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PRESSURE-GRADIENT, VELOCITY-VELOCITY STRUCTURE FUNCTION FOR LOCALLY ISOTROPIC TURBULENCE IN INCOMPRESSIBLE FLUID

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Boulder, Colorado
January 1997

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# Pressure-Gradient, Velocity-Velocity Structure Function For Locally Isotropic Turbulence in Incompressible Fluid 

Reginald J. Hill


#### Abstract

The purpose of this report is to give details such that an associated journal article can be brief. For clarity and for ease of associating topics in the two documents, some of the equations in the journal article appear here as well. However, many equations here do not appear in the journal article. The two-point difference of the pressure gradient correlated with the product of two, two-point differences of velocity components is studied. We call this statistic the pressuregradient, velocity-velocity structure function. An equation is derived that relates this statistic to the third- and fourth-order velocity structure functions. The Navier-Stokes equation, incompressibility, homogeneity, and local isotropy are used; no other assumption is used. We give quantitative criteria for the results to be valid under the less restrictive conditions of local homogeneity and local isotropy. The method of derivation is to use algebraic identities and incompressibility to eliminate those statistics that do not obey local isotropy and local scaling. The results are sensitive to departures from local isotropy and incompressibility. Our equation relates two-point statistics and is analogous to Kolmogorov's [Dokl. Akad. Nauk. SSSR 32:19, 1941] equation that related second- and third-order velocity structure functions. The inertial-range formulas for the pressure-gradient, velocity-velocity structure function are obtained, and the viscous range is discussed. It is shown that the pressure structure function can be expressed in terms of the pressure-gradient, velocity-velocity structure function.


## 1. INTRODUCTION

The purpose of this report is to give details such that an associated journal article can be brief. For clarity and for ease of associating topics in the two documents, some of the
equations in the journal article appear here as well. However, many equations here do not appear in the joufnal article.

The von Kármán-Howarth equation (von Kármán and Howarth, 1938) relates secondand third-order two-point velocity correlations, and Kolmogorov's (1941) version of their equation relates second- and third-order two-point velocity structure functions. The advantage of Kolmogorov's version is that relationships between the structure functions can be derived on the basis of local homogeneity and local isotropy, as distinct from isotropy. One logical extension of these equations is discussed at the end of this Introduction. Another logical extension of Kolmogorov's equation is to consider an infinite hierarchy of equations that relate two-point structure functions, Kolmogorov's equation being the first equation in the hierarchy. Like Kolmogorov's equation, the purpose of each equation in the hierarchy is to relate statistics that obey local isotropy. We derive the next equation in the hierarchy of twopoint equations and use it to study the pressure-gradient, velocity-velocity structure function defined by

$$
\begin{align*}
X_{i j k}(\vec{r}) \equiv & \frac{1}{\rho}\left\langle\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)\left(P_{\mid i}-P_{\mid i^{\prime}}^{\prime}\right)\right. \\
& +\left(u_{i}-u_{i}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)\left(P_{\mid j}-P_{\mid j^{\prime}}^{\prime}\right) \\
& \left.+\left(u_{i}-u_{i}^{\prime}\right)\left(u_{j}-u_{j}^{\prime}\right)\left(P_{\mid k}-P_{\mid k^{\prime}}^{\prime}\right)\right\rangle \tag{1}
\end{align*}
$$

The notations are as follows: $P$ is pressure, $u_{i}$ is a velocity component, angle brackets denote averaging, and a variable is primed or unprimed depending on whether it is obtained at point
$\vec{x}^{\prime}$ or $\vec{x}$, respectively; furthermore, $\vec{r}=\vec{x}-\vec{x}^{\prime}$, the notation $\mid i$ denotes the gradient with respect to $\vec{x}$, $\mid i^{\prime}$ denotes the gradient with respect to $\vec{x}^{\prime}$, and $\rho$ is density. In the Appendix, we relate $X_{i j k}(\vec{r})$ to other statistics for the case of isotropy. Note that the second and third terms in (1) are the same as the first term except for cyclic permutation of the indices. Thus, $X_{i j k}(\vec{r})$ is unchanged by interchange of any pair of indices; it therefore has the simplest isotropic formula for a third-rank tensor.

In sec. 2, the Navier-Stokes equation, incompressibility, and homogeneity are used to derive the next equation in the hierarchy of two-point equations, namely,

$$
\begin{equation*}
\dot{D}_{i j k}(\vec{r})+D_{i j k l}(\vec{r})_{\mid l}=-X_{i j k}(\vec{r})+2 v\left[D_{i j k}(\vec{r})_{\mid l l}+Z_{i j k}(\vec{r})\right], \tag{2}
\end{equation*}
$$

where $X_{i j k}(\vec{r})$ is defined in (1) and

$$
\begin{align*}
& D_{i j k}(\vec{r}) \equiv\left\langle\left(u_{i}-u_{i}^{\prime}\right)\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)\right\rangle  \tag{3a}\\
& D_{i j k l}(\vec{r}) \equiv\left\langle\left(u_{i}-u_{i}^{\prime}\right)\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)\left(u_{l}-u_{l}^{\prime}\right)\right\rangle  \tag{3b}\\
& Z_{i j k}(\vec{r}) \equiv\left\langle\left(u_{i}^{\prime}-u_{i}\right)\left(\zeta_{j k}+\zeta_{j k}^{\prime}\right)+\left(u_{j}^{\prime}-u_{j}\right)\left(\zeta_{i k}+\zeta_{i k}^{\prime}\right)+\left(u_{k}^{\prime}-u_{k}\right)\left(\zeta_{i j}+\zeta_{i j}^{\prime}\right)\right\rangle, \tag{3c}
\end{align*}
$$

where $\zeta_{j k} \equiv u_{j \mid l} u_{k \mid l}$. Where applied to a statistic, the notation $\mid i$ denotes differentiation with respect to $r_{i}$; an example is the second term on the left side of (2). Summation is implied by repeated indices. Thus, the subscript $\mid l l$ in (2) denotes the Laplacian operator, and $D_{i j k l}(\vec{r})_{\mid l}$ is the first-order divergence of $D_{i j k l}(\vec{r})$. The dot above the first term in (2) denotes the derivative with respect to time with $\vec{x}$ and $\vec{x}^{\prime}$ held fixed. In sec. 2 , we give the conditions
for validity of (2) under the less restrictive assumption of local homogeneity, as distinct from homogeneity.

Recently (Hill, 1993; Hill and Wilczak, 1995), we related the pressure structure function to the fourth-order velocity structure function, defined in (3b). The pressure structure function is defined by

$$
\begin{equation*}
D_{P}(\vec{r}) \equiv \frac{1}{\rho^{2}}\left\langle\left(P-P^{\prime}\right)^{2}\right\rangle \tag{4}
\end{equation*}
$$

In sec. 6, we obtain the pressure structure function in terms of components of $X_{i j k}(\vec{r})$ on the basis of local isotropy. Recently (Hill, 1996), we used the Navier-Stokes equation to derive formulas for the statistic $<P u_{i} u_{j}>-<P^{\prime} u_{i} u_{j}>$; in sec. 6 , we relate this statistic to $X_{i j k}(\vec{r})$. In sec 3, we simplify (2) on the basis of local isotropy. We thereby obtain in sec. 4 an inertial-range formula for components of $X_{i j k}(\vec{r})$, and we discuss the viscous range in sec. 5 .

The derivation of (2) given here, as well as the derivations for the pressure structure function in Hill (1993) and Hill and Wilczak (1995) and the pressure-velocity-velocity statistic in Hill (1996), are distinct in an essential way from derivations of pressure-related statistics in books and other papers. The essential simplification used in Hill $(1993,1996)$ and Hill and Wilczak (1995) is the application of incompressibility conditions on fourth-order velocity statistics to eliminate large and extraneous terms from the equations. On the contrary, previous work [cf. Monin and Yaglom (1975), secs. 14.4 and 18.2] derived the
pressure-related statistics in terms of fourth-order velocity correlations without simplifying by use of incompressibility conditions. Subsequently, the assumption of joint Gaussian velocities was used to reduce the fourth-order velocity statistics to second-order statistics. Incompressibility conditions on second-order velocity statistics are then used to simplify the expressions. In fact, incompressibility is the essential condition, and the joint Gaussian assumption produces little additional simplification (Hill, 1993, 1996).

