X}'' \int d\underset{\sim}{r}_a \, \underset{\approx}{G}(\underset{\sim}{r}_1, t; \underset{\sim}{r}_1'', t'') \cdot \underset{\approx}{A}(\underset{\sim}{r}_1'', \underset{\sim}{r}_a) : W(\underset{\sim}{r}_a, t'', t) \, \underset{\sim}{v}(\underset{\sim}{r}_a, t)$$

$$\underset{\approx}{G}(\underset{\sim}{r}_3, t; \underset{\sim}{r}_3'', t'') \cdot \underset{\approx}{G}(\underset{\sim}{r}_2, t; \underset{\sim}{r}_2'', t'') \cdot W(\underset{\sim}{r}_a, t, t'') \, S(\underset{\sim}{r}_a, \underset{\sim}{r}_1'')$$

$$\langle \underset{\approx}{G}^{(3)}(t'';t') \rangle \quad \{1 + O[\tau_c L(\underset{\sim}{r}_a)]\} \quad , \qquad \text{(E15)}$$

where the summation ${\sum_{1}^{1,2,3}}'$ denotes that we sum over the three terms obtained after interchanging 1 with 2, and then interchanging 1 with 3. To simplify the first line on the right side of (E15), we apply the following steps: (1) Substitute (E8) for $\underset{\approx}{G}(\underset{\sim}{r}_1, t; \underset{\sim}{r}_1'', t'')$ in (E15); (2) Use $\int d\underset{\sim}{r}_a \, W(\underset{\sim}{r}_a, t, t'') f(\underset{\sim}{r}_a) = \int d\underset{\sim}{r}_a \, f(\underset{\sim}{r}_a)$, which follows from the facts that $(\partial/\partial t) \, W(\underset{\sim}{r}_a, t, t'') = -[L_o(\underset{\sim}{r}_a) + L(\underset{\sim}{r}_a)] \, W(\underset{\sim}{r}_a, t, t'')$ and $\int d\underset{\sim}{r}_a [L_o(\underset{\sim}{r}_a) + L(\underset{\sim}{r}_a)] \, f(\underset{\sim}{r}_a) = 0$, so that $(\partial/\partial t) \int d\underset{\sim}{r}_a \, W(\underset{\sim}{r}_a, t, t'') \, f(\underset{\sim}{r}_a) = 0$; (3) Substitute the "product" property (E12) into the result of steps 1 and 2; and, (4) Substitute the inverse property (E9). The details of these steps applied to the first line of (E15) are given as follows:

$$\int d\mathbf{X}'' \int d\mathbf{r}_a \; \mathsf{G}(\mathbf{r}_1,t;\mathbf{r}_1'',t'') \cdot \mathsf{A}(\mathbf{r}_1'',\mathbf{r}_a) : W(\mathbf{r}_a,t'',t)$$

$$= \int d\mathbf{X}'' \int d\mathbf{r}_a \; W(\mathbf{r}_a,t,t'') \; W(\mathbf{r}_1,t,t') \; \mathsf{P}^o(\nabla_1) \; \delta(\mathbf{r}_1 - \mathbf{r}_1'') \cdot \mathsf{A}(\mathbf{r}_1'',\mathbf{r}_a) : W(\mathbf{r}_a,t'',t)$$

$$= \int d\mathbf{X}'' \int d\mathbf{r}_a [W(\mathbf{r}_a,t,t'') \; W(\mathbf{r}_1,t,t'') \; \mathsf{P}^o(\nabla_1) \; \delta(\mathbf{r}_1 - \mathbf{r}_1'') \cdot \mathsf{A}(\mathbf{r}_1'',\mathbf{r}_a)]$$

$$: [W(\mathbf{r}_a,t,t'') \; W(\mathbf{r}_a,t'',t)] \; \{1 + O[\tau_c L(\mathbf{r}_a)]\}$$

