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DESCRIPTION AND SOME FORMAL PROPERTIES OF BETA FILTERS; COMPACT SUPPORT QUASI-GAUSSIAN CONVOLUTION OPERATORS WITH APPLICATIONS TO THE CONSTRUCTION OF SPATIAL COVARIANCES.

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Abstract

The beta distributions are a standard class of smooth two-parameter probability densities on a finite line segment, so named because their normalizing constants on a unit interval are the Euler beta functions of the two parameters. Of particular interest and utility are the symmetric distributions obtained when the two parameters are equal and nonnegative integers since, in these cases, the distributions each take the form of a polynomial over the interval. When the shared integer parameter is positive, the function is unimodal, approximating a Gaussian with greater fidelity as the parameter increases. Owing to its computationally convenient form, and the properties we have mentioned, the radially-symmetric beta filter, whose kernel's radial profile is such an integer parameter beta distribution, provides a very attractive choice for synthezing spatial covariances, especially in a multigrid framework where the quasi-Gaussian components of different scales and amplitudes can be superposed to form more general covariance profiles in a controlled way. This note describes the formal properties of the beta filters, including their first few moments, their Fourier or Bessel transforms, and the analytic formulas of the kernels of the homogeneous and self-adjoint covariances that can be constructed by self-convolution of the radially-symmetric filters.

1. Introduction

The beta filter is named after the beta distribution of probability theory (e.g., Dudewicz, 1976) which itself is so named because its normalization factor over a unit interval is the classical Euler beta function (Abramowitz and Stegun, 1972). Our purposes are served by the restriction of the distribution to the cases where the two parameters are equal and are a nonnegative integer, p, and where we take the standard interval to be the double-unit centered interval, $[-1, 1]$, and which we shall directly generalize to the unit disk or ball in any number of dimensions when we consider beta filters acting in these higher spaces.

Many of the formal properties of these distributions, including normalization and moments in any number of dimensions, will be developed in the next section. What do we mean by a multidimensional beta filter? The radially symmetric form of the distribution, that is, the multidimensional radially symmetric distribution whose radial profile through any diameter is the corresponding ordinary beta distribution, can be used in the obvious way to define the kernel of the corresponding radially symmetric smoothing filter. This filter remains spatially homogeneous and is, by construction, isotropic for rigid rotations about any point. In section 3 we study the Fourier and Bessel transforms of these radial beta function.

A trivial generalization allows for the insertion of a constant general linear stretching via the 'aspect tensor' (the aspect tensor for the unstretched standardized case is the identity tensor). The filter then remains homogeneous, but is no longer isotropic. But spatial homogeneity is too restrictive a condition for applications in data assimilation. For the spatially smooth, but inhomogeneous case, there are two distinct modes of filter construction with complementary properties, that serve to generalize further our construction of covariances, and which we mention here for completeness. The 'direct' form of the filters defines the filtered value at a point by the values of the surrounding input, weighted by the filter kernel with the aspect tensor and amplitude factor that are defined at that point of application; we refer to this filter by the operator, $\boldsymbol{C}$. If the kernel has been normalized everywhere (integral equal to one) then this direct application of the filter is a smoother, even for a spatially variable aspect tensor and, for a uniform-valued input field, will deliver the identical output field with no added fluctuations. In contrast, the 'adjoint' of this filter, $\boldsymbol{C}^T$, is not generally a smoother, but possesses instead the complementary property of being conservative (when the kernels are all normalized). That means that, for any initial substance distribution, the total integrated substance after filtering will be the same as before filtering. The component of the background error covariance filter, at a given multigrid generation if we omit the option of modulating the weight using a Helmholtz operator, is then of the form,

$$\boldsymbol{B} = \boldsymbol{C}\boldsymbol{C}^T. \tag{1.1}$$

Since it is always assumed that operators act in the style of matrices acting upon column vectors, this means in practice that we first apply adjoint filter $\boldsymbol{C}^T$ to precision-weighted data (or similar vectors that the conjugate-gradient (CG) algorithms supplies) and *then* apply the direct filter, $\boldsymbol{C}$, to that intermediate result to obtain a smooth analysis increment (or similar vectors that the next stage of the CG algorithm requires). In this way, even though the modulating amplitude scalar field, and the aspect tensor field, may vary spatially, the resulting $\boldsymbol{B}$ truly qualifies as a covariance by being non-negative and symmetric (or 'self-adjoint').

Although the main emphasis of this note concerns the idealized homogeneous forms of our beta filters in the continuum, section 4 does describe how either the radial filters, or carefully orchestrated sequences of line filters, can be made to produce practical anisotropic and spatially inhomogeneous covariance operators on discrete grids, maintaining the required properties of positivity and self-adjointness. These more general covariance filters are based always on overall constructions that are of the form expressible by (1.1).

The fully inhomogeneous filters, with their aspect tensor shapes as well as their amplitudes varying in space (or space-time), are best dealt with using the combinations of variously oriented line beta filters through use of the Triad (2D), Hexad (3D) and Decad (4D) algorithms. The notes, Purser (2020a, 2020b), describe new geometrical and group-theoretic methods by which the Hexad and Decad algorithms respectively can take advantage of the 1D line beta filters applied sequentially along carefully selected sets of generally oblique grid orientations. But in such cases, the amplitudes and exact moments of these inhomogeneous filter responses are not amenable to the exact treatment that we can provide in this note for the special, purely homogeneous, filters formulated as 'radial' beta filters in the full spatial dimensionality and operating on data as true convolutions.

Restricting our focus from here on to only these homogeneous convolution filters, we need to be aware that the radial beta filters are not themselves positive-definite, or even positive semi-definite, operators (although they certainly do at least possess the important attribute of self-adjointness). However, the self-convolution that (1.1) reduces to in the homogeneous case suggests that, upon substituting for $\boldsymbol{C}$ one of the radial beta filters, the resulting self-convolved radial beta filter product *is* guaranteed to be at least positive semi-definite. Since the form of this filter always has a closed analytic expression, it serves as a contender for an idealized quasi-Gaussian covariance operator that could be used in a variational assimilation of the 'physical space' (PSAS) type (da Silva et al., 1995; Cohn et al. 1998). Gaspari and Cohn (1998, 1999) constructed piecewise analytic (in their case, piecewise rational) covariance contenders in essentially this way, but from different starting radial distributions, and with profiles intended to exhibit greater kurtosis (i.e., broader tails and sharper peaks) than the Gaussian. One way to derive the analytic forms of such self-convolution filters is by taking the absolute squares of the Fourier transforms of the starting filter kernel (the '$\boldsymbol{C}$'), and inverse-transforming the product (the '$\boldsymbol{B}$'), exploiting the convolution theorems of Fourier or Fourier-Bessel transforms. In section 5 we describe the self-convolution process in geometrical terms and give the analytic result (without the detailed derivation) for the one-dimensional 'line-filter', and for the radial filters in two and three dimensions. In odd dimensions – one and three – concise generic formulas for these product filters can be given, but there do not appear to be such concise or simple expressions for the even-dimensional cases. In addition to providing piecewise analytic functions of compact support that satisfy the properties we require of a covariance, an alternative role for such functions is to serve as the localization functions of an ensemble variational assimilation scheme.

Some final remarks and conclusions are given in section 6.

2. Definition and elementary properties of the beta filters

The classical beta-distribution of statistics is defined for a variable x in some interval $[a, b]$ with a density distribution with respect to x on that interval given by

$$f(x) = \frac{1}{(b-a)} \frac{\Gamma(z+w)}{\Gamma(z)\Gamma(w)} t^{z-1}(1-t)^{w-1}, \tag{2.1}$$

where

$$t = \frac{x-a}{b-a}, \tag{2.2}$$

and z and w are two positive real parameters. The name comes about because the normalization factor is a rescaled version of the corresponding Euler beta function,

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \int_0^1 t^{z-1}(1-t)^{w-1} dt. \tag{2.3}$$

(see Abramowitz and Stegun 1972, P258). Since we shall not require the full generality of the analytic Γ-function in what follows, we shall replace it by the corresponding factorials of just integers and half-integers, recalling that $\Gamma(z) \equiv (z-1)!$.

