NOAA Technical Memorandum ERL RFC-9

FIRST ORDER AUTOREGRESSIVE LOW-PASS FILTERS: A USER'S QUICK REFERENCE HANDBOOK

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CONTENTS

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ABSTRACT. A collection of formulas, charts and tables describing the properties of first order autoregressive low-pass filters is presented. The collection is structured for utility rather than mathematical sophistication. It provides a quick reference for those using these simple digital low-pass filters. The response characteristics are presented in both the time and frequency domains, and problems associated with record length and initialization are discussed.

1. INTRODUCTION

This describes the properties of digital filters of the form

$$y_K = \alpha y_{K-1} + (1-\alpha)x_K \tag{1}$$

where the series, x_o, x_1,...x_N is the raw data to be filtered and y_o, y_1,...y_N is the filtered output series. The parameter α determines the filter characteristics. Filters of this type are called "autoregressive" because present outputs depend on past outputs as well as the present input. They are called first order because they depend only on one past value. Autoregressive filters are linear filters. Linear means that if the series y_i is related to the series x_i by (1) or (2) and y'_i is related to the series x'_i by the identical equation with the same coefficients, then the series $y''_i = y_i + y'_i$ is related to the series $x''_i = x_i + x'_i$ by the same equation as well. To avoid repetition of the long phrase "first order autoregressive" filters, the shorthand "FOAR filters" will be adopted.

FOAR filters are generally used because of their simplicity. Other linear filters usually require storage in central computer memory of from three to a hundred or more raw data values for each variable to be filtered. Higher order filters require storage of an array of previous output values. The overhead associated with updating these arrays can slow computation and the memory required can sometimes become a problem. In contrast, FOAR filters require no raw data storage at all (only the current value is required) and only the immediately preceeding value of the output needs to be saved. Moreover, the amplitude, phase, and impulse responses of FOAR filters can be calculated quickly from simple equations. Another reason for using FOAR filters is that they may be used in "real-time". That is, immediate results can be displayed because future values of the raw data are not required to compute present values of the filtered data. The asymmetry of the filters in time causes a phase shift in the result but the magnitude of this phase shift is predictable exactly at each frequency of interest.

FOAR filtering is used extensively by the RFC. All data transmitted by the Aircraft Satellite Data Link (ASDL) are filtered in this manner (with α = 0.90490). The vertical wind displayed on our real-time, in-flight displays is filtered by a FOAR filter, the value of α being selected on a case-by-case basis by the flight director (frequently α = 0.60653 is used). The accelerometer drift correction in the vertical wind equation is based on a FOAR smoothed record with α = 0.95150. Users of RFC data have frequently requested information about the filters in the ASDL and real-time display software and much of the information in this paper was collected in response to those requests.

The purpose of this paper is to present the properties of FOAR filters so that those using them or data subjected to them can find all of the practical information needed in one place, and in a less abstract format than that of the mathematics texts from which it is collected. Since these filters are nearly always used in the low-pass mode ($\alpha > 0$), the discussion will be limited to that configuration. A data spacing of one second is assumed throughout for simplicity. For other values $\Delta T \neq 1$ all time values given in the paper should be multiplied by ΔT and all frequencies should be divided by ΔT.

2. BASIC PROPERTIES

1. Form: The FOAR filter is described by

$$y_K = \alpha y_{K-1} + (1-\alpha)x_K \tag{1}$$

where y_i is the output series, x_i the input series, and α the filter parameter. For $\alpha > 0$ the filter is a low-pass filter. For $\alpha < 0$ the filter is a high-pass filter.

2. Stability: The filter is stable in the sense that the impulse response decays with time if $|a| < 1$ (Jenkins and Watts, 1979, p. 162).

3. Impulse Response:

Let $y_o = 0$

Let $x_1 = 1$ and $x_i = 0$ for $i > 1$

Then $y_N = (1-\alpha)\alpha^{N-1}$ for $N > 0$

4. Step Response:

Let $y_o = 0$

Let $x_o = 0$ and $x_i = 1 \quad i > 0$

Then $y_N = (1-\alpha) \sum_{K=0}^{N-1} \alpha^k$ for $N > 0$

5. Autocorrelation: If x_i is a Gaussian white noise process and y_i is generated by (1), then

$$\rho_{yy}(K) = \alpha^{|K|} \quad \text{for } K = 0, \pm 1, \pm 2, \ldots \quad \text{(Ibid).}$$

6. Transfer Function: Let $H(\omega)$ be the transfer function of the filter given by (1). Then for $\alpha > 1$

$$H(\omega) = \frac{(1-\alpha)}{1-\alpha\, e^{-i\omega}}$$

where $\omega = 2\pi f$ is radian frequency and the x_i are assumed spaced one time unit apart. The formula is valid on $|\omega| < \pi$ (Jenkins and Watts, 1969, p. 228). It is the digital equivalent of a low-pass RC filter in the analog domain (Bendat and Piersol, 1971, p. 297). The equivalent $\alpha = \exp(-\Delta T/RC)$ (Ibid).

7. Power Response: If x_i is a white noise process with a unity power spectrum, the power spectrum of y_i is given by $HH^* = P_{yy}(\omega)$, thus for $\alpha > 0$ we have

$$P_{yy}(\omega) = \frac{(1-\alpha)^2}{1+\alpha^2 - 2\alpha \cos \omega}$$

(Bendat and Piersol, 1971, p. 298).

8. Phase Response: The phase response of the filter is given by $\phi_{yx}(\omega) = \tan^{-1}$ (-Imag H/Real H), thus for $\alpha > 0$

$$\phi_{yx}(\omega) = \tan^{-1} \left\{ \frac{-\alpha \sin \omega}{(1-\alpha \cos \omega)} \right\}$$

where a negative phase means output lags input.

3. DERIVED PROPERTIES*

1. E-Folding Time: The e-folding time is the time required for the step response to reach $1-e^{-1}$ of its asymptotic value. This is the same as the time required for the impulse response to decay to e^{-1} times its initial value. It is most convenient to examine this from the point of view, "What value of α is required to place the e-folding point at y_N?" The answer is

$$\alpha = e^{-1/N}$$

That is, if the time between points is one second, then to generate an e-folding time of five seconds choose $\alpha = e^{-1/5} = 0.81873$. The inverse is given by

$$N = -1/\ell n\,(\alpha)$$

2. Half Power Point: The half power frequency $f_{1/2}$ is that at which the power response $P_{yy}(\omega) = 0.5$ where $\omega = 2\pi f$. The result is

$$f_{1/2} = \frac{1}{2\pi} \cos^{-1} \left\{ 2 - (1+\alpha^2)/2\alpha \right\}$$

assuming unity spacing of the y_i. This equation has no solution for $0 \leq \alpha < \alpha_{crit}$ where

$$\alpha_{crit} = (1 - \sqrt{2})/(1 + \sqrt{2}) = 0.1715729$$

because at the critical value $f = 0.5$ ($\omega = \pi$) which is the largest frequency for which the transfer function is valid (it is the Nyquist frequency for the discrete sequence). For values of α larger than α_{crit}, f decreases to 0 as α approaches 1 from below.

4. SOME EXAMPLES AND DISCUSSION

To illustrate the properties discussed thus far in this paper, some examples are presented. Four values of α (0.30, 0.60, 0.90 and 0.98) have been selected to cover a wide range of response characteristics. For each value the equations presented above were used to compute theoretical values for the e-folding time, half power point, power response function and phase response function. For each value, a spike and a step were filtered and the actual time response obtained. For each value, the power response function

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*See Figure 1 and Table 1.

was determined experimentally from an ensemble of realizations of filtered pseudo-white noise. We can compare the application with the theory. Filtered random data is presented to show the smoothing effect of the filter in the time series.

Figure 2 shows the step response of the FOAR filter while Figure 3 shows the impulse response. The calculated values of the e-folding time are indicated on the figures. Since this is a completely deterministic arrangement it is not surprising that the calculated values agree with the experimentally obtained ones.

Figure 4 shows the calculated power and phase response for the selected values of alpha. Figure 5 shows the amplitude response functions actually obtained from an ensemble of 900 trials for each value of α. The calculated response is presented again for ready comparison. The agreement is good but not exact. In depends in part on the number of samples averaged. Figures 6 and 7 show results from ensembles of 99 and 4 sets, respectively. The larger the ensemble the better the agreement, as one would expect for a stochastic process. The agreement also depends on specific data. In general, the mean value of the data used here is close to zero but the mean value of the filtered data is not because the earlier values are weighted heavier in the average. This results in a large value of the transfer function near $f = 0$ since $P_{yy}(0)$ represents essentially the square of the ratio of the means of the two series. Similarly, any member of the ensemble in which the raw data have a spectral "near-zero" point (by statistical accident) contribute a spike to the response function. This disappears when another ensemble is run (compare Figure 8 with Figure 5). The sensitivity of the filter to the data on which it operates on is further illustrated by the initialization problem.

Figures 9-12 show the effect of the filter on random time series when the filter is initialized to the process mean (zero). Figures 13-16 show the effect of initialization to a value signficantly away from the process mean. For large values of α, the initial offset can significantly distort the output. As described above, this distortion also shows up in the transfer function. Examples are shown in Figure 17. Compare Figure 17 ($\alpha = 0.9$) with Figure 6, for example. The effect becomes small as the length of the filtered record becomes large compared with the e-folding time of the filter if the input record is stationary. For non-stationary input records, spectral effects equivalent to "off-mean" initialization may be unavoidable.

5. CONCLUSIONS

FOAR filters provide a simple, well-documented tool for digital low-pass applications. The equations presented in this paper and the curves of Figure 1 enable the user to select an appropriate filter constant, and having selected the constant, the user can determine the complete set of characteristics of the filter. When the cautions mentioned above for use in a real data environment are observed, the filters are well behaved and dependable.