$$= \int d\mathbf{r}_3'' \; d\mathbf{r}_2'' \; d\mathbf{r}_1'' \int d\mathbf{r}_a [W(\mathbf{r}_a,t,t'') \; W(\mathbf{r}_1,t,t'') \; \mathsf{P}^o(\nabla_1) \; \delta(\mathbf{r}_1 - \mathbf{r}_1'') \cdot \mathsf{A}(\mathbf{r}_1'',\mathbf{r}_a)]$$

$$: \{1 + O[\tau_c L(\mathbf{r}_a)]\}$$

$$= \int d\mathbf{r}_3'' \; d\mathbf{r}_2'' \int d\mathbf{r}_a [W(\mathbf{r}_a,t,t'') \; W(\mathbf{r}_1,t,t'') \; \mathsf{P}^o(\nabla_1) \cdot \mathsf{A}(\mathbf{r}_1,\mathbf{r}_a)$$

$$: \{1 + O[\tau_c L(r_a)]\}$$

$$= \int d\mathbf{r}_3'' \; d\mathbf{r}_2'' \int d\mathbf{r}_a [W(\mathbf{r}_a,t,t'') \; W(\mathbf{r}_1,t,t'') \; \mathsf{A}(\mathbf{r}_1,\mathbf{r}_a)]$$

$$: \{1 + O[\tau_c L(\mathbf{r}_a)]\} \quad , \qquad \text{(E16)}$$

where the last line follows from

$$\mathsf{P}^o(\nabla_1) \cdot \mathsf{A}(\mathbf{r}_1,\mathbf{r}_a) = \mathsf{A}(\mathbf{r}_1,\mathbf{r}_a) \quad ,$$

which can be proven from the definitions of P^o and A in Sec. II. According to E(11), the quantity $W(\mathbf{r}_a,t,t'') \; W(\mathbf{r}_1,t,t'')$ in (E16) can be approximated by unity to lowest order in $\tau_c L$:

$$W(\mathbf{r}_a,t,t'') \; W(\mathbf{r}_1,t,t'') = 1 + O[\tau_c L(\mathbf{r}_1) + \tau_c L(\mathbf{r}_a)] \quad . \qquad \text{(E17)}$$

Actually, when used in E(16), this approximation is even better because

$\underset{\approx}{A}(\underset{\sim}{r}_1, \underset{\sim}{r}_a) = \underset{\approx}{A}(\underset{\sim}{r}_1 - \underset{\sim}{r}_a)$, and the propagation of $\underset{\sim}{r}_1$ [within $\underset{\approx}{A}(\underset{\sim}{r}_1 - \underset{\sim}{r}_a)$] by $W(\underset{\sim}{r}_a, t, t'')$ is partially compensated (cancelled out) by the propagation of $\underset{\sim}{r}_a$ in $A(\underset{\sim}{r}_1 - \underset{\sim}{r}_a)$: that is,

$$W(\underset{\sim}{r}_a, t, t'')\, W(\underset{\sim}{r}_1, t, t'')\, \underset{\approx}{A}(\underset{\sim}{r}_1 - \underset{\sim}{r}_a)$$

$$\approx \underset{\approx}{A}[\underset{\sim}{r}_1 + \tau_c L(\underset{\sim}{r}_1) - \underset{\sim}{r}_a - \tau_c L(\underset{\sim}{r}_a)]$$

$$= \underset{\approx}{A}(\underset{\sim}{r}_1 - \underset{\sim}{r}_a) + O[\tau_c L(\underset{\sim}{r}_1) - \tau_c L(\underset{\sim}{r}_a)] \quad , \qquad \text{(E17')}$$

which has merit because the main contribution to the integral in (E15) comes from $\underset{\sim}{r}_a \approx \underset{\sim}{r}_1$. Substituting (E16) and (E17) into (E15), we obtain the almost desired result