Another logical extension of Kolmogorov's equation and the von Kármán-Howarth equation is to test and use closure hypotheses. This extension was studied by Proudman and Reid (1954) and Tatsumi (1957); they derived and studied the three-point equation that relates third- and fourth-order moments. As summarized by Monin and Yaglom (1975, sec. 19.1), a hierarchy of equations, beginning with the von Kármán-Howarth equation, takes its simplest form when one additional spatial point is considered for each additional equation. Compared with our equation that relates four types of two-point statistics, their equation has simpler structure because it involves only two types of three-point statistics. Unfortunately, three-point statistics are difficult to measure. Judging by the results obtained here and in Hill (1993), an equation involving three-point third- and fourth-order velocity structure functions that satisfy local isotropy and local scaling can be derived from the equation by Proudman and Reid (1954) and Tatsumi (1957). Our method of using algebraic identities and incompressibility conditions on fourth-order statistics would be needed for the derivation. The three-point statistics complicate discovery of the necessary identities and conditions. Proudman and Reid (1954) and Tatsumi (1957) used the assumption that velocities at many
points have the joint Gaussian probability distribution, combined with the incompressibility condition on the second-order velocity correlation to eliminate the large and extraneous terms from their equation. Our method obviates the need for these assumptions.

## 2. DERIVATION OF (2)

Using Monin's (1959) equation (6) for velocity difference and following his method for deriving the first equation in the hierarchy of two-point equations, one immediately obtains (2). In doing so, the identity, introduced later in (22), is used for the viscous terms in (2). Then the only approximation used to obtain (2) is use of local homogeneity to commute the Laplacian and divergence to outside the averages and thereby produce the terms $2 \vee D_{i j k}(\vec{r})_{\mid l l}$ and $D_{i j k l}(\vec{r})_{\mid l}$, respectively. No approximations are needed for the other terms in (2). In this derivation, the quantity $Z_{i j k}(\vec{r})$ appears as

$$
Z_{i j k}(\vec{r})=-\left\langle\left(u_{k}-u_{k}^{\prime}\right) \Gamma_{i j}+\left(u_{j}-u_{j}^{\prime}\right) \Gamma_{k i}+\left(u_{i}-u_{i}^{\prime}\right) \Gamma_{j k}\right\rangle
$$

and

$$
\Gamma_{i j} \equiv\left(u_{i}-u_{i}^{\prime}\right)_{\mid l}\left(u_{j}-u_{j}^{\prime}\right)_{\mid l}+\left(u_{i}-u_{i}^{\prime}\right)_{\mid l^{\prime}}\left(u_{j}-u_{j}^{\prime}\right)_{\mid l^{\prime}}
$$

This expression for $Z_{i j k}(\vec{r})$ reduces to (3c).

We also derive (2) using a Eulerian reference frame because such derivation reveals statistics that must vanish. We begin our Eulerian derivation with the Navier-Stokes equation,

$$
\begin{equation*}
\dot{u}_{i}+\left(u_{i} u_{l}\right)_{\mid l}=-\rho^{-1} P_{\mid i}+v u_{i|l|}, \tag{5}
\end{equation*}
$$

and the incompressibility condition,

$$
\begin{equation*}
u_{t \mid l}=0 . \tag{6}
\end{equation*}
$$

The derivation method shows which quantities must vanish under assumptions of incompressibility and homogeneity or local homogeneity. This method gives criteria that can be evaluated using data from experiment or numerical simulation. For specific turbulent flows, one can quantify the magnitude of the discarded terms relative to those retained.

The following notation greatly reduces the number of terms that must be written. $[\ldots+C P]$ means to include all terms like the term or terms given explicitly within the brackets that are obtained by cyclic permutation of the indices $i, j, k$. The index $l$ is never involved in such cyclic permutation; CP is a mnemonic for cyclic permutation. Also, - $\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right)$ means subtract from all previous terms the same terms with $\vec{x}$ and $\vec{x}^{\prime}$ interchanged; that is, unprimed symbols replaced by primed symbols and vice versa. Recall that the subscript notation $\mid l$ and $\mid l^{\prime}$ denotes differentiation with respect to $x_{l}$ and $x_{l}^{\prime}$, respectively, and that summation is implied by repeated indices.

Multiplying the Navier-Stokes equation (5) by $u_{j} u_{k}^{\prime}$, we obtain

$$
\begin{equation*}
\dot{u}_{i} u_{j} u_{k}^{\prime}+\left(u_{i} u_{l}\right)_{\mid l} u_{j} u_{k}^{\prime}=-\frac{1}{\rho} P_{\mid i} u_{j} u_{k}^{\prime}+v u_{i \mid l l} u_{j} u_{k}^{\prime} \tag{7}
\end{equation*}
$$

We obtain another equation from (7) by replacing $u_{j} u_{k}^{\prime}$ with $u_{j}^{\prime} u_{k}$. The resulting two equations give four more equations by interchanging indices $i$ with $j$ and $i$ with $k$. Another equation is obtained by multiplying the Navier-Stokes equation (5), as evaluated at $\vec{x}^{\prime}$, by $u_{j} u_{k}$ :

$$
\begin{equation*}
\dot{u}_{i}^{\prime} u_{j} u_{k}+\left(u_{i}^{\prime} u_{l}^{\prime}\right)_{\mid l^{\prime}}, u_{j} u_{k}=-\frac{1}{\rho} P_{\mid i^{\prime}}^{\prime} u_{j} u_{k}+v u_{i \mid l^{\prime} l^{\prime}}^{\prime} u_{j} u_{k} \tag{8}
\end{equation*}
$$

We obtain two more equations from (8) by interchange of indices $i$ with $j$ and $i$ with $k$.
Adding all nine above-mentioned equations to obtain a single equation, then subtracting the equation obtained by interchanging $\vec{x}^{\prime}$ and $\vec{x}$ within this last equation, gives

$$
\begin{equation*}
\dot{t}_{i j k}+r_{i j k}=h_{i j k}+v y_{i j k}, \tag{9}
\end{equation*}
$$

where we define

$$
\begin{align*}
t_{i j k} & \equiv\left[u_{i}^{\prime} u_{j} u_{k}+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right)  \tag{10a}\\
r_{i j k} & \equiv\left[\left(u_{i}^{\prime} u_{j} u_{k} u_{l}\right)_{\mid l}+\left(u_{i} u_{j}^{\prime} u_{k} u_{l}^{\prime}\right)_{\mid l^{\prime}}+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right)  \tag{10b}\\
h_{i j k} & \equiv \frac{1}{\rho}\left[-P_{\mid i^{\prime}}^{\prime} u_{j} u_{k}-P_{\mid i}\left(u_{j} u_{k}^{\prime}+u_{j}^{\prime} u_{k}\right)+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right)  \tag{10c}\\
y_{i j k} & \equiv\left[u_{i} u_{j} u_{k \mid l^{\prime} l^{\prime}}^{\prime}+\left(u_{i} u_{j}^{\prime}+u_{i}^{\prime} u_{j}\right)\left(u_{k|l|}\right)+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right) . \tag{10d}
\end{align*}
$$

Incompressibility was used to obtain $r_{i j k}$. Define $d_{i j k}$ and $\psi_{i j k}$ as

$$
\begin{align*}
d_{i j k} & \equiv\left(u_{i}-u_{i}^{\prime}\right)\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)  \tag{11}\\
\Psi_{i j k} & \equiv u_{i} u_{j} u_{k}-u_{i}^{\prime} u_{j}^{\prime} u_{k}^{\prime} \tag{12}
\end{align*}
$$

Applying the distributive law of multiplication to (11), we obtain the algebraic identity

$$
\begin{equation*}
t_{i j k}=\psi_{i j k}-d_{i j k} \tag{13}
\end{equation*}
$$