6. REFERENCES

Bendat, J. S., and A. G. Piersol (1971): Random Data: Analysis and Measurement Procedures. Wiley-Interscience, New York, 407 pp.

Jenkins, G. W., and D. G. Watts (1969): Spectral Analyses and its Applications. Holden-Day, San Francisco, 525 pp.

Acknowledgment

The author thanks Dale Martin of NOAA/ERL/AOML/NHRL for preparing the figures for this paper.

RELATED TABLE AND FIGURES

Table 1: E-folding time, half power point and spectral time constant as a function of α. Note that τ_s/τ_e approaches 2π as α approaches 1.

α	E-Folding Time τ_e	Half Power Feqeuency $f_{1/2}$	Half Power Period $\tau_s = 1/f_{1/2}$
0.0	0.0	N/A	N/A
0.1	0.434	N/A	N/A
0.2	0.621	0.352	2.84
0.3	0.831	0.220	4.53
0.4	1.09	0.157	6.36
0.5	1.44	0.115	8.69
0.6	1.96	0.083	12.0
0.7	2.80	0.057	17.4
0.8	4.48	0.035	28.0
0.9	9.49	0.016	59.6
0.95	19.5	0.0082	122.0
0.97	32.8	0.0048	206.0
0.98	49.5	0.0032	311.0
0.99	99.5	0.0016	625.0
1.0	∞	0.0	∞

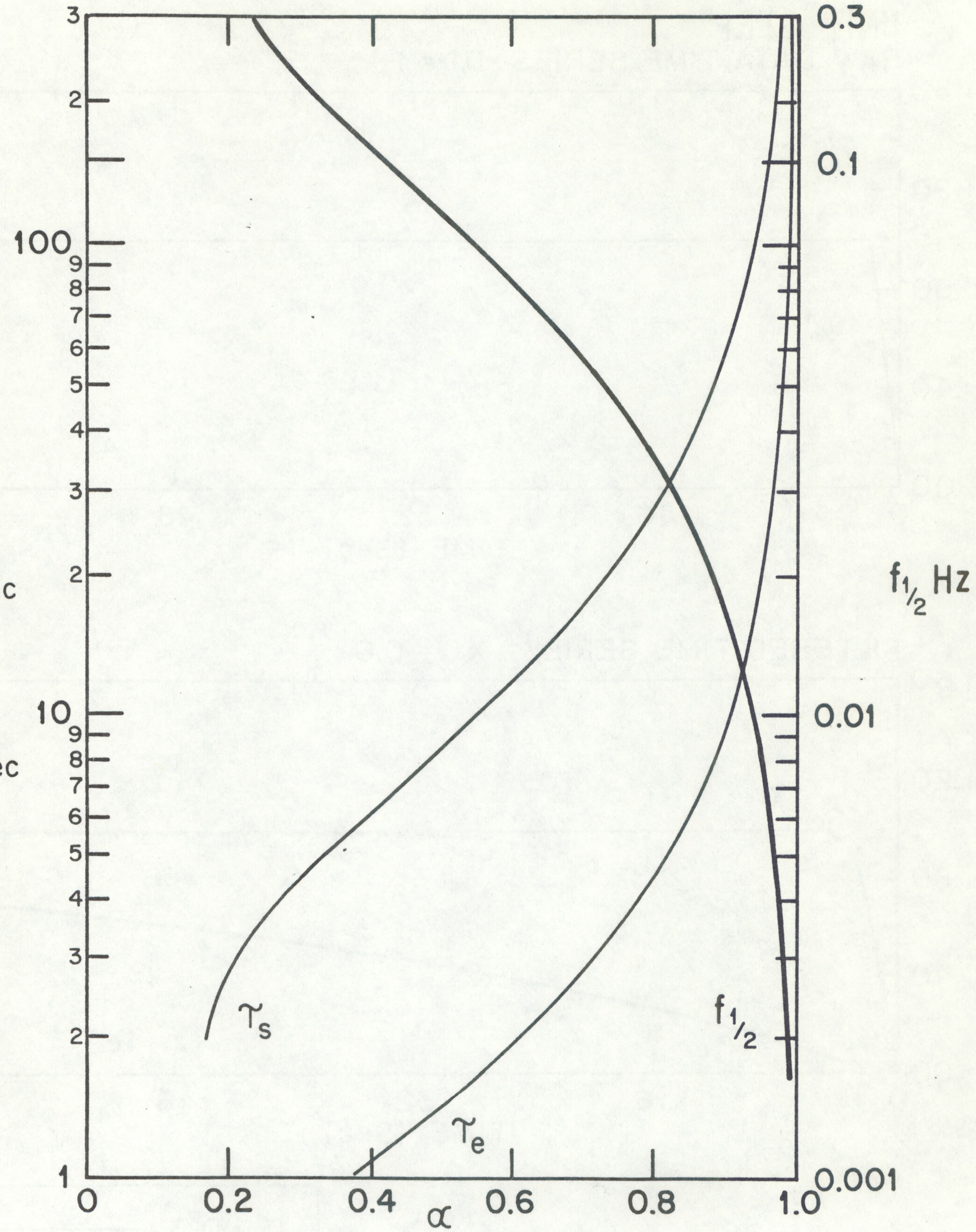

Figure 1. Half power frequency ($f_{1/2}$), spectral response time $\tau_s = 1/f_{1/2}$ and e-folding time (τ_e) as a function of α.

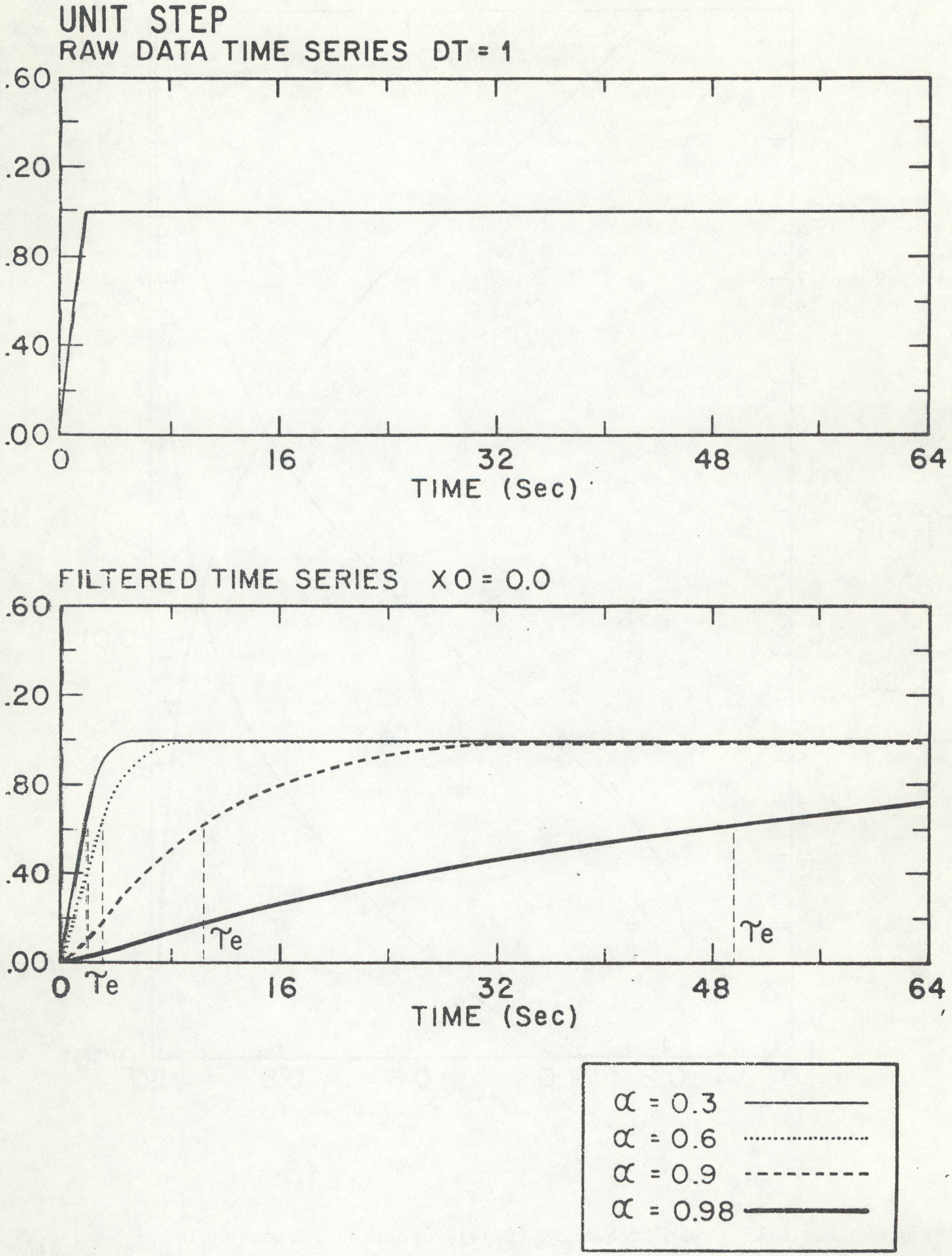

Figure 2. Step response of FOAR filter for α = 0.30, 0.60, 0.90 and 0.98.

