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# PHENOMENOLOGICAL STATISTICAL NON-GAUSSIAN MODEL OF SEA SURFACE WITH ANISOTROPIC SPECTRUM FOR WAVE-SCATTERING THEORY

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## CONTENTS

Abstract . . . . . 1

1. Introduction . . . . . 1

2. Which statistical characteristics of the surface completely describe the scattering cross section? . . . . . 2

3. Joint PDF for upwind and crosswind differences in elevation . . . . . 5

4. Matching with the PDF for slopes . . . . . 9

5. Finding the parameters $\lambda_\mu$, $\kappa_\mu$, and $P_\mu$ . . . . . 12

6. The CF for the arbitrary directed difference $\zeta(\boldsymbol{r}') - \zeta(\boldsymbol{r}'')$ . . . . . 13
   6.1 The CF for an arbitrary directed slope . . . . . 14

7. Multivariate PDF for differences $\zeta(\boldsymbol{r}') - \zeta(\boldsymbol{r}'')$ . . . . . 16

8. Scattering cross section from the absolutely reflecting interface in the Kirchhoff approximation . . . . . 17
   8.1 Geometric optics limit . . . . . 19
   8.2 General approach to geometric optics approximation . . . . . 20

9. Numerical results for the radar cross section for Cox-Munk PDF and 2-D anisotropic spectra . . . . . 21
   9.1 Determining the parameters of the PDF . . . . . 21
   9.2 Calculations of the radar cross sections in the Kirchhoff approximation . . . . . 23

10. Conclusions . . . . . 24
   10.1 Comparison with other methods of statistical description of sea surface . . . . . 24
   10.2 Summary . . . . . 29

Appendix A. Angular dependence of the variance of slope . . . . . 29

References . . . . . 31

# Phenomenological Statistical non-Gaussian Model of Sea Surface with Anisotropic Spectrum for Wave- Scattering Theory

**Valerian I. Tatarskii and Viatcheslav V. Tatarskii**

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**Abstract.** We develop a mathematical model of a statistically homogeneous rough surface that satisfies the following conditions: (1) It has the given two-dimensional (anisotropic) spectrum. (2) It has the given (non-Gaussian) joint probability distribution function (PDF) of two principal slopes at any fixed point. (3) It allows us to obtain in an explicit form the joint probability distribution (characteristic function) for an arbitrary number ($N$) of differences in elevation $\zeta(\mathbf{r}'_k) - \zeta(\mathbf{r}''_k)$ of the surface. (4) It allows us to find in an explicit analytical form any mean scattering cross sections appearing in any theory of wave scattering from rough surface. (5) In describing the non-Gaussian PDF we use, instead of the cumulant expansion, another approach: decomposition of the multivariate non-Gaussian PDF in the sum of multivariate Gaussian PDF having different positions and different variances and correlation coefficients. Using this model, we calculated in the Kirchhoff approximation the scattering from a perfectly reflecting rough surface, having non-Gaussian PDF of slopes and anisotropic spectrum (they correspond to experimental data for wind-driven waves on the water's surface). In the Kirchhoff approximation we obtained significant differences between Gaussian and non-Gaussian cases with the same spectrum, especially in the range of small grazing angles.

## 1. Introduction

Wave scattering from random surfaces depends on different parameters of surface depending on conditions. The important parameter of the scattering process is the Rayleigh parameter $Ra \equiv \nu_0 \sigma$, where $\nu_0 = k \sin\theta_0$, $\theta_0$ is the grazing angle of the incident wave, $k$ is a wave-number, $\nu_0$ is a vertical component of the incident wave-vector, and $\sigma$ is the variance of the surface elevations.

If $Ra \ll 1$, the Bragg scattering mechanism works and in this case the scattering cross section $\Sigma$ depends only on the spectrum of surface

$$\Sigma \sim \Phi(\mathbf{q} - \mathbf{q}_0). \tag{1.1}$$

Here, $\Phi$ is the Fourier transform of the correlation function $B_\zeta(\mathbf{r})$[1]of surface elevations $\zeta(\mathbf{r})$, i.e.,

$$B(\mathbf{r}) \equiv \langle \zeta(\mathbf{r}')\, \zeta(\mathbf{r} + \mathbf{r}') \rangle,$$

($\langle \cdots \rangle$ denotes the mean value) and

$$\begin{aligned} \Phi(\mathbf{q} - \mathbf{q}_0) &= \frac{1}{4\pi^2} \iint \exp\left[-i(\mathbf{q} - \mathbf{q}_0)\mathbf{r}\right] B_\zeta(\mathbf{r})\, d^2 r, \\ B_\zeta(\mathbf{r}) &= \iint \exp(i\mathbf{q}\mathbf{r})\, \Phi(\mathbf{q})\, d^2 q = \iint \cos(\mathbf{q}\mathbf{r})\, \Phi(\mathbf{q})\, d^2 q. \end{aligned} \tag{1.2}$$

In the case of $Ra \ll 1$, the scattering cross section does not depend on the probability distribution of elevations or surface slopes.

If $Ra \gg 1$, the Bragg scattering mechanism does not work and several more complicated scattering theories must be applied. If the curvature radii of the surface are much larger than the wavelength, we can describe the scattering process using the Kirchhoff approximation. In this case, the scattering cross section depends on the characteristic function of *differences* in elevation at two arbitrary points, $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\langle \exp\{i\alpha[\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)]\} \rangle, \quad \alpha = \nu + \nu_0. \tag{1.3}$$

Here, $\nu$ is the vertical wave-number of the scattered wave. In the case of very large $k$, only the linear term of expansion of $\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)$ in powers of $(\mathbf{r}_1 - \mathbf{r}_2)$ is important,

$$\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2) \approx (\mathbf{r}_1 - \mathbf{r}_2)\, \nabla\zeta(\mathbf{r}_2) + \cdots.$$

In this case the Kirchhoff approximation reduces to the geometric optics (GO) approximation and the scattering cross section depends only on the probability distribution function (PDF) of surface slopes $\nabla\zeta(\mathbf{r})$. This result has a simple physical meaning: the scattering cross section is proportional to the number of surface facets having the appropriate slope, i.e., satisfying the condition of specular reflection. Thus, the scattering cross section in the Kirchhoff case depends on a quite different property of the surface: the PDF of differences in elevation (the PDF of slopes, in the GO case) rather than on the surface spectrum.

## 2. Which statistical characteristics of the surface completely describe the scattering cross section?

It is clear that, in general, the scattering cross section may depend not only on both of these parameters (spectrum and $\langle \exp\{i\alpha[\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)]\}\rangle$), but on some more complicated parameters of the surface.

It was shown by Tatarskii [37] that any solution of the scattering problem can be presented as a functional, depending only on the functions of the type

$$\mathcal{L}(\alpha, \mathbf{r}) = \exp[i\alpha\zeta(\mathbf{r})] \tag{2.1}$$

with different $\alpha$ and $\mathbf{r}$. This means that the mean scattering cross section $\Sigma$ can be presented as a functional Taylor series of the form

$$\begin{aligned}
\Sigma = A_0 + \iint d^2r \int d\alpha A_1(\alpha, \mathbf{r}) \langle \mathcal{L}(\alpha, \mathbf{r})\rangle + \\
\iint d^2r' \int d\alpha' \iint d^2r'' \int d\alpha'' A_2(\alpha', \mathbf{r}'; \alpha'', \mathbf{r}'') \langle \mathcal{L}(\alpha', \mathbf{r}')\, \mathcal{L}(\alpha'', \mathbf{r}'')\rangle + \\
\iint d^2r' \int d\alpha' \iint d^2r'' \int d\alpha'' \int d\alpha''' \iint d^2\mathbf{r}''' \times \\
A_3(\alpha', \mathbf{r}'; \alpha'', \mathbf{r}'', \alpha''', \mathbf{r}''') \langle \mathcal{L}(\alpha', \mathbf{r}')\, \mathcal{L}(\alpha'', \mathbf{r}'')\, \mathcal{L}(\alpha''', \mathbf{r}''')\rangle + \cdots.
\end{aligned} \tag{2.2}$$

If we take into account only the four beginning terms of this expansion (up to $A_4$), we obtain as was shown in [37], an approximate formula that includes in particular cases the Bragg scattering, the Kirchhoff approximation, the small-slope approximation[39],[34], the tilt-invariant approximation[4], and the double Kirchhoff approximation[15],[40]. The mean values, appearing in (2.2) are characteristic functions (CF) of one-point, two-point, etc., joint PDF of surface elevations, i.e.,

$$\begin{aligned}
&\langle \mathcal{L}(\alpha, \mathbf{r})\rangle = \chi_1(\alpha, \mathbf{r}) = \langle \exp[i\alpha\zeta(\mathbf{r})]\rangle \\
&\langle \mathcal{L}(\alpha', \mathbf{r}')\, \mathcal{L}(\alpha'', \mathbf{r}'')\rangle = \chi_2(\alpha', \mathbf{r}'; \alpha'', \mathbf{r}'') = \langle \exp[i\alpha'\zeta(\mathbf{r}') + i\alpha''\zeta(\mathbf{r}'')]\rangle \\
&\cdots \\
&\langle \mathcal{L}(\alpha_1, \mathbf{r}_1) \cdots \mathcal{L}(\alpha_n, \mathbf{r}_n)\rangle = \chi_n(\alpha_1, \mathbf{r}_1; \ldots; \alpha_n, \mathbf{r}_n) = \langle \exp[i\alpha_1\zeta(\mathbf{r}_1) + \cdots + i\alpha_n\zeta(\mathbf{r}_n)]\rangle.
\end{aligned} \tag{2.3}$$

Thus, to calculate all these mean values it is enough to know the corresponding CF. The more orders of the scattering iterative term we consider, the more orders of CF are necessary.

The important property of scattering cross section $\Sigma$ is its invariance with respect to translations of the scattering surface as a whole. If we denote the scattering cross section corresponding to the surface $z = \zeta(\mathbf{r})$ as $\Sigma[\zeta(\cdot)]$, this property is expressed by the formula

$$\Sigma[\zeta(\cdot) + h] = \Sigma[\zeta(\cdot)]. \tag{2.4}$$

It follows from this formula, that $\Sigma$ really depends only on such combinations of $\zeta$ that do not change during translations of the surface. In other words, $\Sigma$ may depend only on the *differences* of the type $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$.

In terms of CF (2.3) the invariance property (2.4) takes the form

$$\left\langle \exp\left\{ i\sum_{j=1}^{n} \alpha_j \left[\zeta\left(\mathbf{r}_j\right)+h\right]\right\}\right\rangle = \left\langle \exp\left\{ i\sum_{j=1}^{n} \alpha_j \zeta\left(\mathbf{r}_j\right)+ih\sum \alpha_k\right\}\right\rangle = \left\langle \exp\left\{ i\sum_{j=1}^{n} \alpha_j \zeta\left(\mathbf{r}_j\right)\right\}\right\rangle. \tag{2.5}$$

It follows from this formula that the only CF that may enter in the expression for $\Sigma$, are those for which

$$\sum_{k=1}^{n} \alpha_k = 0. \tag{2.6}$$

The general formula for the characteristic function satisfying the property (2.6) is as follows (we use the special notation $\Theta$ for CF that satisfy the condition (2.6)):

$$\Theta\left(\boldsymbol{\alpha}\right) \equiv \left\langle \exp\left[i\alpha_1\left(\zeta_1-\zeta_1'\right)+\cdots+i\alpha_n\left(\zeta_n-\zeta_n'\right)\right]\right\rangle. \tag{2.7}$$

Here, $\zeta_k \equiv \zeta\left(\mathbf{r}_k\right)$, $\zeta_k' \equiv \zeta\left(\mathbf{r}_k'\right)$, and $\alpha_k' = -\alpha_k$. In fact, (2.7) is the standard CF for *differences* $\left(\zeta_k-\zeta_k'\right)$. Note that some of $\zeta_k$ may coincide with $\zeta_l$ or with $\zeta_l'$; for instance, $\zeta_2=\zeta_1'$. Because of this the total number of different points $\mathbf{r}_k$, $\mathbf{r}_j'$ in formula (2.7) may be either even or odd.

It follows from this analysis that the only statistical characteristics that are necessary to describe the scattering cross section are joint PDF or joint CF for *differences* in elevation at several points of the random surface.[2]Because of this the factor $A_1\left(\alpha,\mathbf{r}\right)$ in (2.2) must be zero.

If we consider the formulae for different terms of expansion of $\Sigma$, obtained in [37] ((1.3) corresponds to the first term of this expansion), we can ascertain that all of them have the form (2.7).

Usually, in all theoretical (analytical and numerical) studies of the rough-surface scattering problem the Gaussian PDF assumption is used. But many scattering surfaces have a PDF that differs from this basic law. As an example, we present in Figure 2.1 the PDF of water surface elevations, obtained in [12] for wind-driven surface waves corresponding to frictional velocity 1.24 $m{\cdot}s^{-1}$. The significant deviation from the Gaussian PDF is evident. In the Longuet-Higgins paper [19] a statistical theory of gravity waves was developed in which it was shown that because of the nonlinearity of equations deviations from Gaussian PDF must appear. The method of cumulants was used in this paper to describe these deviations.

The question arises: How significant is this deviation for wave scattering from the sea surface? This problem was discussed in [25],[26], and [35].

To approach on answer to this question we developed in [36] a statistical model of the surface that possesses the following properties: (1) It has the given PDF of elevations in any fixed point of the surface. (2) It has the given anisotropic spectrum. (3) It is possible to find an explicit analytical formulae for any characteristic function of the type (2.3) for any $n$.

As was shown in [37], using this information we are able to find, in an analytical form, the mean values entering into scattering theories.

The problem of how to construct a statistical model that satisfies conditions 1to3 has an infinite number of solutions because there are no restrictions on the highest (two-point, three-point, etc.) PDF. The model developed in [36] is only one of many possibilities. Some related problems were considered earlier in [2], [27], [28], and [24].

It is clear from the preceding analysis that statistics on *differences* in elevation are more important for the scattering theory than statistics on elevations. On the other hand, it is clear that if the spectrum of surface is anisotropic, that is, if it depends on the angle between the wind direction and the wave-vector direction, we can expect that the PDF of differences also may be anisotropic. The experimental data for the PDF of slopes [5][6] support this conclusion. Because of this, we try in this paper to construct a statistical model of the surface that satisfies the following conditions: (a) It has the given anisotropic spectrum (or correlation or structure functions). (b) It has the given joint PDF of slopes in two principal directions (for the wind-driven surface waves these directions are upwind and crosswind.) (c) It is possible to find the explicit analytical formulae for any characteristic function for differences of the type (2.7) for any $n$.

In most publications devoted to non-Gaussian surfaces, the cumulant expansions (Edgeworth or Gram-Charlier series) were used. This method is standard for describing non-Gaussian distributions. However, it is known that PDF with the final number of cumulants (except Gaussian PDF) does not exist (see, e.g.,[18]). Because of this, the

truncation of Edgeworth or Gram-Charlier expansions *necessarily* leads to the appearance of negative probabilities (see examples in the paper [12]). These negative probabilities may affect the results of calculations of scattering cross sections and violate the energy conservation law.

In describing the non-Gaussian multivariate PDF, we will use decomposition of an arbitrary PDF in the sum of an auxiliary multivariate Gaussian PDF (for a single random variable this method is sometimes used in the Monte-Carlo simulation of non-Gaussian PDF). This approach replaces the conventional cumulant expansion. The method suggested in this paper does not lead to negative probabilities (see Figure 2.1) and, because of its simplicity, it successfully replaces the cumulant expansion. The solution obtained is simple enough to (1) perform all necessary calculations, and (2) obtain the analytical formulae for joint CF of differences in elevation. It allows to obtain the scattering cross section in the Kirchhoff and other approximations for non-Gaussian surfaces with the realistic anisotropic spectrum and the PDF of the principal slopes.

The results obtained show that deviations from the Gaussian PDF may be important and may cause differences in the scattering cross section in several times.

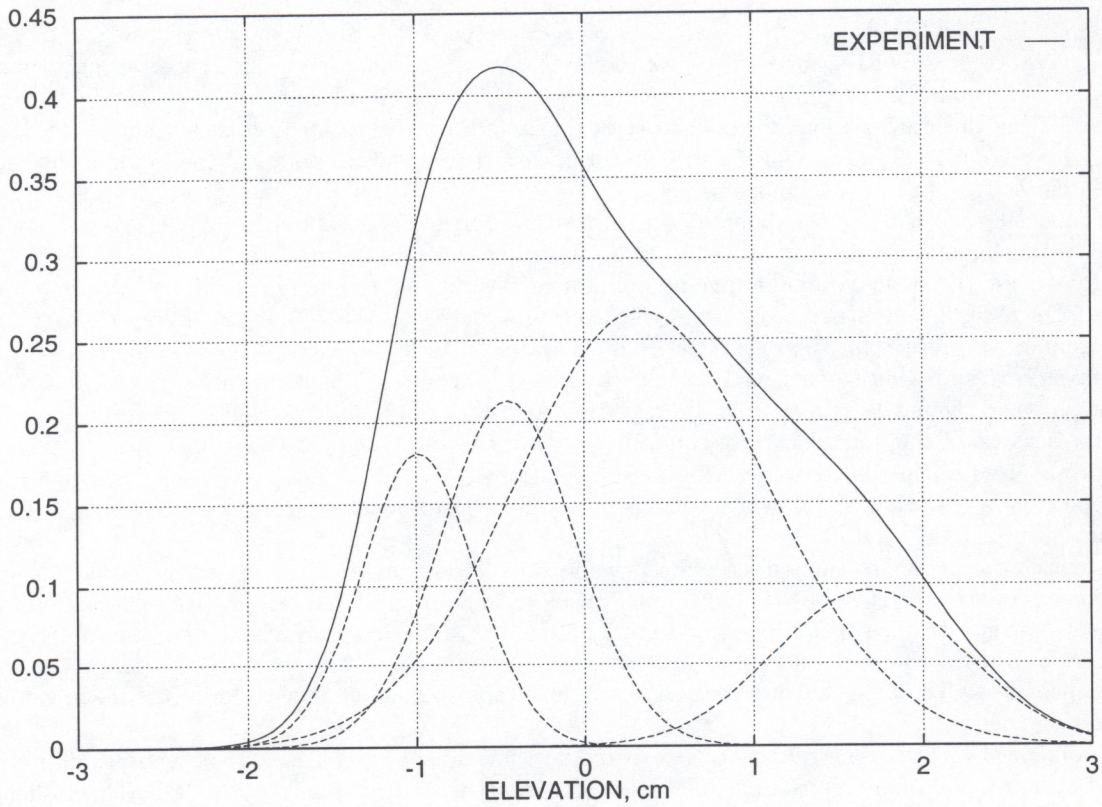


Figure 2.1: Non-Gaussian PDF of elevations for the frictional velocity 1.24m/s taken from the paper [12] and its approximation by the sum of four Gaussian terms. In contrast to the cumulant expansion, no negative probabilities appear in this representation.

To aid the reader's understanding of this paper, we will first describe its logical structure. We then consider the following problems, each of which can be solved once the preceding problem has been solved:

**A. Joint PDF of differences in elevation taken in two principal directions**

We start with an examination of joint non-Gaussian PDF $W(\Delta_1, \Delta_2)$ for two differences in elevation:

$$\Delta_1(l_1) = \zeta(\mathbf{r}+l_1\mathbf{m}_1) - \zeta(\mathbf{r}), \quad \Delta_2(l_2) = \zeta(\mathbf{r}+l_2\mathbf{m}_2) - \zeta(\mathbf{r}),$$

taken in two principal directions: upwind, described by the unit vector $\mathbf{m}_1$, and crosswind, described by the unit

vector $\mathbf{m}_2$. The function $W(\Delta_1, \Delta_2)$ is approximated by the sum of the two-dimensional Gaussian PDF (3.2) having different positions and different matrixes of second moments. The parameters of these auxiliary Gaussian PDF are expressed in terms of the (anisotropic) correlation (or structure) function of the surface, which is assumed to be known from experimental data. The formula (3.39) for characteristic function (CF) of this non-Gaussian PDF thus obtained contains several uncertain numerical parameters: $P_\mu$, $\lambda_\mu$, $\kappa_{1\mu}$, and $\kappa_{2\mu}$ (the index $\mu$ denotes the different Gaussian terms of decomposition).