Define $d_{i j k l}, \gamma_{i j k}, \xi_{i j k}$, and $\theta_{i j k}$ as

$$
\begin{align*}
d_{i j k l} & \equiv\left(u_{i}-u_{i}^{\prime}\right)\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)\left(u_{l}-u_{l}^{\prime}\right)  \tag{14}\\
\gamma_{i j k} & \equiv\left(u_{i} u_{j} u_{k} u_{l}\right)_{\mid l}-\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}^{\prime} u_{l}^{\prime}\right)_{\mid l^{\prime}}  \tag{15}\\
\xi_{i j k} & \equiv-\left(u_{i} u_{j} u_{k} u_{l}^{\prime}\right)_{\mid l}+\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}^{\prime} u_{l}\right)_{\mid l^{\prime}}  \tag{16}\\
\theta_{i j k} & \equiv r_{i j k}+\hat{r}_{i j k} \tag{17}
\end{align*}
$$

where $\hat{r}_{i j k}$ is the same as $r_{i j k}$ in (10b), except $\mid l$ and $\mid l^{\prime}$ are everywhere interchanged.
By applying the distributive law to (14) and differentiating, we have the identity

$$
\begin{equation*}
2 r_{i j k}=d_{i j k l \mid l^{\prime}}-d_{i j k l \mid l}+\gamma_{i j k}+\xi_{i j k}+\theta_{i j k} \tag{18}
\end{equation*}
$$

Define $x_{i j k}$ and $\phi_{i j k}$ as

$$
\begin{align*}
x_{i j k} & \equiv \frac{1}{\rho}\left[\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)\left(P_{\mid i}-P_{\mid i^{\prime}}^{\prime}\right)+\mathrm{CP}\right]  \tag{19}\\
\phi_{i j k} & \equiv \frac{1}{\rho}\left[-P_{\mid i} u_{j} u_{k}+P_{\mid i^{\prime}}^{\prime} u_{j}^{\prime} u_{k}^{\prime}+\mathrm{CP}\right] \tag{20}
\end{align*}
$$

Applying the distributive law to (19), we obtain the algebraic identity

$$
\begin{equation*}
h_{i j k}=x_{i j k}+\phi_{i j k} \tag{21}
\end{equation*}
$$

For any two functions $f$ and $g$, we have the rule

$$
\begin{equation*}
(f g)_{\mid l l}=\left(f_{\mid l l}\right) g+2\left(f_{\mid l}\right)\left(g_{\mid l}\right)+f\left(g_{\mid l l}\right) \tag{22}
\end{equation*}
$$

Applying (22) to $y_{i j k}$, as defined in (10d), and defining $\zeta_{i j}$ as

$$
\begin{equation*}
\zeta_{i j} \equiv u_{i \mid l} u_{j \mid l} \tag{23}
\end{equation*}
$$

we obtain

$$
\begin{align*}
y_{i j k} & =\left[\left(u_{i} u_{j} u_{k}^{\prime}\right)_{\mid l^{\prime} l^{\prime}}+\left(u_{i} u_{j}^{\prime} u_{k}\right)_{\mid l l}-2 u_{k}^{\prime} \zeta_{i j}+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right)  \tag{24a}\\
& =t_{i j k \mid l^{\prime} l^{\prime}}+t_{i j k \mid l l}-2 z_{i j k}, \tag{24b}
\end{align*}
$$

where

$$
\begin{equation*}
z_{i j k} \equiv\left[u_{k}^{\prime} \zeta_{i j}+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right) \tag{24c}
\end{equation*}
$$

Substituting our identities in (9), we have

$$
\begin{gather*}
\dot{\Psi}_{i j k}-\dot{d}_{i j k}+\frac{1}{2}\left(d_{i j k l \mid l^{\prime}}-d_{i j k l \mid l}+\gamma_{i j k}+\xi_{i j k}+\theta_{i j k}\right)= \\
x_{i j k}+\phi_{i j k}+v\left(\Psi_{i j k \mid l^{\prime} l^{\prime}}+\Psi_{i j k \mid l l}-d_{i j k \mid l^{\prime} l^{\prime}}-d_{i j k \mid l l}-2 z_{i j k}\right) . \tag{25}
\end{gather*}
$$

We return to the Navier-Stokes equation (5). Multiply (5) by $u_{j} u_{k}$ and add the three equations obtained from cyclic permutation of indices $i, j$, and $k$. From the resulting equation, subtract the equation obtained by interchanging $\vec{x}$ and $\vec{x}^{\prime}$ to obtain

$$
\begin{equation*}
\dot{\Psi}_{i j k}+\gamma_{i j k}=\phi_{i j k}+v\left(\Psi_{i j k \mid l l}+\Psi_{i j k \mid l^{\prime} l^{\prime}}-2 \tilde{z}_{i j k}\right), \tag{26a}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{z}_{i j k} \equiv\left[u_{k} \zeta_{i j}+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right) \tag{26b}
\end{equation*}
$$

The terms in (26a) follow from algebra and the definitions (12), (15), and (20); the identity (22) was used for the viscous term. Subtracting (26a) from (25) gives

$$
\begin{gather*}
-\dot{d}_{i j k}+\frac{1}{2}\left(d_{i j k l \mid l^{\prime}}-d_{i j k l \mid l}-\gamma_{i j k}+\xi_{i j k}+\theta_{i j k}\right)=  \tag{27}\\
x_{i j k}+v\left(-d_{i j k \mid l^{\prime} l^{\prime}}-d_{i j k \mid l l}-2 z_{i j k}+2 \tilde{z}_{i j k}\right)
\end{gather*}
$$

The Navier-Stokes equation (5) and incompressibility condition (6) are valid at every time and every point. Equations (9), (25), (26a), and (27) are valid at every time and every pair of points $\vec{x}$ and $\vec{x}^{\prime}$ because only laws of algebra and calculus were used to obtain them from (5) and (6).

The complicated form of (27) is for ease of obtaining the following results. Assuming local homogeneity, we have

$$
\begin{equation*}
\left\langle d_{i j k l \mid l^{\prime}}\right\rangle=-\left\langle d_{i j k l \mid l}\right\rangle=-\left\langle d_{i j k l}\right\rangle_{\mid l} . \tag{28}
\end{equation*}
$$

Recall that the differentiations within the left-most and middle expressions in (28) are with respect to $\vec{x}^{\prime}$ and $\vec{x}$, respectively, but on the right side of (28) the differentiation is with respect to components of $\vec{r}=\vec{x}=\vec{x}^{\prime}$. Homogeneity gives

$$
\begin{equation*}
\left\langle\gamma_{i j k}\right\rangle=0 \tag{29}
\end{equation*}
$$

Also,

$$
\begin{align*}
\left\langle\xi_{i j k}\right\rangle & =-\left\langle\left(u_{i} u_{j} u_{k} u_{l}^{\prime}\right)_{\mid l}\right\rangle+\left\langle\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}^{\prime} u_{l}\right)_{\mid l^{\prime}}\right\rangle  \tag{30a}\\
& =-\left\langle u_{i} u_{j} u_{k} u_{l}^{\prime}\right\rangle_{\mid l}-\left\langle u_{i}^{\prime} u_{j}^{\prime} u_{k}^{\prime} u_{l}\right\rangle_{\mid l}  \tag{30b}\\
& =\left\langle\left(u_{i} u_{j} u_{k} u_{l}^{\prime}\right)_{\mid l^{\prime}}\right\rangle-\left\langle\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}^{\prime} u_{l|l|}\right\rangle\right.  \tag{30c}\\
& =\left\langle\left(u_{i} u_{j} u_{k}\right) u_{l \mid l^{\prime}}^{\prime}\right\rangle-\left\langle\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}^{\prime}\right) u_{l \mid l}\right\rangle  \tag{30d}\\
& =0 \tag{30e}
\end{align*}
$$

In (30a, c), the differentiations are with respect to components of $\vec{x}$ and $\vec{x}^{\prime}$, but in (30b) the differentiations are with respect to $\vec{r}$. Passing from (30a) to (30b) to (30c) results from homogeneity; (30d) vanishes because of incompressibility (6). We have on the basis of local homogeneity that

$$
\begin{align*}
\left\langle\theta_{i j k}\right\rangle= & \left\{\left[\left\langle\left[\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}-u_{i} u_{j} u_{k}^{\prime}\right) u_{l}^{\prime}\right]_{\mid l}\right\rangle+\mathrm{CP}\right]\right. \\
& \left.+\left[\left\langle\left[\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}-u_{i} u_{j} u_{k}^{\prime}\right) u_{l}^{\prime}\right]_{\mid l^{\prime}}\right\rangle+\mathrm{CP}\right]\right\}-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right)  \tag{31a}\\
= & {\left[\left\langle\left[\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}-u_{i} u_{j} u_{k}^{\prime}\right) u_{l}^{\prime}\right]\right\rangle_{\mid l}-\left\langle\left[\left(u_{i}^{\prime} u_{j}^{\prime} u_{k}-u_{i} u_{j} u_{k}^{\prime}\right) u_{l}^{\prime}\right]\right\rangle_{\mid l}+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right) }  \tag{31b}\\
= & 0 .
\end{align*}
$$

From local homogeneity, we also have

$$
\begin{equation*}
\left\langle d_{i j k \mid l^{\prime} l^{\prime}}\right\rangle=\left\langle d_{i j k \mid l l}\right\rangle=\left\langle d_{i j k}\right\rangle_{\mid l l} . \tag{32}
\end{equation*}
$$

Therefore, averaging (27) and using homogeneity, we have

$$
\begin{equation*}
-\left\langle\dot{d}_{i j k}\right\rangle-\left\langle d_{i j k l}\right\rangle_{\mid l}=\left\langle x_{i j k}\right\rangle-2 v\left[\left\langle d_{i j k}\right\rangle_{\mid l l}+\left\langle z_{i j k}-\tilde{z}_{i j k}\right\rangle\right], \tag{33}
\end{equation*}
$$

which is the same as (2).