If we standardize the interval to $[-1, +1]$ and choose to restrict the two original parameters, z and w, to being equal to $p+1$ for some small nonnegative integer, p, then we have defined the symmetrical kernels for a convenient family of finite impulse response filters whose shapes go from uniform over the interval in the case $p = 0$ to progressively more Gaussian-approximating bell shapes as the single new integer parameter p increases.

The densities of the distributions, normalized so that their integrals over the active interval $[-1, +1]$ are unity, are given in this basic one-dimensional example by:

$$f_p(x) = C_p(1-x^2)^p, \tag{2.4}$$

with,

$$C_p = \frac{(2p+1)!}{2^{(2p+1)}p!p!} = \frac{(p+\frac{1}{2})!}{p!\sqrt{\pi}}, \tag{2.5}$$

in general, where we have exploited the 'duplication formula' for factorials to get the more succinct second form for C_p in (2.5). The first few of these normalizing coefficients for integer p are conveniently listed:

$$C_0 = \frac{1}{2} \tag{2.6a}$$
$$C_1 = \frac{3}{4} \tag{2.6b}$$
$$C_2 = \frac{15}{16} \tag{2.6c}$$
$$C_3 = \frac{35}{32} \tag{2.6d}$$
$$C_4 = \frac{315}{256}. \tag{2.6e}$$

Since we shall need to consider the cases where these coefficients have a half-integer index when we come to discuss the two-dimensional radial generalizations of our filters, and their attributes, we list some of these as follows:

$$C_{\frac{1}{2}} = \frac{2}{\pi} \tag{2.7a}$$
$$C_{\frac{3}{2}} = \frac{8}{3\pi} \tag{2.7b}$$
$$C_{\frac{5}{2}} = \frac{16}{5\pi} \tag{2.7c}$$
$$C_{\frac{7}{2}} = \frac{128}{35\pi} \tag{2.7d}$$
$$C_{\frac{9}{2}} = \frac{256}{63\pi}. \tag{2.7e}$$

To at least plausibly confirm that these values form a continuous and smooth sequence, we plot their values up to half-integers $p=5$ in Fig. 1. In practice, the choice of p should be made so that the resulting profile at least roughly resembles the 'bell' shape of a Gaussian, but bearing in mind that a larger p implies a greater computational cost; in practical applications a value of $p=2$ is found to be a good choice.

The plotted densities for the first few p (including the boxcar function that corresponds to $p=0$) are given in Fig. 2 and a larger number of the first few cumulative distributions are shown in Fig. 3.

As p grows, the functions f_p have higher and higher peaks and are correspondingly narrower and narrower. The standard measure of their width – actually a measure of the *square* of each function's width, is the second moment of the standardized function:

$$M_2 = \int_{-1}^{1} f_p(x)x^2dx = \frac{1}{2p+3}. \tag{2.8}$$

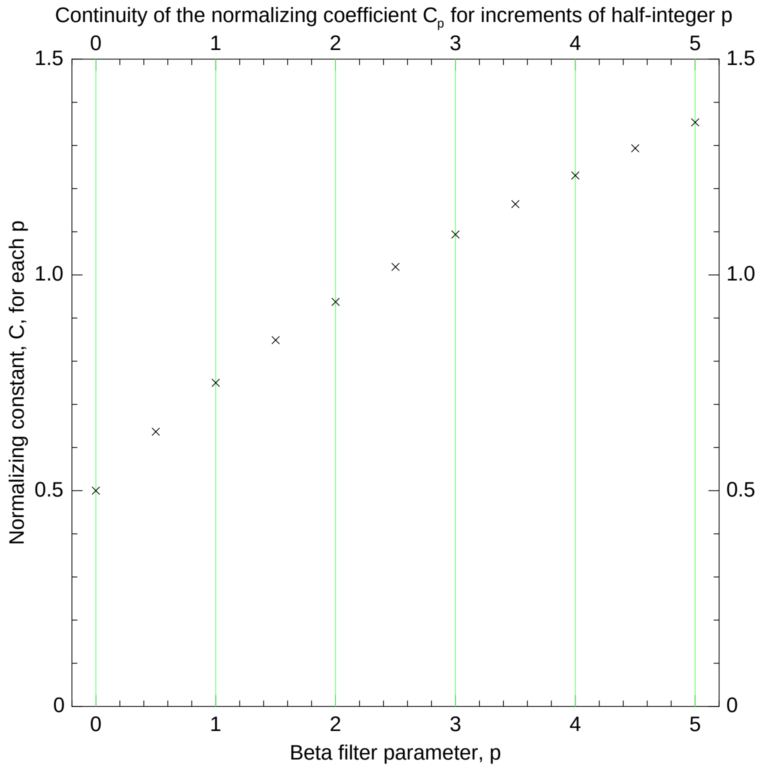

Figure 1. The first few half-integer p normalizing coefficients, C_p, for beta-function filter kernels standardized to the interval $[-1, 1]$.

The virtues of the beta family of filters from a numerical stand point are:

(i) They approximate the Gaussian, especially at larger values of p;

(ii) They have finite support;

(iii) They, and their integrals, are simply polynomials, making them cheap to evaluate;

(iv) Being symmetrical (even powers of x only), they produce polynomials in higher dimensions when generalized as the radial profiles of rotationally-symmetric kernels.

(v) The projected integral, or marginal, density in dimension $d-1$ of the standardized radial beta density of parameter p in dimension d is another standardized radial beta density, but with parameter, $p + \frac{1}{2}$.

The cumulative distributions we obtain by partially integrating the densities are, by this third property, polynomials whose coefficients we tabulate in Table 1.

The proof of property (v) comes from the fact that the marginal integral of the radial d-dimensional beta distribution as a function of the radial distance x in the $d-1$-dimensional projection is an integral of the original density over the coordinate, say y, that we are projecting out:

$$I(x) = \int_{-(1-x^2)^{\frac{1}{2}}}^{(1-x^2)^{\frac{1}{2}}} C(1 - x^2 - y^2)^p dy, \tag{2.9}$$

where the integral bounds represent the edge or surface of the disk or ball in the higher dimen-

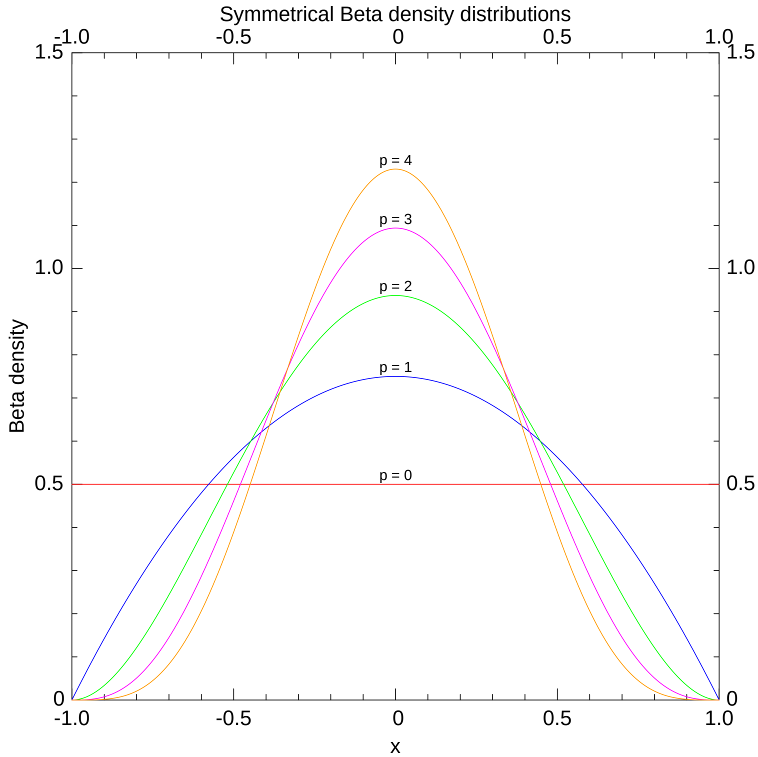

Figure 2. Density profiles for the first few standardized Beta distributions on their interval of support, $[-1, 1]$.

sion, and C is just the requisite normalizing constant. The substitution,

$$z = \frac{y}{(1-x^2)^{\frac{1}{2}}}, \tag{2.10}$$

and hence, $dy = (1-x^2)^{\frac{1}{2}} dz$, implies that the integral becomes:

$$\begin{aligned} I(x) &= \int_{-1}^{1} C(1-x^2)^{p+\frac{1}{2}}(1-z^2)^p dz \\ &= C'(1-x^2)^{p+\frac{1}{2}}, \end{aligned} \tag{2.11}$$

where

$$C' = C/C_p, \tag{2.12}$$

serves as the new normalizing constant for this radial beta of parameter $p+\frac{1}{2}$.

