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AN ANALYSIS OF THE COMPUTATIONAL STABILITY CRITERIA FOR
EXPLICIT AND IMPLICIT INTEGRATION SCHEMES USING
A TWO LAYER MODEL IN PHILLIPS σ COORDINATE

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1. Introduction

The linear computational stability criterion for a multi-level primitive equation model is usually most strongly dependent upon the gravitational modes of oscillation. Recently, A. Robert and others have proposed the use of semi-implicit integration methods for reducing this dependence. In this note, we shall analyze the computational stability of three methods for integration of a simple fluid model which possesses only gravitational modes. The model will be written using the σ coordinate proposed by Phillips and will consist of only two layers. Such a simple partitioning of the vertical structure gives rise to an especially compact problem but is of non-trivial interest.

The three integration methods which are investigated are the explicit, the semi-implicit and a modified semi-implicit scheme. We arrive at a characteristic equation which is formally the same for all three methods. It is evaluated for an isothermal and standard atmosphere.

2. The Linear Equations

We omit the rotation, curvature and irregularity of the Earth's surface and consider the isentropic flow of an ideal, inviscid gas. Slab-symmetry is assumed and the fluid is treated as unbounded in the horizontal. The quasi-static equations are linearized about a barotropic basic state of no-motion. The equations governing the perturbations are then expressed in σ -coordinates:

$$u_t + \phi_x + \bar{\alpha} \sigma p_x^* = 0 \quad (1)$$

$$\phi_\sigma + \bar{\alpha} p^* + \alpha \overline{p^*} = 0 \quad (2)$$

$$p_t^* + \overline{p^*} u_x + \overline{p^*} \dot{\sigma}_\sigma = 0 \quad (3)$$

$$c_p T_t - \bar{\alpha} \sigma p_t^* + c_p \dot{\sigma} \Gamma = 0 \quad (4)$$

$$\overline{p^*} \alpha \sigma + p^* \bar{\alpha} \sigma = RT \quad (5)$$

with
$$\sigma = p/p^* = \overline{p}/\overline{p^*} \quad (6)$$

and
$$\Gamma \equiv \frac{\partial \overline{T}}{\partial \sigma} - \frac{\bar{\alpha} \overline{p^*}}{c_p} \quad (7)$$

The symbols are standard and the overbar represents the basic state value. The unbarred dependent variables are perturbations. The equations are

- (1) the equation of horizontal motion
- (2) the hydrostatic equation
- (3) the continuity equation

- (4) the thermodynamic equation
- (5) the ideal gas law
- (6) the definition of σ (independent variable)
- (7) the static stability.

3. The Finite Difference Equations

We are not concerned with the approximation of horizontal derivatives. The equations will be discretized by introduction of a superscript index to denote the time level at which a parameter is evaluated. The three integration methods differ essentially in how this is done. The time derivatives are in every case evaluated by means of centered differences (leapfrog method). We shall next enumerate the three schemes which are to be investigated:

Explicit:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} + \phi_x^n + \bar{\alpha} \sigma p_x^{*n} = 0 \quad \text{E.1}$$

$$\frac{p^{*n+1} - p^{*n-1}}{2\Delta t} + \bar{p}^* u_x^n + \bar{p}^* \dot{\sigma}_\sigma^n = 0 \quad \text{E.2}$$

$$c_p \frac{T^{n+1} - T^{n-1}}{2\Delta t} - \bar{\alpha} \sigma \frac{p^{*n+1} - p^{*n-1}}{2\Delta t} + c_p \dot{\sigma}_\sigma^n \Gamma = 0 \quad \text{E.3}$$

$$\phi_\sigma^n + \bar{\alpha} p^{*n} + \alpha^n \bar{p}^* = 0 \quad \text{E.4}$$

$$\bar{p}^* \alpha^n \sigma + p^{*n} \bar{\alpha} \sigma = RT^n \quad \text{E.5}$$

Semi-implicit:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} + \frac{1}{2} (\phi_x^{n+1} + \phi_x^{n-1}) + \frac{\bar{\alpha} \sigma}{2} (p_x^{*n+1} + p_x^{*n-1}) = 0 \quad \text{S.1}$$

$$\frac{p^{*n+1} - p^{*n-1}}{2\Delta t} + \frac{\bar{p}^*}{2} (u_x^{n+1} + u_x^{n-1}) + \frac{\bar{p}^*}{2} (\dot{\sigma}_\sigma^{n+1} + \dot{\sigma}_\sigma^{n-1}) = 0 \quad \text{S.2}$$

$$c_p \frac{T^{n+1} - T^{n-1}}{2\Delta t} - \bar{\alpha} \sigma \frac{p^{*n+1} - p^{*n-1}}{2\Delta t} + \frac{c_p}{2} \Gamma (\dot{\sigma}_\sigma^{n+1} + \dot{\sigma}_\sigma^{n-1}) = 0 \quad \text{S.3}$$

The equations 2 and 5 are approximated as in E.4 and E.5.