The purpose of the Eulerian derivation method is now clear. We have established that the validity of (33) under the assumption of homogeneity requires that the quantities $\left\langle\gamma_{i j k}\right\rangle,\left\langle\xi_{i j k}\right\rangle,\left\langle\theta_{i j k}\right\rangle,\left\langle d_{i j k l \mid l^{\prime}}+d_{i j k l \mid l}\right\rangle$, and $v<d_{i j k \mid l^{\prime} l^{\prime}}-d_{i j k|l|}>$ be very much smaller than the terms in (33). As discussed in the first paragraph of this section, (33) is immediately obtained using Monin's (1959) method of derivation. Therefore, all the above quantities must be very much smaller than the term in (33) under the assumption of local homogeneity. Indeed, each of these quantities vanishes for $\vec{r}=0$; therefore, under the less restrictive assumption of local homogeneity, these quantities must become very small relative to terms in (33) as $r$ decreases. This establishes quantitative criteria that can be tested in specific cases by use of data from experiment or from numerical simulation.

Note that homogeneity and (26b) give $<\tilde{z}_{i j k}>=0$. Then, under homogeneity (3c) can be written as

$$
\begin{aligned}
Z_{i j k}(\vec{r}) & =\left\langle z_{i j k}\right\rangle \\
& =\left\langle u_{i}^{\prime} \zeta_{j k}+u_{j}^{\prime} \zeta_{k i}+u_{k}^{\prime} \zeta_{i j}-u_{i} \zeta_{j k}^{\prime}-u_{j} \zeta_{k i}^{\prime}-u_{k} \zeta_{i j}^{\prime}\right\rangle
\end{aligned}
$$

## 3. SIMPLIFICATION BASED ON LOCAL ISOTROPY

Next, we simplify (2) by assuming local isotropy. Local isotropy allows kinematic relationships of both $X_{i j k}(\vec{r})$ and $Z_{i j k}(\vec{r})$ to other statistics. In the Appendix, we give such relationships. We use the preferred Cartesian coordinate system having its 1 -axis along the
vector $\vec{r}$. The subscript $\lambda$ denotes either axis transverse to $\vec{r}$; that is, $\lambda=2$ or 3 . No summation is implied by repeated Greek subscripts.

Each of the third-rank tensors in (2) is symmetric under interchange of every pair of indices, and can therefore be written in terms of two scalar functions [cf. Monin and Yaglom (1975), Eq. (13.80)]. The scalar functions depend only on $r \equiv|\vec{r}|$. For $X_{i j k}(\vec{r})$, the isotropic-tensor formula is

$$
\begin{equation*}
X_{i j k}(\vec{r})=\left[X_{111}(r)-3 X_{1 \lambda \lambda}(r)\right] W_{i j k}+X_{1 \lambda \lambda}(r) \omega_{i j k}, \tag{34}
\end{equation*}
$$

where

$$
\begin{equation*}
W_{i j k} \equiv \frac{r_{i} r_{j} r_{k}}{r^{3}} \tag{35a}
\end{equation*}
$$

and

$$
\begin{equation*}
\omega_{i j k} \equiv \frac{r_{i}}{r} \delta_{j k}+\frac{r_{j}}{r} \delta_{i k}+\frac{r_{k}}{r} \delta_{i j} \tag{35b}
\end{equation*}
$$

Therefore, in the case of local isotropy, (2) gives two equations relating scalar functions.

Of course, $D_{i j k}(\vec{r})$ and $Z_{i j k}(\vec{r})$ also obey (34) with the symbol $X$ everywhere replaced by $D$ and $Z$, respectively. However, incompressibility further simplifies the formulas for $D_{i j k}(\vec{r})$ and $Z_{i j k}(\vec{r})$. The components of $Z_{i j k}$ that are nonzero under local isotropy are

$$
Z_{111}(r)=3\left\langle u_{1}^{\prime} \zeta_{11}-u_{1} \zeta_{11}^{\prime}\right\rangle+3\left\langle u_{1}^{\prime} \zeta_{11}^{\prime}-u_{1} \zeta_{11}\right\rangle
$$

and

$$
\begin{aligned}
Z_{1 \lambda \lambda}(r)= & \left\langle u_{1}^{\prime} \zeta_{\lambda \lambda}-u_{1} \zeta_{\lambda \lambda}^{\prime}\right\rangle+2\left\langle u_{\lambda}^{\prime} \zeta_{1 \lambda}-u_{\lambda} \zeta_{1 \lambda}^{\prime}\right\rangle \\
& +\left\langle u_{1}^{\prime} \zeta_{\lambda \lambda}^{\prime}-u_{1} \zeta_{\lambda \lambda}\right\rangle+2\left\langle u_{\lambda}^{\prime} \zeta_{1 \lambda}^{\prime}-u_{\lambda} \zeta_{1 \lambda}\right\rangle
\end{aligned}
$$

where $\lambda$ is 2 or 3 . Both $\left\langle u_{i}^{\prime} \zeta_{j k}\right\rangle$ and the triple-velocity correlation $\left\langle u_{i}^{\prime} u_{j} u_{k}\right\rangle$ are symmetric under interchange of two indices and solenoidal in the third index. Therefore, $\left\langle u_{i}^{\prime} \zeta_{j k}\right\rangle$ has the same isotropic-tensor formula as $\left\langle u_{i}^{\prime} u_{j} u_{k}\right\rangle$, which is given in Eq. (12.138) by Monin and Yaglom (1975). Adding all such formulas as required in (3c) yields an isotropic formula that is symmetric under interchange of any pair of indices. Since $\left\langle u_{i}^{\prime} \zeta_{j k}\right\rangle$ obeys the same incompressibility conditions as $\left\langle u_{i}^{\prime} u_{j} u_{k}\right\rangle$ [namely, Eq. (12.137) in Monin and Yaglom (1975)], the incompressibility condition on $Z_{i j k}$ is easily derived to be

$$
\begin{equation*}
Z_{1 \lambda \lambda}(r)=\frac{1}{6}\left[r Z_{111}(r)\right]^{(1)} \tag{36}
\end{equation*}
$$

The superscript in parentheses indicates the order of differentiation with respect to $r$.
Substituting (36) in the isotropic formula similar to (34) gives

$$
\begin{align*}
Z_{i j k}(\vec{r})= & \frac{1}{2}\left[Z_{111}(r)-r Z_{111}^{(1)}(r)\right] W_{i j k} \\
& +\frac{1}{6}\left[Z_{111}(r)+r Z_{111}^{(1)}(r)\right] \omega_{i j k} \tag{37}
\end{align*}
$$

The isotropic-tensor formula for $D_{i j k}(\vec{r})$ is well known; it is (37) with the symbol $Z$ replaced everywhere by $D$ [see Monin and Yaglom (1975), Eq. (13.91)]. Likewise, the incompressibility condition on $D_{i j k}(\vec{r})$ is (36) with $Z$ replaced by $D$ :

$$
\begin{equation*}
D_{1 \lambda \lambda}(r)=\frac{1}{6}\left[r D_{111}(r)\right]^{(1)} \tag{38}
\end{equation*}
$$

Performing the Laplacian on the isotropic-tensor formula for $D_{i j k}(\vec{r})$ gives

$$
\begin{equation*}
D_{i j k}(\vec{r})_{\mid l l}=A(r) W_{i j k}+B(r) \omega_{i j k}, \tag{39}
\end{equation*}
$$

where

$$
\begin{align*}
& A(r) \equiv-\frac{6}{r^{2}} D_{111}(r)+\frac{6}{r} D_{111}^{(1)}(r)-\frac{3}{2} D_{111}^{(2)}(r)-\frac{r}{2} D_{111}^{(3)}(r)  \tag{40a}\\
& B(r) \equiv \frac{1}{6}\left[\frac{4}{r^{2}} D_{111}(r)-\frac{4}{r} D_{111}^{(1)}(r)+5 D_{111}^{(2)}(r)+r D_{111}^{(3)}(r)\right] \tag{40b}
\end{align*}
$$

The isotropic-tensor formula for $D_{i j k l}(\vec{r})$ is given in Monin and Yaglom (1975) and Hill (1993). The first-order divergence required in (2) is easily performed on this isotropic tensor formula. The result, i.e., $D_{i j k l}(\vec{r})_{\mid l}$, has the same form as (34).