**B. Joint PDF of two principal slopes**

From consideration of the particular case $l_1, l_2 \to 0$, it is possible to obtain the joint PDF (4.26) of two principal slopes, $\gamma_1$ and $\gamma_2$, in terms of the same uncertain parameters: $P_\mu$, $\lambda_\mu$, $\kappa_{1\mu}$, and $\kappa_{2\mu}$. All these parameters can be determined from a comparison of the approximate formula (4.26) with the experimental data. After finding these parameters we can substitute them in the formula for joint PDF or joint CF of two differences $\Delta_1$ and $\Delta_2$. The result is a formula that agrees with the correlation properties of the surface and with the joint PDF of two principal slopes.

**C. PDF of a single, arbitrary directed, difference in elevations**

The next step is to find the PDF and CF of a single, arbitrary directed, difference in elevation,

$$\Delta(\mathbf{r}', \mathbf{r}'') = \zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$$

that can be easily expressed in terms of the joint CF of two differences, taken in the principal directions. This CF is a superposition of the corresponding Gaussian CF with the same parameters, $P_\mu$, $\lambda_\mu$, $\kappa_{1\mu}$, and $\kappa_{2\mu}$. A particular case of this CF for $l_1, l_2 \to 0$ provides a CF for a slope in an arbitrary direction.

**D. Joint PDF of an arbitrary number of arbitrarily directed differences in elevation**

The joint PDF $\mathcal{P}$ for $M$ arbitrarily directed differences also can be presented in the form of a superposition of $M$-dimensional Gaussian PDF. The coefficients of this superposition do not depend on $M$. Therefore, we can use the same parameters $P_\mu$, $\lambda_\mu$, $\kappa_{1\mu}$, and $\kappa_{2\mu}$ to construct $\mathcal{P}$. The function $\mathcal{P}$, or the corresponding CF (7.3) obtained in this way, describes the random surface with the given anisotropic spectrum (or given structure function) and the given joint PDF of two slopes (derivatives of elevation in two principal directions).

**E. Scattering cross sections**

Scattering cross sections from the absolutely reflecting interface can be obtained for different approximations in terms of the obtained CF. Numerical evaluation of the corresponding integrals in the Kirchhoff approximation shows that deviation from the Gaussian distribution can be very important and can cause significant difference in scattering cross sections, especially in the range of small grazing angles.

**F. Universal angular dependence of the variance of slope**

We show (Appendix A) that only from the symmetry of the spectrum of surface with respect to wind direction it follows the universal dependence ($A$.20) of slope variance $\langle \gamma^2(\psi) \rangle$ on the angle $\psi$ with wind direction.

## 3. Joint PDF for upwind and crosswind differences in elevation

Let us consider the joint PDF for two finite differences, $\Delta_1$ and $\Delta_2$, taken in upwind and crosswind directions:

$$\Delta_1(l_1) \equiv \zeta(\mathbf{r}+l_1\mathbf{m}_1) - \zeta(\mathbf{r}), \quad \Delta_2(l_2) \equiv \zeta(\mathbf{r}+l_2\mathbf{m}_2) - \zeta(\mathbf{r}), \tag{3.1}$$

for arbitrary values of $l_1$ and $l_2$. We assume that the joint PDF for $\Delta_1$ and $\Delta_2$, the function $W(\Delta_1, \Delta_2)$, can be approximated by the sum of two-dimensional Gaussian surfaces of the general type, $W_\mu(\Delta_1, \Delta_2)$:

$$\begin{aligned} W_\mu(\Delta_1, \Delta_2) &= \frac{1}{2\pi\sigma_{1\mu}\sigma_{2\mu}\sqrt{1-\rho_\mu^2}} \times \\ &\exp\left\{ -\frac{\left(\Delta_1 - \overline{\Delta}_{1,\mu}\right)^2}{2\sigma_{1,\mu}^2\left(1-\rho_\mu^2\right)} - \frac{\left(\Delta_2 - \overline{\Delta}_{2,\mu}\right)^2}{2\sigma_{2,\mu}^2\left(1-\rho_\mu^2\right)} + \frac{2\rho_\mu\left(\Delta_1 - \overline{\Delta}_{1,\mu}\right)\left(\Delta_2 - \overline{\Delta}_{2,\mu}\right)}{2\sigma_{1,\mu}\sigma_{2,\mu}\left(1-\rho_\mu^2\right)} \right\}. \end{aligned} \tag{3.2}$$

Each Gaussian surface over the plane $(\Delta_1, \Delta_2)$ described by the function (3.2) is centered in the point

$$\left(\overline{\Delta}_{1,\mu}, \overline{\Delta}_{2,\mu}\right)$$

and is characterized by the parameters $\sigma_{1,\mu}$, $\sigma_{2,\mu}$, and $\rho_\mu$. These parameters are expressed in terms of the mean values calculated with the PDF $W_\mu(\Delta_1, \Delta_2)$ (we call them conditional mean values):

$$\overline{\Delta}_{1,\mu} \equiv \iint W_\mu(\Delta_1, \Delta_2)\, \Delta_1 d\Delta_1 d\Delta_2 = \langle \Delta_1 | \mu \rangle, \tag{3.3}$$

$$\overline{\Delta}_{2,\mu} = \iint W_\mu(\Delta_1, \Delta_2)\, \Delta_2 d\Delta_1 d\Delta_2 = \langle \Delta_2 | \mu \rangle, \tag{3.4}$$

$$\sigma_{1,\mu}^2 = \iint W_\mu(\Delta_1, \Delta_2)\left(\Delta_1 - \overline{\Delta}_{1,\mu}\right)^2 d\Delta_1 d\Delta_2 = \iint W_\mu(\Delta_1, \Delta_2)\, \Delta_1^2 d\Delta_1 d\Delta_2 - \overline{\Delta}_{1,\mu}^2 = \langle \Delta_1^2 | \mu \rangle - \langle \Delta_1 | \mu \rangle^2, \tag{3.5}$$

$$\sigma_{2,\mu}^2 = \iint W_\mu(\Delta_1, \Delta_2)\left(\Delta_2 - \overline{\Delta}_{2,\mu}\right)^2 d\Delta_1 d\Delta_2 = \iint W_\mu(\Delta_1, \Delta_2)\, \Delta_2^2 d\Delta_1 d\Delta_2 - \overline{\Delta}_{2,\mu}^2 = \langle \Delta_2^2 | \mu \rangle - \langle \Delta_2 | \mu \rangle^2, \tag{3.6}$$

$$\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu = \iint W_\mu(\Delta_1, \Delta_2)\left(\Delta_1 - \overline{\Delta}_{1,\mu}\right)\left(\Delta_2 - \overline{\Delta}_{2,\mu}\right) d\Delta_1 d\Delta_2 = \langle \Delta_1\Delta_2 | \mu \rangle - \langle \Delta_1 | \mu \rangle \langle \Delta_2 | \mu \rangle. \tag{3.7}$$

We seek an approximation of the joint PDF of two differences $\Delta_1$ and $\Delta_2$, the function $W(\Delta_1, \Delta_2)$, in the form[3]

$$W(\Delta_1, \Delta_2) \approx \sum_\mu P_\mu W_\mu(\Delta_1, \Delta_2), \tag{3.8}$$

where $P_\mu > 0$. Because each function $W_\mu$ is normalized, the normalization condition for $W_\mu$ takes the form

$$\sum_\mu P_\mu = 1. \tag{3.9}$$

Thus, we can consider the positive numbers $P_\mu$ as some probabilities and the functions $W_\mu(\Delta_1, \Delta_2)$ as conditional PDF for fixed $\mu$.

Let us consider the joint CF for $\Delta_1$ and $\Delta_2$:

$$\Theta_\Delta(\alpha_1, l_1; \alpha_2, l_2) \equiv \langle \exp i\left[\alpha_1 \Delta_1(l_1) + i\alpha_2 \Delta_2(l_2)\right]\rangle. \tag{3.10}$$

If we use the approximation (3.8) for $W(\Delta_1, \Delta_2)$, we obtain the corresponding approximation for CF:

$$\begin{aligned} &\Theta_\Delta(\alpha_1, l_1; \alpha_2, l_2) \approx \sum_\mu P_\mu \Theta_{\Delta,\mu}(\alpha_1, l_1; \alpha_2, l_2) = \\ &\sum_\mu P_\mu \exp\left[i\alpha_1\overline{\Delta}_{1,\mu} + i\alpha_2\overline{\Delta}_{2,\mu} - \frac{1}{2}\left(\alpha_1^2\sigma_{1,\mu}^2 + 2\alpha_1\alpha_2\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu + \alpha_2^2\sigma_{2,\mu}^2\right)\right]. \end{aligned} \tag{3.11}$$

Here,

$$\Theta_{\Delta,\mu}(\alpha_1, l_1; \alpha_2, l_2) = \exp\left[i\alpha_1\overline{\Delta}_{1,\mu} + i\alpha_2\overline{\Delta}_{2,\mu} - \frac{1}{2}\left(\alpha_1^2\sigma_{1,\mu}^2 + 2\alpha_1\alpha_2\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu + \alpha_2^2\sigma_{2,\mu}^2\right)\right] \tag{3.12}$$

is a CF corresponding to the conditional Gaussian PDF (3.2).

To determine the unknown coefficients and functions, entering in (3.11) and (3.12), we compare the expansions of $\Theta(\alpha_1, l_1; \alpha_2, l_2)$ that follow from the definition (3.10) and from the representation (3.11). The expansion of (3.10) in powers of $\alpha_1$ and $\alpha_2$ has the form:

$$\Theta(\alpha_1, l_1; \alpha_2, l_2) = 1 + i\alpha_1\langle\Delta_1\rangle + i\alpha_2\langle\Delta_2\rangle - \frac{1}{2}\left[\alpha_1^2\langle\Delta_1^2\rangle + 2\alpha_1\alpha_2\langle\Delta_1\Delta_2\rangle + \alpha_2^2\langle\Delta_2^2\rangle\right] + \cdots. \tag{3.13}$$

Because $\langle\zeta\rangle = 0$, we obtain $\langle\Delta_1\rangle = \langle\Delta_2\rangle = 0$ and

$$\Theta(\alpha_1, l_1; \alpha_2, l_2) = 1 - \frac{1}{2}\left[\alpha_1^2\langle\Delta_1^2\rangle + 2\alpha_1\alpha_2\langle\Delta_1\Delta_2\rangle + \alpha_2^2\langle\Delta_2^2\rangle\right] + \cdots. \tag{3.14}$$

The expansion of (3.11) in powers of $\alpha_1$, $\alpha_2$ has the form

$$\begin{aligned}&\Theta_\Delta(\alpha_1, l_1; \alpha_2, l_2) \approx \\ &\sum_\mu P_\mu\Big\{1 + i\alpha_1\overline{\Delta}_{1,\mu} + i\alpha_2\overline{\Delta}_{2,\mu} - \frac{1}{2}\left(\alpha_1^2\sigma_{1,\mu}^2 + 2\alpha_1\alpha_2\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu + \alpha_2^2\sigma_{2,\mu}^2\right) \\ &-\frac{1}{2}\left(\alpha_1^2\overline{\Delta}_{1,\mu}^2 + 2\alpha_1\alpha_2\overline{\Delta}_{1,\mu}\overline{\Delta}_{2,\mu} + \alpha_2^2\overline{\Delta}_{2,\mu}^2\right) + \cdots\Big\}.\end{aligned} \tag{3.15}$$

From comparison of the zero-order in $\alpha$-s terms of expansions (3.14) and (3.15) we obtain the same relation (3.9). From comparison of the linear in $\alpha_1$ and $\alpha_2$ terms we obtain

$$\sum_\mu P_\mu\overline{\Delta}_{1,\mu} = 0, \ \sum_\mu P_\mu\overline{\Delta}_{2,\mu} = 0 \tag{3.16}$$

From comparison of the coefficients in front of $\alpha_1^2$, $\alpha_2^2$, and $\alpha_1\alpha_2$ we obtain:

$$\sum_\mu P_\mu\left[\sigma_{1,\mu}^2 + \overline{\Delta}_{1,\mu}^2\right] = \langle\Delta_1^2\rangle, \tag{3.17}$$

$$\sum_\mu P_\mu\left[\sigma_{2,\mu}^2 + \overline{\Delta}_{2,\mu}^2\right] = \langle\Delta_2^2\rangle, \tag{3.18}$$

$$\sum_\mu P_\mu\left[\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu + \overline{\Delta}_{1,\mu}\overline{\Delta}_{2,\mu}\right] = \langle\Delta_1\Delta_2\rangle. \tag{3.19}$$

Note that all of the values $\Delta_1$, $\Delta_2$, $\overline{\Delta}_{1,\mu}$, $\overline{\Delta}_{2,\mu}$, $\sigma_{1,\mu}^2$, $\sigma_{2,\mu}^2$, and $\rho_\mu$ depend on $l_1$ or $l_2$. If we substitute $\overline{\Delta}_{1,\mu}$, $\overline{\Delta}_{2,\mu}$, $\sigma_{1,\mu}^2$, $\sigma_{2,\mu}^2$, and $\rho_\mu$ in (3.17) to (3.19) in terms of conditional mean values (3.3) to (3.7), we obtain:

$$\sum_\mu P_\mu\left\langle\Delta_1^2(l_1)\middle|\,\mu\right\rangle = \left\langle\Delta_1^2(l_1)\right\rangle, \tag{3.20}$$

$$\sum_\mu P_\mu\left\langle\Delta_2^2(l_2)\middle|\,\mu\right\rangle = \left\langle\Delta_2^2(l_2)\right\rangle, \tag{3.21}$$

$$\sum_\mu P_\mu\left\langle\Delta_1(l_1)\Delta_2(l_2)\middle|\,\mu\right\rangle = \left\langle\Delta_1(l_1)\Delta_2(l_2)\right\rangle. \tag{3.22}$$

We can satisfy all of the equations (3.20) to (3.22) if we set

$$\left\langle\Delta_1^2(l_1)\middle|\,\mu\right\rangle = \lambda_\mu\left\langle\Delta_1^2(l_1)\right\rangle, \tag{3.23}$$

$$\left\langle\Delta_2^2(l_2)\middle|\,\mu\right\rangle = \lambda_\mu\left\langle\Delta_2^2(l_2)\right\rangle, \tag{3.24}$$

$$\left\langle\Delta_1(l_1)\Delta_2(l_2)\middle|\,\mu\right\rangle = \lambda_\mu\left\langle\Delta_1(l_1)\Delta_2(l_2)\right\rangle. \tag{3.25}$$

In other words, all of the conditional second moments of differences are proportional to corresponding known unconditional second moments with the same coefficient $\lambda_\mu$. In this case, all of the equations (3.20) to (3.22), formulated in terms of functions of $l_1, l_2$, reduce to a single equation with respect to the numbers $\lambda_\mu$:

$$\sum_\mu P_\mu\lambda_\mu = 1. \tag{3.26}$$

Note that in terms of the structure function of the surface,

$$D(\mathbf{r}' - \mathbf{r}'') = \left\langle\left[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')\right]^2\right\rangle, \tag{3.27}$$

the functions (3.23) and (3.24) take the form

$$\left\langle \Delta_1^2 (l_1) \middle| \mu \right\rangle = \lambda_\mu D (l_1 \mathbf{m}_1), \tag{3.28}$$

$$\left\langle \Delta_2^2 (l_2) \middle| \mu \right\rangle = \lambda_\mu D (l_2 \mathbf{m}_2). \tag{3.29}$$

The expression

$$\langle \Delta_1 (l_1) \Delta_2 (l_2) \rangle = \langle [\zeta (\mathbf{r} + l_1 \mathbf{m}_1) - \zeta (\mathbf{r})] [\zeta (\mathbf{r} + l_2 \mathbf{m}_2) - \zeta (\mathbf{r})] \rangle$$

can be transformed using the Yaglom identity [41]:

$$(A - B)(C - D) = \frac{1}{2} \left[ (A - D)^2 + (B - C)^2 - (A - C)^2 - (B - D)^2 \right] \tag{3.30}$$

as follows:

$$\langle \Delta_1 (l_1) \Delta_2 (l_2) \rangle = \frac{1}{2} [D (l_1 \mathbf{m}_1) + D (l_2 \mathbf{m}_2) - D (l_1 \mathbf{m}_1 - l_2 \mathbf{m}_2)]. \tag{3.31}$$

Thus, we can rewrite (3.25) in the form

$$\langle \Delta_1 (l_1) \Delta_2 (l_2) | \mu \rangle = \frac{\lambda_\mu}{2} [D (l_1 \mathbf{m}_1) + D (l_2 \mathbf{m}_2) - D (l_1 \mathbf{m}_1 - l_2 \mathbf{m}_2)]. \tag{3.32}$$

Let us consider now the equations (3.16). The derivatives of $\overline{\Delta}_{1,\mu} (l_1)$ and $\overline{\Delta}_{2,\mu}$ with respect to $l_1$ or $l_2$ in the points $l_1 = 0$ or $l_2 = 0$ are equal to the conditional mean values of slopes. We will find a bit later that these values must be non-zero. Because of this we cannot set these functions $\overline{\Delta}_{1,\mu} (l_1)$ and $\overline{\Delta}_{2,\mu} (l_2)$ to be zero, in spite of such choice is consistent with (3.16).

The typical value of the difference

$$\overline{\Delta}_{1,\mu} = \langle \zeta (\mathbf{r} + l_1 \mathbf{m}_1) - \zeta (\mathbf{r}) | \mu \rangle$$

is on the order of $\sqrt{\langle \Delta_1^2 (l_1) \rangle}$. On the other hand, $\overline{\Delta}_{1,\mu}$ as a function of $l_1$ must be an odd function, that is,

$$\overline{\Delta}_{1,\mu} (-l_1) = -\overline{\Delta}_{1,\mu} (l_1). \tag{3.33}$$

Because of this we can seek $\overline{\Delta}_{1,\mu}$ and $\overline{\Delta}_{2,\mu}$ in the form[4]

$$\begin{aligned} \overline{\Delta}_{1,\mu} (l_1) &= \kappa_{1,\mu} \frac{l_1}{|l_1|} \sqrt{\langle \Delta_1^2 (l_1) \rangle} = \kappa_{1,\mu} l_1 \sqrt{\frac{\langle \Delta_1^2 (l_1) \rangle}{l_1^2}}, \\ \overline{\Delta}_{2,\mu} (l_2) &= \kappa_{2,\mu} \frac{l_2}{|l_2|} \sqrt{\langle \Delta_2^2 (l_2) \rangle} = \kappa_{2,\mu} l_2 \sqrt{\frac{\langle \Delta_2^2 (l_2) \rangle}{l_2^2}}. \end{aligned} \tag{3.34}$$

Note that if $l_1 \to 0$, the function $\left\langle \Delta_1^2 (l_1) \right\rangle$ is proportional to $l_1^2$ and

$$\sqrt{\langle \Delta_1^2 (l_1) \rangle / l_1^2} \sim \text{Constant}.$$

Thus, the function $\overline{\Delta}_{1,\mu} (l_1)$ is proportional to $l_1$ for small $l_1$, i.e., it has a continuous derivative in the point $l_1 = 0$.