Modified Semi-implicit;

$$\frac{p^{*n+1} - p^{*n-1}}{2\Delta t} + \frac{\bar{p}^*}{2} \left\{ \frac{u^{n+1}}{x} + \frac{u^{n-1}}{x} \right\} + \bar{p}^* \dot{\sigma}_\sigma^n = 0 \quad \text{M.2}$$

$$c_p \frac{T^{n+1} - T^{n-1}}{2\Delta t} - \bar{\alpha} \sigma \frac{p^{*n+1} - p^{*n-1}}{2\Delta t} + c_p \Gamma \dot{\sigma}^n = 0 \quad \text{M.3}$$

Equations 1, 2 and 5 are approximated as in (S.1), (E.4) and (E.5).

We now assume solutions of the form

$$q^n = q \zeta^n e^{ikx} \quad (8)$$

and substitute this form for all of the dependent variables. From this point on, we note that the dependent variables are functions of σ alone. The stability of the numerical schemes will hinge on the existence of non-trivial solutions for which $|\zeta| < 1$.

4. The Characteristic Equation

Carrying out the indicated substitution, we find the following general form for the equations governing the vertical dependence of the solutions:

$$u + \beta \phi + \bar{\alpha} \sigma \beta p^* = 0 \quad (9)$$

$$p^* + \bar{p}^* \beta u + \bar{p}^* \mu \dot{\sigma}_\sigma = 0 \quad (10)$$

$$c_p T - \bar{\alpha} \sigma p^* + c_p \Gamma \mu \dot{\sigma} = 0 \quad (11)$$

$$\phi_\sigma + \bar{\alpha} p^* + \alpha \bar{p}^* = 0 \quad (12)$$

$$\bar{p}^* \alpha \sigma + p^* \bar{\alpha} \sigma = RT \quad (13)$$

In these equations, the definition of β and μ varies with the integration method.

Explicit:	$\beta = \frac{2ik\Delta t \zeta}{\zeta^2 - 1}$	$\mu = \frac{2\Delta t \zeta}{\zeta^2 - 1}$	
Semi-implicit:	$\beta = \frac{ik\Delta t (\zeta^2 + 1)}{(\zeta^2 - 1)}$	$\mu = \frac{\Delta t (\zeta^2 + 1)}{(\zeta^2 - 1)}$	(14)
Modified Semi-implicit:	$\beta = \frac{(ik\Delta t) (\zeta^2 + 1)}{(\zeta^2 - 1)}$	$\mu = \frac{2\Delta t \zeta}{\zeta^2 - 1}$	

We have assumed that $\zeta^2 \neq 1$, and note that β has dimensions $(L^{-1}T)$.

5. The Two Layer Model

We next consider a two-layer model for the vertical resolution of the equations. The sketch below is self-explanatory:

$\sigma = 0$	$\dot{\sigma} = 0$	$p = 0$	ϕ_2
$\sigma = .25$	u_2, α_2, T_2	$\bar{p}_2 = \frac{\bar{p}^*}{4}$	
$\sigma = .5$	$\dot{\sigma}$	$\bar{p} = \frac{\bar{p}^*}{2}$	ϕ_1
$\sigma = .75$	u_1, α_1, T_1	$\bar{p}_1 = \frac{3\bar{p}^*}{4}$	
$\sigma = 1$	$\dot{\sigma} = 0$	$\bar{p} = \bar{p}^*$	$\phi = 0$

$$\Gamma = \left[\frac{\bar{T}_1 - \bar{T}_2}{0.5} - \frac{\bar{\alpha}_1 + \bar{\alpha}_2}{2} \frac{\bar{p}^*}{c_p} \right] \quad (15)$$