From the foregoing results, the two scalar equations implied by (2) are

$$
\begin{equation*}
\dot{D}_{1 \lambda \lambda}(r)+D_{11 \lambda \lambda}^{(1)}(r)-\frac{4}{3 r}\left[D_{\lambda \lambda \lambda \lambda}(r)-3 D_{11 \lambda \lambda}(r)\right]=-X_{1 \lambda \lambda}(r)+2 v\left[B(r)+Z_{1 \lambda \lambda}(r)\right] \tag{41a}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{D}_{111}(r)+D_{1111}^{(1)}(r)+\frac{2}{r}\left[D_{1111}(r)-3 D_{11 \lambda \lambda}(r)\right]=-X_{111}(r)+2 v\left[C(r)+Z_{111}(r)\right] \tag{41b}
\end{equation*}
$$

where

$$
\begin{equation*}
C(r)=A(r)+3 B(r)=-\frac{4}{r^{2}} D_{111}(r)+\frac{4}{r} D_{111}^{(1)}(r)+D_{111}^{(2)}(r) \tag{42}
\end{equation*}
$$

and (36) and (38) can be used to eliminate $D_{1 \lambda \lambda}(r)$ and $Z_{1 \lambda \lambda}(r)$. Multiplying (41b) by $r / 6$ and differentiating the resultant equation, then substituting (36) and (38) and subtracting (41a), gives

$$
\begin{aligned}
&-\frac{1}{6}\left[r X_{111}(r)\right]^{(1)}+ X_{1 \lambda \lambda}(r)= \\
& \frac{r}{6} D_{1111}^{(2)}(r)+\frac{1}{2} D_{1111}^{(1)}(r)-2 D_{11 \lambda \lambda}^{(1)}(r) \\
&-\frac{4}{r} D_{11 \lambda \lambda}(r)+\frac{4}{3 r} D_{\lambda \lambda \lambda \lambda}(r)
\end{aligned}
$$

Both the time-derivative and viscous terms have been eliminated from this result. It can be shown that this result can be obtained from the incompressibility condition (53) in sec. 6 .

## 4. INERTIAL RANGE OF $X_{i j k}$

We now consider the inertial range using (41a,b). Kolmogorov's (1941) analysis established that

$$
\begin{equation*}
D_{111}(r)=-\frac{4}{5} \varepsilon r+6 v D_{11}^{(1)}(r), \tag{43}
\end{equation*}
$$

where $\varepsilon$ is the rate of dissipation of turbulence kinetic energy per unit mass of fluid, and $D_{11}(r)$ is the longitudinal component of the second-order structure function of velocity.

Kolmogorov showed that (43) is valid for large Reynolds numbers under the assumptions of local homogeneity and local isotropy, from which local stationarity follows. Equation (43) will be used to quantify the viscous terms in (41a,b).

For the inertial range (Hill, 1993; Hill and Wilczak, 1995),

$$
\begin{equation*}
D_{\alpha \alpha \beta \beta}(r)=C_{\alpha \beta} \varepsilon^{4 / 3} r^{q}, \tag{44}
\end{equation*}
$$

where subscripts $\alpha$ and $\beta$ may take the values 1,2 , or 3 and

$$
\begin{equation*}
q \equiv \frac{4}{3}-\frac{2 \mu}{9} \tag{45}
\end{equation*}
$$

For purposes of calculation, we will use the value $\mu=0.25$ as recommended in the review by Sreenivasan and Kailasnath (1993). Following Hill (1993), we define the dimensionless universal constants

$$
\begin{align*}
& H_{1 \lambda} \equiv C_{1 \lambda} / C_{11}  \tag{46a}\\
& H_{\lambda \lambda} \equiv C_{\lambda \lambda} / C_{11} . \tag{46b}
\end{align*}
$$

Using (38), (43), (44), and (45), we consider the term $\dot{D}_{1 \lambda \lambda}(r)$ in (41a) and $\dot{D}_{111}(r)$ in (41b) relative to the other terms on the left side of (41a,b). Following Monin and Yaglom (1975), we take $\varepsilon$ to be of the order of the cube of the root-mean-squared velocity divided by an external scale, and the time scale for changes in $\varepsilon$ to be of the order of the external scale divided by the root-mean-squared velocity. Then, we see that the terms $\dot{D}_{1 \lambda \lambda}(r)$ and $\dot{D}_{111}(r)$ are negligible in the inertial range. Specifically, these terms are negligible for values of $r$ that are very much smaller than the external scale. The viscous terms in $(41 \mathrm{a}, \mathrm{b})$ must be negligible for $r$ in an inertial range; we prove this fact later in this section. Thus, for the inertial range, (41a,b) become

$$
\begin{equation*}
X_{1 \lambda \lambda}(r)=-D_{11 \lambda \lambda}^{(1)}+\frac{4}{3 r}\left[D_{\lambda \lambda \lambda \lambda}(r)-3 D_{11 \lambda \lambda}(r)\right] \tag{47a}
\end{equation*}
$$

and

$$
\begin{equation*}
X_{111}(r)=-D_{1111}^{(1)}(r)-\frac{2}{r}\left[D_{1111}(r)-3 D_{11 \lambda \lambda}(r)\right] \tag{47b}
\end{equation*}
$$

Substituting (44), (45), and (46a,b) in (47a,b) gives

$$
\begin{align*}
& X_{1 \lambda \lambda}(r)=\frac{2}{r} D_{11 \lambda \lambda}(r)\left[-\left(\frac{q}{2}+2\right)+\frac{2}{3} \frac{H_{\lambda \lambda}}{H_{1 \lambda}}\right]  \tag{48a}\\
& X_{111}(r)=\frac{2}{r} D_{1111}(r)\left[-\left(\frac{q}{2}+1\right)+3 H_{1 \lambda}\right] \tag{48b}
\end{align*}
$$

Using the measured values $H_{\lambda \lambda}=1.5$ and $H_{1 \lambda}=0.43$ from Hill and Wilczak (1995), the quantity in square brackets in (48a) is -0.3 , which is only $13 \%$ of the largest term in those brackets. The error in the measurements of $H_{\lambda \lambda}$ is at least $13 \%$, but $H_{1 \lambda}$ is more accurately measured. We showed (Hill and Wilczak, 1995) that the values $H_{\lambda \lambda}=1.5$ and $H_{1 \lambda}=0.43$ are not sufficiently accurate to obtain the level of the inertial range of $D_{P}(r)$. On the basis of pressure spectra obtained from both experimentation and numerical simulation, we obtained $H_{P} \propto 1 / 3$ (Hill and Wilczak, 1995), where $H_{P}$ is the universal constant determining the inertial-range level of $D_{P}(r)$. Using $H_{P}=1 / 3$ in (41a) of Hill and Wilczak (1995) that relates $H_{P}$ to $H_{\lambda \lambda}$ and $H_{1 \lambda}$, we obtain $H_{\lambda \lambda}=-\left(q^{2} / 4\right)+(3 / 2)(2+q) H_{1 \lambda}$, which gives $H_{\lambda \lambda}=1.706$ if $H_{1 \lambda}=0.43$. Using these values, the quantity in square brackets in (48a) is only $1 \%$ of the largest term within the brackets. We conclude that the values of $H_{\lambda \lambda}$ and $H_{1 \lambda}$ are too poorly known to obtain the quantity in square brackets in (48a). Therein lies an opportunity. If numerical simulation can produce $X_{1 \lambda \lambda}(r)$ and $D_{11 \lambda \lambda}(r)$ within the inertial range, then (48a) gives a stringent constraint on the value of $H_{\lambda \lambda} / H_{1 \lambda}$. This constraint can be used to better determine $H_{\lambda \lambda}$ and $H_{1 \lambda}$, and hence helps to determine the inertial-range level of $D_{P}(r)$.