If we denote the normalizing coefficient for the two-dimensional radial beta kernel of parameter p in the unit disk by $C_p^{(2)}$, that is,

$$f_p^{(2)}(x, y) = C_p^{(2)}(1-x^2-y^2)^p, \quad x^2+y^2 < 1, \tag{2.13}$$

then we find from the aforementioned properties that,

$$C_p^{(2)} = C_p C_{p+\frac{1}{2}} = \frac{p+1}{\pi}. \tag{2.14}$$

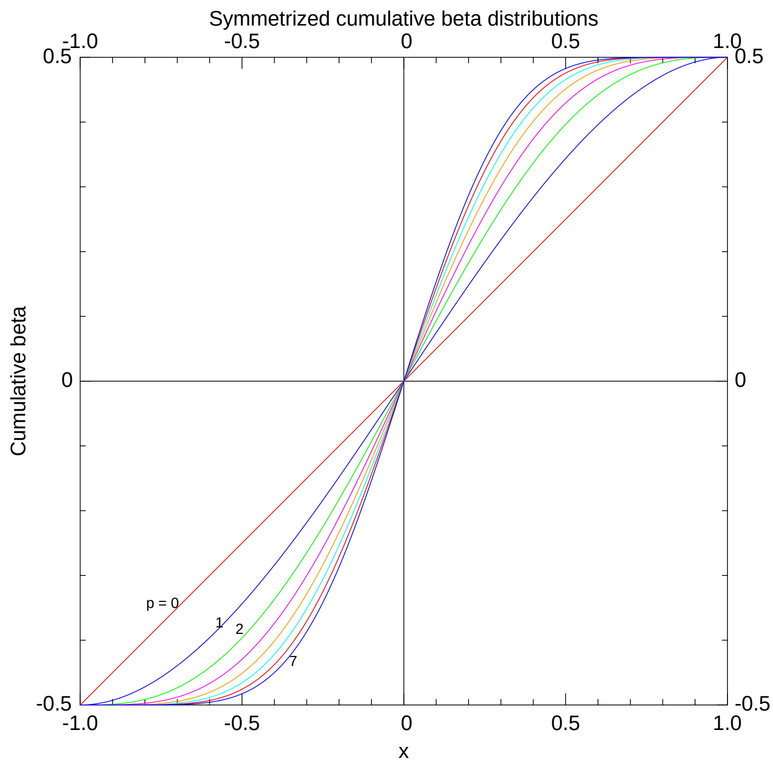

Figure 3. First few standardized beta cumulative distributions on their interval of support, $[-1, 1]$.

In a corresponding way, if we denote the normalizing coefficient for the three-dimensional standardized filter, $C_p^{(3)}$, of parameter p in the unit sphere, then,

$$C_p^{(3)} = C_p C_{p+\frac{1}{2}} C_{p+1}. \tag{2.15}$$

The first few of these give:

$$C_0^{(3)} = \frac{3}{4\pi} \tag{2.16a}$$

$$C_1^{(3)} = \frac{15}{8\pi} \tag{2.16b}$$

$$C_2^{(3)} = \frac{105}{32\pi} \tag{2.16c}$$

$$C_3^{(3)} = \frac{315}{64\pi}, \tag{2.16d}$$

and so on.

If we examine the unidirectional second moments for our standardized radial beta filter kernels, we find that, in two dimensions, the integrals corresponding to (2.8) are:

$$M_2^{(2)} = \iint_{x^2+y^2<1} f_p^{(2)}(x, y) x^2 dx\, dy = \frac{1}{2p+4}, \tag{2.17}$$

and in three dimensions,

$$M_2^{(3)} = \iiint_{x^2+y^2+z^2<1} f_p^{(3)}(x,y,z)x^2 dx\, dy\, dz = \frac{1}{2p+5}. \tag{2.18}$$

TABLE 1. Polynomial coefficients of the cumulative beta distributions expressed as antisymmetric integrals from their midpoints. Each polynomial in this form can be expressed as the sum of terms of the form, $a_q x^q/b$, where only odd exponents, q appear and where the numerators a_q, and denominators, b, are integers, the b being common for each filter order, p.

p	b	a_1	a_3	a_5	a_7	a_9	a_{11}	a_{13}	a_{15}
0	2	1							
1	4	3	-1						
2	16	15	-10	3					
3	32	35	-35	21	-5				
4	256	315	-420	378	-180	35			
5	512	693	-1155	1386	-990	385	-63		
6	2048	3003	-6006	9009	-8580	5005	-1638	231	
7	4096	6435	-15015	27027	-32175	25025	-12285	3465	-429

3. Spectral analysis.

In one dimension, the Fourier analysis of the normalized and standardized beta filter with parameter p gives us the transform in wave number k:

$$F_p(k) = A_p(k)\sin(k) + B_p(k)\cos(k), \tag{3.1}$$

where the $A_p(k)$ and $B_p(k)$ are the following polynomials in $1/k$:

$$A_p(k) = \sum_{q=0}^{\lfloor \frac{p}{2} \rfloor} \frac{(-1)^q(2p+1)!(2p-2q)!}{2^{(2p-2q)}p!(2q)!(p-2q)!k^{(2p+1-2q)}}, \tag{3.2}$$

$$B_p(k) = \sum_{q=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{-(-1)^q(2p+1)!(2p-1-2q)!}{2^{(2p-1-2q)}p!(2q+1)!(p-1-2q)!k^{(2p-2q)}}. \tag{3.3}$$

For the first few p (starting at $p=0$) the resulting spectra are shown in Fig. 4. If the filters are applied twice, it is the spectrum of their convolution that results, or simply the square of the previous spectrum (by the usual convolution theorem). The sidelobes then become very much less prominent, as Fig. 5 shows.

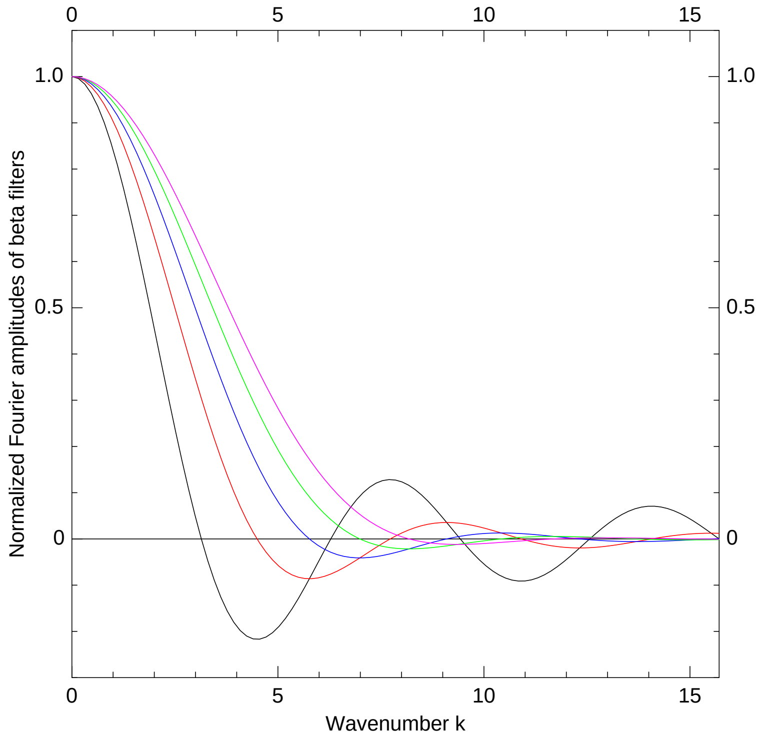

Figure 4. Fourier response of the one-dimensional standardized beta filters for integer increments of p starting with $p = 0$. The 'Gibbsing' oscillatory sidelobes in these spectra clearly decrease in amplitude as p increases, as well as decaying faster with k, as expected from the progressively higher degrees of continuity possessed by the original density distributions at ± 1 as p increases.