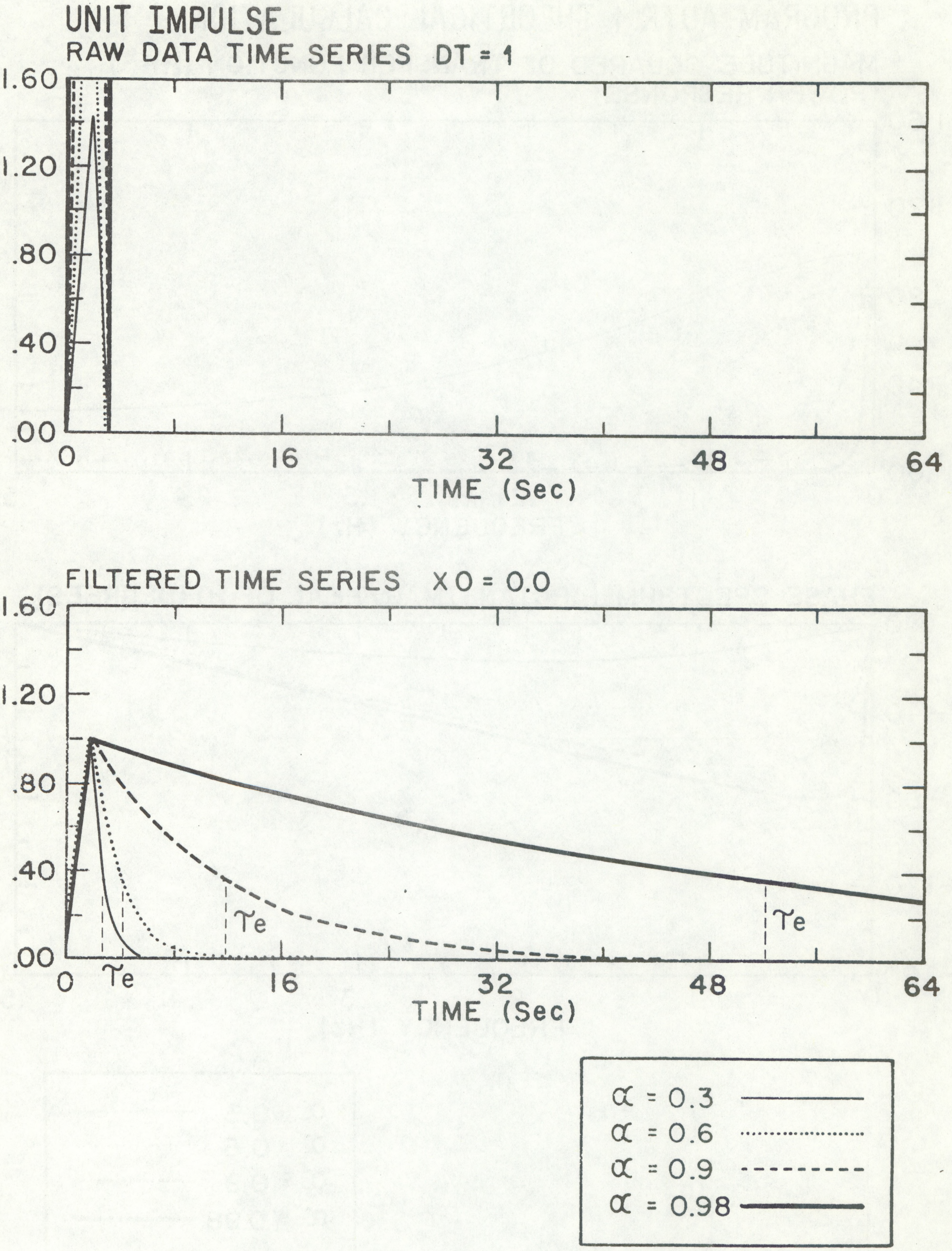

Figure 3. Impulse response of FOAR filter for α = 0.30, 0.60, 0.90 and 0.98.

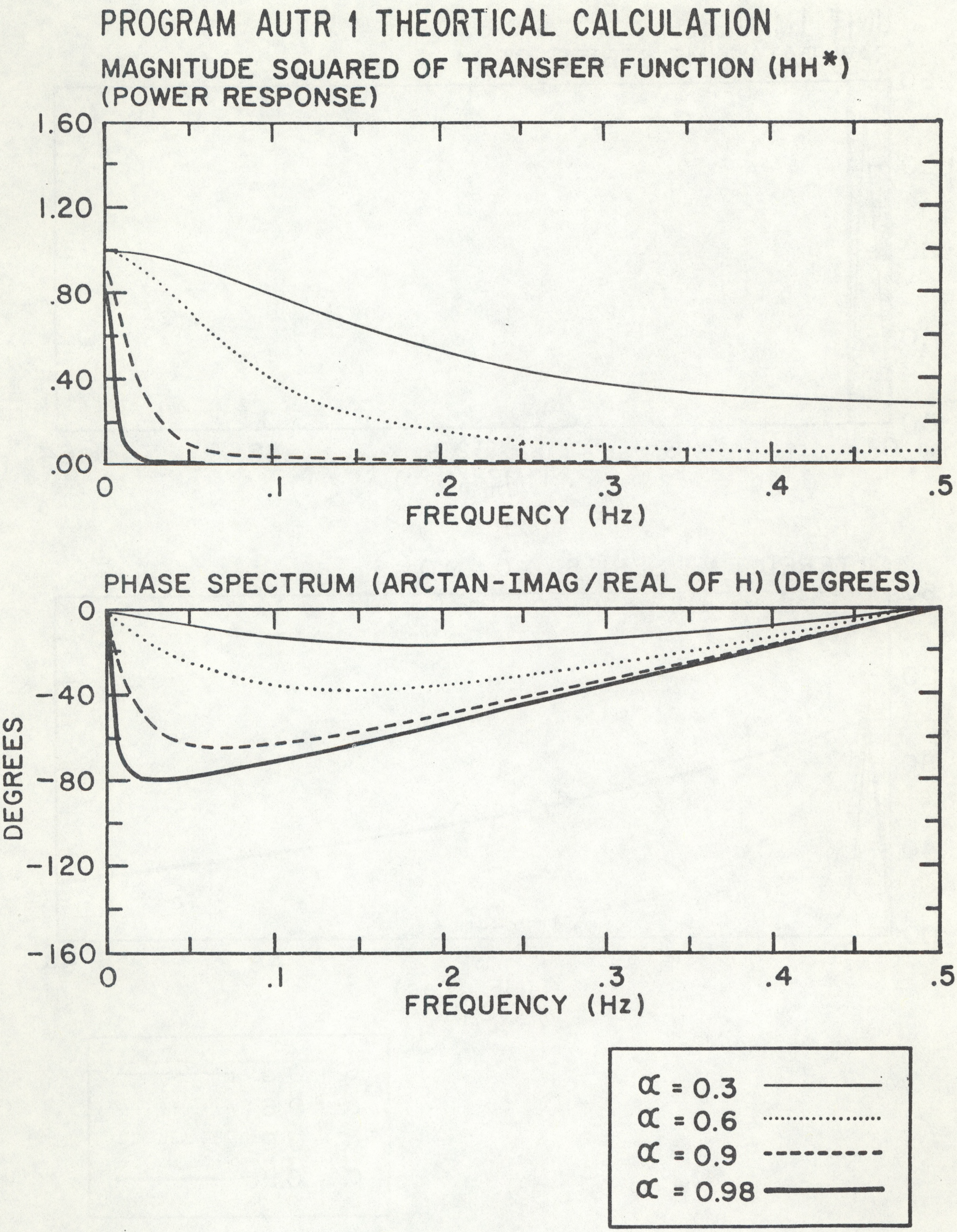

Figure 4. Power and phase responses calculated for FOAR filter when α = 0.30, 0.60, 0.90 and 0.98.

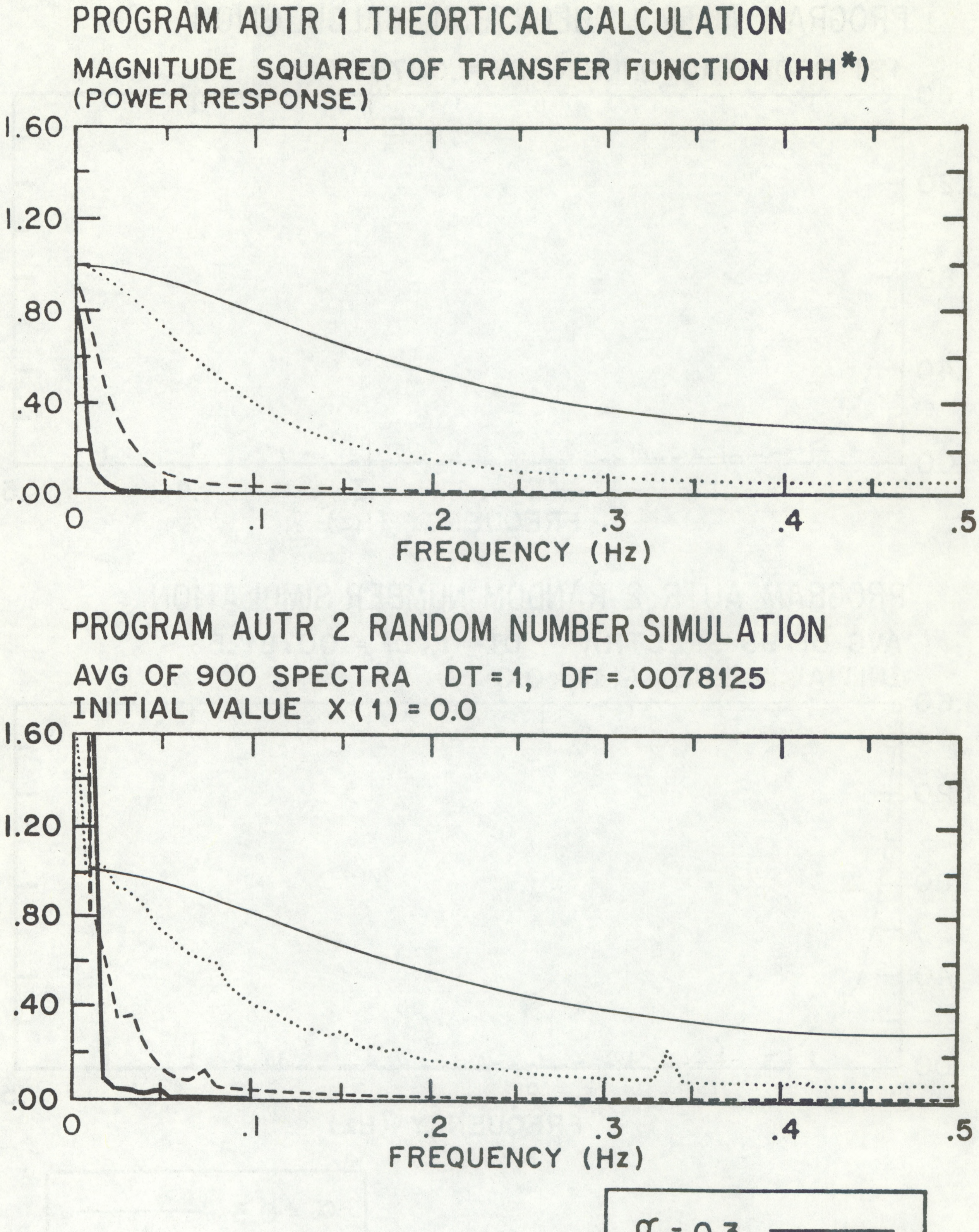

Figure 5. Calculated and observed power responses for an ensemble of 900 realizations for $\alpha = 0.30$, 0.60, 0.90 and 0.98.