On the other hand, substituting $H_{1 \lambda}=0.43$ in the quantity in square brackets in (48b) gives a value that is $21 \%$ of the largest term within the square brackets, so the error is not worse than $50 \%$. Therefore, from $(48 a, b)$ we obtain

$$
\begin{align*}
& X_{1 \lambda \lambda}(r) \propto \frac{1}{r} D_{11 \lambda \lambda}(r) \propto \varepsilon^{4 / 3} r^{q-1}  \tag{49a}\\
& X_{111}(r) \propto-0.7 \frac{1}{r} D_{1111}(r)=-0.7 C_{11} \varepsilon^{4 / 3} r^{q-1} \tag{49b}
\end{align*}
$$

We express (49a) as a proportionality because the coefficient is completely uncertain at present. The coefficient in (49b) is uncertain by not more than $50 \%$. With these uncertainties in mind, in an inertial range, $\left|X_{1 \lambda \lambda}(r)\right|<\left|X_{111}(r)\right| ; X_{111}(r)$ and $X_{1 \lambda \lambda}(r)$ have an approximately $r^{1 / 3}$ power law, and $X_{111}(r)$ is negative.

The inertial-range contributions of the viscous terms in $(41 \mathrm{a}, \mathrm{b})$ can now be estimated. From (43), the inertial-range asymptotic formula is $D_{111}(r)=-4 / 5 \varepsilon r$, substitution of which into (40a,b) or (42) gives $A(r)=B(r)=C(r)=0$. Thus, it is only the very small viscous term in (43) that gives the nonzero evaluation of $A(r), B(r)$, and $C(r)$ for the inertial range. Substituting (43) into (42) and using the inertial-range formula $D_{11}(r)=K \varepsilon^{2 / 3} r^{2 / 3}$, where $K \approx 2$ is Kolmogorov's constant and the intermittency parameter $\mu$ is neglected, we have for the inertial range

$$
C(r) \approx 40 \vee \varepsilon^{2 / 3} r^{-7 / 3}
$$

The same result applies to $B(r)$ with 40 replaced by 8.6. In (41b), one viscous term is $2 \mathrm{v} C(r)$, which is to be compared with the term $-X_{111}(r)$; using (49b), the ratio of these terms is

$$
\frac{2 v C(r)}{-X_{111}(r)} \propto(28 / F)(r / \eta)^{-8 / 3},
$$

where $\eta=\left(v^{3} / \varepsilon\right)^{1 / 4}$ is Kolmogorov's microscale and $F \equiv C_{11} / K^{2}$ is the inertial-range flatness factor. Since $F$ has the value of about 10 at large Reynolds numbers, it follows that $2 \vee C(r) \ll-X_{111}(r)$ for $r \gg \eta$. The same result applies to $2 \vee B(r)$ relative to $-X_{1 \lambda \lambda}(r)$ in (41a), although the proportionality constant is uncertain because of (49a).

It is important to note that $C(r)$ and $B(r)$ are not correctly obtained on the basis of dimensional analysis using inertial-range parameters $\varepsilon$ and $r$; in fact, this gives $C(r) \propto \varepsilon r^{-1}$.

Standard dimensional analysis for the inertial range uses $\varepsilon$ and $r$ as the only relevant quantities and yields $Z_{111}(r) \propto \varepsilon r^{-1}$ and $Z_{1 \lambda \lambda}(r) \propto \varepsilon r^{-1}$. Then, in comparison with (49b),

$$
\frac{2 \vee Z_{111}(r)}{-X_{111}(r)} \propto(r / \eta)^{-4 / 3}
$$

where the intermittency parameter $\mu$ has been neglected. Comparing $2 v Z_{111}(r)$ with any of the terms in (41b) that contain fourth-order velocity structure functions also leads to this same
estimate. Therefore, in (41b), and similarly in (41a), the viscous terms are negligible in the inertial range because $(r / \eta)^{-4 / 3}$ becomes very small for $r \gg \eta$.

In the derivation of $(41 \mathrm{a}, \mathrm{b})$, the terms $2 \vee B(r)$ and $2 \vee C(r)$ are obtained by operations analogous to those that gave the term $6 v D_{11}^{(1)}(r)$ in Kolmogorov's equation (43). Similarly, the terms $2 \vee Z_{1 \lambda \lambda}(r)$ and $2 \vee Z_{111}(r)$ are analogous to the term $-4 / 5 \varepsilon r$ in (43); by this analogy, one wonders if $2 \vee Z_{1 \lambda \lambda}(r)$ and $2 \vee Z_{111}(r)$ have essential roles in the inertial range since $-4 / 5 \varepsilon r$ does have an essential role in (43) for the inertial range. However, (41a,b) differ from (43) in an essential way; namely, (41a,b) can balance at inertial-range scales as in $(47 \mathrm{a}, \mathrm{b})$ without the terms in question, but (43) cannot balance without $-4 / 5 \varepsilon r$.

In the next section, we show that the analogy between $(41 \mathrm{a}, \mathrm{b})$ and $(43)$ does hold in the asymptotic viscous range. In (41b), for example, $2 v C(r)$ and $2 \vee Z_{111}(r)$ are asymptotically (as $r \rightarrow 0$ ) equal but opposite as are $6 v D_{11}^{(1)}(r)$ and $-4 / 5 \varepsilon r$, with the remaining terms in these equations contributing only in the next order of smallness.

## 5. VISCOUS RANGE AND RELATIONSHIPS BETWEEN DERIVATIVE MOMENTS

As $r \rightarrow 0$ in the viscous range, all terms in (2) and (41a,b) are of order $r^{3}$. This is obvious for all terms except the viscous terms [as can be seen from definitions of (1) and $(3 \mathrm{a}, \mathrm{b})]$. For the viscous term, we return to the last term in (9) and use (10d). To this we must subtract the viscous term that produces the viscous term in (26a), namely,

$$
\begin{equation*}
v \mathrm{Y}_{i j k}=\left[u_{i} u_{j}\left(u_{k \mid l l}\right)+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right) . \tag{50}
\end{equation*}
$$

That is, the viscous terms that produce the entire viscous term in (27) are

$$
\begin{equation*}
v\left(y_{i j k}-\mathrm{Y}_{i j k}\right) \equiv v\left\{\left[u_{i} u_{j} u_{k \mid l^{\prime} l^{\prime}}^{\prime}+\left(u_{i} u_{j}^{\prime}+u_{i}^{\prime} u_{j}\right)\left(u_{k|l|}\right)-u_{i} u_{j}\left(u_{k \mid l l}\right)+\mathrm{CP}\right]-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right)\right\} . \tag{51}
\end{equation*}
$$

Consider the Taylor series expansion of the right side of (51). The terms of even order in $r_{n}$ vanish because the coefficients arising from the terms in (51) denoted by $-\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right)$ cancel those from the quantity in square brackets in (51). Also, these terms vanish by local isotropy. The nonvanishing terms are of odd order in $r_{n}$. The Taylor series expansion of the term given explicitly in (51) minus the Taylor series expansion of the corresponding term from $\left(\vec{x} \leftrightarrow \vec{x}^{\prime}\right.$ ) gives zero identically for the term proportional to $r_{n}$. Hence, the viscous term is of order $r^{3}$ as $r \rightarrow 0$. Note that no averaging is performed in (51), so this result is independent of any assumptions of local homogeneity or local isotropy. The quantities $B(r)$, $Z_{1 \lambda \lambda}(r), C(r)$, and $Z_{111}(r)$ in the viscous terms of $(41 \mathrm{a}, \mathrm{b})$ are all of order $r$ as $r \rightarrow 0$. Hence, the vanishing of the viscous terms of (41a,b) to order $r^{3}$ implies relationships between third-order derivative moments. These relationships are easily obtained and are not given here.