$$\underset{\approx}{J} = \sum_{1}^{1,2,3}{}' \int d\underset{\sim}{r}_3''\, d\underset{\sim}{r}_2''\, d\underset{\sim}{r}_a\, A(\underset{\sim}{r}_1, \underset{\sim}{r}_a) : \underset{\sim}{v}(\underset{\sim}{r}_a, t)\, \underset{\approx}{G}(\underset{\sim}{r}_3, t; \underset{\sim}{r}_3'', t'') \cdot \underset{\approx}{G}(\underset{\sim}{r}_2, t; \underset{\sim}{r}_2'', t'')$$

$$\cdot\, W(\underset{\sim}{r}_a, t, t'')\, S(\underset{\sim}{r}_a, \underset{\sim}{r}_1'') \left\langle \underset{\approx}{G}^{(3)}(t''; t') \right\rangle$$

$$\times \{1 + O[\tau_c L(\underset{\sim}{r}_a)] + O[\tau_c L(\underset{\sim}{r}_1)] \quad . \qquad \text{(E18)}$$

To complete the derivation, we have, mainly, to express $W(\underset{\sim}{r}_a, t; t'')$ in terms of $\underset{\approx}{G}(\underset{\sim}{r}_a, t; \underset{\sim}{r}_1'', t'')$ within (E18), as follows:

$$W(\underset{\sim}{r}_a, t; t'')\, S(\underset{\sim}{r}_a, \underset{\sim}{r}_1'') \left\langle \underset{\approx}{G}^{(3)}(t''; t) \right\rangle$$

$$= \int d\underset{\sim}{r}_1''\, W(\underset{\sim}{r}_a, t; t'')\, \delta(\underset{\sim}{r}_a - \underset{\sim}{r}_1'') \left\langle \underset{\approx}{G}^{(3)}(t''; t) \right\rangle$$

$$= \int d\underset{\sim}{r}_1''\, W(\underset{\sim}{r}_a, t, t'')\, \delta(\underset{\sim}{r}_a - \underset{\sim}{r}_1'')\, \underset{\approx}{P}^{o}\left(\frac{\partial}{\partial r_1''}\right) \cdot \left\langle \underset{\approx}{G}^{(3)}(t''; t) \right\rangle \quad , \qquad \text{(E19)}$$

where we have used

$$\underset{\approx}{P}^{o}\left(\frac{\partial}{\partial \underset{\sim}{r}_1''}\right)\cdot\left\langle \underset{\approx}{G}(\underset{\sim}{r}_1'',t'';\underset{\sim}{r}_1',t')\right\rangle = \left\langle \underset{\approx}{G}(\underset{\sim}{r}_1'',t'';\underset{\sim}{r}_1',t')\right\rangle ,$$

together with (19a), which gives

$$\underset{\approx}{P}^{o}\left(\frac{\partial}{\partial \underset{\sim}{r}_1''}\right)\cdot\left\langle \underset{\approx}{G}^{(3)}(t'';t')\right\rangle = \left\langle \underset{\approx}{G}^{(3)}(t'';t')\right\rangle$$

Now, (E19) becomes

$$\int d\underset{\sim}{r}_1''\; W(\underset{\sim}{r}_a,t;t'')\;\delta(\underset{\sim}{r}_a-\underset{\sim}{r}_1'')\;\underset{\approx}{P}^{o}\left(\frac{\partial}{\partial \underset{\sim}{r}_1''}\right)\cdot\left\langle \underset{\approx}{G}^{(3)}(t'';t')\right\rangle$$

$$=\int d\underset{\sim}{r}_1''\; W(\underset{\sim}{r}_a,t;t'')\;\underset{\approx}{P}^{o}\left(\frac{\partial}{\partial \underset{\sim}{r}_a'}\right)\delta(\underset{\sim}{r}_a-\underset{\sim}{r}_1'')\cdot\left\langle \underset{\approx}{G}^{(3)}(t'';t')\right\rangle$$

$$=\int d\underset{\sim}{r}_1''\;\underset{\approx}{G}(\underset{\sim}{r}_a,t;\underset{\sim}{r}_1'',t'')\cdot\left\langle \underset{\approx}{G}^{(3)}(t'';t')\right\rangle \quad . \qquad \text{(E20)}$$