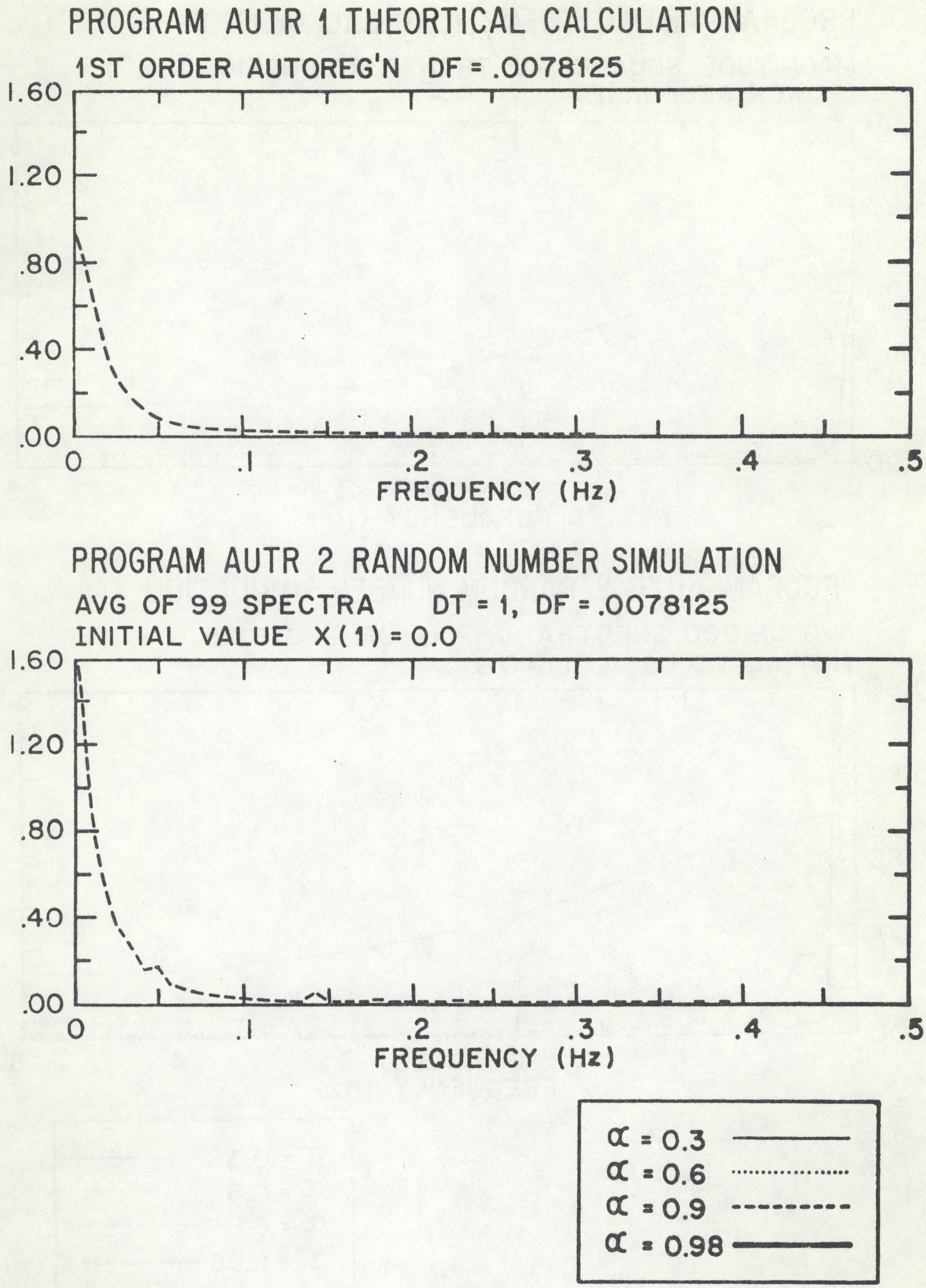

Figure 6. Calculated and observed power responses for α = 0.9 for an ensemble of 99 realizations.

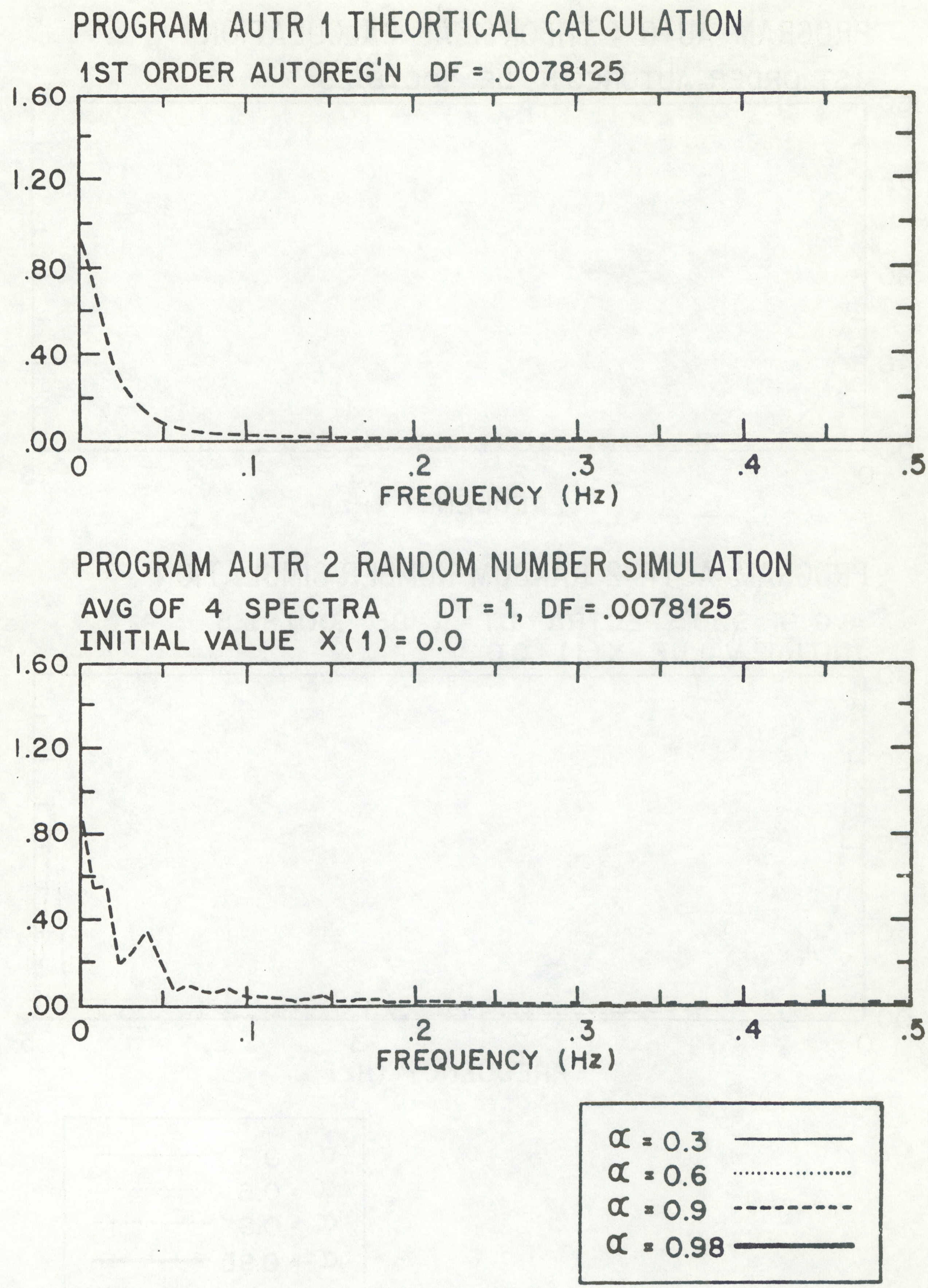

Figure 7. Calculated and observed power responses for $\alpha = 0.9$ for an ensemble of 4 realizations.