Since all terms in (41a,b) are of order $r^{3}$ as $r \rightarrow 0$, we can obtain two complicated relationships between third-order and fourth-order velocity-derivative moments and moments
involving a product of velocity derivatives and pressure derivatives. These relationships are easily obtained and are not given here.

## 6. RELATIONSHIP TO THE PRESSURE STRUCTURE FUNCTION

We can use (2) to derive formulas for $D_{P}(r)$ in terms of $X_{111}(r)$ and $X_{1 \lambda \lambda}(r)$. The first step is to prove on the basis of local homogeneity that

$$
Z_{i j k}(\vec{r})_{\mid i j k}=D_{i j k}(\vec{r})_{\mid i j k}=0,
$$

which, as shown by Hill (1997), leads immediately to the incompressibility results (36) and (38) on the further basis of local isotropy.

The relationship $D_{i j k}(\vec{r})_{\mid i j k}=0$ has been proven by Hill (1997) using only steps that are valid under local homogeneity. The following analogous steps easily obtain $Z_{i j k}(\vec{r})_{\mid i j k}=0$. Commuting the three derivatives to inside the average defined in (3c) gives, for example,

$$
\begin{equation*}
Z_{i j k}(\vec{r})_{\mid i j k}=\left\langle\left(u_{i}^{\prime} \zeta_{j k}+u_{i}^{\prime} \zeta_{j k}^{\prime}-u_{i} \zeta_{j k}-u_{i} \zeta_{j k}^{\prime}+\mathrm{CP}\right)_{\mid i j k}\right\rangle, \tag{52a}
\end{equation*}
$$

where the distributive law of multiplication has been used on the first term in (3c) and the other two terms are implied by the CP notation. The second explicit term in (52a) vanishes because the derivatives are with respect to $\vec{x}$. The fourth explicit term in (52a) vanishes
when operated on by $\mid i$ because of incompressibility, i.e., $u_{i \mid i}=0$. Now commute the derivatives to outside the average on the basis of local homogeneity and return them to inside the average as derivatives with respect to $\vec{x}^{\prime}$; then (52a) becomes

$$
\begin{equation*}
Z_{i j k}(\vec{r})_{\mid i j k}=-\left\langle\left(u_{i}^{\prime} \zeta_{j k}-u_{i} \zeta_{j k}+\mathrm{CP}\right)_{\mid i^{\prime} j^{\prime} k^{\prime}}\right\rangle \tag{52b}
\end{equation*}
$$

As with (52a), the first explicit term in (52b) vanishes by incompressibility and the second term vanishes because it does not depend on $\vec{x}^{\prime}$. Thus, we have $Z_{i j k}(\vec{r})_{\mid i j k}=0$ on the basis of local homogeneity. Hence (36) and (38) are valid on the basis of local homogeneity, local isotropy, and incompressibility.

Performing the third-order divergence of (2) then gives

$$
\begin{equation*}
D_{i j k l}(\vec{r})_{\mid i j k l}=-X_{i j k}(\vec{r})_{\mid i j k} . \tag{53}
\end{equation*}
$$

We showed (Hill and Wilczak, 1995) that the pressure structure function, as defined in (4), satisfies

$$
\begin{equation*}
D_{P}(\vec{r})_{\mid i i k k}=-2 Q(\vec{r}), \tag{54}
\end{equation*}
$$

where

$$
\begin{align*}
Q(\vec{r}) & \equiv\left\langle\left(u_{i} u_{j}\right)_{\mid i j}\left(u_{k}^{\prime} u_{l}^{\prime}\right)_{\mid k^{\prime} l^{\prime}}\right\rangle \\
& =\frac{1}{6} D_{i j k l}(\vec{r})_{\mid i j k l} \tag{55}
\end{align*}
$$

The Laplacian is applied twice on the left side of (54). Local homogeneity is needed for (53), (54), and (55); local isotropy is not needed here (Hill and Wilczak, 1995). Consequently, on the basis of local homogeneity, we have

$$
\begin{equation*}
D_{P}(\vec{r})_{\mid i i k k}=-2 Q(\vec{r})=\frac{1}{3} X_{i j k}(\vec{r})_{\mid i j k} . \tag{56}
\end{equation*}
$$

Assuming isotropy, Batchelor (1951) showed how to solve such an equation for $D_{P}(r)$ in terms of $Q(r)$. We showed (Hill and Wilczak, 1995) that Batchelor's solution applies in the case of local isotropy. Performing the third-order divergence of the isotropic formula (34), we have

$$
\begin{align*}
Q(r) & =-\frac{1}{6} X_{i j k}(\vec{r})_{\mid i j k}  \tag{57a}\\
& =-\frac{1}{6} X_{111}^{(3)}(r)-\frac{1}{r} X_{111}^{(2)}(r)-\frac{1}{r^{2}} X_{111}^{(1)}(r)+\frac{1}{r} X_{1 \lambda \lambda}^{(2)}(r)+\frac{3}{r^{2}} X_{1 \lambda \lambda}^{(1)}(r) . \tag{57b}
\end{align*}
$$

Therefore, the pressure structure function can be expressed in terms of the scalar functions $X_{111}(r)$ and $X_{1 \lambda \lambda}(r)$. Substituting (57b) into the equation relating $D_{P}(r)$ and $Q(r)$ [Hill and

Wilczak, 1995, Eq. (5)], as obtained from Batchelor (1951)] and integrating by parts using the technique in Hill (1993), we have

$$
\begin{equation*}
D_{P}(r)=\int_{0}^{r} X_{1 \lambda \lambda}(y) d y+\frac{r^{2}}{3} \int_{r}^{\infty} y^{-2}\left[3 X_{1 \lambda \lambda}(y)-X_{111}(y)\right] d y \tag{58a}
\end{equation*}
$$

In addition, we recall that

$$
\begin{align*}
D_{P}(r)= & -\frac{1}{3} D_{1111}(r)+\frac{4}{3} r^{2} \int_{r}^{\infty} y^{-3}\left[D_{1111}(y)+D_{\lambda \lambda \lambda \lambda}(y)-6 D_{11 \gamma \gamma}(y)\right] d y \\
& +\frac{4}{3} \int_{0}^{r} y^{-1}\left[D_{\lambda \lambda \lambda \lambda}(y)-3 D_{11 \gamma \gamma}(y)\right] d y \tag{58b}
\end{align*}
$$

and

$$
\begin{equation*}
D_{P}(r)=-2 V_{11}(r)-\frac{4}{r} \int_{0}^{r}\left[V_{11}(y)-V_{\lambda \lambda}(y)\right] d y \tag{58c}
\end{equation*}
$$

In (58b), we repeat the result in Hill (1993) and Hill and Wilczak (1995), and (58c) is given in Hill (1996) in terms of components of

$$
\begin{align*}
V_{i j}(\vec{r}) & \equiv \frac{1}{2 \rho}\left\langle\left(P-P^{\prime}\right)\left(u_{i} u_{j}-u_{i}^{\prime} u_{j}^{\prime}\right)\right\rangle  \tag{59a}\\
& =\left\langle P u_{i} u_{j}\right\rangle-\left\langle P^{\prime} u_{i} u_{j}\right\rangle \tag{59b}
\end{align*}
$$

Isotropy implies that $(59 \mathrm{a}, \mathrm{b})$ are equal (Hill, 1996). Results in (58a-c) complete the above equalities between integrals of three types of statistics. In addition, (58a, c) can be used to
relate $D_{P}(r)$ to a combination of the statistics $U_{i j}(\vec{r})$ and $E_{i j k}(\vec{r})$, which are defined in the Appendix. Whereas (58c) requires isotropy, (58a,b) require only local homogeneity and local isotropy. The pressure structure function can be used to derive the mean-squared pressure gradient, pressure-gradient correlation, pressure variance, and pressure spectrum. These quantities are obtained using (58b) in Hill and Wilczak (1995). Use of (58a,c) in the equations in Hill and Wilczak (1995) easily yields these quantities in terms of $V_{11}(r)$ and $V_{\lambda \lambda}(r)$, as well as $X_{111}(r)$ and $X_{1 \lambda \lambda}(r)$.