In two dimensions, the Fourier analysis of the normalized and standardized radial beta filter with parameter p gives us a rotationally symmetric transform whose radial form in absolute wavenumber k is defined in terms of the Bessel functions:

$$F_p^{(2)}(k) = A_p^{(2)}(k) J_1(k) + B_p^{(2)}(k) J_0(k), \tag{3.4}$$

where, again, the coefficients $A_p^{(2)}(k)$ and $B_k^{(2)}(k)$ are polynomials in $1/k$ but the functions they multiply, $J_1(k)$ and $J_0(k)$, are the standard Bessel functions of orders 1 and 0. In this case, the relevant definitions are:

$$A_p^{(2)}(k) = \sum_{q=0}^{\lfloor \frac{p}{2} \rfloor} \frac{(-1)^q 2^{(2p+1-2q)} (p+1)!(p-q)!(p-q)!}{q!q!(p-2q)!k^{(2p+1-2q)}}, \tag{3.5}$$

$$B_p^{(2)}(k) = \sum_{q=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{-(-1)^q 2^{(2p-2q)} (p+1)!(p-q)!(p-1-q)!}{q!(q+1)!(p-1-2q)!k^{(2p-2q)}}. \tag{3.6}$$

These calculations rely on the following standard integrals (see Abramowitz and Stegun 1972, articles (3,752, 3.753):

$$I_{-\frac{1}{2}} = \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} \cos(kx) dx = \pi J_0(k), \tag{3.7}$$

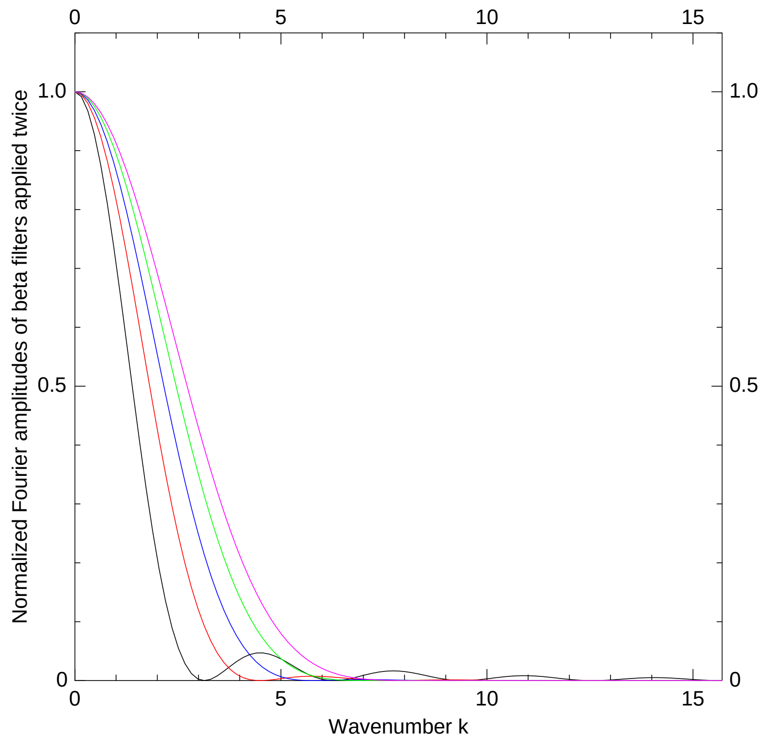

Figure 5. Like Fig. 4, but for the filters applied twice.

and

$$I_{\frac{1}{2}} = \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} \cos(kx) dx = \frac{\pi J_1(k)}{k}, \tag{3.8}$$

together with the following recurrence, valid for half-odd-integer $p \geq \frac{3}{2}$, obtained by integrating by parts twice:

$$I_p = \frac{(4p^2 - 2p)}{k^2} I_{p-1} - \frac{(4p^2 - 4p)}{k^2} I_{p-2}. \tag{3.9}$$

Corresponding to the spectra we showed for the one-dimensional cases, the radial amplitudes of the two-dimensional Fourier transforms of these planar radial beta filters, standardized to the unit disk, are given in Fig. 6 for a single filter application, and Fig. 7 for the filters applied twice.

In three dimensions, the unidirectional projection of the radial beta filter kernel of parameter p is, as we have noted, identical to the one-dimensional beta filter kernel of parameter $p+1$, and so the unidirectional Fourier transform has the form:

$$F_p^{(3)}(k) = A_p^{(3)}(k) \sin(k) + B_p^{(3)}(k) \cos(k), \tag{3.10}$$

in which,

$$A_p^{(3)} = A_{p+1} \tag{3.11a}$$

$$B_p^{(3)} = B_{p+1}. \tag{3.11b}$$

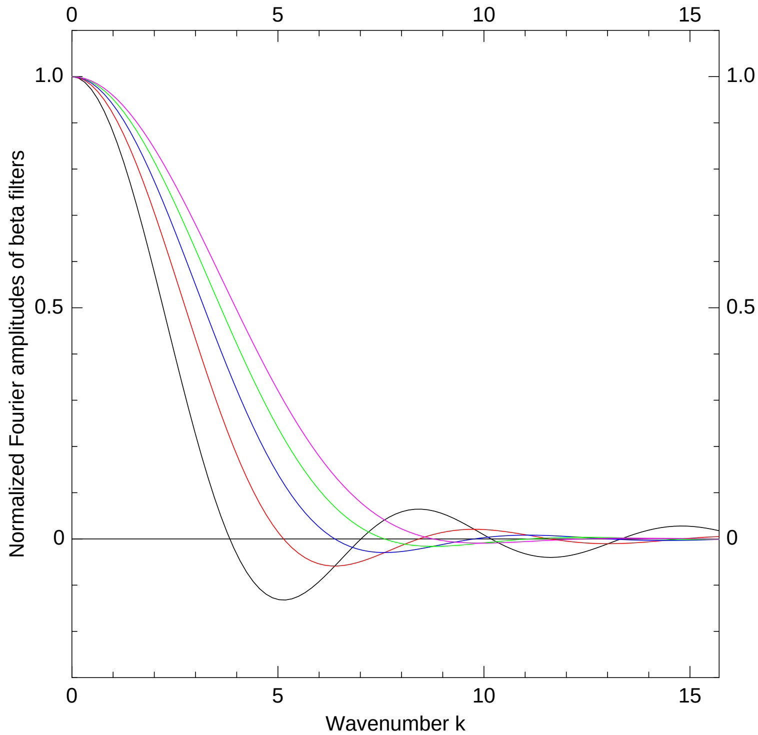

Figure 6. Fourier response of the two-dimensional standardized beta filters for integer increments of p starting with $p = 0$.