The equations will be written out:

$$u_1 + \beta \frac{\phi_1}{2} + \beta \bar{\alpha}_1 \frac{3}{4} p^* = 0 \quad (16)$$

$$u_2 + \beta \frac{\phi_1}{2} + \beta \frac{\phi_2}{2} + \beta \bar{\alpha}_2 \frac{1}{4} p^* = 0 \quad (17)$$

$$p^* + \bar{p}^* \beta u_1 - \bar{p}^* 2 \mu \dot{\sigma} = 0 \quad (18)$$

$$p^* + \bar{p}^* \beta u_2 + \bar{p}^* 2 \mu \dot{\sigma} = 0 \quad (19)$$

$$c_p \bar{T}_1 - \bar{\alpha}_1 \frac{3}{4} p^* + \frac{c_p \Gamma}{2} \mu \dot{\sigma} = 0 \quad (20)$$

$$c_p \bar{T}_2 - \bar{\alpha}_2 \frac{1}{4} p^* + \frac{c_p \Gamma}{2} \mu \dot{\sigma} = 0 \quad (21)$$

$$-2 \phi_1 + \bar{\alpha}_1 p^* + \alpha_1 \bar{p}^* = 0 \quad (22)$$

$$2 \phi_1 - 2 \phi_2 + \bar{\alpha}_2 p^* + \alpha_2 \bar{p}^* = 0 \quad (23)$$

$$\bar{p}^* \alpha_2 + p^* \bar{\alpha}_2 = 4 R T_2 \quad (24)$$

$$\bar{p}^* \alpha_1 + p^* \bar{\alpha}_1 = \frac{4R}{3} T_1 \quad (25)$$

This set of simultaneous, homogeneous equations will possess non-trivial solutions only if the determinant of the coefficient matrix is zero. Before deriving the determinant, it is convenient to eliminate certain of the dependent variables. One may eliminate u_1 and u_2 between (16) and (18) and (17) and (19) to get

$$\begin{aligned} (1 - \bar{p}_2 \bar{\alpha}_2 \beta^2) p^* - \frac{\bar{p}^* \beta^2}{2} (\phi_1 + \phi_2) + 2 \mu \bar{p}^* \dot{\sigma} &= 0 \\ (1 - \bar{p}_1 \bar{\alpha}_1 \beta^2) p^* - \frac{\bar{p}^* \beta^2}{2} \phi_1 - 2 \mu \bar{p}^* \dot{\sigma} &= 0 \end{aligned} \quad (26)$$

The temperature and specific volume may be eliminated by combining eqs. (20-25) into the pair,

$$\begin{aligned} 2 \phi_2 - 2 \phi_1 - \bar{\alpha}_2 x p^* + 2 R \Gamma \mu \dot{\sigma} &= 0 \\ 2 \phi_1 - \bar{\alpha}_1 x p^* + \frac{2R\Gamma}{3} \mu \dot{\sigma} &= 0 \end{aligned} \quad (27)$$

We have introduced $x = R/c_p$, $\bar{p}_1 = 3 \bar{p}^*/4$ and $\bar{p}_2 = \bar{p}^*/4$.

When the determinant of the coefficients of (26) and (27) is worked out, one has upon setting it to zero the frequency equation:

$$\begin{aligned} (12 - R\Gamma\beta^2) [\bar{p}^* x (2 \bar{\alpha}_1 + \bar{\alpha}_2) \beta^2 - 4 (1 - \bar{\alpha}_2 \bar{p}_2 \beta^2)] \\ + (12 + 5 R\Gamma\beta^2) [\bar{p}^* x \bar{\alpha}_1 \beta^2 - 4 (1 - \bar{\alpha}_1 \bar{p}_1 \beta^2)] = 0 \end{aligned} \quad (28)$$

It is most noteworthy that the factor μ divides out the equation. It therefore follows that the semi-implicit and the modified semi-implicit methods have identical stability criteria in the case studied herein.

6. The Isothermal Basic State

In order to determine the stability criterion, we must provide a basic state. In this section, we investigate the result obtained for an *isothermal* basic state. Let the isothermal temperature be \bar{T} . Then

$$\begin{aligned} \bar{p}_1 \bar{\alpha}_1 &= R \bar{T}, & \bar{p}^* \bar{\alpha}_1 &= \frac{4R}{3} \bar{T} \\ \bar{p}_2 \bar{\alpha}_2 &= R \bar{T}, & \bar{p}^* \bar{\alpha}_2 &= 4 R \bar{T} \end{aligned} \quad (29)$$

$$\begin{aligned} \Gamma &\equiv \frac{\partial \bar{T}}{\partial \sigma} - \frac{\bar{\alpha} p^*}{c_p} = - \frac{\alpha_1 + \alpha_2}{2} \frac{\bar{p}^*}{c_p} = - \frac{R\bar{T}}{2 c_p} \left[\frac{4}{3} + 4 \right] \\ &= - x \bar{T} \left(\frac{8}{3} \right) \end{aligned}$$