## 7. SENSITIVITY TO INCOMPRESSIBILITY AND LOCAL ISOTROPY

In assessing the sensitivity of the foregoing results to the assumptions of local isotropy and incompressibility, we consider the asymptotic case of Reynolds number approaching infinity such that we can let $r / L$ be as small as we like with $r$ in the inertial range and $L$ a scale of the energy-containing range. As shown in Hill (1993), the relationship in (58b) is very sensitive to departures from isotropy and incompressibility. This is because the terms in (58b) are significantly larger than $D_{P}(r)$ and because incompressibility eliminates yet other terms that grow without bound relative to the remaining terms as the asymptotic case is approached. For the same reasons, it is clear that our results are also sensitive to departures from isotropy and incompressibility. The sensitivity to departures from isotropy of our results for $X_{i j k}(\vec{r})$ are illustrated in $(48 \mathrm{a}, \mathrm{b})$ and the discussion of these equations.

The two statistics $\left.<u_{i} u_{j} u_{k} u_{l}^{\prime}\right\rangle$ and $\left\langle u_{i}^{\prime} u_{j}^{\prime} u_{k}^{\prime} u_{l}\right\rangle$ in (30b) are asymptotically much greater than $D_{i j k l}(\vec{r})$. Consequently, the vanishing of the derivative (30b-e) by incompressibility causes sensitivity of the final result (2) to the accuracy of incompressibility in the asymptotic case. Experimentation or numerical simulation would have to determine the correlation of dilatation and the triple-velocity product that appears in (30d); otherwise, a nonvanishing value for $\left\langle\xi_{i j k}>\right.$ would indicate inaccuracy of either homogeneity or incompressibility because homogeneity is also needed for (30a-e).

## 8. SUMMARY

Equation (2) is derived from the Navier-Stokes equation, local homogeneity, and incompressibility. Monin's method and a more lengthy Eulerian derivation both give (2). Criteria for validity of (2) on the basis of local homogeneity are given below (33) at the end of sec. 2. These criteria can be tested by using data from experiments or from numerical simulation. The two scalar equations (41a,b) are obtained from (2) on the basis of local isotropy. These latter equations give the inertial-range formulas, (48a,b) and (49a,b), for the components $X_{111}(r)$ and $X_{1 \lambda \lambda}(r)$, where 1 and $\lambda$ denote axes parallel and transverse to $\vec{r}$. These components have a $(1 / 3)-(2 \mu / 9)$ inertial-range power law and are proportional to $\varepsilon^{4 / 3}$; this result can also be derived on the basis of dimensional analysis followed by averaging over local fluctuations in the $\varepsilon$. However, dimensional analysis does not give the relationship in (48a,b) between inertial-range proportionality constants of $X_{111}(r), X_{1 \lambda \lambda}(r), D_{1111}(r)$,
$D_{\lambda \lambda \lambda \lambda}(r)$, and $D_{11 \lambda \lambda}(r) . X_{111}(r)$ is negative in the inertial range, and $\left|X_{1 \lambda \lambda}(r)\right|<\left|X_{111}(r)\right|$. Numerical simulations that determine $X_{1 \lambda \lambda}(r)$ and $D_{11 \lambda \lambda}(r)$ in the inertial range can determine $H_{\lambda \lambda} / H_{1 \lambda}$ and hence help determine the inertial-range level of $D_{P}(r)$. The leadingorder viscous-range behavior of components of $X_{i j k}(\vec{r})$ is $r^{3}$. For the case of local isotropy, (58a) relates the pressure structure function to integrals of components of $X_{i j k}(\vec{r})$. Therefore, $X_{i j k}(\vec{r})$ is also related to the mean-squared pressure gradient, the pressure-gradient correlation, and the pressure spectrum. Equations (58a-c) relate integrals of components of the fourth-order velocity structure function, of the pressure-velocity-velocity correlation, and of $X_{i j k}(\vec{r})$. Our results are demonstrated to be sensitive to the accuracy of the assumptions of local isotropy and incompressibility.

## 9. ACKNOWLEDGMENT

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## 10. REFERENCES

Batchelor, G. K., 1951. Pressure fluctuations in isotropic turbulence. Proc. Cambridge Philos. Soc. 47:359-374.

Hill, R. J., 1993. Relationships between several fourth-order velocity statistics and the pressure structure function for isotropic, incompressible turbulence. NOAA Tech. Rep. ERL 449-ETL 65, 60 pp. (Available from the author or the National Technical Information Service, 5285 Port Royal Rd., Springfield, VA 22161.)

Hill, R. J., 1996. Pressure-velocity-velocity statistics in isotropic turbulence. Phys. Fluids 8:3085-3093.

Hill, R. J., 1997. Applicability of Kolmogorov's and Monin's equations of turbulence. J. Fluid Mech., submitted.

Hill, R. J., and J. M. Wilczak, 1995. Pressure structure functions and spectra for locally isotropic turbulence. J. Fluid Mech. 296:247-269.

Kolmogorov, A. N., 1941. Energy dissipation in locally isotropic turbulence. Dok. Akad. Nauk SSSR 32:19-21.

Monin, A. S., 1959. The theory of locally isotropic turbulence. Dok. Akad. Nauk. SSSR 125(3):515-518.

Monin, A. S., and A. M. Yaglom, 1975. Statistical Fluid Mechanics: Mechanics of Turbulence. The MIT Press, Cambridge, Mass., vol. 2, 874 pp .

Proudman, I., and W. H. Reid, 1954. On the decay of a normally distributed and homogeneous turbulent velocity field. Philos. Trans. R. Soc. London, Ser. A 247:163-189.

Sreenivasan, K. R., and P. Kailasnath, 1993. An update on the intermittency exponent in turbulence. Phys. Fluids A 5:512-514.

Tatsumi, T., 1957. The theory of the decay process of incompressible isotropic turbulence. Proc. R. Soc. London, Ser. A 239:16-45.
von Kármán, T., and L. Howarth, 1938. On the statistical theory of isotropic turbulence. Proc. R. Soc. London A 164:192-215.

## Appendix

Isotropy allows relationships between different statistics. We establish such relationships for $X_{i j k}$ and $Z_{i j k}$. Isotropy gives

$$
\begin{aligned}
\left\langle u_{i} \zeta_{j k}\right\rangle & =0 \\
\left\langle-P_{\mid i^{\prime}}^{\prime}\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)\right\rangle & =\left\langle P_{\mid i}\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right)\right\rangle \\
\left\langle P_{\mid i} u_{j} u_{k}\right\rangle & =0
\end{aligned}
$$

Applying these relationships to the definitions in (1) and (3c), we have the following relationships allowed by isotropy,

$$
\begin{aligned}
X_{i j k} & =\frac{2}{\rho}\left\langle\left[\left(u_{j}-u_{j}^{\prime}\right)\left(u_{k}-u_{k}^{\prime}\right) P_{\mid i}+\mathrm{CP}\right]\right\rangle \\
& =2\left[-V_{i j \mid k}+U_{i j \mid k}+\mathrm{CP}\right]+2 E_{i j k}
\end{aligned}
$$

where

$$
\begin{aligned}
U_{i j} & \equiv \frac{1}{\rho}\left[2\left\langle P u_{i} u_{j}\right\rangle-\left\langle P\left(u_{i} u_{j}^{\prime}+u_{i}^{\prime} u_{j}\right)\right\rangle\right] \\
& =\frac{1}{\rho}\left\langle P\left[u_{i}\left(u_{j}-u_{j}^{\prime}\right)+u_{j}\left(u_{i}-u_{i}^{\prime}\right)\right]\right\rangle \\
& =\frac{1}{2 \rho}\left\langle\left(P u_{i}-P^{\prime} u_{i}^{\prime}\right)\left(u_{j}-u_{j}^{\prime}\right)+\left(P u_{j}-P^{\prime} u_{j}^{\prime}\right)\left(u_{i}-u_{i}^{\prime}\right)\right\rangle, \\
E_{i j k} & \equiv \frac{1}{\rho}\left\langle P\left(u_{i \mid j}+u_{j \mid i}\right) u_{k}^{\prime}+\mathrm{CP}\right\rangle,
\end{aligned}
$$

and $V_{i j}$ is defined as in (59) and has several alternative formulas allowed by isotropy (Hill, 1995).