After substituting (3.34) in the equations (3.16) they reduce to the equations with respect to the numbers $\kappa_{1,\mu}$ and $\kappa_{2,\mu}$:

$$\sum_\mu P_\mu \kappa_{1,\mu} = 0, \ \sum_\mu P_\mu \kappa_{2,\mu} = 0. \tag{3.35}$$

We expressed all the functions

$$\overline{\Delta}_{1,\mu}, \overline{\Delta}_{2,\mu}, \left\langle \Delta_1^2 (l_1) \middle| \mu \right\rangle, \left\langle \Delta_2^2 (l_2) \middle| \mu \right\rangle, \text{ and } \langle \Delta_1 (l_1) \Delta_2 (l_2) | \mu \rangle$$

in terms of the known structure function $D (\mathbf{r}' - \mathbf{r}'')$ of the surface and the unknown numbers $\lambda_\mu$, $\kappa_{1,\mu}$, and $\kappa_{2,\mu}$. If we substitute the formulae obtained in expressions (3.5) to (3.7), we obtain

$$\sigma_{1,\mu}^2 = \left\langle \Delta_1^2 \middle| \mu \right\rangle - \langle \Delta_1 | \mu \rangle^2 = \left( \lambda_\mu - \kappa_{1,\mu}^2 \right) D (l_1 \mathbf{m}_1), \tag{3.36}$$

$$\sigma_{2,\mu}^2 = \langle \Delta_2^2 | \mu \rangle - \langle \Delta_2 | \mu \rangle^2 = \left(\lambda_\mu - \kappa_{2,\mu}^2\right) D\left(l_2 \mathbf{m}_2\right), \tag{3.37}$$

$$\begin{aligned} &\sigma_{1,\mu}\sigma_{2,\mu}\rho_\mu = \langle \Delta_1\Delta_2 | \mu \rangle - \langle \Delta_1 | \mu \rangle \langle \Delta_2 | \mu \rangle = \\ &\frac{\lambda_\mu}{2}\left[D\left(l_1\mathbf{m}_1\right) + D\left(l_2\mathbf{m}_2\right) - D\left(l_1\mathbf{m}_1 - l_2\mathbf{m}_2\right)\right] - \kappa_{1,\mu}\kappa_{2,\mu} l_1 l_2 \sqrt{\frac{D\left(l_1\mathbf{m}_1\right) D\left(l_2\mathbf{m}_2\right)}{l_1^2 l_2^2}}. \end{aligned} \tag{3.38}$$

For the joint CF of the differences in the elevation of surface, substituting (3.36) to (3.38) in (3.11) we obtain:

$$\begin{aligned} &\Theta_\Delta\left(\alpha_1, l_1; \alpha_2, l_2\right) = \left\langle \exp\left\{i\alpha_1\Delta_1\left(l_1\right) + i\alpha_2\Delta_2\left(l_2\right)\right\}\right\rangle \approx \\ &\sum_\mu P_\mu \exp\left\{ i\left[\alpha_1\kappa_{1,\mu} l_1 \sqrt{\frac{D\left(\mathbf{m}_1 l_1\right)}{l_1^2}} + \alpha_2\kappa_{2,\mu} l_2 \sqrt{\frac{D\left(\mathbf{m}_2 l_2\right)}{l_2^2}}\right] - \right. \\ &-\frac{1}{2}\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\alpha_1^2 D\left(\mathbf{m}_1 l_1\right) - \frac{1}{2}\left(\lambda_\mu - \kappa_{2,\mu}^2\right)\alpha_2^2 D\left(\mathbf{m}_2 l_2\right) - \\ &-\alpha_1\alpha_2\left[\frac{\lambda_\mu}{2}\left[D\left(l_1\mathbf{m}_1\right) + D\left(l_2\mathbf{m}_2\right) - D\left(l_1\mathbf{m}_1 - l_2\mathbf{m}_2\right)\right] - \right. \\ &\left.\left. -\kappa_{1,\mu}\kappa_{2,\mu} l_1 l_2 \sqrt{\frac{D\left(l_1\mathbf{m}_1\right) D\left(l_2\mathbf{m}_2\right)}{l_1^2 l_2^2}}\right]\right\}. \end{aligned} \tag{3.39}$$

Formula (3.39) does not contain any unknown functions, but only unknown numerical parameters $P_\mu$, $\lambda_\mu$, $\kappa_{1,\mu}$, and $\kappa_{2,\mu}$. To find these parameters, we consider the particular case of (3.39) while $l_1, l_2 \to 0$. In this case we obtain from CF for differences the CF for *derivatives* of these differences, i.e., for the slopes of the surface.

## 4. Matching with the PDF for slopes

The slope of a surface in a point $\mathbf{r}$ taken in a direction determined by the unit vector $\mathbf{n}$ is given by the formula

$$\gamma\left(\mathbf{n}, \mathbf{r}\right) \equiv \mathbf{n}\nabla\zeta\left(\mathbf{r}\right). \tag{4.1}$$

We assume that the spectrum of surface $\Phi\left(\mathbf{q}\right)$ is symmetrical with respect to the wind direction determined by the unit vector $\mathbf{m}_1$. If we choose the $x$-axis along the vector $\mathbf{m}_1$, we obtain

$$\mathbf{m}_1 = \left(1, 0\right). \tag{4.2}$$

The vector $\mathbf{q}$ can be presented in the form

$$\mathbf{q} = \left(q\cos\varphi, q\sin\varphi\right), \tag{4.3}$$

where $\varphi$ is the angle between $\mathbf{q}$ and the wind direction. The symmetry of the spectrum with respect to the wind direction means that

$$\Phi\left(q, \varphi\right) = \Phi\left(q, -\varphi\right). \tag{4.4}$$

The structure function of the surface,

$$D\left(\mathbf{r}' - \mathbf{r}''\right) \equiv \left\langle \left[\zeta\left(\mathbf{r}'\right) - \zeta\left(\mathbf{r}''\right)\right]^2 \right\rangle, \tag{4.5}$$

in terms of the spectrum $\Phi$, has the form (compare with (1.2)):

$$D\left(r, \psi\right) = 2\iint \left[1 - \cos\left(\mathbf{qr}\right)\right] \Phi\left(q, \varphi\right) q dq d\varphi. \tag{4.6}$$

Let us consider in (3.39) the case $l_1, l_2 \to 0$ and substitute

$$\begin{aligned} \Delta_1 &= \zeta\left(\mathbf{r} + l_1\mathbf{m}_1\right) - \zeta\left(\mathbf{r}\right) \to l_1\mathbf{m}_1\nabla\zeta\left(\mathbf{r}\right) = l_1\gamma_1\left(\mathbf{r}\right) \\ \Delta_2 &= \zeta\left(\mathbf{r} + l_2\mathbf{m}_2\right) - \zeta\left(\mathbf{r}\right) \to l_2\mathbf{m}_2\nabla\zeta\left(\mathbf{r}\right) = l_2\gamma_2\left(\mathbf{r}\right). \end{aligned} \tag{4.7}$$

Here,

$$\gamma_1(\mathbf{r}) \equiv \mathbf{m}_1 \nabla \zeta(\mathbf{r}), \ \gamma_2(\mathbf{r}) \equiv \mathbf{m}_2 \nabla \zeta(\mathbf{r}) \tag{4.8}$$

are the slopes of the surface at the point $\mathbf{r}$, taken in the upwind direction $\mathbf{m}_1$ and in the crosswind direction $\mathbf{m}_2$. We obtain:

$$\Theta_\Delta(\alpha_1, l_1; \alpha_2, l_2) \to \langle \exp\{i(\alpha_1 l_1)\gamma_1(\mathbf{r}) + i(\alpha_2 l_2)\gamma_2(\mathbf{r})\}\rangle \equiv \Theta_\gamma(\alpha_1 l_1, \alpha_2 l_2). \tag{4.9}$$

If we denote

$$\beta_1 = \alpha_1 l_1, \ \beta_2 = \alpha_2 l_2, \tag{4.10}$$

and consider the case $l_1, l_2 \to 0$, $\beta_1, \beta_2 =$ Constant, we obtain the relation between $\Theta_\Delta$ and $\Theta_\gamma$:

$$\Theta_\gamma(\beta_1, \beta_2) = \lim_{l_1, l_2 \to 0} \Theta_\Delta\left(\frac{\beta_1}{l_1}, l_1; \frac{\beta_2}{l_2}, l_2\right). \tag{4.11}$$

According to the definitions of slopes,

$$\lim_{l_1 \to 0} \frac{D(l_1 \mathbf{m}_1)}{l_1^2} = \langle \gamma_1^2 \rangle, \ \lim_{l_2 \to 0} \frac{D(l_2 \mathbf{m}_2)}{l_2^2} = \langle \gamma_2^2 \rangle. \tag{4.12}$$

Therefore, the following limiting formulae are true for the values entering in (3.39):

$$\frac{\beta_1}{l_1}\kappa_{1,\mu} l_1 \sqrt{\frac{D(l_1 \mathbf{m}_1)}{l_1^2}} \to \beta_1 \kappa_{1,\mu}\sqrt{\langle \gamma_1^2 \rangle}, \ \frac{\beta_2}{l_2}\kappa_{2,\mu} l_2 \sqrt{\frac{D(l_2 \mathbf{m}_2)}{l_2^2}} \to \beta_2 \kappa_{2,\mu}\sqrt{\langle \gamma_2^2 \rangle}. \tag{4.13}$$

The term

$$D(l_1 \mathbf{m}_1) + D(l_2 \mathbf{m}_2) - D(\mathbf{m}_1 l_1 - \mathbf{m}_2 l_2) = 2\langle \Delta_1(l_1)\Delta_2(l_2)\rangle,$$

following the product $\alpha_1 \alpha_2$ in (3.39), needs more attention. Using spectral representation (4.6), we find

$$\langle \Delta_1 \Delta_2 \rangle = \iint \Phi(\mathbf{q})\, d^2 q \{1 - \cos(\mathbf{q}\mathbf{m}_1 l_1) - \cos(\mathbf{q}\mathbf{m}_2 l_2) + \cos(\mathbf{q}\mathbf{m}_1 l_1 - \mathbf{q}\mathbf{m}_2 l_2)\} \tag{4.14}$$

But

$$\{\cdots\} = 4\sin\frac{\mathbf{q}\mathbf{m}_1 l_1}{2}\sin\frac{\mathbf{q}\mathbf{m}_2 l_2}{2}\cos\frac{\mathbf{q}\mathbf{m}_1 l_1 - \mathbf{q}\mathbf{m}_2 l_2}{2},$$

and

$$\langle \Delta_1 \Delta_2 \rangle = 4\iint \Phi(\mathbf{q})\sin\frac{\mathbf{q}\mathbf{m}_1 l_1}{2}\sin\frac{\mathbf{q}\mathbf{m}_2 l_2}{2}\cos\frac{\mathbf{q}\mathbf{m}_1 l_1 - \mathbf{q}\mathbf{m}_2 l_2}{2} d^2 q. \tag{4.15}$$

For $l_1, l_2 \to 0$ we obtain

$$4\sin\frac{\mathbf{q}\mathbf{m}_1 l_1}{2}\sin\frac{\mathbf{q}\mathbf{m}_2 l_2}{2}\cos\frac{\mathbf{q}\mathbf{m}_1 l_1 - \mathbf{q}\mathbf{m}_2 l_2}{2} \to l_1 l_2 (\mathbf{q}\mathbf{m}_1)(\mathbf{q}\mathbf{m}_2)$$

and

$$\lim_{l_1, l_2 \to 0} \frac{\langle \Delta_1 \Delta_2 \rangle}{l_1 l_2} = \langle \gamma_1 \gamma_2 \rangle =$$
$$\iint \Phi(\mathbf{q})(\mathbf{q}\mathbf{m}_1)(\mathbf{q}\mathbf{m}_2)\, d^2 q = \int_0^\infty q^3 dq \int_{-\pi}^{\pi} \Phi(q, \varphi)\sin\varphi\cos\varphi d\varphi = 0$$

because of $\Phi(q, \varphi) = \Phi(q, -\varphi)$ (the integrand is an odd function with respect to $\varphi$). Thus, we proved that the term

$$\frac{\langle \Delta_1 \Delta_2 \rangle}{l_1 l_2} \to 0 \text{ while } l_1, l_2 \to 0,$$

or

$$\lim_{l_1, l_2 \to 0} \frac{D(l_1 \mathbf{m}_1) + D(l_2 \mathbf{m}_2) - D(\mathbf{m}_1 l_1 - \mathbf{m}_2 l_2)}{l_1 l_2} = 0, \tag{4.16}$$

vanishes while $l_1, l_2 \to 0$. This relation also can be written in the form

$$\langle \gamma_1 \gamma_2 \rangle = 0. \tag{4.17}$$

It follows from (4.17) that two principal slopes in the same point on a surface are uncorrelated (it is shown in Appendix A that this relation follows only from the symmetry of spectrum with respect to wind direction).

Thus, using (4.11), (4.12), (4.13), (4.13), and (4.16), we obtain from (3.39):

$$\begin{aligned} &\Theta_\gamma(\beta_1, \beta_2) = \langle \exp\{i\beta_1\gamma_1(\mathbf{r}) + i\beta_2\gamma_2(\mathbf{r})\}\rangle \approx \\ &\sum_\mu P_\mu \exp\left\{ i\beta_1\kappa_{1,\mu}\sqrt{\langle\gamma_1^2\rangle} + i\beta_2\kappa_{2,\mu}\sqrt{\langle\gamma_2^2\rangle} - \right. \\ &-\frac{\beta_1^2}{2}\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\langle\gamma_1^2\rangle - \frac{\beta_2^2}{2}\left(\lambda_\mu - \kappa_{2,\mu}^2\right)\langle\gamma_2^2\rangle \\ &\left. +\beta_1\beta_2\kappa_{1,\mu}\kappa_{2,\mu}\sqrt{\langle\gamma_1^2\rangle\langle\gamma_2^2\rangle}\right\}. \end{aligned} \tag{4.18}$$

It is easy to verify by direct differentiation of the right-hand side of (4.18) that

$$-\left.\frac{\partial^2\Theta_\gamma(\beta_1,\beta_2)}{\partial\beta_1\partial\beta_2}\right|_{\beta_1=\beta_2=0} = \langle\gamma_1\gamma_2\rangle = 0 \tag{4.19}$$

for any values of the parameters. The mean value of the slope $\langle\gamma_1\rangle$,

$$\langle\gamma_1\rangle = \frac{1}{i}\left.\frac{\partial\Theta_\gamma(\beta_1,\beta_2)}{\partial\beta_1}\right|_{\beta_1=\beta_2=0} = \sqrt{\langle\gamma_1^2\rangle}\sum_\mu P_\mu\kappa_{1,\mu} = 0 \tag{4.20}$$

because of (3.35). The similar formula is true for $\gamma_2$. Thus, the principal slopes $\gamma_1$ and $\gamma_2$ are statistically dependent, but uncorrelated.[5]

The CF of the marginal PDF,

$$W_\gamma(\gamma_1) \equiv \int W_\gamma(\gamma_1,\gamma_2)\,d\gamma_2, \tag{4.21}$$

can be obtained from (4.18) if we set $\beta_2 = 0$. We obtain

$$\Theta_{\gamma_1}(\beta_1) = \langle\exp(i\beta_1\gamma_1)\rangle = \sum_\mu P_\mu \exp\left\{ i\beta_1\kappa_{1,\mu}\sqrt{\langle\gamma_1^2\rangle} - \frac{\beta_1^2}{2}\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\langle\gamma_1^2\rangle\right\}. \tag{4.22}$$

The corresponding marginal PDF is given by the formula

$$W_\gamma(\gamma_1) = \sum_\mu \frac{P_\mu}{\sqrt{2\pi\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\langle\gamma_1^2\rangle}} \exp\left\{ -\frac{\left[\gamma_1 - \kappa_{1,\mu}\sqrt{\langle\gamma_1^2\rangle}\right]^2}{2\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\langle\gamma_1^2\rangle}\right\}. \tag{4.23}$$

Similarly,

$$W_\gamma(\gamma_2) = \sum_\mu \frac{P_\mu}{\sqrt{2\pi\left(\lambda_\mu - \kappa_{2,\mu}^2\right)\langle\gamma_2^2\rangle}} \exp\left\{ -\frac{\left[\gamma_2 - \kappa_{2,\mu}\sqrt{\langle\gamma_2^2\rangle}\right]^2}{2\left(\lambda_\mu - \kappa_{2,\mu}^2\right)\langle\gamma_2^2\rangle}\right\}. \tag{4.24}$$

It follows from (4.18) that the conditional Gaussian distribution, marked by subscript $\mu$, has the following parameters:

$$\begin{aligned} &\langle\gamma_1|\,\mu\rangle = \kappa_{1,\mu}\sqrt{\langle\gamma_1^2\rangle},\ \ \langle\gamma_2|\,\mu\rangle = \kappa_{2,\mu}\sqrt{\langle\gamma_2^2\rangle} \\ &\sigma_{\gamma_1,\mu}^2 = \left(\lambda_\mu - \kappa_{1,\mu}^2\right)\langle\gamma_1^2\rangle,\ \ \sigma_{\gamma_2,\mu}^2 = \left(\lambda_\mu - \kappa_{2,\mu}^2\right)\langle\gamma_2^2\rangle,\ \ \sigma_{\gamma_1,\mu}\sigma_{\gamma_2,\mu}\rho_\mu = -\kappa_{1,\mu}\kappa_{2,\mu}\sqrt{\langle\gamma_1^2\rangle\langle\gamma_2^2\rangle} \\ &\rho_\mu = -\frac{\kappa_{1,\mu}\kappa_{2,\mu}}{\sqrt{\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\left(\lambda_\mu - \kappa_{2,\mu}^2\right)}},\ \ 1 - \rho_\mu^2 = \frac{\lambda_\mu\left(\lambda_\mu - \kappa_{1,\mu}^2 - \kappa_{2,\mu}^2\right)}{\left(\lambda_\mu - \kappa_{1,\mu}^2\right)\left(\lambda_\mu - \kappa_{2,\mu}^2\right)} \end{aligned} \tag{4.25}$$

$$\sigma_{1,\mu}^2\sigma_{2,\mu}^2\left(1-\rho_\mu^2\right)=\left\langle\gamma_1^2\right\rangle\left\langle\gamma_2^2\right\rangle\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right).$$

The PDF that corresponds to CF (4.18), is given by the formula

$$W_\gamma\left(\gamma_1,\gamma_2\right)=$$
$$\sum\nolimits_\mu\frac{P_\mu}{2\pi\sqrt{\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)\left\langle\gamma_1^2\right\rangle\left\langle\gamma_2^2\right\rangle}}\exp\left\{-\frac{\left(\lambda_\mu-\kappa_{2,\mu}^2\right)\left[\gamma_1-\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle}\right]^2}{2\left\langle\gamma_1^2\right\rangle\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}-\right.$$
$$\left.-\frac{\left(\lambda_\mu-\kappa_{1,\mu}^2\right)\left[\gamma_2-\kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}\right]^2}{2\left\langle\gamma_2^2\right\rangle\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}-\frac{\kappa_{1,\mu}\kappa_{2,\mu}\left[\gamma_1-\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle}\right]\left[\gamma_2-\kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}\right]}{\sqrt{\left\langle\gamma_1^2\right\rangle\left\langle\gamma_2^2\right\rangle}\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}\right\}. \tag{4.26}$$

If we integrate (4.26) over $\gamma_2$ or $\gamma_1$, we correspondingly obtain (4.23) or (4.24).