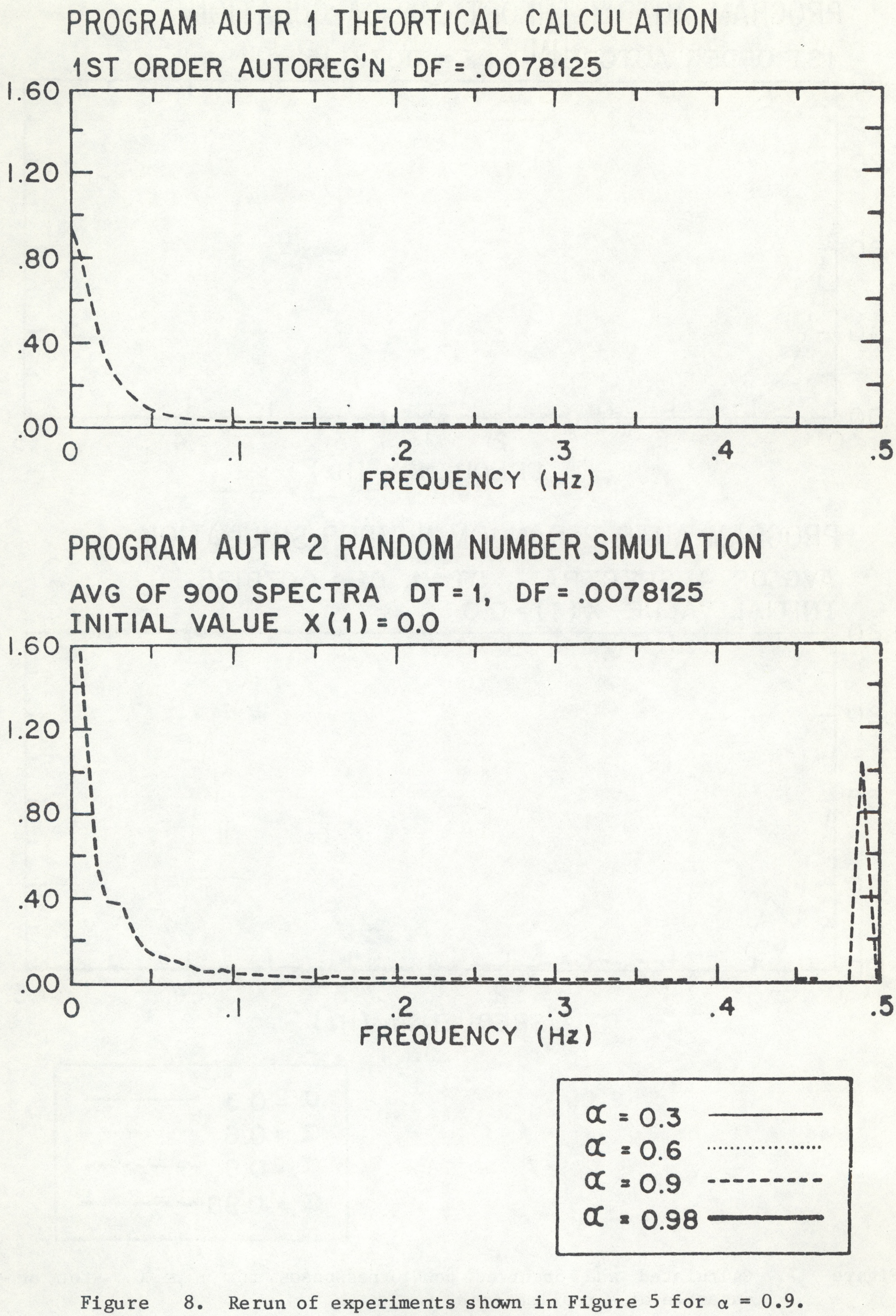

Figure 8. Rerun of experiments shown in Figure 5 for $\alpha = 0.9$.

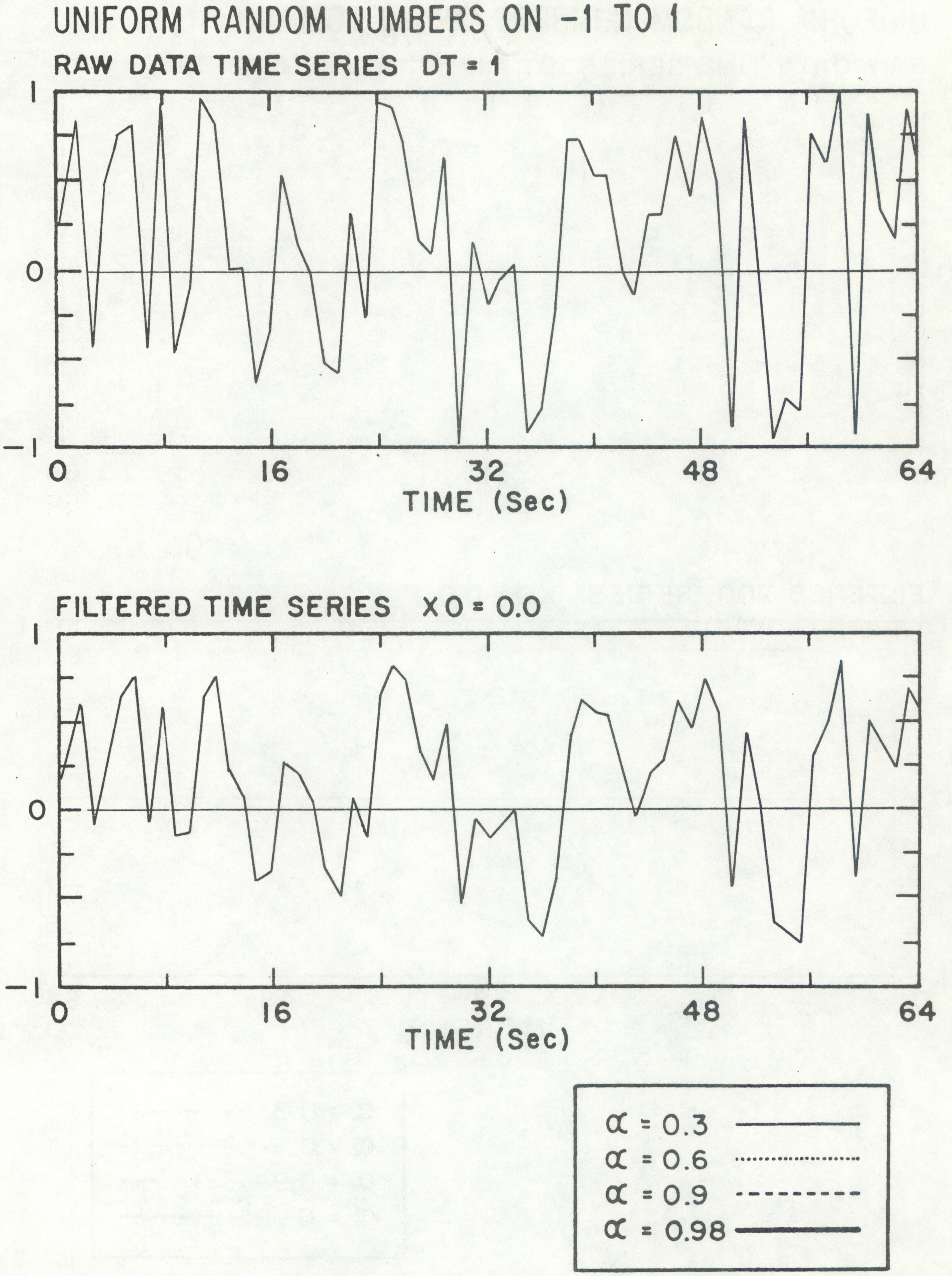

Figure 9. Effect of FOAR filter on random sequence, $\alpha = 0.30$ when initialized to process mean (zero).

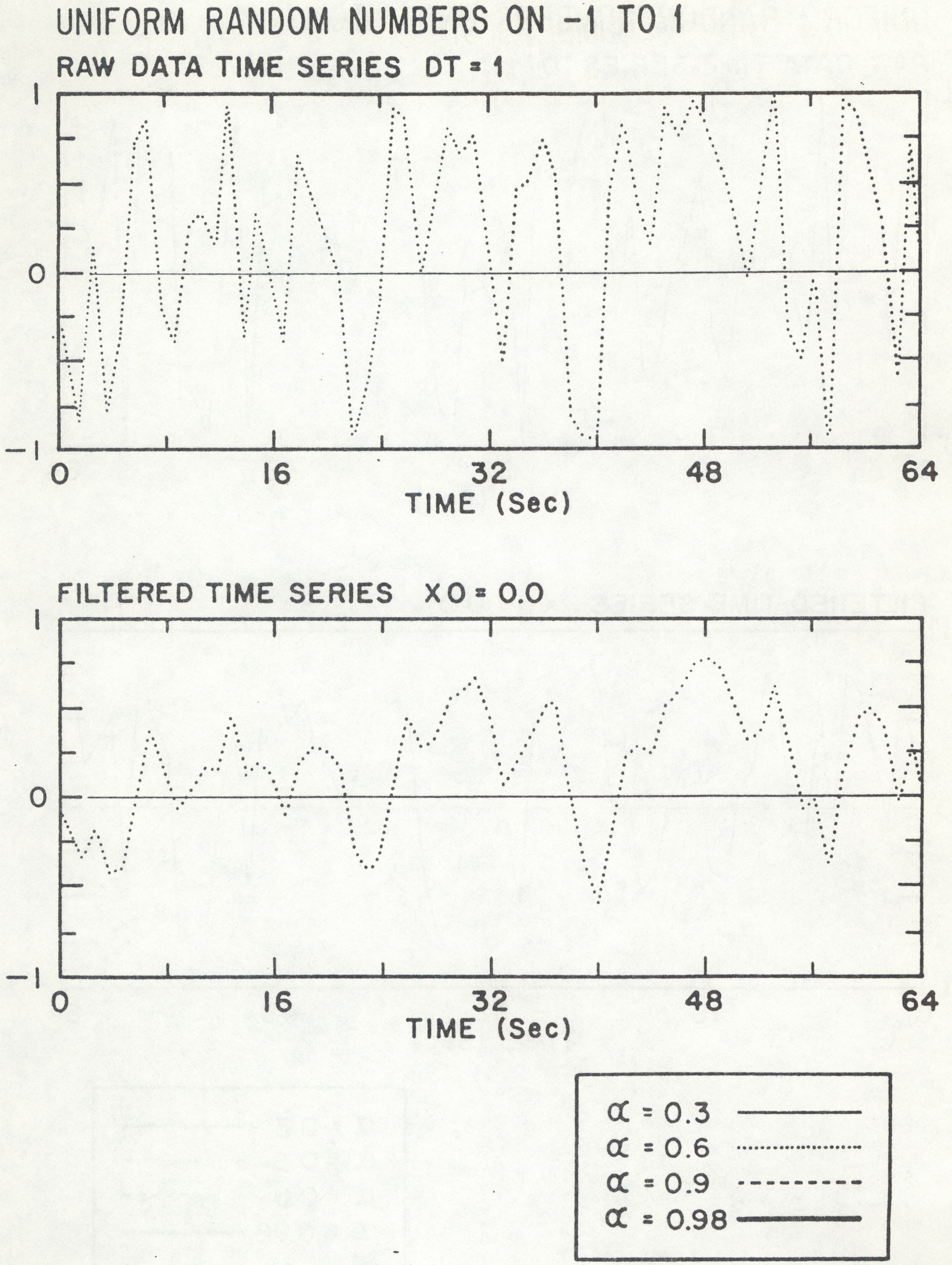

Figure 10. Effect of FOAR filter on random sequence, $\alpha = 0.60$ when initialized to process mean (zero).