These quantities may be introduced into equation (28). We shall define first

$$c_s^2 \equiv x R \bar{T}, \quad (30)$$

$$\begin{aligned} & \left(12 + \frac{8}{3} c_s^2 \beta^2 \right) \left[\left(\frac{8}{3} + \frac{12}{3} \right) c_s^2 \beta^2 - 4 \left(1 - \frac{c_s^2 \beta^2}{x} \right) \right] \\ & + \left(12 - \frac{40}{3} c_s^2 \beta^2 \right) \left[\left(\frac{4}{3} \right) c_s^2 \beta^2 - 4 \left(1 - \frac{c_s^2 \beta^2}{x} \right) \right] = 0 \end{aligned} \quad (31)$$

We now set $z = (c_s \beta)^2$, and use $x = .287$; the eq. 31 is a quadratic in z with roots

$$z_1 = 3.9 \quad (32)$$

$$\text{and } z_2 = 0.2$$

Using these roots, we may now investigate the stability of the explicit scheme. Referring to eq. 14 for the appropriate form for β and using z_j to denote one of the roots (32), we have

$$- \frac{4(k\Delta t)^2 \zeta^2}{(\zeta^2 - 1)^2} c_s^2 = z_j \quad (33)$$

If we set

$$c_j^2 \equiv \frac{c_s^2}{z_j} \quad (34)$$

eq. (33) may be written

$$(\zeta^2 - 1)^2 = - (2k\Delta t c_j)^2 \zeta^2 \quad (35)$$

or

$$(\zeta^2 - 1) = \pm i (2k\Delta t c_j) \zeta \quad (36)$$

Equation (36) represents two quadratic equations in ζ . One obtains

$$\zeta = \mp i (k\Delta t c_j) \pm \left[1 - (k\Delta t c_j)^2 \right]^{\frac{1}{2}} \quad (37)$$

Thus if

$$(k\Delta t c_j) < 1 \quad (38)$$

one finds $|\zeta| = 1$, and concludes that given (38) the explicit method will be neutral.

The two roots (32) give rise to two critical velocities whose values may be calculated from the definition of c_s . One has

$$c_1^2 = \frac{\bar{xRT}}{3.9} = \left(\frac{\bar{x}}{3.9\gamma}\right) \cdot \gamma\bar{RT}$$

$$c_1^2 = \frac{.287}{(3.9)(1.405)} \gamma\bar{RT} \approx 0.05 \gamma\bar{RT} \quad (39)$$

and

$$c_2^2 = \frac{\bar{xRT}}{0.2} = \frac{.287}{(0.2)(1.405)} \gamma\bar{RT}$$

$$c_2^2 \approx \frac{.287}{.281} \gamma\bar{RT} \approx \gamma\bar{RT} \quad (40)$$

Since $\sqrt{\gamma\bar{RT}}$ is well-known as the phase speed of the Lamb wave, we conclude that c_2 is the phase speed associated with the horizontal acoustic mode in the two layer model. The phase speed c_1 is calculated to be about 22% of the speed of the Lamb wave. It is therefore categorized as an internal gravity mode phase speed.

Finally, the implicit schemes' stability is investigated by referring to eq. (14) for the appropriate form of β . Using the same notation, we have

$$-(k\Delta t)^2 \left(\frac{\zeta^2 + 1}{\zeta^2 - 1} \right)^2 c_s^2 = z_i \quad (41)$$

This yields the quadratics

$$\zeta^2 = + \left(\frac{c_j k\Delta t + i}{-c_j k\Delta t + i} \right) \quad (42)$$

and

$$\zeta^2 = + \left(\frac{i - c_j k\Delta t}{i + c_j k\Delta t} \right) \quad (43)$$

Since if $|\zeta| \leq 1$ so is $|\zeta^2| \leq 1$, it follows that the implicit methods are unconditionally neutral.