Because the values $\sigma_{1,\mu}^2$, $\sigma_{2,\mu}^2$, and $1-\rho_\mu^2$ must be non-negative, we obtain the following restrictions for the parameters $\lambda_\mu$, $\kappa_{1,\mu}$, and $\kappa_{2,\mu}$:

$$\lambda_\mu-\kappa_{\alpha,\mu}^2\geq 0,\ \ \alpha=1,2;\ \ \lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\geq 0. \tag{4.27}$$

## 5. Finding the parameters $\lambda_\mu$, $\kappa_\mu$, and $P_\mu$

The next step in our consideration is to find parameters $\lambda_\mu$, $\kappa_\mu$, and $P_\mu$. If the function $W_\gamma\left(\gamma_1,\gamma_2\right)$ is known (for example, from the experimental data), we can approximate this function by the formula (4.26) (see footnote [3] after the formula (3.7)). It follows from the statistical symmetry of slopes with respect to wind direction that distribution in the crosswind direction must be symmetrical, i.e.,

$$W_\gamma\left(\gamma_1,\gamma_2\right)=W_\gamma\left(\gamma_1,-\gamma_2\right). \tag{5.1}$$

A similar condition must hold for the marginal PDF $W_\gamma\left(\gamma_2\right)$:

$$W_\gamma\left(\gamma_2\right)=W_\gamma\left(-\gamma_2\right). \tag{5.2}$$

The top of each conditional Gaussian PDF numbered by the subscript $\mu$ has on the plane $\left(\gamma_1,\gamma_2\right)$ the coordinates

$$\left(\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle},\ \kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}\right). \tag{5.3}$$

It follows from the formula (4.24) and the symmetry condition (5.2) that each point numbered by $\mu$, and having coordinates (5.3), must be accompanied by the dissymmetric point, numbered by some $\mu'$, and having the coordinates

$$\kappa_{1,\mu'}\sqrt{\left\langle\gamma_1^2\right\rangle}=\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle},\ \ \kappa_{2,\mu'}\sqrt{\left\langle\gamma_2^2\right\rangle}=-\kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}, \tag{5.4}$$

and the same values of $P_{\mu'}=P_\mu$ and $\lambda_{\mu'}=\lambda_\mu$. It is convenient to numerate this point by $\mu'=-\mu$. In this case,[6]

$$\kappa_{1,-\mu}=\kappa_{1,\mu};\ \ \kappa_{2,-\mu}=-\kappa_{2,\mu};\ \ P_\mu=P_{-\mu},\ \lambda_\mu=\lambda_{-\mu}. \tag{5.5}$$

Therefore, we must approximate the experimental joint PDF of two principal slopes by the formula (4.26) with the additional conditions

$$\begin{gathered}\sum_\mu P_\mu=1,\ P_\mu>0,\ \sum_\mu P_\mu\kappa_{1,\mu}=\sum_\mu P_\mu\kappa_{2,\mu}=0,\ \sum_\mu P_\mu\lambda_\mu=1\\ P_{-\mu}=P_\mu,\ \ \lambda_{-\mu}=\lambda_\mu,\ \ \kappa_{1,-\mu}=\kappa_{1,\mu};\ \ \kappa_{2,-\mu}=-\kappa_{2,\mu},\ \ \ \lambda_\mu\geq\kappa_{1,\mu}^2+\kappa_{2,\mu}^2.\end{gathered} \tag{5.6}$$

The quantities $\left\langle\gamma_1^2\right\rangle$ and $\left\langle\gamma_2^2\right\rangle$ are known from the known joint experimental PDF $W_\gamma\left(\gamma_1,\gamma_2\right)$. Therefore, only the numbers $\kappa_{1,\mu}$, $\kappa_{2,\mu}$, $\lambda_\mu$, and $P_\mu$ are the subject of finding. Note that, in general, the conditional 2-D Gaussian PDF is characterized by five independent parameters: two shifts and three coefficients of the quadratic form. In our case, only three independent parameters $\kappa_{1,\mu}$, $\kappa_{2,\mu}$, $\lambda_\mu$ remain; the two other coefficients of the quadratic form are some functions of $\kappa_{1,\mu}$, $\kappa_{2,\mu}$, $\lambda_\mu$.

The procedure of approximation can be performed by minimization of the integrated squared difference between the given joint PDF and its approximation by the formula (4.26). In the process of approximation we find all the numerical parameters $P_\mu$, $\kappa_{1,\mu}$, $\kappa_{2,\mu}$, and $\lambda_\mu$ (see the example in the section "Numerical results...").[7]

## 6. The CF for the arbitrary directed difference $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$

We have already obtained the formula (3.39) for the joint CF of two differences, taken in the two perpendicular principal directions. In this section we generalize this formula for the arbitrary directed difference of the type

$$\Delta(\mathbf{r}', \mathbf{r}'') = \zeta(\mathbf{r}') - \zeta(\mathbf{r}''). \tag{6.1}$$

It is easy to formulate the problem of finding the CF for such differences in terms of the solved problem. Let us draw through the point $\mathbf{r}'$ a straight line in the direction of the vector $\mathbf{m}_1$ (the upwind direction) and draw through the point $\mathbf{r}''$ a straight line in the direction of the vector $\mathbf{m}_2$ (the crosswind direction). These two lines intersect at some point $\mathbf{r}$, depending on $\mathbf{r}'$ and $\mathbf{r}''$. This point is determined by the equations

$$\mathbf{r} = \mathbf{r}' + \mathbf{m}_1 l_1 = \mathbf{r}'' - \mathbf{m}_2 l_2. \tag{6.2}$$

If we multiply (6.2) by the vector $\mathbf{m}_1$ and take into account that

$$\mathbf{m}_1^2 = \mathbf{m}_2^2 = 1, \ \mathbf{m}_1\mathbf{m}_2 = 0, \tag{6.3}$$

we obtain

$$\mathbf{r}'\mathbf{m}_1 + l_1 = \mathbf{r}''\mathbf{m}_1, \ \ l_1 = l_1(\mathbf{r}', \mathbf{r}'') = (\mathbf{r}'' - \mathbf{r}')\,\mathbf{m}_1 = x'' - x'. \tag{6.4}$$

Similarly, multiplying (6.2) by the vector $\mathbf{m}_2$, we obtain

$$\mathbf{r}'\mathbf{m}_2 = \mathbf{r}''\mathbf{m}_2 - l_2, \ \ l_2 = l_2(\mathbf{r}', \mathbf{r}'') = (\mathbf{r}'' - \mathbf{r}')\,\mathbf{m}_2 = y'' - y'. \tag{6.5}$$

Thus, we obtain for $\mathbf{r}$ two equivalent formulae[8]

$$\mathbf{r} = \mathbf{r}(\mathbf{r}', \mathbf{r}'') = \mathbf{r}' + [(\mathbf{r}'' - \mathbf{r}') \cdot \mathbf{m}_1]\,\mathbf{m}_1 = \mathbf{r}'' - [(\mathbf{r}'' - \mathbf{r}') \cdot \mathbf{m}_2]\,\mathbf{m}_2. \tag{6.6}$$

In coordinate form,

$$\mathbf{r}' = (x', y'), \ \mathbf{r}'' = (x'', y''), \ \mathbf{r} = (x'', y') \tag{6.7}$$

We can present the difference $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$ as follows:

$$\begin{aligned} &\zeta(\mathbf{r}') - \zeta(\mathbf{r}'') = \Delta(\mathbf{r}', \mathbf{r}'') = \\ &[\zeta(\mathbf{r}') - \zeta(\mathbf{r}(\mathbf{r}', \mathbf{r}''))] - [\zeta(\mathbf{r}'') - \zeta(\mathbf{r}(\mathbf{r}', \mathbf{r}''))] = \Delta_1(l_1(\mathbf{r}', \mathbf{r}'')) - \Delta_2(l_2(\mathbf{r}', \mathbf{r}'')). \end{aligned} \tag{6.8}$$

Thus, for the CF of $\Delta(\mathbf{r}', \mathbf{r}'')$ we obtain

$$\begin{aligned} &\Theta_\Delta(\alpha) = \langle \exp\{i\alpha\Delta(\mathbf{r}', \mathbf{r}'')\}\rangle = \\ &\langle \exp\{i\alpha\Delta_1(l_1(\mathbf{r}', \mathbf{r}'')) - i\alpha\Delta_2(l_2(\mathbf{r}', \mathbf{r}''))\}\rangle = \Theta_\Delta(\alpha, l_1; -\alpha, l_2), \end{aligned} \tag{6.9}$$

where $\Theta_\Delta(\alpha, l_1; -\alpha, l_2)$ is given by the formula (3.39). Substitution of (3.39) in (6.9) leads to the formula

$$\Theta_\Delta(\alpha) = \langle \exp\{i\alpha[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')]\}\rangle \approx \exp\{i\alpha\mathcal{L} - \alpha^2\mathcal{Q}\}, \tag{6.10}$$

where

$$\mathcal{L} = \kappa_{1,\mu} l_1(\mathbf{r}', \mathbf{r}'') \sqrt{\frac{D(\mathbf{m}_1 l_1(\mathbf{r}', \mathbf{r}''))}{l_1^2(\mathbf{r}', \mathbf{r}'')}} - -\kappa_{2,\mu} l_2(\mathbf{r}', \mathbf{r}'') \sqrt{\frac{D(\mathbf{m}_2 l_2(\mathbf{r}', \mathbf{r}''))}{l_2^2(\mathbf{r}', \mathbf{r}'')}} \tag{6.11}$$

and

$$\begin{aligned} \mathcal{Q} = &\frac{1}{2}(\lambda_\mu - \kappa_{1,\mu}^2)\, D(\mathbf{m}_1 l_1(\mathbf{r}', \mathbf{r}'')) + +\frac{1}{2}(\lambda_\mu - \kappa_{2,\mu}^2)\, D(\mathbf{m}_2 l_2(\mathbf{r}', \mathbf{r}'')) - \\ &-\left[\frac{\lambda_\mu}{2}\Big[D(\mathbf{m}_1 l_1(\mathbf{r}', \mathbf{r}'')) + D(\mathbf{m}_2 l_2(\mathbf{r}', \mathbf{r}'')) + D(\mathbf{m}_1 l_1(\mathbf{r}', \mathbf{r}'') - \mathbf{m}_2 l_2(\mathbf{r}', \mathbf{r}''))\Big] + \right. \\ &\left. \kappa_{1,\mu}\kappa_{2,\mu} l_1(\mathbf{r}', \mathbf{r}'')\, l_2(\mathbf{r}', \mathbf{r}'') \sqrt{\frac{D(\mathbf{m}_1 l_1(\mathbf{r}', \mathbf{r}''))\, D(\mathbf{m}_2 l_2(\mathbf{r}', \mathbf{r}''))}{l_1^2(\mathbf{r}', \mathbf{r}'')\, l_2^2(\mathbf{r}', \mathbf{r}'')}}\right]. \end{aligned} \tag{6.12}$$

After cancelation of several terms following the factor $\alpha^2\lambda_\mu$, we obtain

$$\Theta_\Delta(\alpha) = \langle \exp\{i\alpha[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')]\}\rangle \approx \sum_\mu P_\mu \exp\left\{i\alpha\mathcal{L} - \frac{1}{2}\alpha^2\lambda_\mu D(\mathbf{m}_1 l_1(\mathbf{r}',\mathbf{r}'') - \mathbf{m}_2 l_2(\mathbf{r}',\mathbf{r}'')) + \frac{\alpha^2\mathcal{L}^2}{2}\right\}. \tag{6.13}$$

It is easy to show that the vector

$$\mathbf{m}_1 l_1(\mathbf{r}',\mathbf{r}'') - \mathbf{m}_2 l_2(\mathbf{r}',\mathbf{r}'') = (\mathbf{r}-\mathbf{r}') + (\mathbf{r}-\mathbf{r}'') = (x''-x', y'-y'')$$

is dissymmetric to the vector $\mathbf{r}''-\mathbf{r}'$ with respect to the direction of $\mathbf{m}_1$ (i.e., to the wind direction). Because we assumed the symmetry of the spectrum (and the structure function) with respect to wind direction, we obtain from this symmetry:

$$D(\mathbf{m}_1 l_1(\mathbf{r}',\mathbf{r}'') - \mathbf{m}_2 l_2(\mathbf{r}',\mathbf{r}'')) = D(\mathbf{r}''-\mathbf{r}'). \tag{6.14}$$

Thus, we can simplify formula (6.13) and write

$$\Theta_\Delta(\alpha) = \langle \exp\{i\alpha[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')]\}\rangle \approx \sum_\mu P_\mu \exp\left\{i\alpha\mathcal{L} - \frac{1}{2}\alpha^2\left[\lambda_\mu D(\mathbf{r}''-\mathbf{r}') - \mathcal{L}^2\right]\right\}. \tag{6.15}$$

It follows from (6.15) that the coefficient following the factor $\alpha$ in the exponent presents the conditional mean value of $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$:

$$\langle[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')]|\,\mu\rangle = \mathcal{L} = \kappa_{1,\mu} l_1(\mathbf{r}',\mathbf{r}'')\sqrt{\frac{D(\mathbf{m}_1 l_1(\mathbf{r}',\mathbf{r}''))}{l_1^2(\mathbf{r}',\mathbf{r}'')}} - \kappa_{2,\mu} l_2(\mathbf{r}',\mathbf{r}'')\sqrt{\frac{D(\mathbf{m}_2 l_2(\mathbf{r}',\mathbf{r}''))}{l_2^2(\mathbf{r}',\mathbf{r}'')}}, \tag{6.16}$$

and the coefficient following the factor $\alpha^2/2$ presents the conditional variance of the same difference:

$$\left\langle[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')]^2\middle|\,\mu\right\rangle - \langle[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')]|\,\mu\rangle^2 = \lambda_\mu D(\mathbf{r}''-\mathbf{r}') - \mathcal{L}^2 \geq 0. \tag{6.17}$$

From comparison of (6.17) and (6.16) it follows that

$$\left\langle[\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')]^2\middle|\,\mu\right\rangle = \lambda_\mu D(\mathbf{r}''-\mathbf{r}'). \tag{6.18}$$

This result extends (3.28), (3.29) to an arbitrary directed argument of the conditional structure function.

### 6.1. The CF for an arbitrary directed slope

Let us set in (6.15)

$$\mathbf{r}' = \mathbf{r} + \frac{\boldsymbol{\rho}}{2},\ \mathbf{r}'' = \mathbf{r} - \frac{\boldsymbol{\rho}}{2}, \tag{6.19}$$

and consider the case $|\boldsymbol{\rho}| \to 0$. For the difference $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$ we obtain:

$$\zeta(\mathbf{r}') - \zeta(\mathbf{r}'') \approx \boldsymbol{\rho}\boldsymbol{\nabla}\zeta(\mathbf{r}) + \cdots = \boldsymbol{\rho}\boldsymbol{\gamma}(\mathbf{r}) + \cdots. \tag{6.20}$$

Because in the chosen coordinate system we have

$$\boldsymbol{\rho} = (l_1, l_2) \text{ and } \boldsymbol{\gamma} = (\gamma_1, \gamma_2)$$

, we can also write

$$\zeta(\mathbf{r}') - \zeta(\mathbf{r}'') = l_1\gamma_1 + l_2\gamma_2 + \cdots. \tag{6.21}$$

For the values entering in (6.15) for $l_1 \to 0$, $l_2 \to 0$ we obtain

$$\frac{D\left(\mathbf{m}_1 l_1\left(\mathbf{r}',\mathbf{r}''\right)\right)}{l_1^2\left(\mathbf{r}',\mathbf{r}''\right)} \to \left\langle \gamma_1^2 \right\rangle, \quad \frac{D\left(\mathbf{m}_2 l_2\left(\mathbf{r}',\mathbf{r}''\right)\right)}{l_2^2\left(\mathbf{r}',\mathbf{r}''\right)} \to \left\langle \gamma_2^2 \right\rangle \tag{6.22}$$

and (because $\langle \gamma_1 \gamma_2 \rangle = 0$),

$$D\left(\mathbf{r}''-\mathbf{r}'\right) \to \left\langle \left[l_1\gamma_1 + l_2\gamma_2\right]^2 \right\rangle = l_1^2\left\langle \gamma_1^2 \right\rangle + l_2^2\left\langle \gamma_2^2 \right\rangle. \tag{6.23}$$

Thus, for $l_1 \to 0$, $l_2 \to 0$ we can write denoting

$$\mathcal{A}\left(\alpha,\beta\right) \equiv \alpha\kappa_{1,\mu}\sqrt{\left\langle \gamma_1^2 \right\rangle} - \beta\kappa_{2,\mu}\sqrt{\left\langle \gamma_2^2 \right\rangle}, \tag{6.24}$$

$$\begin{aligned} &\left\langle \exp\left[i\alpha\left(l_1\gamma_1 + l_2\gamma_2\right)\right]\right\rangle = \Theta_\gamma\left(\alpha\boldsymbol{\rho}\right) \approx \\ &\sum_\mu P_\mu \exp\left\{ i\alpha\mathcal{A}\left(l_1,l_2\right) - \frac{1}{2}\alpha^2\left\{ \lambda_\mu\left[l_1^2\left\langle \gamma_1^2 \right\rangle + l_2^2\left\langle \gamma_2^2 \right\rangle\right] - \mathcal{A}^2\left(l_1,l_2\right) \right\}\right\}. \end{aligned} \tag{6.25}$$

The function $\Theta_\gamma\left(\alpha\boldsymbol{\rho}\right)$ really depends on the product $\alpha\boldsymbol{\rho} = \boldsymbol{\beta} = \left(\alpha l_1, \alpha l_2\right)$:

$$\begin{aligned} &\left\langle \exp\left[i\boldsymbol{\beta}\boldsymbol{\gamma}\left(\mathbf{r}\right)\right]\right\rangle \equiv \Theta_\gamma\left(\boldsymbol{\beta}\right) \approx \\ &\sum_\mu P_\mu \exp\left\{ i\mathcal{A}\left(\beta_1,\beta_2\right) - \frac{1}{2}\left\{ \lambda_\mu\left[\beta_1^2\left\langle \gamma_1^2 \right\rangle + \beta_2^2\left\langle \gamma_2^2 \right\rangle\right] - \mathcal{A}^2\left(\beta_1,\beta_2\right) \right\}\right\}. \end{aligned} \tag{6.26}$$

We can present (6.26) in another form, if we set

$$\beta_1 = \beta\cos\psi, \;\; \beta_2 = \beta\sin\psi, \tag{6.27}$$

where $\psi$ is the angle with respect to wind direction. In this case,

$$\gamma_1\cos\psi + \gamma_2\sin\psi \equiv \gamma\left(\psi\right) \tag{6.28}$$

is the slope in $\psi$-direction and

$$\beta_1^2\left\langle \gamma_1^2 \right\rangle + \beta_2^2\left\langle \gamma_2^2 \right\rangle = \beta^2\left[\cos^2\psi\left\langle \gamma_1^2 \right\rangle + \sin^2\psi\left\langle \gamma_2^2 \right\rangle\right] = \beta^2\left\langle \gamma^2\left(\psi\right)\right\rangle. \tag{6.29}$$

(The last equality is true because of $\langle \gamma_1\gamma_2 \rangle = 0$). We emphasize that formula (6.29) describes the dependence of the rms. of slope on the direction. This dependence is universal; it follows only from the symmetry of spectrum with respect to wind direction and does not depend on PDF (see Appendix A for a derivation of (6.29) that is based only on the symmetry of spectrum).

The scalar product $\boldsymbol{\beta}\boldsymbol{\gamma}\left(\mathbf{r}\right)$ takes the form

$$\boldsymbol{\beta}\boldsymbol{\gamma}\left(\mathbf{r}\right) = \beta_1\gamma_1 + \beta_2\gamma_2 = \beta\left[\gamma_1\cos\psi + \gamma_2\sin\psi\right] = \beta\gamma\left(\psi\right). \tag{6.30}$$

Substituting (6.27) - (6.30) in (6.26), we obtain:

$$\left\langle \exp\left[i\beta\gamma\left(\psi\right)\right]\right\rangle \equiv \Theta_{\gamma(\psi)}\left(\beta\right) \approx \sum_\mu P_\mu \exp\left\{ i\beta\mathcal{A}\left(\psi\right) - \frac{\beta^2}{2}\left\{ \lambda_\mu\left\langle \gamma^2\left(\psi\right)\right\rangle - \mathcal{A}^2\left(\psi\right)\right\}\right\}, \tag{6.31}$$

where

$$\mathcal{A}\left(\psi\right) \equiv \mathcal{A}\left(\cos\psi,\sin\psi\right) = \kappa_{1,\mu}\sqrt{\left\langle \gamma_1^2 \right\rangle}\cos\psi - \kappa_{2,\mu}\sqrt{\left\langle \gamma_2^2 \right\rangle}\sin\psi. \tag{6.32}$$

## 7. Multivariate PDF for differences $\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')$

For many problems it is necessary to know the joint CF for several differences of the type

$$\Delta_1(\mathbf{r}_1', \mathbf{r}_1'') = \zeta(\mathbf{r}_1') - \zeta(\mathbf{r}_1''), ..., \Delta_n(\mathbf{r}_n', \mathbf{r}_n'') = \zeta(\mathbf{r}_n') - \zeta(\mathbf{r}_n''). \tag{7.1}$$

For example, such CF's appear in the theory of wave scattering from rough surfaces; they contain all the information necessary to calculat the scattering cross sections.