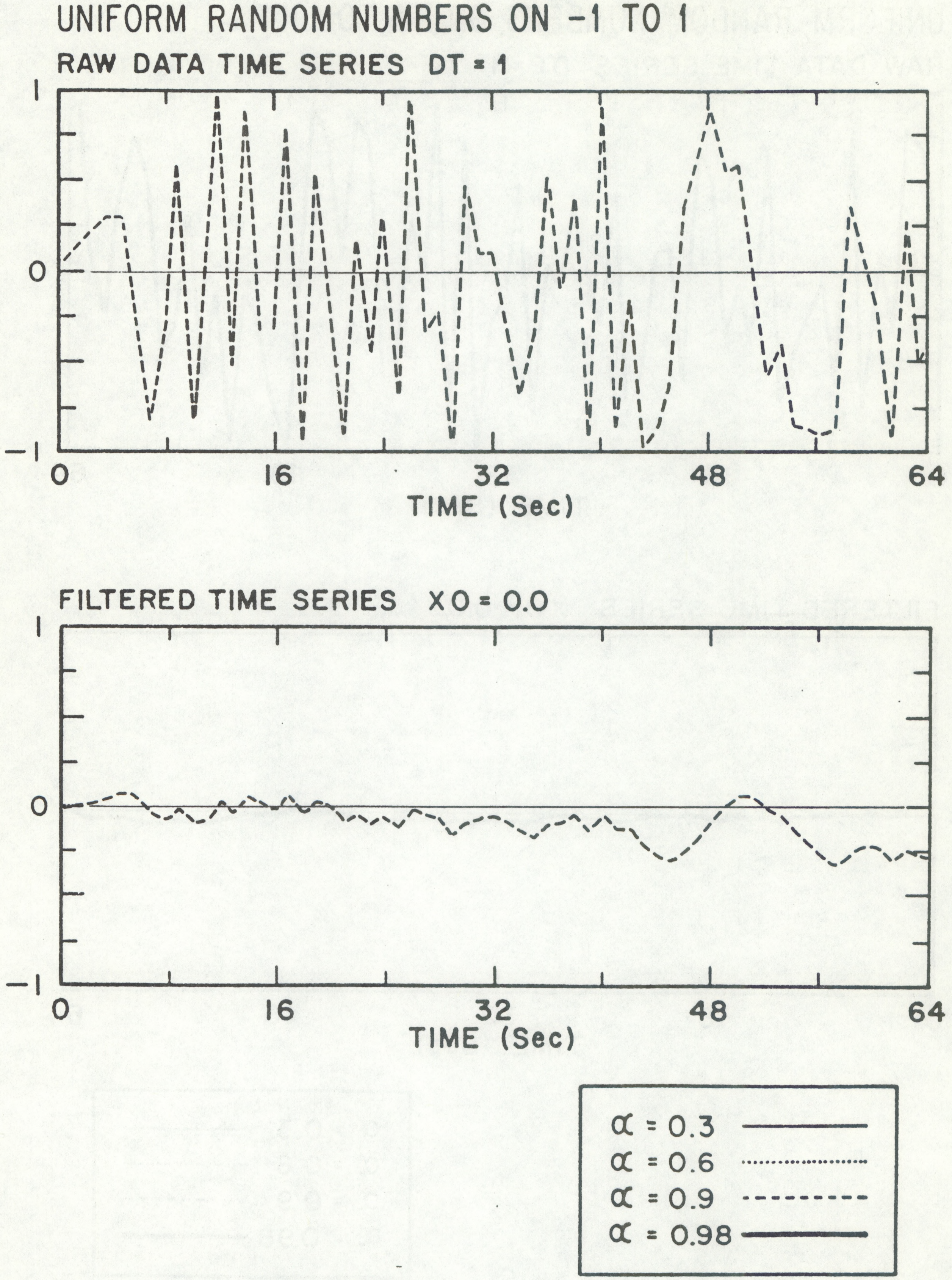

Figure 11. Effect of FOAR filter on random sequence, $\alpha = 0.90$ when initialized to process mean (zero).

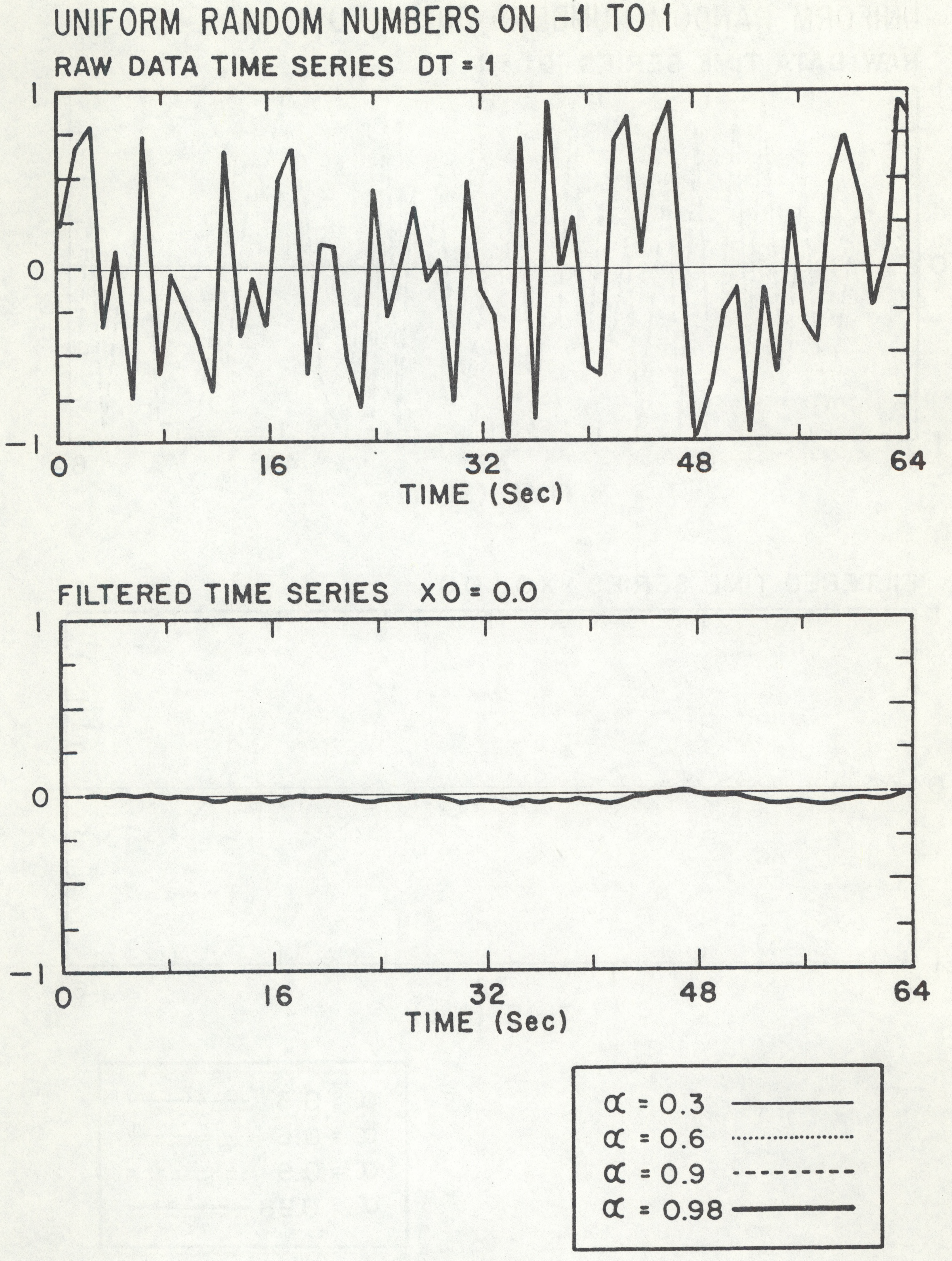

Figure 12. Effect of FOAR filter on random sequence, α = 0.98 when initialized to process mean (zero).