Substituting (E20) into (E18) gives

$$\underset{\approx}{J} = \sum_{1}^{1,2,3}{}' \int d\underset{\sim}{X}''\; d\underset{\sim}{r}_a\;\underset{\approx}{A}(\underset{\sim}{r}_1,\underset{\sim}{r}_a):\underset{\sim}{v}(\underset{\sim}{r}_a,t)\;\underset{\approx}{G}(\underset{\sim}{r}_3,t;\underset{\sim}{r}_3'',t'')\cdot\underset{\approx}{G}(\underset{\sim}{r}_2,t;\underset{\sim}{r}_2'',t'')$$

$$\cdot\,\underset{\approx}{G}(\underset{\sim}{r}_a,t;\underset{\sim}{r}_1'',t'')\cdot\left\langle \underset{\approx}{G}^{(3)}(t'',t')\right\rangle$$

$$\times\,\{1+O[\tau_c L(\underset{\sim}{r}_a)]+O[\tau_c L(\underset{\sim}{r}_1)]\}\ , \qquad \text{(E21)}$$

which becomes, with the definition of $L(\underset{\sim}{r}_1,t)$ in (10), followed by the definition of $\underset{\approx}{G}^{(3)}$ in (19a):

$$\underset{\approx}{J} = \sum^{1,2,3}{}' \int d\underset{\sim}{X}''\;\; L(\underset{\sim}{r}_1,t)\;\underset{\approx}{G}(\underset{\sim}{r}_3,t;\underset{\sim}{r}_3'',t'')\cdot\underset{\approx}{G}(\underset{\sim}{r}_2,t;\underset{\sim}{r}_2'',t'')$$

$$\cdot\,\underset{\approx}{G}(\underset{\sim}{r}_1,t;\underset{\sim}{r}_1'';t'')\cdot\left\langle \underset{\approx}{G}^{(3)}(t'';t')\right\rangle$$

$$\times\,\{1+O[\tau_c L(\underset{\sim}{r}_1)+O[\tau_c L(\underset{\sim}{r}_1'')]\}$$

$$= \sum_{1}^{1,2,3}{}' \int d\underset{\sim}{X}'' \, L(\underset{\sim}{r}_1,t) \; \underset{\approx}{G}^{(3)}(t;t'') \cdot \langle \underset{\approx}{G}^{(3)}(t'';t') \rangle$$

$$\times \{1 + O[\tau_c L(\underset{\sim}{r}_1)] + O[\tau_c L(\underset{\sim}{r}_1'')]\} \quad . \qquad \text{(E22)}$$

Finally, we use the summation

$$\sum_{1}^{1,2,3}{}' L(\underset{\sim}{r}_1,t) = \sum_{i}^{1,2,3} L(\underset{\sim}{r}_i,t) \equiv L^{(3)}(t)$$

in (E22) to obtain

$$\underset{\approx}{J} = \int d\underset{\sim}{X}'' \, L^{(3)}(t) \; \underset{\approx}{G}^{(3)}(t;t'') \cdot \langle \underset{\approx}{G}^{(3)}(t'';t') \rangle$$

$$\times [1 + O(\tau_c L^{(3)})] \qquad \text{(E23)}$$

Equation (E23) together with (E1) gives (66c)

$$\int d\underset{\sim}{X}'' \; \underset{\approx}{G}^{(3)}(t;t'') \cdot L^{(3)}(t'') \langle \underset{\approx}{G}^{(3)}(t'';t') \rangle$$

$$= \int d\underset{\sim}{X}'' \, L^{(3)}(t) \; \underset{\approx}{G}^{(3)}(t;t'') \cdot \langle \underset{\approx}{G}^{(3)}(t'';t') \rangle$$

$$+ O[\tau_3 (L^{(3)})^2] \qquad \text{(E24)}$$

as we set out to prove.