ENVIRONMENTAL ASSESSMENT
and
ECONOMIC ANALYSIS
for
44 TO THE FISHERY MANAGEMENT PLAN FOR
THE GROUNDFISH FISHERY OF THE BERING SEA AND ALEUTIAN ISLANDS
AREA
AND
AMENDMENT 44 TO THE FISHERY MANAGEMENT PLAN FOR
THE GROUNDFISH FISHERY OF THE GULF OF ALASKA

TO REDEFINE ACCEPTABLE BIOLOGICAL CATCH AND OVERFISHING

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Executive Summary

Reviews by NMFS' Overfishing Definitions Review Panel (ODRP) and the Council's Scientific and Statistical Committee (SSC) have indicated that the definitions of "acceptable biological catch" and "overfishing" contained in the fishery management plans for groundfish of the Bering Sea/Aleutian Islands and Gulf of Alaska could and should be improved. Suggestions for improvement include the following: A) greater imprecision in parameter estimates should correspond to more conservative fishing mortality rates; B) for a stock below its target abundance level, fishing mortality rates should vary directly with biomass and ultimately fall to zero should the stock become critically depleted; and C) a buffer should be maintained between acceptable biological catch and the overfishing level.

This plan amendment proposal contains two alternatives: Alternative 1 (No Action) maintains the current definitions, and Alternative 2 (Proposed Revision) modifies the current definitions in response to the suggestions made by the ODRP and SSC. The differences between the two alternatives can perhaps best be illustrated by considering the case in which a point estimate of the fishing mortality rate at maximum sustainable yield (F_{MSY}) is available together with a reliable description of the amount of uncertainty surrounding that estimate. Under the current definitions, the target fishing mortality rate (F_{ABC}) and the maximum allowable fishing mortality rate (F_{OFL}, the rate above which overfishing is defined to occur) are both set equal to the point estimate of F_{MSY}, regardless of the level of uncertainty associated with that estimate. Under the proposed revision, the ratio between F_{ABC} and F_{OFL} varies inversely with the level of uncertainty (i.e., the greater the uncertainty in the estimate of F_{MSY}, the lower F_{ABC} is in relation to F_{OFL}).

Even in cases where reliable descriptions of the level of uncertainty associated with a point estimate of F_{MSY} are not available, the proposed revision maintains an appropriate buffer between F_{ABC} and F_{OFL}. Also, whenever a target abundance level can be reasonably identified, the proposed revision reduces fishing mortality rates as stock size falls below that target level. The current definitions do neither of these.

Because the proposed revision institutes new safeguards against overly aggressive harvest rates, particularly under conditions of high uncertainty or low stock size, the revision is expected to result in positive environmental impacts. The revision would also be expected to result in positive long-term economic impacts in those cases where the objective of optimizing long-term average yield on a species-by-species basis is a suitable proxy for maximizing long-term economic impacts. However, it is possible that negative economic impacts could be generated in the short term for a few fisheries, particularly Bering Sea and Aleutian Islands fisheries targeting on rockfish species other than Pacific ocean perch, where total allowable catch might be reduced by about 15 percent. The assumptions or conditions under which the net economic impacts of such short-term costs might outweigh those of the expected long-term benefits have not been determined.

1.0 INTRODUCTION

The groundfish fisheries in the Exclusive Economic Zone (EEZ, 3 to 200 miles offshore) off Alaska are managed under the Fishery Management Plan for the Groundfish Fisheries of the Gulf of Alaska and the Fishery Management Plan for the Groundfish Fisheries of the Bering Sea and Aleutian Islands Area. Both fishery management plans (FMPs) were developed by the North Pacific Fishery Management Council (Council) under the Magnuson Fishery Conservation and Management Act (Magnuson Act). The Gulf of Alaska (GOA) FMP was approved by the Secretary of Commerce and became effective in 1978 and the Bering Sea and Aleutian Islands Area (BSAI) FMP become effective in 1982.

Actions taken to amend FMPs or implement other regulations governing the groundfish fisheries must meet the requirements of applicable Federal laws and regulations. In addition to the Magnuson Act, the most important of these are the National Environmental Policy Act (NEPA), the Endangered Species Act (ESA), and the Marine Mammal Protection Act (MMPA).

NEPA requires a description of the purpose and need for the proposed action as well as a description of alternative actions which may address the problem. This information is included in Section 1 of this document. Section 2 contains information on the biological and environmental impacts of the alternatives as required by NEPA. Impacts on endangered species and marine mammals are also addressed in this section. Section 3 contains an economic analysis which considers the economic impacts of the alternatives as compared to no action.

This Environmental Assessment (EA) and related economic analysis addresses a pair of plan amendments (one each for the BSAI and GOA Groundfish FMPs) to redefine "acceptable biological catch" (ABC) and "overfishing."