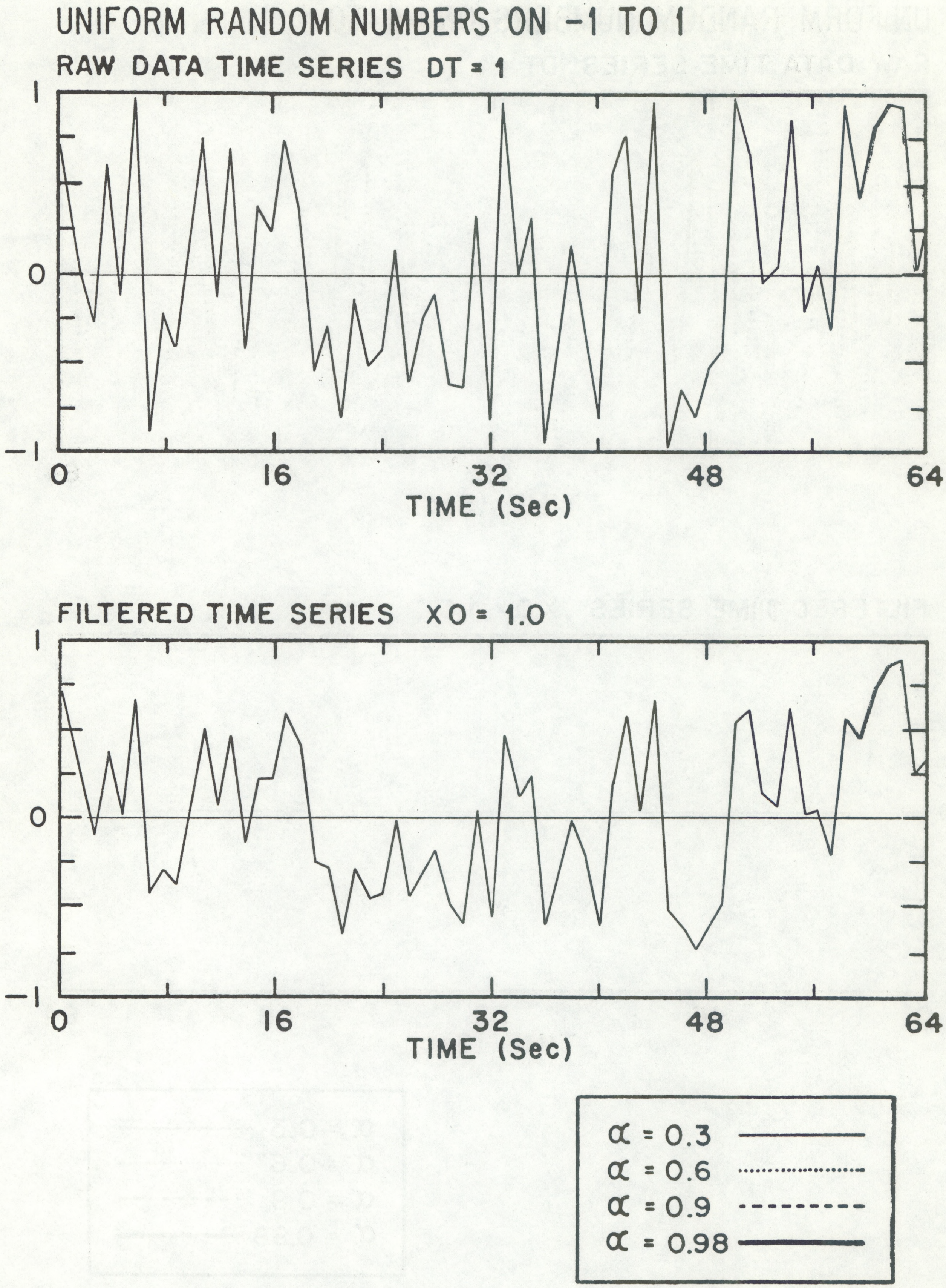

Figure 13. Effect of FOAR filter on random sequence initialized away from process (here $x_o = 1$ while $\bar{x} \approx 0$), $\alpha = 0.30$.

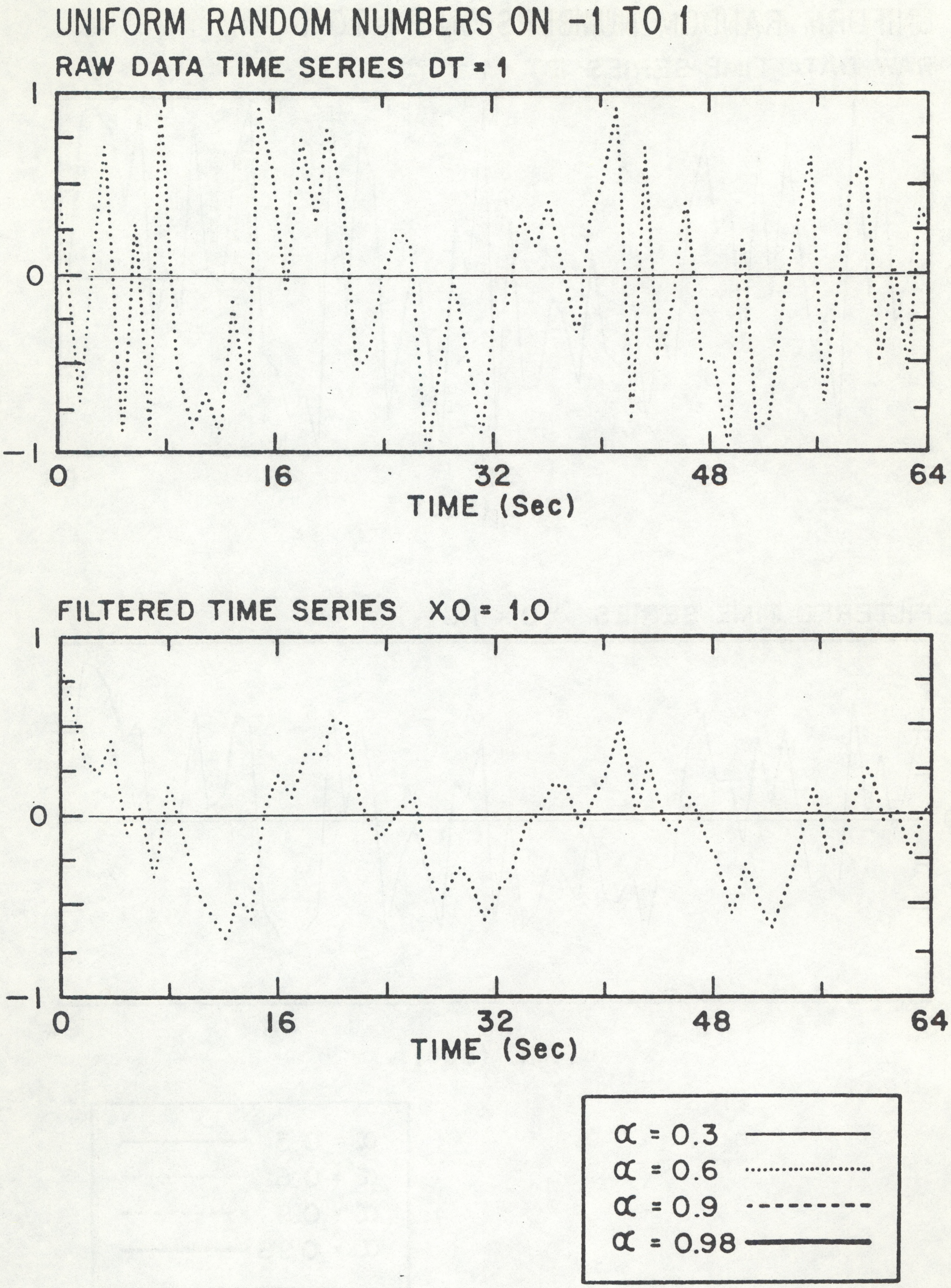

Figure 14. Effect of FOAR filter on random sequence initialized away from process (here $x_o = 1$ while $\bar{x} \approx 0$), $\alpha = 0.60$.