7. The Standard Atmosphere

The following values of basic state parameters are assumed representative of the standard atmosphere:

$$\begin{aligned} \bar{p}^* &= 1000 \text{ mb} \\ \bar{p}_1 &= 750 \text{ mb} & \bar{T}_1 &= 268.5^\circ\text{K} \\ \bar{p}_2 &= 250 \text{ mb} & \bar{T}_2 &= 228.5^\circ\text{K} \end{aligned}$$

$$\Gamma = \left(2 - \frac{2x}{3} \right) \bar{T}_1 - (2 + 2x) \bar{T}_2$$

$$\Gamma = \left(\frac{2}{3x\gamma} \right) x \bar{T}_1 - \left(\frac{2(1+x)}{x} \right) x \bar{T}_2$$

$$\bar{p}^* \bar{\alpha}_1 = \frac{4}{3} R \bar{T}_1 \quad \bar{p}^* \bar{\alpha}_2 = 4 R \bar{T}_2$$

$$\bar{p}_1 \bar{\alpha}_1 = R \bar{T}_1 \quad \bar{p}_2 \bar{\alpha}_2 = R \bar{T}_2$$

We define,

$$c_1^2 = x R \bar{T}_1$$

$$c_2^2 = x R \bar{T}_2$$

Using these values in eq. (28), that equation may be written,

$$\begin{aligned} & \left(12 - \beta^2 \left[\frac{2}{3x\gamma} c_1^2 - \frac{2(1+x)}{x} c_2^2 \right] \right) \left(\beta^2 \left[\frac{8}{3} c_1^2 + 4 c_2^2 \right] - 4 \left[1 - \frac{\beta^2}{x} c_2^2 \right] \right) \\ & + \left(12 + 5\beta^2 \left[\frac{2}{3x\gamma} c_1^2 - \frac{2(1+x)}{x} c_2^2 \right] \right) \left(\beta^2 \frac{4}{3} c_1^2 - 4 \left[1 - \frac{\beta^2}{x} c_1^2 \right] \right) = 0 \end{aligned} \quad (44)$$

We now define \bar{c}^2 and ϵ by

$$\bar{c}^2 \equiv xR \frac{\bar{T}_1 + \bar{T}_2}{2} \quad (45)$$

$$\epsilon \equiv \frac{\bar{T}_1 - \bar{T}_2}{\bar{T}_1 + \bar{T}_2} \quad (46)$$

so that

$$c_1^2 = \bar{c}^2 (1+\epsilon) \quad (47)$$

$$c_2^2 = \bar{c}^2 (1-\epsilon)$$

We also define

$$z = \bar{c}^2 \beta^2 \quad (48)$$

The equation (44) can then be written

$$\begin{aligned} & \left[18 x\gamma - [(1+\epsilon) - 3\gamma(1+x)(1-\epsilon)]z \right] \left[[(5x+3) - \epsilon(x+3)]z - 3x \right] \\ & + \left[18 x\gamma + 5[(1+\epsilon) - 3\gamma(1+x)(1-\epsilon)]z \right] \left[(3+x)(1+\epsilon)z - 3x \right] \end{aligned} \quad (49)$$

We now evaluate the parameters in eq. (49), using the numerical values,

$$x = .287$$

$$\gamma = 1.405$$

$$\varepsilon = \frac{40}{497} \approx .08$$

$$x\gamma \approx 0.40$$

$$\begin{aligned} (7.2 + 4.z)(4.18z - .86) + (7.2 - 20.z)(3.64z - .86) \\ = 56.08z^2 - 70.06z + 12.38 = 0 \end{aligned} \quad (50)$$

The roots of (50) are approximately

$$z_1 = 1.02 \quad (51)$$

$$z_2 = .22$$

Proceeding as in section 6, we find for the explicit method the conditional criteria

$$(k\Delta t c_j) < 1$$

with

$$c_j^2 = \frac{\bar{c}^2}{z_j}$$

The implicit method once more is unconditionally neutral.

Finally, we calculate the critical phase speeds,

$$c_1^2 = \frac{x}{\gamma (1.02)} \left[\gamma R \left(\frac{\bar{T}_1 + \bar{T}_2}{2} \right) \right] \approx .2 \gamma R \left(\frac{\bar{T}_1 + \bar{T}_2}{2} \right)$$

and

$$c_2^2 = \frac{x}{\gamma (.22)} \left[\gamma R \left(\frac{\bar{T}_1 + \bar{T}_2}{2} \right) \right] \approx .92 \gamma R \left(\frac{\bar{T}_1 + \bar{T}_2}{2} \right).$$

If we set

$$c_s^2 = \gamma R \left(\frac{\bar{T}_1 + \bar{T}_2}{2} \right)$$

we have

$$c_1 \approx .45 c_s$$

$$c_2 \approx .96 c_s$$

Thus, in the standard atmosphere, the two critical phase speeds are approximately that of the Lamb wave at the average temperature and one-half that speed. This holds for our particular choice of vertical structure.