4. Practical algorithms for combining inhomogeneous beta filters to form covariance operators

(a) *The radial beta filters*

If we have a description of the scaling and stretching of the covariance components in terms of partial 'aspect tensors' $\boldsymbol{A}$ at each generation of a multigrid structure, then we can relate the 'effective radial distance', r, or its square, to the actual vector separation, in grid units, $\boldsymbol{v}$, by computing a quadratic form. For example, in three dimensions:

$$r^2 = \boldsymbol{v}^T \boldsymbol{A}^{-1} \boldsymbol{v}/(2p+5), \tag{4.1}$$

where the normalizing constant, $(2p+5)$ ensures the correct scaling of the filter's second moment. First, we can use this r^2 to tell us whether we are inside the ellipsoid of influence of the filter, $r^2 < 1$. Then the radial beta filter kernel of parameter p using this value of r^2 in the formula will have the intended $\boldsymbol{A}$ as its normalized second spatial moment. However, we do not advocate employing the formal normalizing coefficient, $C_p^{(3)}$, when we apply a filter constructed directly, because then the inevitable discretization effects will always make the numerical sum of gridded values different from the formal integral. Rather, we numerically sum the array elements of the 'matrix' that this direct filter implies in order that the filter weights can have, after appropriate empirical normalization, exactly the desired conserving property at each point of application. In the case where the radial filter contributes to the 'substance-

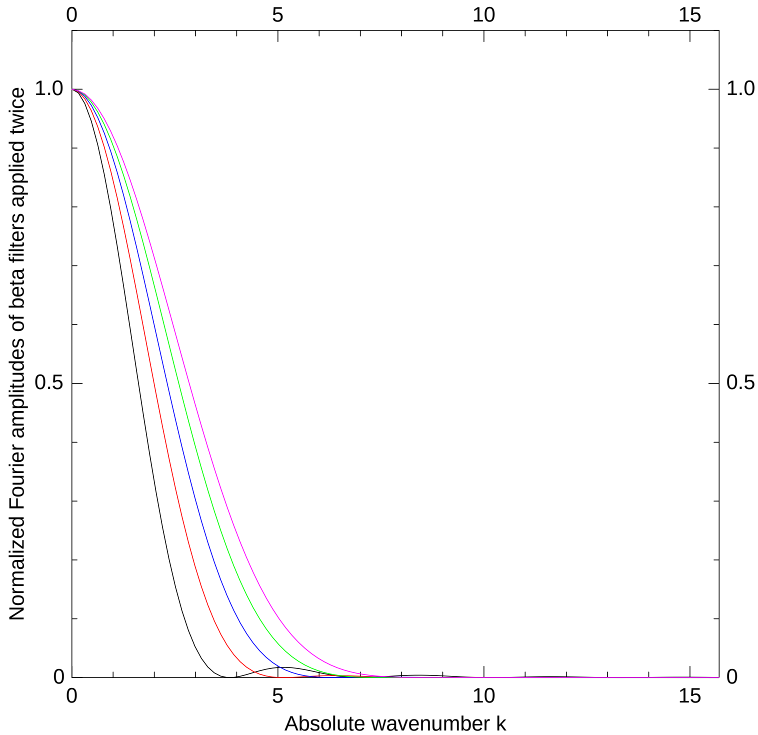

Figure 7. Like Fig. 6, but for the filters applied twice.

respecting' (conservative) half, $\boldsymbol{C}^T$, of the covariance, $\boldsymbol{B} = \boldsymbol{C}\boldsymbol{C}^T$, it is the column sums of the matrix representation of the filter that must sum to one; for the filters in the 'value averaging' (smoothing) half, $\boldsymbol{C}$, it is conversely the row sums that must be one, of course. Although algorithmically straightforward to implement, we can anticipate that these radial filters are liable to become very costly to apply, in terms of the amount of computations, in situations where a large number of gridpoints are covered by the 'footprint' of each impulse response of the filter. In order to remedy this defect, we might turn to the sequential application of 'line filters' coordinated in their effects through the use of the 'Triad' (two dimensions), 'Hexad' (three dimensions), or 'Decad' (four dimensions) algorithms (Purser et al., 2003; Purser, 2020a, 2020b).

(*b*) *The line beta filters*

In the simplest application, the one-dimensional version of the radial filters we have already described could serve as the line filters composing the elementary steps of the polyad (Triad or Hexad) algorithms. But a concern here is that, when the one-dimensional projection of the aspect tensor dictated by the polyad algorithm goes to zero, as often occurs, there is a tendency for the leading edge of the successive kernel sections of the filter to have footprints that override each other as the location of zero aspect is approached†.

† This defect of the original polyad algorithms with recursive filters was discovered by Dave Parrish

A solution, that avoids the possibility of violations of monotonicity, is to take each section of the one dimensional grid where the aspect projection is positive, and to stretch it in inverse proportion to the square-root of the projected aspect value so that, on the stretched grid, the scale of the beta filter is rendered exactly constant, and it becomes the grid points whose images in this new framework becomes nonuniformly distributed. But note that the grid point images retain their correct order and, more to the point, their influence domains under the application of the beta filter, being all of identical width now, also remain strictly in their correct order even as they overlap. Now we can define a weighted cumulative distribution in this stretched domain consisting of the superposition of all the input grid points' centered beta-function cumulative distributions, weighted by their input values. By construction, this grand cumulative function must be a non-decreasing piecewise polynomial, with the difference between its asymptotic 'right' value and its asymptotic 'left' value exactly equal to the total mass of the all the contributing inputs. If we read off the values of this function at all the stretched-grid images of the half-way-staggered grid point locations (i.e., the one-dimensional 'grid cell boundaries') , and then further take the differences between consecutive pairs of these, we end up with the output values of this mass-conserving variant of the line beta filter. Technically, this is a 'fast' algorithm because the computations scale no faster than the product of the number of grid points times the squared index p^2 (since the degree of each segment of the piecewise polynomial we are sequentially maintaining and evaluating in this algorithm is just $2p + 1$, which requires $\mathcal{O}(p^2)$ operations to adjust as each kernel comes into play or drops out). This is a description of the contribution to the $\boldsymbol{C}^T$ half of the covariance filter; the corresponding contribution to $\boldsymbol{C}$ is just the adjoint of the same sequence of operations in reverse order. When the line beta filters are consistently modified in this way, we find that the numerical noise at the geographical loci of the polyad boundaries, which was troublesome in the case of the self-adjoint recursive filters, essentially diappears.

(c) *Remarks*

These line filters can be incorporated into the polyad algorithms where they cost essentially the same regardless of the filtering scale; in this respect they behave very much like the recursive filters that our new beta filters are designed to replace. However, the pure (unblended) polyad algorithms are known to generate numerical roughnesses at the transition points of the triads or hexads when the line filters are the self-adjoint recursive filters of the kind used in the GSI, requiring some ameliorating adjustments to the pure algorithms. In the case of the Triad algorithm used in the present Real-Time Mesoscale Analysis (RTMA), the adjustment adopted is to replace the pure Triad algorithm by a blended 4-line variant that causes each triad to switch on and switch off more gradually than the pure Triad method would require. But in the case of the beta filters, it appears that the numerical noise at the loci of polyad transitions do *not* appear, provided the factorization of covariance $\boldsymbol{B}$ into factors $\boldsymbol{C}\boldsymbol{C}^T$ is such that the initially applied factors $\boldsymbol{C}^T$ contain the conserving form of the line filters, and the subsequently applied $\boldsymbol{C}$ factor contains the smoothing line filters, as discussed in the introduction.

5. Analytic formulas for the response functions of self-convolved beta filters

(a) One-dimensional line beta filters

As a reminder, the way in which the unnormalized radial beta distribution is defined on the standard unit disc or sphere, $|r| < 1$, is by defining the squared-radius

$$\rho = |r|^2, \tag{5.1}$$

and, for a beta function parameter p, defining the kernel function,

$$f_p(r) = (1-\rho)^p, \quad \rho = r^2 < 1. \tag{5.2}$$

The convolution of $f_p(r)$ with itself in the 1D case involves an integral in that single dimension. We can express the function describing the convolution of the unnormalized beta distributions f_p by the function $g_p(r)$ and it is convenient to let x be $r/2$, half the separation of the pair of overlapping beta distributions, when we perform the integral over the overlapping portion.