We seek this CF,

$$\Theta_\Delta(\alpha_1, ..., \alpha_n) \equiv \left\langle \exp\left\{ i \sum \alpha_i \Delta_i(\mathbf{r}_i', \mathbf{r}_i'') \right\} \right\rangle, \tag{7.2}$$

in the form

$$\Theta_\Delta(\alpha_1, ..., \alpha_n) \approx \sum_\mu P_\mu \exp\left\{ i \sum_{i=1}^n \alpha_i \langle \Delta_i(\mathbf{r}_i', \mathbf{r}_i'') | \mu \rangle - \frac{1}{2} \sum_{i,k=1}^n B_{ik}(\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_k', \mathbf{r}_k'' | \mu) \alpha_i \alpha_k \right\}, \tag{7.3}$$

with the same values $P_\mu$, $\kappa_{1,\mu}$, $\kappa_{2,\mu}$, and $\lambda_\mu$ that have already been determined. Here,

$$B_{ik}(\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_k', \mathbf{r}_k'' | \mu) = \langle \Delta_i(\mathbf{r}_i', \mathbf{r}_i'') \Delta_k(\mathbf{r}_k', \mathbf{r}_k'') | \mu \rangle - \langle \Delta_i(\mathbf{r}_i', \mathbf{r}_i'') | \mu \rangle \langle \Delta_k(\mathbf{r}_k', \mathbf{r}_k'') | \mu \rangle. \tag{7.4}$$

Using the Yaglom identity (3.30), we obtain:

$$\begin{aligned} &\langle \Delta_i(\mathbf{r}_i', \mathbf{r}_i'') \Delta_j(\mathbf{r}_j', \mathbf{r}_j'') | \mu \rangle \equiv \\ &\left\langle \left[ \zeta(\mathbf{r}_i') - \zeta(\mathbf{r}_i'') \right] \left[ \zeta(\mathbf{r}_j') - \zeta(\mathbf{r}_j'') \right] \middle| \mu \right\rangle = \\ &\frac{1}{2} \left\{ \left\langle \left[ \zeta(\mathbf{r}_i') - \zeta(\mathbf{r}_j'') \right]^2 \middle| \mu \right\rangle + \left\langle \left[ \zeta(\mathbf{r}_j') - \zeta(\mathbf{r}_i'') \right]^2 \middle| \mu \right\rangle - \right. \\ &\left. - \left\langle \left[ \zeta(\mathbf{r}_i') - \zeta(\mathbf{r}_j') \right]^2 \middle| \mu \right\rangle - \left\langle \left[ \zeta(\mathbf{r}_i'') - \zeta(\mathbf{r}_j'') \right]^2 \middle| \mu \right\rangle \right\} \end{aligned} \tag{7.5}$$

But for the arbitrary directed, conditional, mean value and structure function we have already obtained formulae (6.16) and (6.18):

$$\langle [\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')] | \mu \rangle = \langle \Delta(\mathbf{r}', \mathbf{r}'') | \mu \rangle = \mathcal{L}(\mathbf{r}', \mathbf{r}''), \tag{7.6}$$

$$\left\langle [\zeta(\mathbf{r}') - \zeta(\mathbf{r}'')]^2 \middle| \mu \right\rangle = \lambda_\mu D(\mathbf{r}'' - \mathbf{r}'), \tag{7.7}$$

where, according to (6.4) and (6.5),

$$l_1(\mathbf{r}', \mathbf{r}'') = (\mathbf{r}'' - \mathbf{r}') \mathbf{m}_1 = x'' - x', \; l_2(\mathbf{r}', \mathbf{r}'') = (\mathbf{r}'' - \mathbf{r}') \mathbf{m}_2 = y'' - y', \tag{7.8}$$

(i.e., the arguments of the anisotropic structure functions in (7.6) are the upwind and the crosswind components of the vector $\mathbf{r}'' - \mathbf{r}'$).

Substituting (7.7) in (7.5), we obtain

$$\begin{aligned} &\langle \Delta_i(\mathbf{r}_i', \mathbf{r}_i'') \Delta_j(\mathbf{r}_j', \mathbf{r}_j'') | \mu \rangle \equiv \\ &\frac{\lambda_\mu}{2} \left\{ D(\mathbf{r}_i' - \mathbf{r}_j'') + D(\mathbf{r}_j' - \mathbf{r}_i'') - D(\mathbf{r}_i' - \mathbf{r}_j') - D(\mathbf{r}_i'' - \mathbf{r}_i'') \right\}, \end{aligned} \tag{7.9}$$

and the formula (7.4) for $B_{ij}(\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_j', \mathbf{r}_j'' | \mu)$ takes the form

$$\begin{aligned} &B_{ij}(\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_j', \mathbf{r}_j'' | \mu) = \\ &\frac{\lambda_\mu}{2} \left[ D(\mathbf{r}_i' - \mathbf{r}_j'') + D(\mathbf{r}_j' - \mathbf{r}_i'') - D(\mathbf{r}_i' - \mathbf{r}_j') - D(\mathbf{r}_i'' - \mathbf{r}_i'') \right] - \\ - \quad &\mathcal{L}(\mathbf{r}_i', \mathbf{r}_i'') \mathcal{L}(\mathbf{r}_j', \mathbf{r}_j''). \end{aligned} \tag{7.10}$$

Formula (7.3) where

$$\langle \Delta_i (\mathbf{r}_i', \mathbf{r}_i'') | \mu \rangle$$

is determined by (7.6),

$$B_{ij} (\mathbf{r}_i', \mathbf{r}_i''; \mathbf{r}_j', \mathbf{r}_j'' | \mu)$$

is determined by (7.10), and

$$l_1 (\mathbf{r}_j', \mathbf{r}_j''), l_2 (\mathbf{r}_j', \mathbf{r}_j'')$$

are determined by (7.8), presents the joint multivariate CF for several, arbitrarily directed, differences in elevation.

## 8. Scattering cross section from the absolutely reflecting interface in the Kirchhoff approximation

For the scattering cross section $\Sigma (\mathbf{q}, \mathbf{q}_0)$ for both the Dirichlet and Neuman cases (in the Kirchhoff approximation these two cases coincide), we have (see, e.g., [37]):

$$\Sigma (\mathbf{q}, \mathbf{q}_0) = 4\pi^2 \left[ \frac{k^2 + \nu\nu_0 - \mathbf{q}\mathbf{q}_0}{\nu + \nu_0} \right]^2 \left\langle |\mathcal{J} (\mathbf{q} - \mathbf{q}_0, \nu + \nu_0)|^2 \right\rangle . \tag{8.1}$$

Here, the wave-vector of the incident plane wave $\mathbf{k}_0 = (\mathbf{q}_0, -\nu_0)$ has the horizontal component $\mathbf{q}_0$ and the vertical component $\nu_0 = \sqrt{k^2 - \mathbf{q}_0^2}$, and the wave-vector of the scattered wave has the form $\mathbf{k} = (\mathbf{q}, \nu)$, where $\nu = \sqrt{k^2 - \mathbf{q}^2}$. The value $\left\langle |\mathcal{J}|^2 \right\rangle$ is given by the formula

$$\begin{aligned} &\left\langle |\mathcal{J} (\mathbf{q} - \mathbf{q}_0, \nu + \nu_0)|^2 \right\rangle = \\ &\frac{1}{16\pi^4} \iint d^2 r' \iint d^2 r'' \exp \left[ i (\mathbf{q} - \mathbf{q}_0) (\mathbf{r}' - \mathbf{r}'') + i (\nu + \nu_0) [\zeta (\mathbf{r}') - \zeta (\mathbf{r}'')] \right] \end{aligned} \tag{8.2}$$

The mean value, appearing in this formula:

$$\langle \exp \{ i (\nu + \nu_0) [\zeta (\mathbf{r}') - \zeta (\mathbf{r}'')] \} \rangle ,$$

is the CF for the differences in elevation that can be expressed in terms of (6.15):

$$\begin{aligned} &\langle \exp \{ i (\nu + \nu_0) [\zeta (\mathbf{r}') - \zeta (\mathbf{r}'')] \} \rangle \approx \\ &\sum_\mu P_\mu \exp \left\{ i (\nu + \nu_0) \mathcal{L} (\mathbf{r}', \mathbf{r}'') - \frac{1}{2} (\nu + \nu_0)^2 \left\{ \lambda_\mu D (\mathbf{r}'' - \mathbf{r}') - [\mathcal{L} (\mathbf{r}', \mathbf{r}'')]^2 \right\} \right\}. \end{aligned} \tag{8.3}$$

Here,

$$\mathbf{r}' = (x', y') , \ \mathbf{r}'' = (x'', y'') , \ l_1 = x'' - x', \ l_2 = y'' - y', \ \mathbf{r}'' - \mathbf{r}' = (l_1, l_2) , \tag{8.4}$$

and

$$\mathbf{m}_1 l_1 (\mathbf{r}', \mathbf{r}'') = (l_1, 0) , \ \mathbf{m}_2 l_2 (\mathbf{r}', \mathbf{r}'') = (0, l_2) . \tag{8.5}$$

For (8.2) we obtain, choosing

$$\mathbf{r}' \text{ and } \mathbf{r}'' - \mathbf{r}' \equiv \mathbf{r} = (l_1, l_2)$$

as a new variables of integration:

$$\begin{aligned} &\left\langle |\mathcal{J} (\mathbf{q} - \mathbf{q}_0, \nu + \nu_0)|^2 \right\rangle = \\ &\frac{1}{16\pi^4} \sum_\mu P_\mu \iint d^2 r' \int dl_1 \int dl_2 \exp \left[ i (q_1 - q_{10}) l_1 + i (q_2 - q_{20}) l_2 \right] \times \\ &\exp \left\{ i (\nu + \nu_0) \widetilde{\mathcal{L}} (l_1, l_2) - \frac{1}{2} (\nu + \nu_0)^2 \left\{ \lambda_\mu D (l_1, l_2) - \left[ \widetilde{\mathcal{L}} (l_1, l_2) \right]^2 \right\} \right\}; \end{aligned} \tag{8.6}$$

where we denoted the value $\mathcal{L}$ depending on new variables $l_1$ and $l_2$ as $\widetilde{\mathcal{L}}(l_1, l_2)$:

$$\widetilde{\mathcal{L}}(l_1, l_2) \equiv \kappa_{1,\mu} l_1 \sqrt{\frac{D(l_1, 0)}{l_1^2}} - \kappa_{2,\mu} l_2 \sqrt{\frac{D(0, l_2)}{l_2^2}}. \tag{8.7}$$

Taking into account that

$$\iint d^2 r' = A$$

where $A \to \infty$ is the total scattering area, we obtain

$$\begin{aligned} &\frac{1}{A}\left\langle \left|\mathcal{J}(\mathbf{q}-\mathbf{q}_0, \nu+\nu_0)\right|^2 \right\rangle = \\ &\frac{1}{16\pi^4}\sum_\mu P_\mu \int dl_1 \int dl_2 \exp\left\{ i(q_1-q_{10}) l_1 + i(q_2-q_{20}) l_2 + i(\nu+\nu_0)\widetilde{\mathcal{L}}(l_1, l_2) - \right. \\ &\left. -\frac{1}{2}(\nu+\nu_0)^2 \left\{ \lambda_\mu D(l_1, l_2) - \left[\widetilde{\mathcal{L}}(l_1, l_2)\right]^2 \right\} \right\}. \end{aligned} \tag{8.8}$$

For the scattering cross section from the unit of area, $\Sigma_0 \equiv \Sigma/A$, we obtain

$$\begin{aligned} &\Sigma_0(\mathbf{q}, \mathbf{q}_0) = \\ &\left[\frac{k^2+\nu\nu_0-\mathbf{q}\mathbf{q}_0}{2\pi(\nu+\nu_0)}\right]^2 \sum_\mu P_\mu \int dl_1 \int dl_2 \exp\left[i(q_1-q_{10}) l_1 + i(q_2-q_{20}) l_2\right] \times \\ &\exp\left\{ i(\nu+\nu_0)\widetilde{\mathcal{L}}(l_1, l_2) - \frac{1}{2}(\nu+\nu_0)^2 \left\{ \lambda_\mu D(l_1, l_2) - \left[\widetilde{\mathcal{L}}(l_1, l_2)\right]^2 \right\} \right\} \end{aligned} \tag{8.9}$$

The function $D(\mathbf{r})$ saturates while $r \to \infty$, and $D(\infty) = 2\sigma_0^2$, where $\sigma_0^2 = \langle \zeta^2 \rangle$ is the rms. of the surface elevations. Because of this, it is useful to separate the singular integrals in (8.9).

Let us denote

$$\mathcal{F}(l_1, l_2) = \frac{1}{2}(\nu+\nu_0)^2 \left\{ \lambda_\mu D(l_1, l_2) - \left[\widetilde{\mathcal{L}}(l_1, l_2)\right]^2 \right\}. \tag{8.10}$$

In terms of $\mathcal{F}(l_1, l_2)$ formula (8.9) takes the form

$$\begin{aligned} &\Sigma_0(\mathbf{q}, \mathbf{q}_0) = \left[\frac{k^2+\nu\nu_0-\mathbf{q}\mathbf{q}_0}{2\pi(\nu+\nu_0)}\right]^2 \times \\ &\sum_\mu P_\mu \int dl_1 \int dl_2 \exp\left\{ i\left[(q_1-q_{10}) l_1 + (q_2-q_{20}) l_2\right] + i(\nu+\nu_0)\widetilde{\mathcal{L}}(l_1, l_2) \right\} \exp\left[-\mathcal{F}(l_1, l_2)\right]. \end{aligned} \tag{8.11}$$

We present $\exp\left[-\mathcal{F}(l_1, l_2)\right]$ in the form

$$\exp\left[-\mathcal{F}(l_1, l_2)\right] = A(l_1, l_2) + B(l_1, l_2), \tag{8.12}$$

where

$$A(l_1, l_2) \equiv \exp\left[-\mathcal{F}(l_1, l_2)\right] - \exp\left[-\mathcal{F}(l_1, \infty)\right] - \exp\left[-\mathcal{F}(\infty, l_2)\right] + \exp\left[-\mathcal{F}(\infty, \infty)\right], \tag{8.13}$$

and

$$B(l_1, l_2) = \exp\left[-\mathcal{F}(l_1, \infty)\right] + \exp\left[-\mathcal{F}(\infty, l_2)\right] - \exp\left[-\mathcal{F}(\infty, \infty)\right]. \tag{8.14}$$

The function $A(l_1, l_2)$ satisfies the conditions

$$A(\infty, l_2) = A(l_1, \infty) = A(\infty, \infty) = 0. \tag{8.15}$$

Thus, if we substitute in (8.11) the sum (8.12) instead of $\exp(-\mathcal{F})$, the term containing $A$ will converge. The second term containing $B$ leads to sum of $\delta$-functions and contributes only to the specular directions. Thus, the diffuse part of the scattering cross section is given by the formula

$$\Sigma_0 = \left[\frac{k^2 + \nu\nu_0 - \mathbf{q}\mathbf{q}_0}{2\pi(\nu + \nu_0)}\right]^2 \sum_\mu P_\mu \int dl_1 \int dl_2 \exp\left[i(q_1 - q_{10})l_1 + i(q_2 - q_{20})l_2\right] A(l_1, l_2). \tag{8.16}$$

From (8.10) we obtain

$$\mathcal{F}(l_1, \pm\infty) = \frac{1}{2}(\nu + \nu_0)^2 \left\{2\lambda_\mu \sigma_0^2 - \left[\kappa_{1,\mu}\frac{l_1}{|l_1|}\sqrt{D(l_1, 0)} \mp \kappa_{2,\mu}\sigma_0\sqrt{2}\right]^2\right\}, \tag{8.17}$$

$$\mathcal{F}(\pm\infty, l_2) = \frac{1}{2}(\nu + \nu_0)^2 \left\{2\lambda_\mu \sigma_0^2 - \left[\pm\kappa_{1,\mu}\sigma_0\sqrt{2} - \kappa_{2,\mu}\frac{l_2}{|l_2|}\sqrt{D(0, l_2)}\right]^2\right\}, \tag{8.18}$$

$$\mathcal{F}(\pm\infty, \pm\infty') = (\nu + \nu_0)^2 \left[\lambda_\mu - (\pm\kappa_{1,\mu} - (\pm')\kappa_{2,\mu})^2\right]\sigma_0^2. \tag{8.19}$$

These formulae were used in our numerical calculations[6].

### 8.1. Geometric optics limit

The geometric optics (GO) limit corresponds to the expansion of the function

$$F(l_1, l_2) \equiv i(\nu + \nu_0)\widetilde{\mathcal{L}}(l_1, l_2) - \mathcal{F}(l_1, l_2)$$

in powers of $l_1$ and $l_2$ and keeping in this expansion the terms up to the second order. Because $l_1$ and $l_2$ correspond to the principal directions, the cross-term $l_1 l_2$ does not appear in this expansion. Using (6.23), we obtain

$$D(l_1, l_2) = a_1 l_1^2 + a_2 l_2^2 + O(l^4).$$

It follows from (4.12) that $a_1 = \langle\gamma_1^2\rangle$ and $a_2 = \langle\gamma_2^2\rangle$:

$$D(l_1, l_2) = \langle\gamma_1^2\rangle l_1^2 + \langle\gamma_2^2\rangle l_2^2 + O(l^4). \tag{8.20}$$

Thus, the expansion of the function $F$ after combining similar terms has the form

$$\begin{aligned} F(l_1, l_2) \approx F_0(l_1, l_2) \equiv i(\nu + \nu_0)\left[\kappa_{1,\mu} l_1\sqrt{\langle\gamma_1^2\rangle} - \kappa_{2,\mu} l_2\sqrt{\langle\gamma_2^2\rangle}\right] - \\ -\frac{1}{2}(\nu + \nu_0)^2\left\{(\lambda_\mu - \kappa_{1,\mu}^2)\langle\gamma_1^2\rangle l_1^2 + (\lambda_\mu - \kappa_{2,\mu}^2)\langle\gamma_2^2\rangle l_2^2 + 2\kappa_{1,\mu}\kappa_{2,\mu}\sqrt{\langle\gamma_1^2\rangle\langle\gamma_2^2\rangle} l_1 l_2\right\}. \end{aligned} \tag{8.21}$$

Let us compare the formula (8.21) with the formula (6.25) for the joint CF of two principal slopes. This CF has the form

$$\langle\exp[i\alpha(l_1\gamma_1 + l_2\gamma_2)]\rangle = \Theta_{\gamma_1,\gamma_2}(\alpha l_1, \alpha l_2) \approx \sum_\mu P_\mu \Theta_{\mu;\gamma_1,\gamma_2}(\alpha l_1, \alpha l_2)$$

where $\Theta_{\mu;\gamma_1,\gamma_2}(\alpha l_1, \alpha l_2)$ is the conditional CF:

$$\begin{aligned} &\Theta_{\mu;\gamma_1,\gamma_2}(\alpha l_1, \alpha l_2) = \\ &\exp\left\{i\alpha\left[\kappa_{1,\mu} l_1\sqrt{\langle\gamma_1^2\rangle} - \kappa_{2,\mu} l_2\sqrt{\langle\gamma_2^2\rangle}\right] - \right. \\ &\left. -\frac{1}{2}\alpha^2\left\{\lambda_\mu\left[l_1^2\langle\gamma_1^2\rangle + l_2^2\langle\gamma_2^2\rangle\right] - \left[\kappa_{1,\mu} l_1\sqrt{\langle\gamma_1^2\rangle} - \kappa_{2,\mu} l_2\sqrt{\langle\gamma_2^2\rangle}\right]^2\right\}\right\}. \end{aligned} \tag{8.22}$$

[6]The additional restriction for the parameters $\lambda_\mu$, $\kappa_{1\mu}$, and $\kappa_{2\mu}$ appears from (8.19): $\lambda_\mu \geq (\kappa_{1\mu} \pm \kappa_{2\mu})^2$. This restriction differs from (4.27).