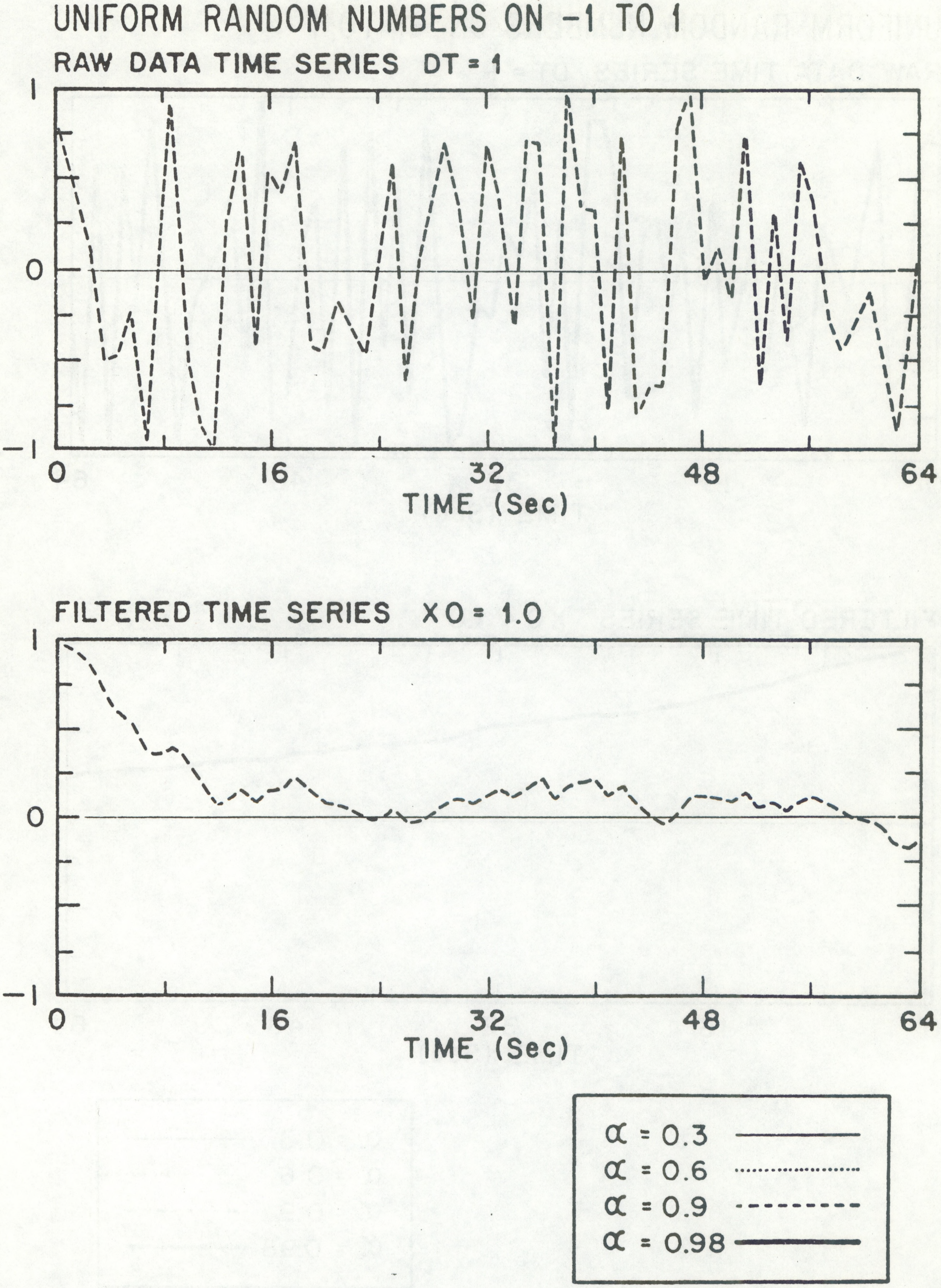

Figure 15. Effect of FOAR filter on random sequence intialized away from process (here x_o = 1 while $\bar{x} \approx 0$), α = 0.90.

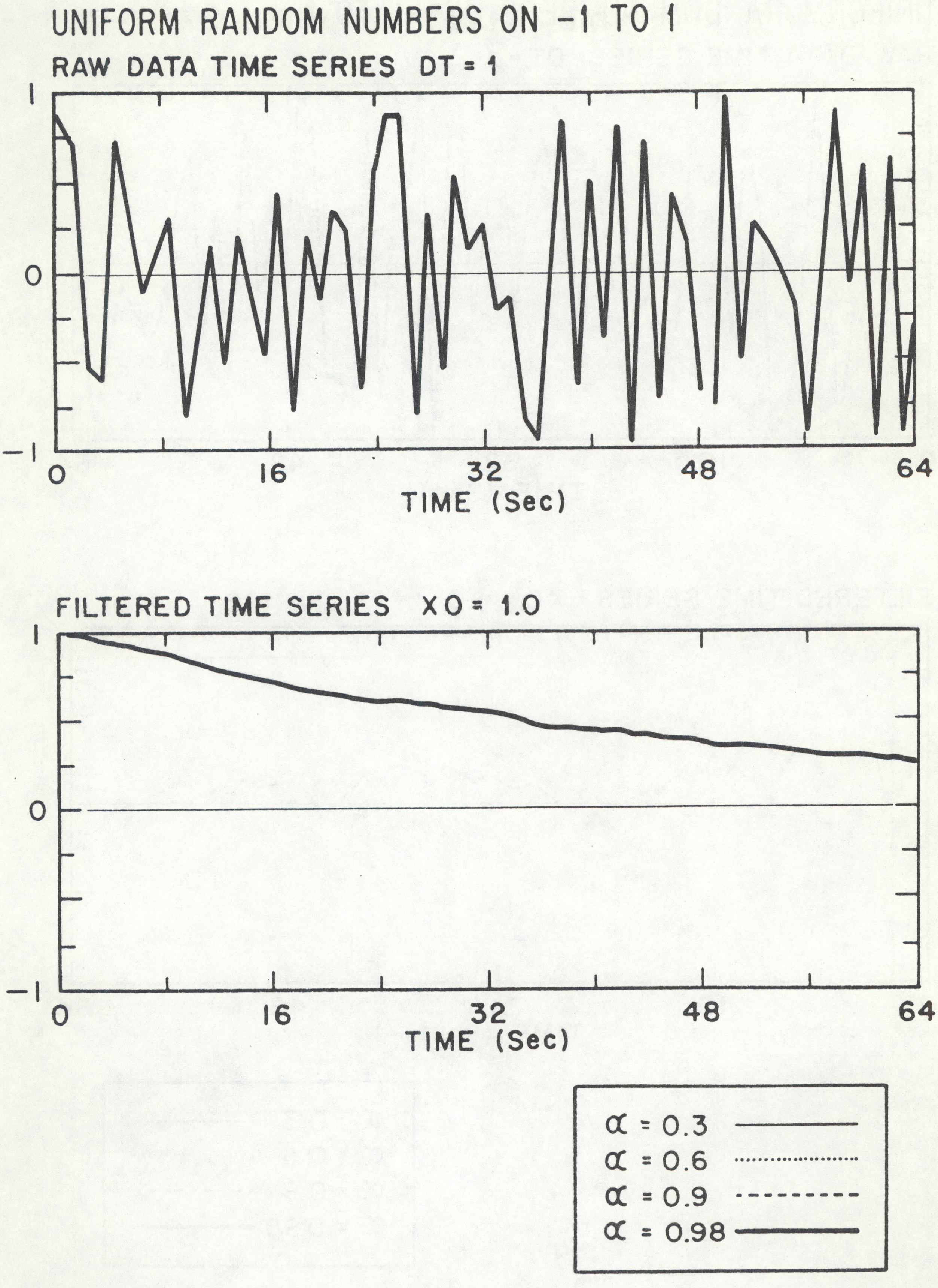

Figure 16. Effect of FOAR filter on random sequence initialized away from process (here $x_o = 1$ while $\bar{x} \approx 0$), $\alpha = 0.98$

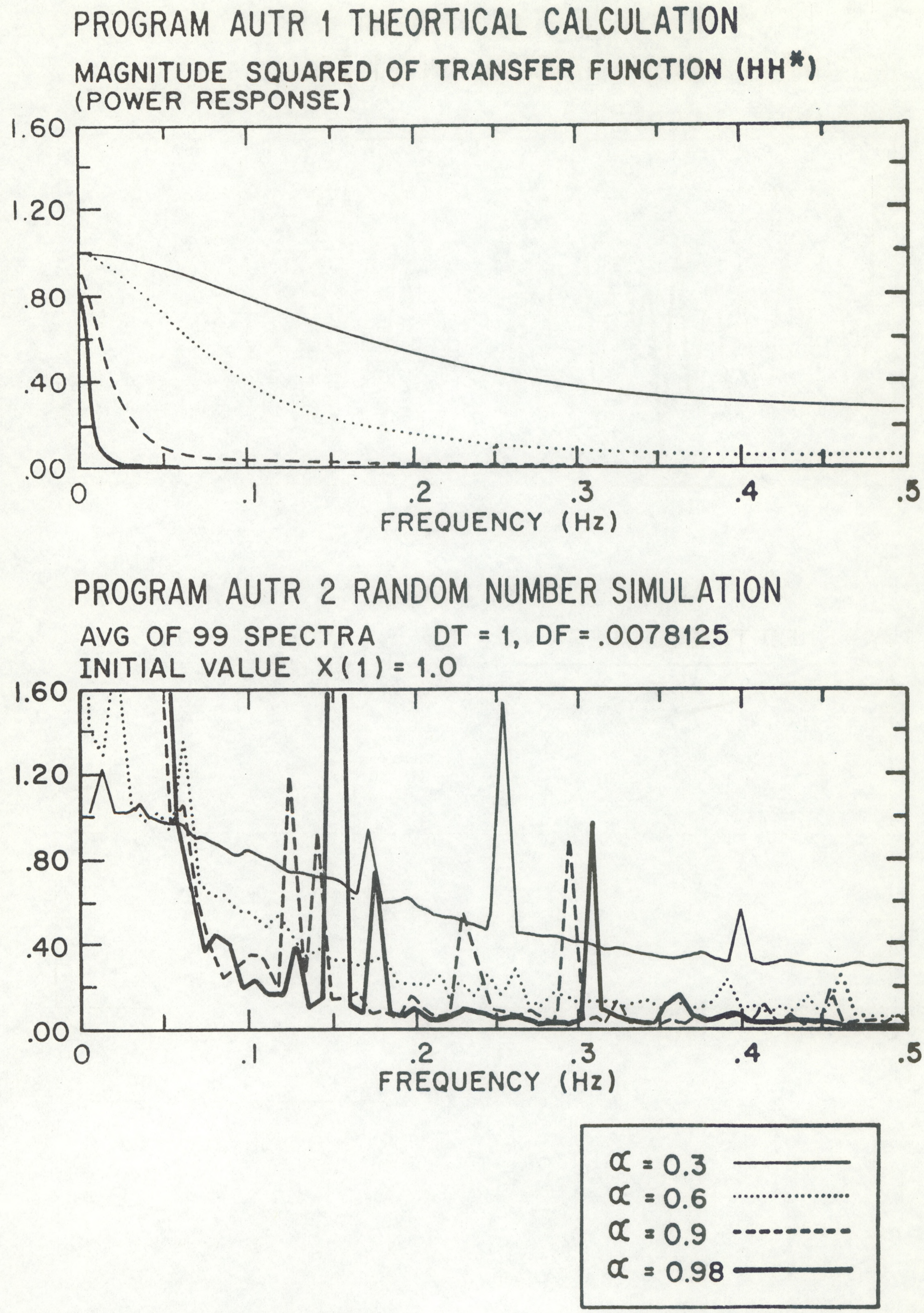

Figure 17. Spectra of ensemble of 99 samples initialized with $x_o = 1$ (process mean $\bar{x} \approx 0$), α = 0.30, 0.60, 0.90 and 0.98.