For the 1D beta distributions, the self-convolution is clearly going to result in another polynomial in each of the pieces in which the structure of the integral remains geometrically continuous. In this case, the two nontrivial pieces are simply the intervals $r \in [-2, 0]$ and $r \in [0, 2]$, with the function, g_p, in these intervals being mutual mirror images. If we work with the separation variable $x = r/2$, we need therefore only consider the unit-interval range $x \in [0, 1]$. When we begin to look at the set of the terms that make up the integral of the product of the separated copies of f_p (similar to the picture shown in Fig. 8, we encounter algebraic symmetries that simplify the expression of the final polynomial. For each p the degree of the polynomial in x in the active interval $[0, 1]$ is $4p + 1$ and the coefficient of the term in x^j is always the negative of the coefficient in the term in x^{4p+1-j} as a consequence of the aforementioned algebraic symmetry. When the integrations are carried out we find that the first half of all the polynomial coefficients are only nonzero for even powers of x and, consistent with the algebraic symmetry, the second half are all nonzero only for the odd powers of x. As a consequence of this property, self-convolved function g_p exhibits a continuity at the origin of its first $2p$ derivatives (and p derivatives vanish at the end points).

The convolution integral is performed on one symmetric half of the product function, and the result multiplied by two. Alternatively, we can use the Fourier transform results of section 3 and take the absolute square at each wavenumber to define the Fourier transform of the self-convolution; this is a consequence of the Convolution Theorem. The result we want is then obtained by taking the inverse transform of this wavenumber domain product. We omit the detailed derivation here and merely state the final results. The explicit generic form for g_p in 1D is found to be:

$$g_p(2x) = 2\sum_{j=0}^{p}(-1)^j \binom{p}{j} p! 2^{3p} \frac{(2p-1-2j)!!}{(4p+1-2j)!!}(x^{2j} - x^{4p+1-2j}). \tag{5.3}$$

The first few examples of these unnormalized but otherwise valid 'correlations functions' are:

$$g_0(2x) \;=\; 2 - 2x \tag{5.4a}$$

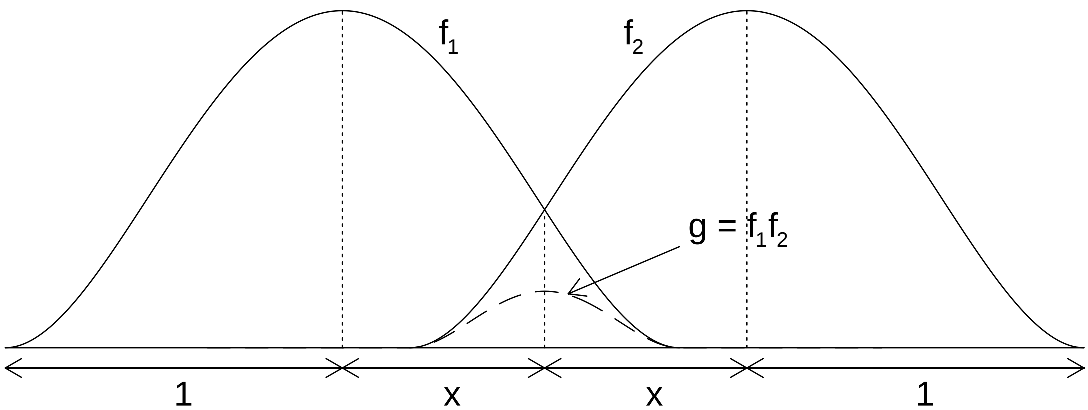

Figure 8. The idea of the self-convolution of a bell-shaped distribution, f, is depicted here when the separation of its two copies, f_1 and f_2, is $r = 2x$. The convolution product, $g(r)$, is computed as the integral under the graph (dashed curve) of the product of f_1 and f_2 at each separation.

$$g_1(2x) = \frac{16}{15} - \frac{16}{3}x^2 + \frac{16}{3}x^3 - \frac{16}{15}x^5 \tag{5.4b}$$

$$g_2(2x) = \frac{256}{315} - \frac{512}{105}x^2 + \frac{256}{15}x^4 - \frac{256}{15}x^5 + \frac{512}{105}x^7 - \frac{256}{315}x^9 \tag{5.4c}$$

$$g_3(2x) = \frac{2048}{3003} - \frac{2048}{385}x^2 + \frac{2048}{105}x^4 - \frac{2048}{35}x^6 + \frac{2048}{35}x^7 - \frac{2048}{105}x^9 + \frac{2048}{385}x^{11} - \frac{2048}{3003}x^{13}. \tag{5.4d}$$

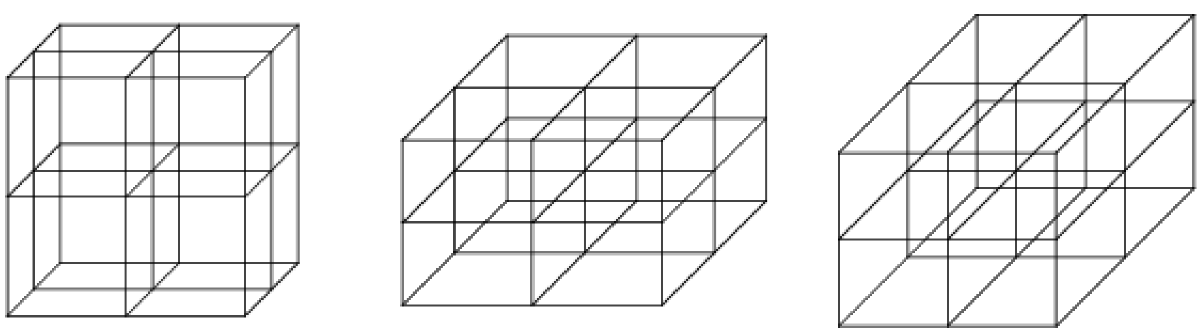

Figure 9. The generic (structurally-stable) configurations of triads that employ finite-support line filters. The left panel shows an example of the configuration possessing 52 polygonal subregions; the center panel shows one with 56 subregions; the right panel shows an example of the configuration with 58 subregions. In each subregion, the self-convolved homogeneous Triad line beta filter would exhibit some polynomial form and the pieces would join continuously and with a degree of smoothness determined by the filter index, p.

In the Triad and Hexad algorithms, the 1D line filters can be applied in three (triad) or six (hexad) generalized grid line directions in sequence to form each part, $\boldsymbol{C}$ or $\boldsymbol{C}^T$, of the final covariance contribution, $\boldsymbol{B}$. Therefore, in the case of the Triad algorithm, the final resulting response function for homogeneous statistics will be a collection of polynomials jointly in the two Euclidean coordinates in a collection of 'pieces' separated by the line segments that are the projections back into the 2D domain of the Cartesian product of the three triad member two-piece correlation functions like the form (5.3) but stretched by an appropriate amount along each of the triad's three directions. The generic configurations depend upon the relative magnitudes of the stretching (relative to the triad grid generators); if these grid-relative spans (proportional to the square-roots of the usual triad 'weights', which denote the projected aspect tensors) are denoted S_1, S_2 and S_3 in increasing order, then the three generic cases illustrated in Fig. 9 correspond to: the 52-piece configuration in the left panel with $2S_1 < S_2 < S_3$; the 56-piece configuration in the center panel with $S_1 < S_2 < 2S1 < S_3$; the 58-piece configuration in the right panel with $S_1 < S_2 < S_3 < 2S_1$. In principle, it is possible, knowing the amplitude, the S_i and corresponding triad directions, and the chosen filter index p, to compute the exact polynomial describing the kernel of the self-adjoint $\boldsymbol{B}$ in each of these polygonal pieces. However, it must be obvious by now that, for any reasonable value of p, the degrees of such polynomials will typically be very large indeed, making this exercise one of possible diagnostic value in code development, but not a practical option in operations.