From comparison of (8.21) and (8.22) we see that

$$\exp\left[F_0\left(l_1,l_2\right)\right]=\Theta_{\mu;\gamma_1,\gamma_2}\left(\left(\nu+\nu_0\right)l_1,\left(\nu+\nu_0\right)l_2\right). \tag{8.23}$$

Let us substitute (8.23) in (8.11). We obtain:

$$\begin{aligned}\Sigma_0=&\left[\frac{k^2+\nu\nu_0-\mathbf{q}\mathbf{q}_0}{2\pi\left(\nu+\nu_0\right)}\right]^2\sum_\mu P_\mu\int dl_1\int dl_2\times\\&\exp\left[i\left(q_1-q_{10}\right)l_1+i\left(q_2-q_{20}\right)l_2\right]\Theta_{\mu;\gamma_1,\gamma_2}\left(\left(\nu+\nu_0\right)l_1,\left(\nu+\nu_0\right)l_2\right).\end{aligned} \tag{8.24}$$

If we change the variables of integration according to the formulae

$$\left(\nu+\nu_0\right)l_1=\beta_1,\ \left(\nu+\nu_0\right)l_2=\beta_2;\ dl_1dl_2=\frac{d\beta_1d\beta_2}{\left(\nu+\nu_0\right)^2},$$

we obtain:

$$\begin{aligned}\Sigma_0=&\frac{\left(k^2+\nu\nu_0-\mathbf{q}\mathbf{q}_0\right)^2}{4\pi^2\left(\nu+\nu_0\right)^4}\sum_\mu P_\mu\int d\beta_1\int d\beta_2\Theta_{\mu;\gamma_1,\gamma_2}\left(\beta_1,\beta_2\right)\times\\&\exp\left[i\frac{q_1-q_{10}}{\nu+\nu_0}\beta_1+i\frac{q_2-q_{20}}{\nu+\nu_0}\beta_2\right].\end{aligned} \tag{8.25}$$

But, according to definition,

$$\frac{1}{4\pi^2}\iint\exp\left(-i\beta_1\gamma_1-i\beta_2\gamma_2\right)\Theta_{\mu;\gamma_1,\gamma_2}\left(\beta_1,\beta_2\right)d\beta_1d\beta_2=W_{\mu,\gamma_1,\gamma_2}\left(\gamma_1,\gamma_2\right) \tag{8.26}$$

is the joint conditional PDF of two principal slopes. Thus, (8.25) takes the form

$$\Sigma_0=\frac{\left(k^2+\nu\nu_0-\mathbf{q}\mathbf{q}_0\right)^2}{\left(\nu+\nu_0\right)^4}\sum_\mu P_\mu W_{\mu,\gamma_1,\gamma_2}\left(\frac{q_{10}-q_1}{\nu+\nu_0},\frac{q_{20}-q_2}{\nu+\nu_0}\right). \tag{8.27}$$

The sum on the right-hand side is equal to the initial joint PDF of two principal slopes. Thus we can finally write the formula

$$\Sigma_0=\frac{\left(k^2+\nu\nu_0-\mathbf{q}\mathbf{q}_0\right)^2}{\left(\nu+\nu_0\right)^4}W_{\gamma_1,\gamma_2}\left(\frac{q_{10}-q_1}{\nu+\nu_0},\frac{q_{20}-q_2}{\nu+\nu_0}\right). \tag{8.28}$$

The function $W_{\mu,\gamma_1,\gamma_2}\left(\gamma_1,\gamma_2\right)$ was obtained above (see (4.26)):

$$\begin{aligned}&W_\gamma\left(\gamma_1,\gamma_2\right)=\\&\sum\nolimits_\mu\frac{P_\mu}{2\pi\sqrt{\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)\left\langle\gamma_1^2\right\rangle\left\langle\gamma_2^2\right\rangle}}\exp\left\{-\frac{\left(\lambda_\mu-\kappa_{2,\mu}^2\right)\left[\gamma_1-\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle}\right]^2}{2\left\langle\gamma_1^2\right\rangle\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}-\right.\\&\left.-\frac{\left(\lambda_\mu-\kappa_{1,\mu}^2\right)\left[\gamma_2-\kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}\right]^2}{2\left\langle\gamma_2^2\right\rangle\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}-\frac{\kappa_{1,\mu}\kappa_{2,\mu}\left[\gamma_1-\kappa_{1,\mu}\sqrt{\left\langle\gamma_1^2\right\rangle}\right]\left[\gamma_2-\kappa_{2,\mu}\sqrt{\left\langle\gamma_2^2\right\rangle}\right]}{\sqrt{\left\langle\gamma_1^2\right\rangle\left\langle\gamma_2^2\right\rangle}\lambda_\mu\left(\lambda_\mu-\kappa_{1,\mu}^2-\kappa_{2,\mu}^2\right)}\right\}.\end{aligned}$$

### 8.2. General approach to geometric optics approximation

We show in this subsection that the formula (8.28) obtained by using decomposition of the PDF in the sum of Gaussian terms has a universal meaning (this is a well-known fact; we include the following derivation only for completeness of the paper). We start with the formula

$$\Sigma\left(\mathbf{q},\mathbf{q}_0\right)=4\pi^2\left[\frac{k^2+\nu\nu_0-\mathbf{q}\mathbf{q}_0}{\nu+\nu_0}\right]^2\left\langle\left|\mathcal{J}\left(\mathbf{q}-\mathbf{q}_0,\nu+\nu_0\right)\right|^2\right\rangle,$$

where

$$\left\langle \left|\mathcal{J}\left(\mathbf{q}-\mathbf{q}_0,\nu+\nu_0\right)\right|^2\right\rangle = \frac{1}{16\pi^4}\iint d^2r' \iint d^2r'' \times$$
$$\exp\left[i\left(\mathbf{q}-\mathbf{q}_0\right)\left(\mathbf{r}'-\mathbf{r}''\right)\right]\left\langle \exp\left\{i\left(\nu+\nu_0\right)\left[\zeta\left(\mathbf{r}'\right)-\zeta\left(\mathbf{r}''\right)\right]\right\}\right\rangle.$$

In GO limit we expand the difference $\zeta(\mathbf{r}')-\zeta(\mathbf{r}'')$ as follows:

$$\begin{aligned}&\mathbf{r}'=\mathbf{r}+\frac{\boldsymbol{\rho}}{2};\ \mathbf{r}''=\mathbf{r}-\frac{\boldsymbol{\rho}}{2};\ \zeta(\mathbf{r}')=\zeta(\mathbf{r})+\frac{\boldsymbol{\rho}}{2}\boldsymbol{\nabla}\zeta(\mathbf{r});\ \zeta(\mathbf{r}'')=\zeta(\mathbf{r})-\frac{\boldsymbol{\rho}}{2}\boldsymbol{\nabla}\zeta(\mathbf{r});\\ &\zeta(\mathbf{r}')-\zeta(\mathbf{r}'')=\boldsymbol{\rho}\boldsymbol{\nabla}\zeta(\mathbf{r})=\boldsymbol{\rho}\boldsymbol{\gamma},\end{aligned} \tag{8.29}$$

where $\boldsymbol{\gamma}=\boldsymbol{\nabla}\zeta(\mathbf{r})$. Then,

$$\begin{aligned}&\left\langle \left|\mathcal{J}\left(\mathbf{q}-\mathbf{q}_0,\nu+\nu_0\right)\right|^2\right\rangle = \frac{1}{16\pi^4}\iint d^2r \iint d^2\rho \exp\left[i\left(\mathbf{q}-\mathbf{q}_0\right)\boldsymbol{\rho}\right]\left\langle \exp\left\{i\left(\nu+\nu_0\right)\boldsymbol{\rho}\boldsymbol{\gamma}\right\}\right\rangle = \\ &\frac{A}{16\pi^4}\iint d^2\rho \exp\left[i\left(\mathbf{q}-\mathbf{q}_0\right)\boldsymbol{\rho}\right]\left\langle \exp\left\{i\left(\nu+\nu_0\right)\boldsymbol{\rho}\boldsymbol{\gamma}\right\}\right\rangle,\end{aligned} \tag{8.30}$$

where $A$ is the total scattering area. For the last exponential we have:

$$\left\langle \exp\left\{i\left(\nu+\nu_0\right)\boldsymbol{\rho}\boldsymbol{\gamma}\right\}\right\rangle = \iint \exp\left\{i\left(\nu+\nu_0\right)\boldsymbol{\rho}\boldsymbol{\gamma}\right\} W_\gamma\left(\boldsymbol{\gamma}\right) d^2\gamma, \tag{8.31}$$

and substituting (8.31) in (8.30) we obtain:

$$\begin{aligned}&\left\langle \left|\mathcal{J}\left(\mathbf{q}-\mathbf{q}_0,\nu+\nu_0\right)\right|^2\right\rangle = \\ &\frac{A}{16\pi^4}\iint d^2\rho \exp\left[i\left(\mathbf{q}-\mathbf{q}_0\right)\boldsymbol{\rho}\right]\iint \exp\left\{i\left(\nu+\nu_0\right)\boldsymbol{\rho}\boldsymbol{\gamma}\right\} W_\gamma\left(\boldsymbol{\gamma}\right) d^2\gamma = \\ &\frac{A}{16\pi^4}\iint W_\gamma\left(\boldsymbol{\gamma}\right) d^2\gamma \iint d^2\rho \exp\left[i\left(\mathbf{q}-\mathbf{q}_0+\left(\nu+\nu_0\right)\boldsymbol{\gamma}\right)\boldsymbol{\rho}\right].\end{aligned} \tag{8.32}$$

Because the last integral over $\boldsymbol{\rho}$ presents the 2-D $\delta$-function, we obtain using the known formula $\delta_2(a\mathbf{x}) = |a|^{-2}\delta_2(\mathbf{x})$:

$$\left\langle \left|\mathcal{J}\left(\mathbf{q}-\mathbf{q}_0,\nu+\nu_0\right)\right|^2\right\rangle = \frac{A}{4\pi^2\left(\nu+\nu_0\right)^2} W_\gamma\left(\frac{\mathbf{q}_0-\mathbf{q}}{\nu+\nu_0}\right)$$

Thus,

$$\Sigma_{0,\mathrm{geom}} = \frac{\left(k^2+\nu\nu_0-\mathbf{q}\mathbf{q}_0\right)^2}{\left(\nu+\nu_0\right)^4} W_\gamma\left(\frac{\mathbf{q}_0-\mathbf{q}}{\nu_0+\nu}\right) \tag{8.33}$$

According to this formula, in GO limit the scattering cross section is proportional to the PDF of slopes of surface satisfying the specular reflection condition. This result has a simple physical meaning: the scattering cross section is proportional to the number of surface facets having the appropriate slope.

## 9. Numerical results for the radar cross section for Cox-Munk PDF and 2-D anisotropic spectra

### 9.1. Determining the parameters of the PDF

We now consider an example, the joint PDF of slopes, taken from papers [5] and [6] for the wind speed $u_{10} = 10m/\sec$. We seek the parameters of approximation, $\kappa_{1,\mu}$, $\kappa_{2,\mu}$, $\lambda_\mu$, and $P_\mu$, by minimizing the integrated square of the difference between Cox and Munk function $W_{\mathrm{CM}}(\gamma_1,\gamma_2)$ and its approximation $W_\gamma(\gamma_1,\gamma_2)$, given by the formula (4.26):

$$E^2 \equiv \iint \left[W_{\mathrm{CM}}\left(\gamma_1,\gamma_2\right)-W_\gamma\left(\gamma_1,\gamma_2\right)\right]^2 d\gamma_1 d\gamma_2.$$

We used the N. Metropolis ("Simulated Annealing") minimization algorithm for searching the global minimum [22]. The parameters, obtained for this example, are presented in the following tables:

Table 1. Parameters of approximation of the Cox-Munk joint PDF for slopes for $u = 10m/s$

| $\mu$ | $P_\mu$ | $\lambda_\mu$ | $\kappa_{1\mu}$ | $\kappa_{2\mu}$ |
|---|---|---|---|---|
| $\pm 1$ | 0.2219658 | 1.1135315 | −0.04151202 | ±0.4965694 |
| $\pm 2$ | 0.2058264 | 0.6431118 | +0.01032697 | ±0.3410244 |
| $\pm 3$ | 0.0722078 | 1.5560527 | −0.09817264 | ±0.7710721 |

The difference between the joint PDF $W_{\mathrm{CM}}(\gamma_1, \gamma_2)$ [5] and the approximation $W_\gamma(\gamma_1, \gamma_2)$, given by (4.26), is presented in Figure 9.1. The relative accuracy of this approximation,

$$\delta = \frac{\sqrt{\iint \left[W_{\mathrm{CM}}(\gamma_1, \gamma_2) - W_\gamma(\gamma_1, \gamma_2)\right]^2 d\gamma_1 d\gamma_2}}{\sqrt{\iint W_\gamma^2(\gamma_1, \gamma_2)\, d\gamma_1 d\gamma_2}}$$

is about 7.7 %.[7]

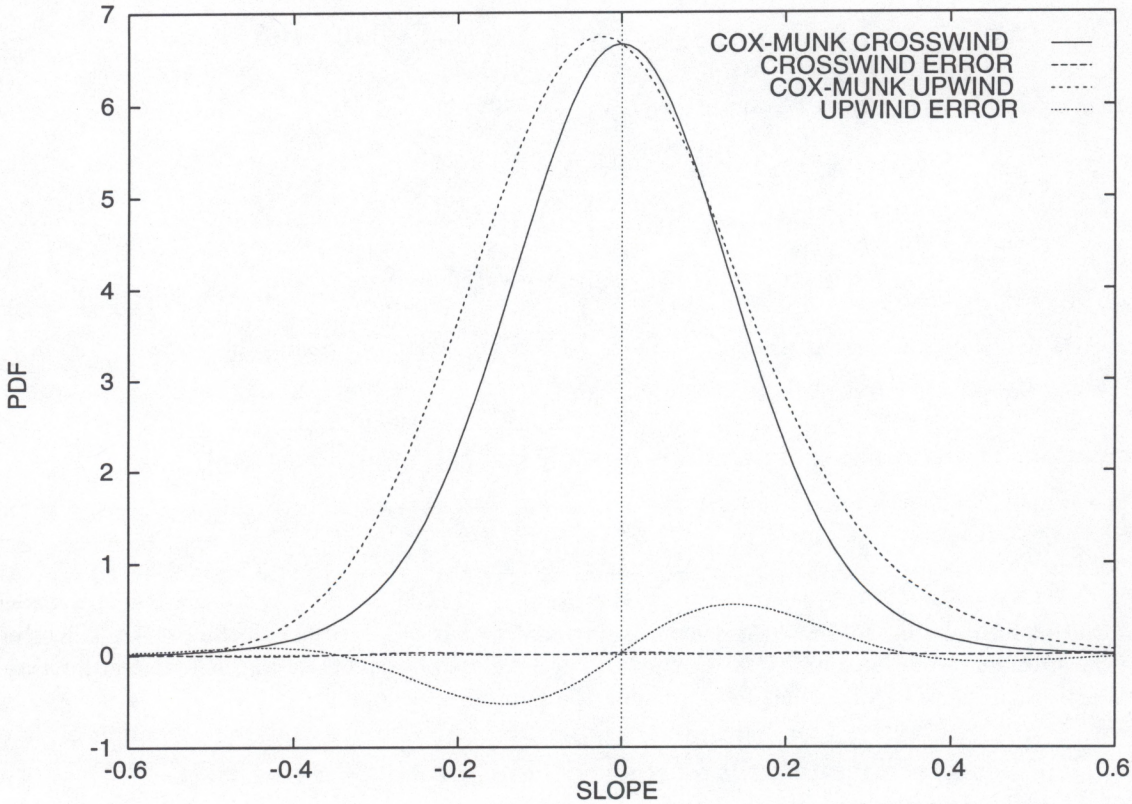


Figure 9.1: Approximation of the Cox-Munk upwind and crosswind PDF for the wind-speed $u_{10} = 10m \cdot s^{-1}$ and errors of approximations. Real approximation was performed for 2-d PDF, but to make the results clearer we presented only two cross sections of the 2-d PDF.

Similar measurements made with a scanning laser slope gauge were published in [3] and [30].

In applications not only the PDF is important but also the spectrum, or correlation (structure) function. In our previous paper [36] we used the generalized experimental spectrum of surface presented in Apel's paper [1]. But this spectrum does not agree with the PDF of slopes based on Cox and Munk data.[5]

[7] We choose such normalization that the ratio is dimensionless.

According to (4.12), the values $\langle\gamma_1^2\rangle$, $\langle\gamma_2^2\rangle$ can be expressed in terms of $D(\mathbf{r})$ by the formulae

$$\langle\gamma_1^2\rangle = \lim_{l_1\to 0}\frac{D(l_1\mathbf{m}_1)}{l_1^2},\ \langle\gamma_2^2\rangle = \lim_{l_2\to 0}\frac{D(l_2\mathbf{m}_2)}{l_2^2}. \tag{9.1}$$

Substituting the spectral representation (4.6),

$$D(r,\psi) = 2\iint[1-\cos(\mathbf{qr})]\,\Phi(q,\varphi)\,qdqd\varphi,$$

we find

$$\langle\gamma_1^2\rangle = \lim_{l\to 0}\frac{2}{l^2}\int_0^\infty qdq\int_0^{2\pi}[1-\cos(ql\cos\varphi)]\,\Phi(q,\varphi).$$

Because

$$\lim_{l\to 0}\frac{2}{l^2}[1-\cos(ql\cos\varphi)] = \lim_{l\to 0}\frac{2}{l^2}\cdot 2\sin^2\left(\frac{ql\cos\varphi}{2}\right) = q^2\cos^2\varphi$$

we obtain:

$$\int W_1(\gamma_1)\,\gamma_1^2 d\gamma_1 = \langle\gamma_1^2\rangle = \int_0^\infty q^3dq\int_0^{2\pi}\Phi(q,\varphi)\cos^2\varphi d\varphi. \tag{9.2}$$

Similarly,

$$\int W_2(\gamma_2)\,\gamma_2^2 d\gamma_2 = \langle\gamma_2^2\rangle = \int_0^\infty q^3dq\int_0^{2\pi}\Phi(q,\varphi)\sin^2\varphi d\varphi. \tag{9.3}$$

If we find the values $\langle\gamma_1^2\rangle$ and $\langle\gamma_2^2\rangle$ from the Cox and Munk data (the integrals on the left-hand sides of (9.2) and (9.3)), we obtain significant differences from the values calculated via the integrals on the right-hand sides of (9.2) and (9.3).[8] In the paper of Elfouhaily et al. [9] this controversy was resolved by incorporating the slope data in the spectrum. Because of this, we used in our calculations the generalized spectrum suggested in [9].[9]

### 9.2. Calculations of the radar cross sections in the Kirchhoff approximation

We used the formula (8.16) for calculations of the scattering cross sections. The functions entering in (8.16) are determined by the formulae (8.10) (for $\mathcal{F}(l_1,l_2)$), (8.13) (for $A$), (8.17-8.18) (the values of $\mathcal{F}(l_1,\pm\infty)$, $\mathcal{F}(\pm\infty,l_2)$, and $\mathcal{F}(\pm\infty,\pm\infty')$), and ($A$.8) (for structure function of elevations $D(r,\psi)$). The function $D(r,\psi)$ was calculated by the formula ($A$.8) on the basis of the spectrum published in [9]. Special attention was paid to such approximation of the function $D(r,\psi)$, which provides the correct value of the second derivatives in the point $r=0$ (the correct values of $\langle\gamma_1^2\rangle$ and $\langle\gamma_2^2\rangle$). In this case the results of the Kirchhoff approximation match with the limiting case of GO approximation that is given by the formula (8.33).

In Figure 9.2 we present the results of calculations of the radar cross section as a function of the grazing incident angle $\theta$ for different values of $k$. We used the anisotropic 2-D spectrum of wind-driven waves taken from [9] for the wind speed $u = 10m\cdot s^{-1}$ and the Cox-Munk PDF of slopes for the same wind speed.

To estimate how the Cox-Munk PDF influences the scattering cross section, we performed the calculations for the same anisotropic spectrum of surface waves but used the Gaussian PDF of slopes. The results are presented in Figure 9.3. To eliminate the possible influence of computational errors on the results in the region of small grazing angles, we present the results for the limiting case of GO approximation $k=\infty$ when we can use analytical formula (8.33).

The same result is presented in the Figure 9.4 in the logarithmic scale where the ratio of the Cox-Munk and the Gaussian cases is clearer in the region of small grazing angles.

Similar results were obtained for the finite values of wavenumber. In Figure 9.5 we present the angular dependence of the radar cross section for $k = 30cm^{-1}$ for the Cox-Munk and the Gaussian PDF of slopes.

[8] The necessity to match the PDF of slopes and the spectrum was noted in paper [26].

[9] We are grateful to the authors of [9] who supplied us with the program for numerically calculating the spectrum.