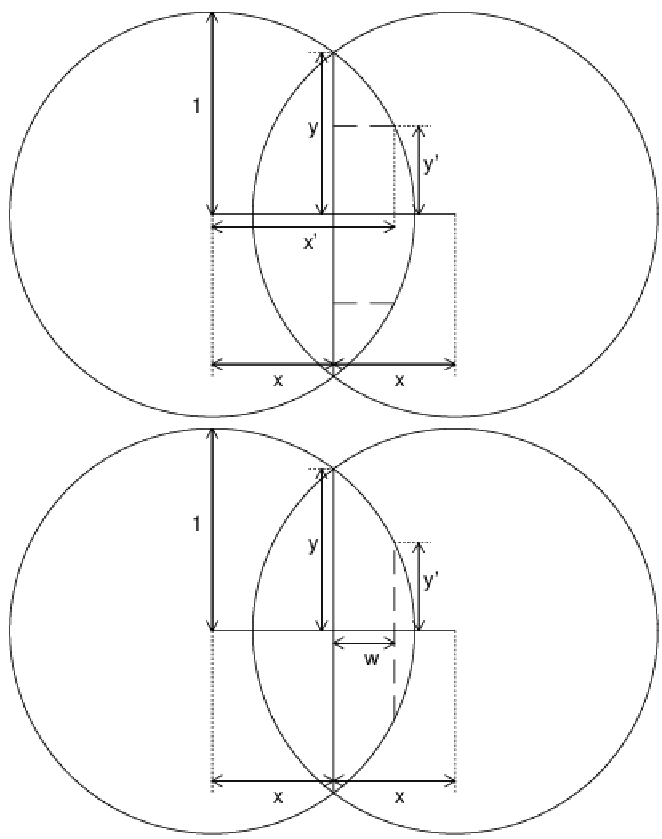

Figure 10. Configuration of spheres to show how the convolution of a radial beta distribution with itself can be computed by the integral of the product of the kernel in the lune of overlap. The upper panel shows where a 'cylinder' (dashed line) might be placed to supply the intermediate surface on which to initially integrate the product of the beta distributions. The resulting intermediate integral would then itself be integrated as y' goes from 0 to y. Alternatively, the lower panel shows how the intermediate surface might be made a 'disk' (dashed line, again) displaced from the median plane by amount w, and this intermediate integral could then be itself integrated as w goes from 0 to $1 - x$. As in the 1D case, the final tally would be doubled to account for the mirror symmetry (left to right) of the lune or lenticular overlap volume itself.

(*b*) *Self-convolution of radial beta functions in higher dimensions*

In higher dimensions, we have two natural choices for the way in which the convolution integral can proceed, as illustrated in the two panels of Fig. 10. The first case, illustrated in the top panels, can be termed the 'integration by cylinders', since the intermediate surface on which the self-product of the beta distribution is a 'cylinder' occupying the region of the overlap lens to the right of the median plane (i.e., with $x' > x$) and of radius $y' = \sqrt{1 - x'^2}$. The second case, illustrated in the lower panel of the figure, can be termed the 'integration by disks', since the intermediate surface integral is over the disk shown a distance w to the right of the median plane. In both cases, the respective intermediate integrals are themselves integrated over the rest of the half-lens, and the final result doubled to account for this mirror-symmetry. The terms 'cylinder' and 'disk' take their usual geometrical meanings only when we are working in 3D; in the 2D case, the 'cylinder' projects down into two line segments (both shown in the upper panel as dashed lines above and below the symmetry axis) while the 'disk' projects down to a single vertically-aligned line segment (the dashed line in the lower panel). In dimensions D higher than three, the hypersurface of what we have called a 'cylinder' has a circumferential section forming a $(D-2)$-spherical surface, while what we have called a 'disk' becomes spherical ball of dimension $D - 1$.

In cases where we wish to work in a number of dimensions D for our radial beta functions where D is odd (i.e., in 3D) the integrals are relatively straightforward by either the cylinder or disk approach and lead to a polynomial solution in the radial distance r of the resulting function $g_p(r)$ for all p. In the cases where the dimensionality D is even (i.e., in 2D and 4D, as practical examples), the final function $g_p(r)$ involves inverse trigonometric and square-root terms, and generic expressions are much harder to obtain. However, what we find is that the method of integrating cylinders does at least provide a way of deferring the difficult integrations until the cylinder-surface intermediate calculations, which involve only polynomial expressions, have been done. Also, we find that the integration of the product of the pair of beta distributions along one of the dashed segments parallel to the symmetry axis but displaced $y' \leq y$ from it, involves a formula very closely related to the one we already derived for the full 1D convolution integral (5.3). Therefore we shall take the approach here of using the integration of cylinders method.

The coordinate in the direction parallel to the symmtry axis will be denoted 'x' (or variants of it) as measured from the origin at the center of the distribution to the left of the figures 10. We have already taken "$2x$" to measure the separation of the two distribution's centers, so we take y' to be the cylinder radius and $x' = (1 - y'^2)^{1/2}$ the x coordinate of its intersection with the unit sphere containing the left distribution. If x'' is the coordinate of a general point on this cylinder, with $x \leq x'' \leq x'$, our first problem is to evaluate the integral,

$$h_p(x, x') = \int_x^{x'} f_p((y'^2 + x''^2)^{1/2}) f_p((y'^2 + (2x - x'')^2)^{1/2}) dx''. \tag{5.5}$$

Since the f_p are all even polynomial functions of their radial argument, and $y'^2 = 1 - x'^2$, the square-roots go away to leave polynomial expressions homogeneous in the variables x and x' for these intermediate integrals h. Moreover, these integrals, h_p, apart from trivial rescalings, are of the same form as the corresponding integrals, at each p, that were solved for the 1D convolution case. The result will be stated:

$$h_p(x,x') = \sum_{j=0}^{p}(-1)^j\binom{p}{j}p!2^{3p}\frac{(2p-1-2j)!!}{(4p+1-2j)!!}\left(x^{2j}x'^{4p+1-2j} - x^{4p+1-2j}x'^{2j}\right). \qquad (5.6)$$

To proceed, we must multiply this intermediate h_p by the measure associated with the 'circumference' of this 'cylinder', which is a factor, $\sigma_{D-2}y'^{D-2}$, where σ_d denotes the measure of the unit d-sphere. For $d=0,1,2$ these $\sigma_d = 2, 2\pi, 4\pi$ respectively. Then we must integrate with respect to y', remembering that x' is just a function of this y', from 0 to $y=(1-x^2)^{1/2}$. And finally, multiply this result by 2 to account for the other half of the lens-shaped overlap region. To summarize this second and final integration is:

$$g_p(2x) = 2\sigma_{D-2}\int_0^{(1-x^2)^{1/2}} h_p(x,(1-y'^2)^{1/2})y'^{D-2}dy'. \qquad (5.7)$$

(c) The case in $D=2$ dimensions

In two dimensions, the convolution integration for g_p looks like:

$$g_p(x) = 4\int_0^{(1-x^2)^{1/2}} h_p(x,x')dy', \qquad (5.8)$$

where, as before, $x' = (1-y'^2)^{1/2}$. The factors of the terms in h_p imply the requirement to integrate arbitrary integer powers of x' with respect to y'. The generic results we need are:

$$\int_0^y x'^{2m}dy' = \frac{(2m)!!}{(2m+1)!!}\left[y\sum_{j=0}^{m}\frac{(2j-1)!!}{(2j)!!}x^{2j}\right], \qquad m \geq 0 \qquad (5.9)$$

$$\int_0^y x'^{2m-1}dy' = \frac{(2m-1)!!}{(2m)!!}\left[\theta + xy\sum_{j=0}^{m-1}\frac{(2j)!!}{(2j+1)!!}x^{2j}\right], \qquad m > 0 \qquad (5.10)$$

where $\theta = \arcsin y = \arccos(x)$.

The formula for $g_p(2x)$ in 2D does not seem to have a clean and concise generic expression. The first few explicit formulas are:

$$g_0(2x) = 2\theta - 2xy \qquad (5.11a)$$

$$g_1(2x) = \theta\left(\frac{2}{3} - 4x^2\right) + xy\left(\frac{2}{3} + \frac{32}{9}x^2 - \frac{8}{9}x^4\right) \qquad (5.11b)$$

$$g_2(2x) = \theta\left(\frac{2}{5} - \frac{8}{3}x^2 + \frac{32}{3}x^4\right) + xy\left(\frac{2}{5} - \frac{12}{5}x^2 - \frac{2048}{225}x^4 + \frac{736}{225}x^6 - \frac{128}{225}x^8\right) \qquad (5.11c)$$

$$g_3(2x) = \theta\left(\frac{2}{7} - \frac{12}{5}x^2 + \frac{48}{5}x^4 - 32x^6\right) + xy\left(\frac{2}{7} - \frac{232}{105}x^2 + \frac{856}{105}x^4 + \frac{32768}{1225}x^6 - \frac{41344}{3675}x^8 + \frac{11776}{3675}x^{10} - \frac{512}{1225}x^{12}\right). \qquad (5.11d)$$

(*d*) *The case in $D = 3$ dimensions*

In the case where we are working with 3D beta distributions, the result of the second stage of integration (of h_p is a polynomial expression in x. The coefficients inherit some of the algebraic symmetry characteristics of the intermediate integral, but with minor modifications, as we see when we state the generic result as follows:

$$\begin{aligned} \frac{g_p(2x)}{\pi} &= \sum_{j=0}^{p} \binom{p}{j} \frac{(-1)^j 2^{3p+2}(2p-1-2j)!!x^{2j}}{(4p+3-2j)!!} \\ &\quad + \sum_{j=0}^{p+1} \binom{p+1}{j} \frac{(-1)^j p! 2^{3p+1}(2p+1-2j)!!x^{4p+3-2j}}{(p+1)(4p+3-2j)!!}. \end{aligned} \tag{5.12}$$

The first few of these formulas written out explicitly are:

$$g_0(2x) = \pi\left(\frac{4}{3} - 2x + \frac{2}{3}x^3\right) \tag{5.13a}$$

$$g_1(2x) = \pi\left(\frac{32}{105} - \frac{32}{15}x^2 + \frac{8}{3}x^3 - \frac{16}{15}x^5 + \frac{8}{35}x^7\right) \tag{5.13b}$$

$$\begin{aligned} g_2(2x) &= \pi\left(\frac{512}{3465} - \frac{1024}{945}x^2 + \frac{512}{105}x^4 - \frac{256}{45}x^5 + \frac{256}{105}x^7\right. \\ &\quad \left. -\frac{256}{315}x^9 + \frac{256}{2079}x^{11}\right) \end{aligned} \tag{5.13c}$$

$$\begin{aligned} g_3(2x) &= \pi\left(\frac{4096}{45045} - \frac{4096}{5005}x^2 + \frac{4096}{1155}x^4 - \frac{4096}{315}x^6\right. \\ &\quad \left. +\frac{512}{35}x^7 - \frac{2048}{35}x^9 + \frac{1024}{385}x^{11} - \frac{2048}{3003}x^{13} + \frac{512}{6435}x^{15}\right). \end{aligned} \tag{5.13d}$$

(*e*) *Remarks*

It is clear that the complications in evaluating the second integral (5.7) in the 2D case come from the need to integrate, with respect to y', odd powers of $x' \equiv (1 - y'^2)^{1/2}$, from $y' = 0$ to $y' = y = (1 - x^2)^{1/2}$, and that these integrals then always lead to the inclusion of the angle variable, $\theta = \arccos(x)$, possibly multiplied by even powers of x. The singular behavior of the derivatives of these terms at $x = 1$ is partially canceled by the derivatives of the other terms that all involve a factor of $y = (1 - x^2)^{1/2}$ with the same kind of 'square-root' characteristic at this end point, so that the resulting g_p, especially for larger values of the index p, can remain quite smooth. If we were to calculate them, the expressions for g_p in the 4D case would also exhibit the same characteristics, since θ appears in these formulas too.

In the 3D case, the integration of each term in h_p to get the final g_p in direct calculations is made much simpler by the presence of the factor, y', multiplying each power of x' in the integrand, which ensures that the result is then a polynomial. In the case where spectral transforms are used and the convolution theorems invoked, the relative simplicity of the analytic formulas for the odd-dimensioned cases seems to occur as a result of the spectral transforms being of the pure Fourier kind; for even-dimensional cases, the additional complications can be interpretted as coming about owing to the need to use the more complicated Fourier-Bessel transforms.

The moments of these self-convolved beta distributions can all be deduced from the known moments (discussed in section 3) of the basic beta distributions we began with, using the standard rules for the composition of moments under the application of a convolution. We can also then use these explicit formulas for the self-convolved distributions in combination with self-adjoint differential operators of the Helmholtz type (or their generalizations to several variables) to inspect the forms that the modified covariance components would take, even before we proceed to include these additional differential operators in the multigrid code. In this way, we can more easily and conveniently 'prototype' such generalizations, which should allow an acceleration of the development of this method. The inclusion of nontrivial stretching by means of an anisotropic aspect tensor is a trivial generalization of the isotropic assumptions we have made here.

We might also consider using these non-negative and self-adjoint kernel functions within the hybrid ensemble GSI as alternative localization functions, as they very closely approximate the Gaussian form while remaining very convenient to compute and possessing finite support.

6. Conclusion

The beta filter, based on the incomplete beta function (or beta distribution) with equal and integer parameters p, seems to be the simplest way to approximate the Gaussian function using a polynomial of degree $2p$ confined to a finite interval. Being an even function, it immediately generalizes to radially symmetric polynomials in higher dimensions confined to a support consisting of a disk or (spherical) ball. For the continuous domains, we have provided the formulas for these beta filter profiles, their moments, and their Fourier or Fourier-Bessel spectral transforms. We have also provided some analytic formulas for the profiles of the self-convolutions of these radial filters. These product filters, unlike the radial beta filters themselves, are positive semi-definite (having no negative eigenvalues) and so can serve as idealized covariances, easily computed for any pair of spatial locations. There are many applications where such functions become useful, including the debugging of the more complex (but more efficient) algorithms that use the line filters within an appropriate polyad algorithm, or using them as the localization functions within an ensemble-based variational assimilation scheme.

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References

Abramowitz, M., and I. A. Stegun 1972 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.*, Dover, New York. 1046 pp.

Cohn, S. E., A. da Silva, J. Guo, M. Sienkiewicz, and D. Lamich 1998 Assessing the effects of data selection with the DAO physical-space statistical analysis system. *Mon. Wea. Rev.,* **126,** 2913–2926.

Da Silva, A., J. Pfaendtner, J. Guo, M. Sienkiewicz, and S. Cohn 1995 Assessing the effects of data selection with DAO's physical-space statistical analysis system. *Second International Symposium on Assimilation of Observations in Meteorology and Oceanography*, Tokyo, 13–17 March 1995. WMO/TD., **651,** 273–278.

Dudewicz, E. J. 1976 *Introduction to Statistics and Probability.* Holt, Rinehart and Winston, 512 pp.

Gaspari, G., and S. E. Cohn 1998 Construction of correlation functions in two and three dimensions. Office Note Series on Global Modeling and Data Assimilation, DAO Office Note 96-03R1, DAO, GSFC, 53 pp.

Gaspari, G., and S. E. Cohn 1999 Construction of correlation functions in two and three dimensions. *Quart. J. Roy. Meteor. Soc.,* **125,** 723–757.

Hamming, R. W. 1989 *Digital Filters*, Third Edition: Prentice Hall, New Jersey. 284 pp.

Purser, R. J. 2020a A formulation of the Hexad algorithm using the geometry of the Fano projective plane. NOAA/NCEP Office Note 499.

Purser, R. J. 2020b A formulation of the Decad algorithm using the symmetries of the Galois field, $GF(16)$. NOAA/NCEP Office Note 500.

Purser, R. J., W.-S. Wu, D. F. Parrish, and N. M. Roberts 2003 Numerical aspects of the application of recursive filters to variational statistical analysis. Part II: Spatially inhomogeneous and anisotropic general covariances. *Mon. Wea. Rev.,* **131,** 1536–1548.