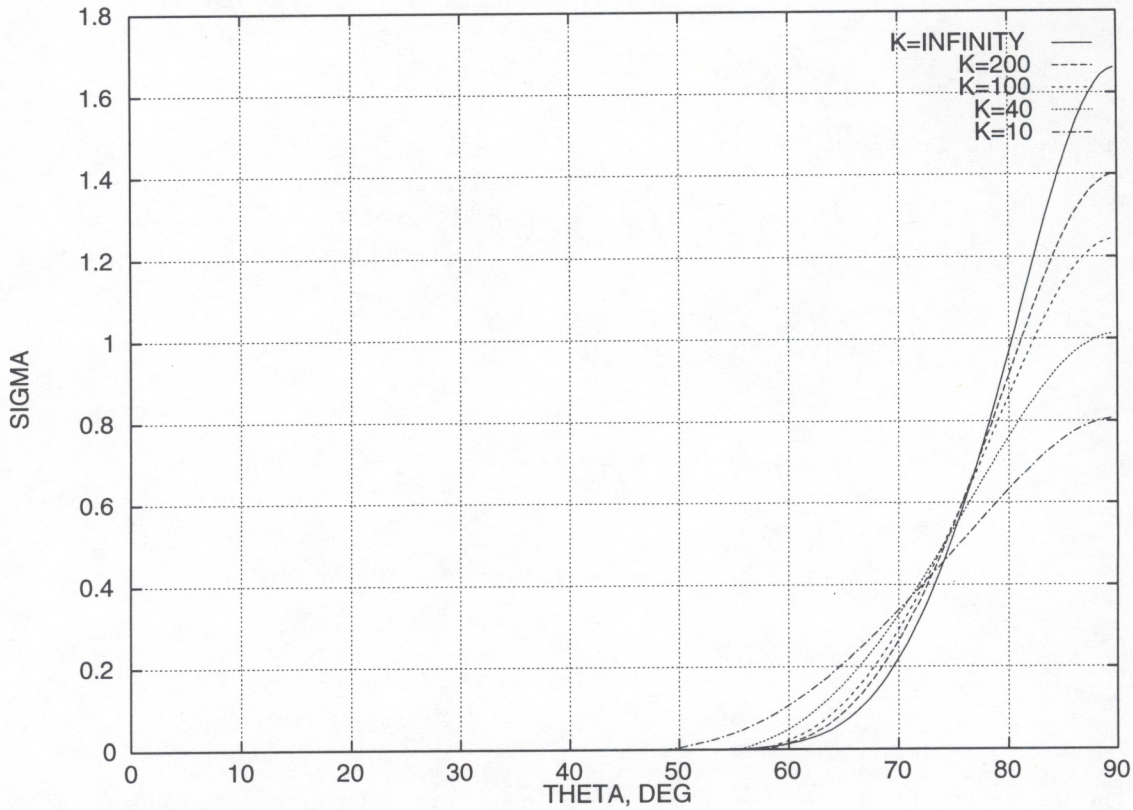


Figure 9.2: Backscattering cross section as a function of the grazing incident angle $\theta$ for different values of $k$.

In Figure 9.6 we present the radar cross section as a function of azimuthal angle. Both the anisotropy of the spectrum and anisotropy of the PDF affect this angular dependence in the case of the finite $k$ (i.e., in the Kirchhoff approximation). The difference between the curves corresponding to the Cox-Munk and the Gaussian PDF is caused by the anisotropy of PDF. In the case of GO limit, only the anisotropy of the PDF is important.

Experimental data on the azimuthal dependence of the radar cross section (see, e.g., [17], [31], [21]) are in qualitative agreement with the results presented in 9.6.[10]

## 10. Conclusions

### 10.1. Comparison with other methods of statistical description of sea surface

There are several different approaches to the problem of statistical description of sea surfaces. All of these approaches are based on the general theory of random functions (see, e.g., [41], [25], [33], and [29]). The paper of Longuet-Higgins[19], devoted to random surfaces, served as a starting point for works, describing the statistics of nonlinear surface waves. The paper [19] is based on a special model of the rough surface. This model is equivalent to the following representation of a random 2-D field:

$$\zeta(\mathbf{r}) = \iint \xi(\mathbf{q}) \exp(i\mathbf{q}\mathbf{r})\, d^2q. \tag{10.1}$$

[10]The measurements described in [17], [31], and [21] were performed in the range of Bragg scattering. Because of this, we cannot expect quantitative agreement of these results with the results of the calculations we performed in the Kirchhoff approximation (for much larger values of $k$).

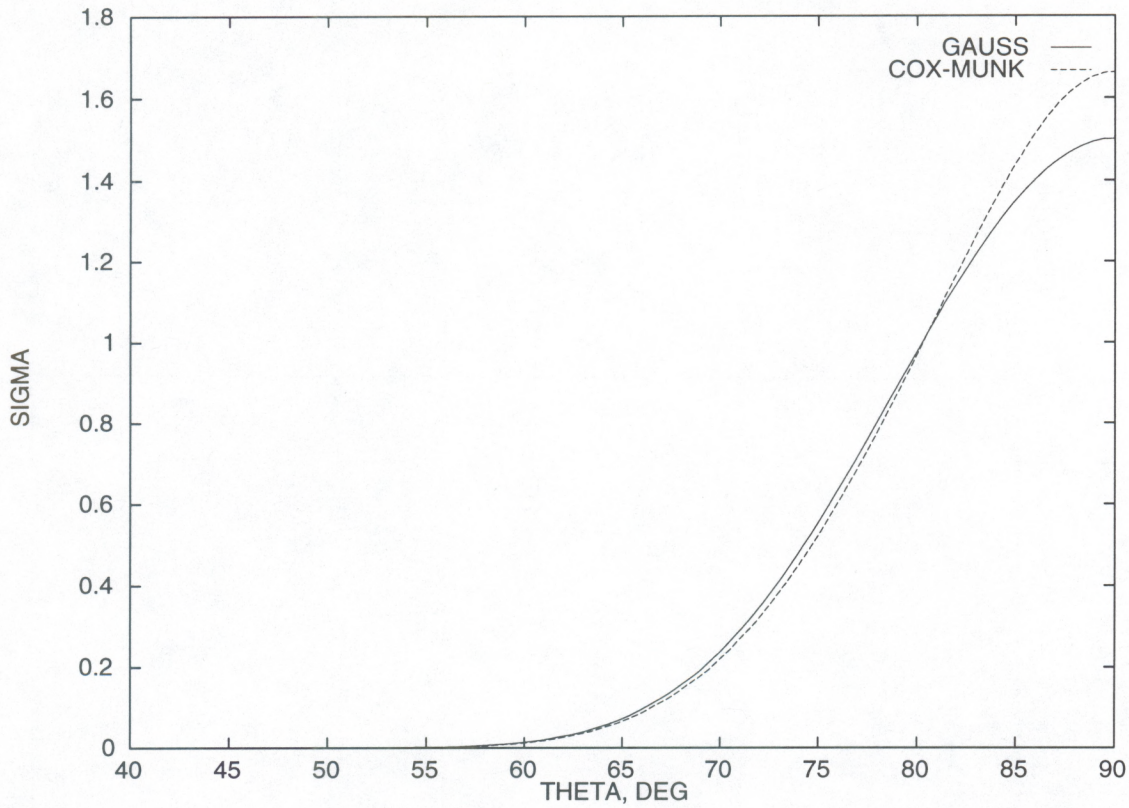


Figure 9.3: Angular dependence of the back-scattering cross sections for Cox-Munk PDF and for Gaussian approximation of PDF in the geometric optics limit ($k = \infty$).

Here, the random spectral density $\xi(\mathbf{q})$ is determined by the following relations:

$$\begin{aligned} &\xi(\mathbf{q}) \text{ is Gaussian random function} \\ &\langle \xi(\mathbf{q}) \rangle = 0 \\ &\langle \xi(\mathbf{q}') \xi^*(\mathbf{q}'') \rangle = E(\mathbf{q}') \delta(\mathbf{q}' - \mathbf{q}'') \\ &\langle \xi(\mathbf{q}') \xi(\mathbf{q}'') \rangle = \langle \xi^*(\mathbf{q}') \xi^*(\mathbf{q}'') \rangle = 0. \end{aligned} \tag{10.2}$$

Representation (10.2) is widely used in the theory of turbulence and wave propagation in random media [33], [29].[11] Thus, the random surfaces considered in [19] are Gaussian.

In the following paper of Longuet-Higgins [20], the model of random functions developed in [19], was applied to nonlinear surface gravity waves. In this case, the surface is non-Gaussian and the following decomposition (in terms of (10.2)) was used:

$$\begin{aligned} \zeta(\mathbf{r}) = &\iint \xi(\mathbf{q}) \exp(i\mathbf{q}\mathbf{r})\, d^2q + \iint d^2q_1 \iint d^2q_2 \exp[i(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{r}]\, C_2(\mathbf{q}_1, \mathbf{q}_2)\, \xi(\mathbf{q}_1)\, \xi(\mathbf{q}_2) + \\ &\iint d^2q_1 \iint d^2q_2 \iint d^2q_3 \exp[i(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)\mathbf{r}]\, C_3(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)\, \xi(\mathbf{q}_1)\, \xi(\mathbf{q}_2)\, \xi(\mathbf{q}_3) + \cdots. \end{aligned} \tag{10.3}$$

[11] In paper [19] the more cumbersome representation that includes the finite sum

$$\zeta(\mathbf{r}) = \mathrm{Re} \sum_{n=1}^{N} c_n \exp(i\mathbf{k}_n \mathbf{r})$$

with the random coefficients $c_n$ and the following limiting process $N \to \infty$, was used. But all of the results of this approach are obtainable from a more compact representation (10.2).

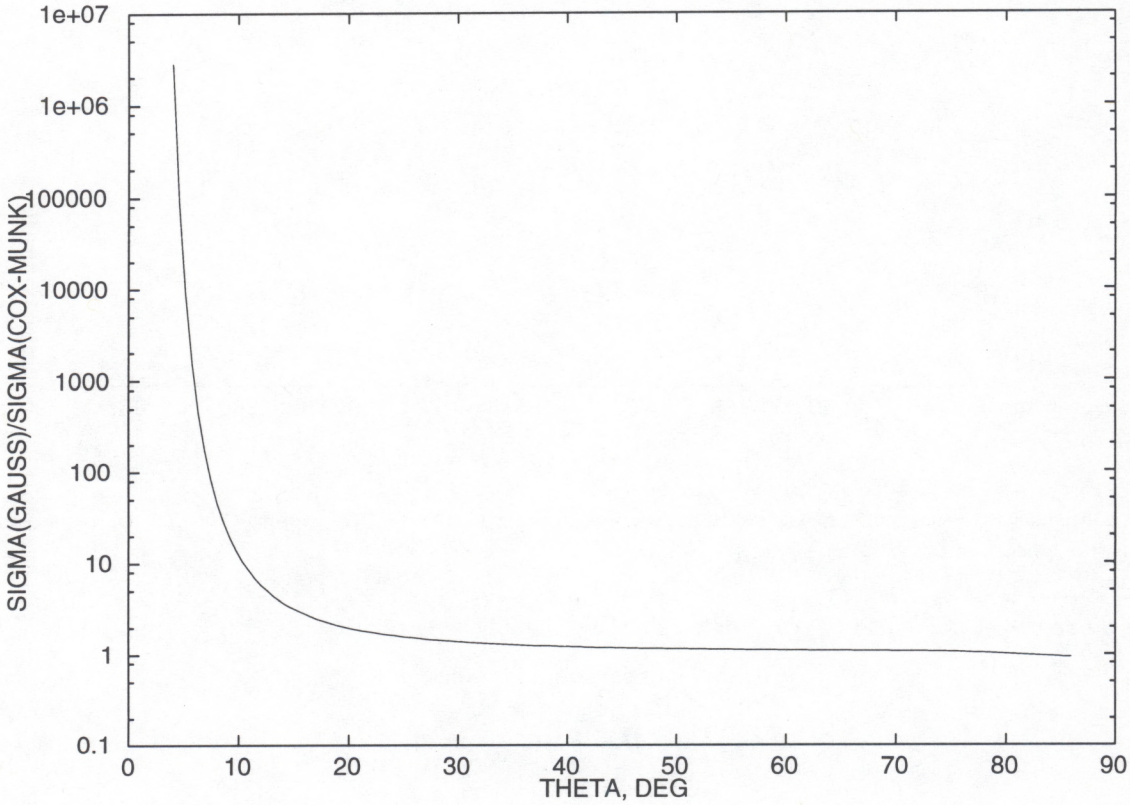


Figure 9.4: The ratio of the backscattering cross sections for Gaussian and Cox-Munk cases in the limit of geometrical optics ($k = \infty$).

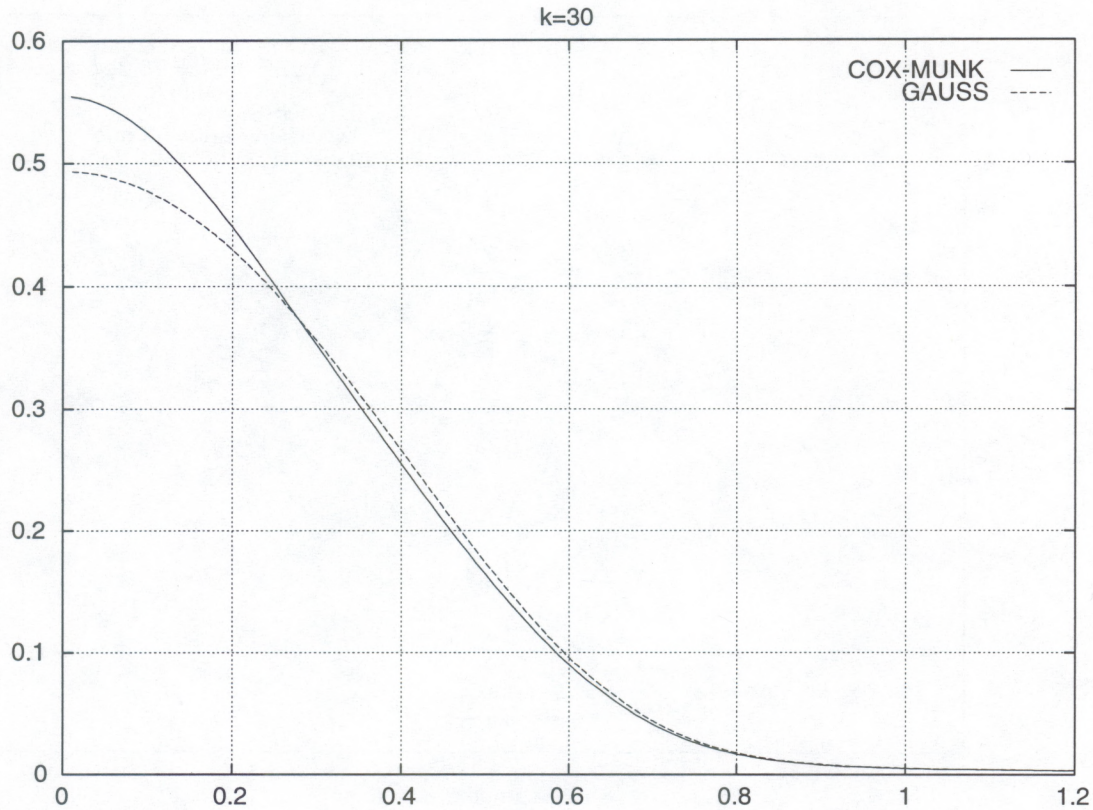


Figure 9.5: The radar cross section as a function of the incident angle $\alpha = 90^0 - \theta$ for $k = 30cm^{-1}$. Similar to the geometric optic limit, the Gaussian approximation reduces the radar cross section in the region near the nadir and increases it in the region of small grazing angles.

The coefficients $C_k(\mathbf{q}_1, ..., \mathbf{q}_k)$ were determined by the substitution of (10.3) in the hydrodynamic equations, expanded in the perturbation series in powers of $\zeta$. As a result, the expansion of the non-Gaussian PDF in the Gram-Charlier series was obtained. This method describes only small deviations from the Gaussian distribution, because it uses the perturbation expansion. The random Gaussian field $\xi(\mathbf{q})$, entering in (10.3), is completely auxiliary and has no direct meaning.

The model of [20] was used in [16] for description of radar impulses from the sea surface in GO approximation. This model was extended in [32] for the joint PDF of elevation and two slopes, and applied to radar altimetry.

In [13] the method of [20] was generalized for random Stokes waves. This work also starts from the auxiliary Gaussian field, but the field undergoes some nonlinear transforming, induced by the shape of the Stokes wave. As a result an explicit formula for the PDF of elevations was obtained. In paper [14] the same method was applied to the joint PDF of elevation and slope for the random Stokes waves. This was possible because of the dynamic relationship between the elevation and slope for the Stokes wave. In [38] the restrictions related to the appearance of breaking waves were included in the consideration.[12]

The main goal of the present paper is to develop a model of the sea surface that will allow us to calculate the CF of the surface of the arbitrary order, and at the same time satisfy necessary conditions for the second-order PDF

[12]The assumption that the first-order solution is Gaussian, used in [20] and in the many following papers, is an additional assumption. It is rather difficult to ground this assumption, because in the presence of nonlinear effects, the first-order component by itself has no physical meaning. It is applied to a nonexistent physical object. If the waves are really linear (amplitudes are very small), the Gaussian PDF seems to be natural, but this fact has no relation to the first-order component of the nonlinear waves. Thus, this assumption can be considered as a convenient and effective working hypothesis, but only successful comparison with the experimental data can serve as a justification for its use.

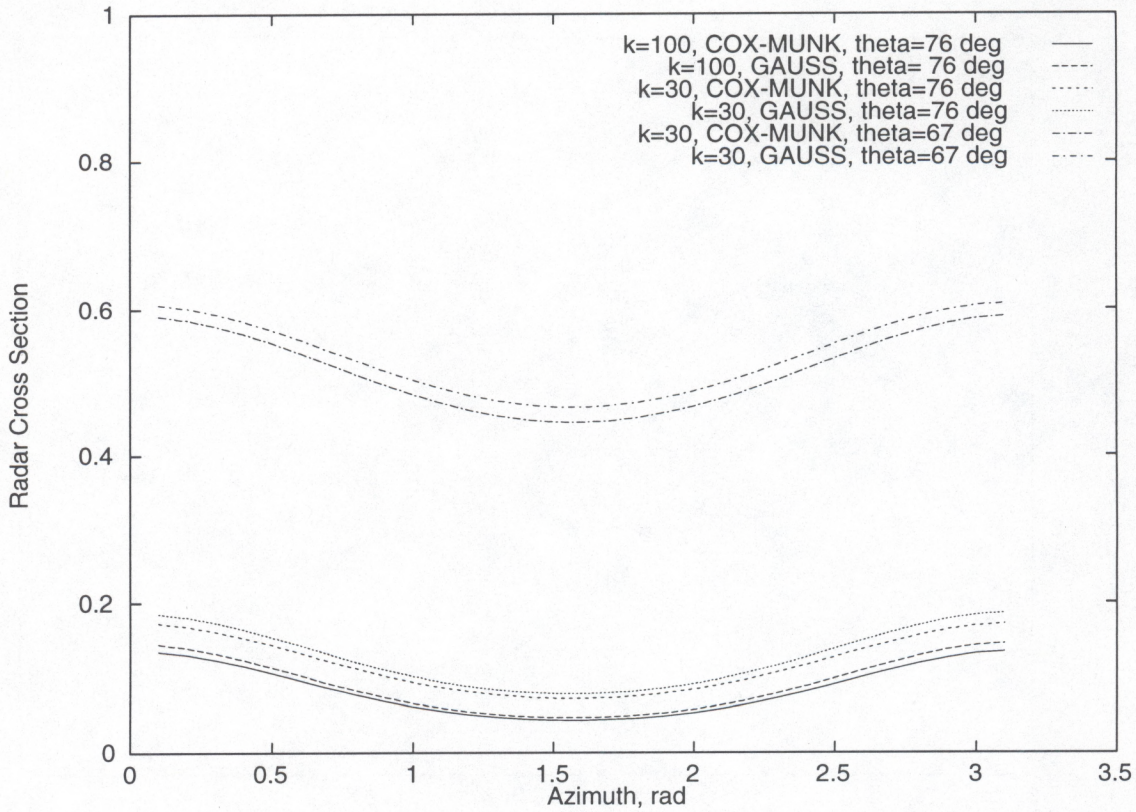


Figure 9.6: Azimuthal dependence of the radar cross section for the wind speed $u_{10} = 10m/s$ for different values of $k$ and different grazing angles $\theta$.

of slopes and for the spectrum. Such CF appear in the modern theories of rough-surface scattering (see section 2 of this paper). The above mentioned sea surface models do not allow us to achieve this goal. For instance, the approach of papers [20] and [32] leads to the truncated Gram-Charlier series that necessarily entail negative probabilities. The method of papers [13], [14], and [38] is free from this disadvantage, but it does not allow us to find high-order CF, as well as the method in papers [20] and [32]. For instance, non of the above-mentioned methods allow us to calculate the scattering cross sections even in the Kirchhoff approximation; only the simplest GO approximation can be considered. The method developed in this paper allows us calculating a scattering cross section in any scattering theory (see section 2).

Another difference in the method presented in this paper is its phenomenological nature. We did not try to utilize dynamical equations of motion, but used the experimental data instead. However, we could have used not only experimental data, but also any results of theoretical consideration.

Usage of the decomposition of the multivariate non-Gaussian PDF in the sum of Gaussian PDF allows us to describe such non-Gaussian PDF without the difficulties related to the truncation of the Gram-Charlier series. This method can be applied to various problems dealing with non-Gaussian distributions.

We should emphasize that the model of the random surface developed is *not ergodic*. This means that it is impossible to create a single surface that is large enough that the averaging over this single surface leads to the same mean values as statistical averaging. Each realization of the surface has the (conditional) Gaussian PDF. If we want to use this mathematical model for some numerical method of calculation of the scattering field, and apply the Monte-Carlo simulation method (instead of analytical averaging) we must first prepare the set of models of Gaussian random surfaces. Each of these surfaces must have its specific values of $\lambda_\mu$, $\kappa_{\mu,1}$, and $\kappa_{\mu,2}$, and the

total number of surfaces having these parameters and included in the ensemble must be proportional to $P_\mu$.[13]

### 10.2. Summary

The main results of this paper are as follows:

**1.** For the mathematical description of the (multivariate) non-Gaussian probability distributions we used a decomposition of an arbitrary PDF in the sum of auxiliary Gaussian PDF, having different parameters. This method can successfully replace the standard representation of non-Gaussian distributions in terms of the Edgeworth or Gram-Charlier series and is free from the main disadvantage of these approaches – negative probabilities. This method is very simple in application and it easily allows one to find different mean values as a sum of corresponding partial Gaussian mean values.

**2.** We obtained the multivariate characteristic function for an arbitrary number of differences in elevation of a random surface. The corresponding probability distribution satisfies the following conditions: (a) the spectrum (or correlation function) of surface is the given (anisotropic) function, and (b) the joint probability distribution of two principal slopes of the surface is the given (anisotropic) function.

**3.** We used the generalized experimental data for the spectrum of sea surface from paper [9], and the data for joint PDF of slopes from papers [5] and [6] for the windspeed $10\,m \cdot s^{-1}$.

**4.** We calculated the scattering cross sections for the absolutely reflecting air-sea interface in the Kirchhoff approximation for the Gaussian and non-Gaussian (Cox-Munk) joint PDF of slopes, and found a significant difference between these two cases, especially in the range of small grazing angles.

**5.** We obtained the universal angular dependence of the variance of slope for the case in which the spectrum is symmetrical with respect to some direction (wind direction in our case). This result agrees well with the experimental data; it follows only from the symmetry of spectrum and does not depend on the probability distribution (see Appendix A).

## A. Angular dependence of the variance of slope

The slope of a surface at a point $\mathbf{r}$ in a direction determined by the unit vector $\mathbf{n}$, is given by the formula

$$\gamma(\mathbf{n},\mathbf{r}) \equiv \mathbf{n}\boldsymbol{\nabla}\zeta(\mathbf{r}). \tag{A.1}$$

We assume that the spectrum of surface $\Phi(\mathbf{q})$ is symmetrical with respect to wind direction, determined by the unit vector $\mathbf{m}_1$. If we choose the $x$-axis along the vector $\mathbf{m}$, we obtain

$$\mathbf{m}_1 = (1,0)\,;\ \ \mathbf{q} = (q\cos\varphi, q\sin\varphi)\,, \tag{A.2}$$

where $\varphi$ is the angle between $\mathbf{q}$ and the wind direction. The symmetry of the spectrum with respect to the wind direction means that

$$\Phi(q,\varphi) = \Phi(q,-\varphi)\,. \tag{A.3}$$

The structure function of the surface in terms of the spectrum $\Phi$ has the form (see (4.6))

$$D(r,\psi) = 2\iint [1-\cos(\mathbf{qr})]\,\Phi(q,\varphi)\,qdqd\varphi, \tag{A.4}$$

where $\mathbf{r} = \mathbf{r}' - \mathbf{r}'' = \mathbf{n}r$. We present the vector $\mathbf{r}$ in the form

$$\mathbf{r} = \mathbf{n}r;\ \ \mathbf{n} = (\cos\psi, \sin\psi)\,, \tag{A.5}$$

where $\psi$ is the angle between $\mathbf{r}$ and the wind direction. For the scalar product $\mathbf{qr}$, entering in $(A.4)$, we have

$$\mathbf{qr} = qr\cos(\varphi-\psi)\,. \tag{A.6}$$

[13] If we put all of these samples on the single joint surface, we obtain a nonhomogeneous random surface with the *Gaussian* PDF and the correlation function, depending not only on the differences of coordinates, but on both coordinates. The sum of the partial PDF of the form $W = \sum_\mu P_\mu W_\mu$ corresponds to the random choice of the surface numbered by $\mu$ with the probability $P_\mu$.

Using the known formula [11], **8.511.4**,

$$\cos\left[qr\cos\left(\varphi-\psi\right)\right]=J_0\left(qr\right)+2\sum_{n=1}^{\infty}\left(-1\right)^n J_{2n}\left(qr\right)\cos\left(2n\varphi-2n\psi\right), \tag{A.7}$$

we obtain for $D\left(r,\psi\right)$:

$$\begin{aligned} D\left(r,\psi\right)=D\left(\mathbf{r}\right)=2\int_0^{\infty}\left[1-J_0\left(qr\right)\right]qdq\int_0^{2\pi}\Phi\left(q,\varphi\right)d\varphi- \\ -4\sum_{n=1}^{\infty}\left(-1\right)^n\int_0^{\infty}J_{2n}\left(qr\right)qdq\int_0^{2\pi}\Phi\left(q,\varphi\right)\cos\left(2n\varphi-2n\psi\right)d\varphi. \end{aligned} \tag{A.8}$$

The slope of the surface, taken in the direction $\mathbf{n}$, according to $(A.1)$ can be presented in the form

$$\gamma\left(\mathbf{r},\mathbf{n}\right)=\lim_{l\to 0}\frac{\zeta\left(\mathbf{r}+\mathbf{n}l\right)-\zeta\left(\mathbf{r}\right)}{l}. \tag{A.9}$$

Therefore,

$$\left\langle\gamma^2\left(\mathbf{r},\mathbf{n}\right)\right\rangle=\lim_{l\to 0}\frac{\left\langle\left[\zeta\left(\mathbf{r}+\mathbf{n}l\right)-\zeta\left(\mathbf{r}\right)\right]^2\right\rangle}{l^2}=\lim_{l\to 0}\frac{D\left(\mathbf{n}l\right)}{l^2}, \tag{A.10}$$

where $\mathbf{n}l$, the argument of the structure function, is equal to $(l\cos\psi, l\sin\psi)$. If we substitute $(A.8)$ in $(A.10)$ and set $r=l\to 0$, the known limits

$$\lim_{l\to 0}\frac{1-J_0\left(ql\right)}{l^2}=\frac{q^2}{4},\ \lim_{l\to 0}\frac{J_2\left(ql\right)}{l^2}=\frac{q^2}{8},\ \text{and}\ \lim_{l\to 0}\frac{J_{2n}\left(ql\right)}{l^2}=0,\ n=2,3,... \tag{A.11}$$

appear.

Therefore, only two beginning terms of expansion survive while $l\to 0$ and the result is

$$\lim_{l\to 0}\frac{D\left(l,\psi\right)}{l^2}=\frac{1}{2}\int q^3dq\int\Phi\left(q,\varphi\right)d\varphi++\frac{1}{2}\int q^3dq\int\Phi\left(q,\varphi\right)\cos\left(2\varphi-2\psi\right)d\varphi. \tag{A.12}$$

But

$$\cos\left(2\varphi-2\psi\right)=\cos\left(2\varphi\right)\cos\left(2\psi\right)+\sin\left(2\varphi\right)\sin\left(2\psi\right), \tag{A.13}$$

and after integration over $\varphi$ in $(A.12)$ the term containing $\sin\left(2\varphi\right)$ vanishes because of $\Phi\left(q,\varphi\right)=\Phi\left(q,-\varphi\right)$. Thus, the general result is

$$\begin{aligned} \left\langle\gamma^2\left(\psi\right)\right\rangle=\lim_{l\to 0}\frac{D\left(l,\psi\right)}{l^2}=\frac{1}{2}\int q^3dq\int\Phi\left(q,\varphi\right)d\varphi+ \\ \frac{1}{2}\cos\left(2\psi\right)\int q^3dq\int\Phi\left(q,\varphi\right)\cos\left(2\varphi\right)d\varphi==a+b\cos\left(2\psi\right), \end{aligned} \tag{A.14}$$

where

$$a=\frac{1}{2}\int q^3dq\int\Phi\left(q,\varphi\right)d\varphi,\ b=\frac{1}{2}\int q^3dq\int\Phi\left(q,\varphi\right)\cos\left(2\varphi\right)d\varphi. \tag{A.15}$$

If we substitute in $(A.14)$

$$\cos 2\psi=\cos^2\psi-\sin^2\psi,$$

we obtain

$$\left\langle\gamma^2\left(\psi\right)\right\rangle=A\cos^2\psi+B\sin^2\psi \tag{A.16}$$

where

$$\begin{aligned} A=a+b=\int_0^{\infty}q^3dq\int_0^{2\pi}\Phi\left(q,\varphi\right)\cos^2\varphi d\varphi\geq 0, \\ B=a-b=\int_0^{\infty}q^3dq\int_0^{2\pi}\Phi\left(q,\varphi\right)\sin^2\varphi d\varphi\geq 0. \end{aligned} \tag{A.17}$$

If we set $\psi = 0$ in $(A.16)$, we obtain the variance of the slope in the upwind direction:

$$\langle \gamma^2 (0) \rangle \equiv \langle \gamma_1^2 \rangle = A. \tag{A.18}$$

If we set $\psi = \pi/2$ in $(A.16)$, we obtain the variance of the slope in the crosswind direction:

$$\langle \gamma^2 (\pi/2) \rangle \equiv \langle \gamma_2^2 \rangle = B. \tag{A.19}$$

Therefore, formula $(A.16)$ can be presented as follows:

$$\langle \gamma^2 (\psi) \rangle = \langle \gamma_1^2 \rangle \cos^2 \psi + \langle \gamma_2^2 \rangle \sin^2 \psi. \tag{A.20}$$

From $(A.1)$ we find that the random value of the slope can be presented in the form

$$\gamma (\mathbf{n}, \mathbf{r}) = \gamma (\mathbf{r}, \psi) = \gamma_1 \cos \psi + \gamma_2 \sin \psi \tag{A.21}$$

where

$$\gamma_1 = \frac{\partial \zeta (\mathbf{r})}{\partial x}, \ \gamma_2 = \frac{\partial \zeta (\mathbf{r})}{\partial y} \tag{A.22}$$

are the random values of the slopes in the two principal (upwind and crosswind) directions. If we calculate the mean square of $(A.21)$, we obtain

$$\langle \gamma^2 (\psi) \rangle = \langle \gamma_1^2 \rangle \cos^2 \psi + + \langle \gamma_2^2 \rangle \cos^2 \psi + 2 \langle \gamma_1 \gamma_2 \rangle \sin \psi \cos \psi. \tag{A.23}$$

From comparison of this formula with $(A.20)$ we find that

$$\langle \gamma_1 \gamma_2 \rangle \equiv 0. \tag{A.24}$$

We emphasize that the main results of Appendix A, formulae $(A.20)$ and $(A.24)$, are the precise consequences of the symmetry of the spectrum $(A.3)$ and do not depend on the PDF of surface. The symmetry condition is enough to derive these formulae.

Note that the angular dependence of the form $(A.14)$ was widely used in many experimental works.

## References


[1] Apel, J.R., 1994. An improved model of the ocean surface wave vector spectrum and its effects on radar backscatter. J. Geophys. Res./Oceans **99**.

[2] Beckmann, P., 1973. Orthogonal Polynomials for Engineers and Physicist (Section 6.2). The Golem Press, Boulder, Col.

[3] Bock, E. J. and T. Hara, 1995. Optical measurements of capillary-gravity wave spectra using a scanning laser slope gauge. J. Atmos. Ocean. Technol. **12**(2): 395-403.

[4] Charnotskii, M.I. and V.I. Tatarskii, 1995. Tilt-invariant theory of rough-surface scattering: I. *Waves in Random Media* **5**: 361-80.

[5] Cox, C. and W. Munk, 1954. Statistics of the sea surface derived from sun glitter. J. Mar. Res. **13:** 198-227.

[6] Cox, C. and W. Munk, 1954. Measurement of the roughness of the sea surface from photographs of the sun's glitter. J. Opt. Soc. Am. **44**(11): 838-850.

[7] DeWolf, D.A., 1984. Private communication.

[8] DeWolf, D.A., 1989. Gaussian decomposition of beams and other functions. J. Appl. Phys. **65**(12): 5166-5169.

[9] Elfouhaily, T., B. Chapton, K. Katsaros, and D. Vandemark, 1997. A unified directional spectrum for long and short wind-driven waves. J. Geophys. Res. **102**(C7): 15,781-15,796.

[10] Fung, A.K., and K.K. Lee, 1982. A semi-empirical sea-spectrum model for scattering coefficient estimation. IEEE J. Oceanic Eng., **7**(4): 166-176.

[11] Gradshteyn, I.S., and I.M. Ryzhik, 1980. Table of Integrals, Series, and Products. Corrected and enlarged ed. prepared by A. Jeffrey. Academic Press, Inc., San Diego, CA.

[12] Huang, N.E., and S.R. Long, 1980. An experimental study of the surface elevation probability distribution and statistics of wind-generated waves. J. Fluid Mech. **101**(1):179-200.

[13] Huang, N.E., S.R. Long, C. C. Tung, and Y. Yuan, 1983. A non-Gaussian statistical model for surface elevation of nonlinear random wave field. J. Geophys. Res. **88**(C12): 7,597-7,606.

[14] Huang, N.E., S.R. Long, L.F. Bliven, and C.C. Tung, 1984. The non-Gaussian joint probability density function of slope and elevation for nonlinear gravity wave field. J. Geophys. Res. **89**:1961-1972.

[15] Ishimaru, A., and J.S. Chen, 1990. Scattering from very rough surfaces based on the modified second-order Kirchhoff approximation with angular and propagation shadowing. *J. Acoust. Soc. Am.* **88**(4):1877-83.

[16] Jackson, F.C., 1979. The reflection of impulses from a nonlinear random sea. J. Geophys. Res., **84**: 4939-4943.

[17] Jones, W.L., and L.C. Schroeder, 1978. Radar backscatter from the ocean: dependence on surface friction velocity. Boundary-Layer Meteorol., **13**, :133-149.

[18] Kendall, M., and A. Stuart, 1977. The Advanced Theory of Statistics, 4th ed., **1**, section 4.8, p.105 (Marcinkiewicz theorem) and Exerc. 4.11, p. 125.

[19] Longuet-Higgins, M.S., 1962. The statistical geometry of random surfaces. In: Proceedings of Symposia in Applied Mathematics, Vol. XIII, "Hydrodynamic Instability." Amer. Math. Soc., Providence RI:105-143.

[20] Longuet-Higgins, M.S., 1963. The effect of non-linearities on statistical distributions in the theory of sea waves. J. Fluid Mech. **17**:459-480.

[21] Masuko H., K. Okamoto, M. Shimada, and S. Niwa, 1986. Measurement of microwave signatures of the ocean surface using X and Ka band airborne scatterometers. J. Geophys. Res. **91**(C11):13,065-13,083.

[22] Minimizing Multimodal Functions of Continuous Variables with the "Simulated Annealing" Algorithm. In the September 1987 (**13**(3):262-280) issue of the ACM Transactions on Mathematical Software.

[23] Mitsuyasu, H., F. Taasi, T. Suhara, S. Misuno, M. Ohkusu, T. Honda, and K. Rikiishi, 1975. Observations of the directional spectrum of the ocean waves using a cloverleaf buoy. J. Phys. Oceanogr. **5**(4):750-760.

[24] Newsam, G.N., and M. Wegener, 1994. Generating Non-Gaussian Random Fields for Sea Surface Simulations. Proceedings of the 1994 International Conference on Acoustics, Speech and Signal Processing, Adelaide, South Australia,April 19-22, 1994.

[25] Nickolaev, N.I., O.I. Yordanov, and M.A. Mickalev, 1991. Radar backscattering from non-Gaussian sea surface. Proceedings of IGARSS'91, Espoo, Finland, June 3-6, 1991, **3**: 1275-1278.

[26] Nickolaev, N.I., O.I. Yordanov and M.A. Michalev, 1992. Non-Gaussian effects in two-scale model for rough surface scattering. Journal Geophys. Res. **97**(C10): 15.617-15.624.

[27] Rosenfeld, S. 1986. Modeling a pseudo-random sequence which has specified correlation and probability structure. Meteorol. and Hydrol. **3**:9-14, (in English and Russian).

[28] Rosenfeld, S. 1988. Some new methods for generating non-Gaussian autocorrelated sequences. Advances of Soviet Scientific Center for Hydro-meteorological Information (World Data Center "B") **153**. Gidrometeoizdat, Moscow, Russia (in Russian; abstract in English)

[29] Rytov, S.M., Yu.A. Kravtsov, and V.I. Tatarskii. Principles of Statistical Radiophysics. vol. 1: Elements of Random Processes Theory (1987); vol. 2: Correlation Theory of Random Processes (1988); vol. 3: Elements of Random Fields (1989); vol. 4: Wave Propagation Through Random media (1989). Springer-Verlag, Berlin.

[30] Shaw, J.A., and J.H. Churnside, 1997. Scanning-laser glint measurements of sea-surface slope statistics. Appl. Opt. **36**(18): 4202-4213.

[31] Schroeder, L.C., P.R. Schaffner, J.L. Mitchell, and W.L. Jones, 1985. AAFE RADSCAT 13.9 GHz measurements and analysis: wind-speed signature of the ocean. IEEE J. Oceanic Eng. , **10**(4):346-357.

[32] Srokosz, M.A., 1986. On the joint distribution of surface elevation and slopes for a nonlinear random sea, with an application to radar altimetry. J. Geophys. Res.**91**(C1): 995-1006.

[33] Tatarskii, V.I., 1971. The Effects of the Turbulent Atmosphere on wave Propagation. Published for NOAA and NSF by the Israel Program for Scientific Translations, Jerusalem.

[34] Tatarskii, V.I., 1993. The expansion of the solution of the rough-surface scattering problem in powers of quasi-slopes. Waves in Random Media **3**:127-146.

[35] Tatarskii, V.I., and G.A. Voronovich, 1994. The role of non-Gaussian probability distribution in the wave scattering by rough surface theory. Proceedings of PIERS'94, July 11-15, 1994, Noordwijk, The Netherlands, :354.

[36] Tatarskii, V.V., and V.I. Tatarskii, 1996. Non-Gaussian statistical model of the ocean surface for wave-scattering theories. Waves in Random Media **6**:419-435.

[37] Tatarskii, V.I., 1997. Formulation of rough-surface scattering theory in terms of phase factors and approximate solutions based on this formulation. Waves in Random Media **7**:557-578.

[38] Tong, C.C., N.E. Huang, Y. Yuan, and S.R. Long, 1989. Probability function of breaking -limited surface elevation. J. Geophys. Res. **94**(C1): 967-972.

[39] Voronovich, A.G., 1985. Small-slope approximation in wave scattering by rough surface. Sov. Phys.-JETP, **62**: 65-70.

[40] Voronovich, A.G., 1996. Non-local small-slope approximation for wave scattering from rough surfaces Waves in Random Media **6**:151-167.

[41] Yaglom, A.M., 1962. An Introduction to the Theory of Stationary Random Functions. Prentice-Hall (Dover).

[42] Yaglom, A.M., 1987. Correlation Theory of Stationary and Related Random Functions. Vol. 1, 2. Springer-Verlag, NY.



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