1.1 Purpose of and Need for the Action

The Magnuson Act includes a set of "national standards" with which all fishery management plans and implementing regulations must be consistent. The first national standard states,

> "Conservation and management measures shall prevent overfishing while achieving, on a continuing basis, the optimum yield from each fishery for the United States fishing industry."

Thus, the Magnuson Act places a high priority on the prevention of overfishing. However, nowhere in the Magnuson Act is overfishing defined. In 50 CFR part 600, subpart D, the National Oceanic and Atmospheric Administration (NOAA) presented its

Guidelines for Fishery Management Plans (the "National Standards Guidelines"), which contain the following general definition:

> "Overfishing is a level or rate of fishing mortality that jeopardizes the long-term capacity of a stock or stock complex to produce maximum sustainable yield (MSY) on a continuing basis."

Because of the generality of this definition, NOAA believed that it would be difficult to apply unambiguously. Therefore, the National Standards Guidelines also contain the following directive:

> "Each FMP must specify, to the maximum extent possible, an objective and measurable definition of overfishing for each stock or stock complex covered by that FMP, and provide an analysis of how the definition was determined and how it relates to reproductive potential."

In response to that directive, the BSAI and GOA Groundfish FMPs were amended to include an objective and measurable definition of overfishing, effective in 1991.

The National Standards Guidelines also make allowance for the use of ABC as a step in the total allowable catch (TAC) specification process. The definition of ABC contained in the BSAI and GOA Groundfish FMPs was last amended in 1987.

In the years since the Council's current ABC and overfishing definitions were implemented, it has been possible to examine how well these definitions have served their intended purpose. In addition, there has been opportunity for the development of increased understanding within the fishery science community as to desirable properties of reference fishing mortality rates such as those used to define ABC and overfishing. As a result, several concerns regarding the definitions used in the BSAI and GOA Groundfish FMPs have been raised, particularly by NMFS' Overfishing Definitions Review Panel (ODRP, Rosenberg et al. 1994) and the Council's Scientific and Statistical Committee (SSC, minutes from January 1992). These concerns are paraphrased below, where the following notation is used: OFL is the overfishing level (i.e., the catch during the coming year that would correspond to the overfishing definition), MSY is maximum sustainable yield, B represents projected biomass at the start of the coming harvest year, B_{MSY} is the biomass corresponding to MSY, B_{thr} is a "threshold" biomass level greater than zero, B_{pre} is a "precautionary" biomass level greater than B_{thr}, F represents fishing mortality rate, F_{ABC} is the F used to set ABC for the coming harvest year, F_{OFL} is the F corresponding to the overfishing definition, and F_{MSY} is the F corresponding to MSY.

ODRP concerns (paraphrased):

1) F_{OFL} should vary directly with biomass when the latter is between B_{thr} and B_{pre}.

Currently, B_{thr} is set equal to zero, and B_{pre} is defined only for those few cases in which a reliable estimate of B_{MSY} is available.

2) For healthy stocks, F_{OFL} should exceed F_{MSY}. Currently, F_{OFL} is set equal to F_{MSY} whenever biomass exceeds B_{MSY} (provided that reliable estimates for F_{MSY} and B_{MSY} exist).

3) The responsibility for determining reliability of information and recommending it for Secretarial approval should be specified. Currently, the definition of overfishing is cast in terms of the "sufficiency" of the available data to estimate various quantities, but no single authority (e.g., Council, SSC, Plan Team) is given specific responsibility for determining what constitutes "sufficient."

4) Ambiguity should be eliminated in any text relating spawning per recruit (SPR) to exploitable biomass. Currently, language describing the measurement of SPR could potentially be misconstrued as referring to absolute biomass.

SSC concerns (paraphrased):

5) F_{ABC} should be reduced when $B<B_{MSY}$. Currently, F_{ABC} is not tied to biomass except in the (hypothetical) case where a "threshold" has been identified for a particular stock.

6) More caution should be required when less information is available. Currently, the level of uncertainty surrounding an estimate (e.g., an estimate of F_{MSY}) has no explicit relationship to the value of F_{ABC}, so long as the problematic "sufficiency" criterion referenced in Concern #3 (above) is satisfied.

7) F_{OFL} should exceed F_{ABC}. Currently, there is no requirement for a buffer between F_{ABC} and F_{OFL}.

8) OFL should remain constant over time when catch history is the only information available. Currently, in cases where catch history is the only information available, OFL is set equal to the average catch since 1977, meaning that OFL should tend to decrease over time (assuming that catch never exceeds OFL).

It is in response to the above concerns that the present amendment proposal was developed. At the June 1996 meeting, based on its review of the draft EA and related economic analysis and input from its advisory bodies and public testimony, the Council adopted Alternative 2, which revises the ABC and overfishing definitions for BSAI and GOA groundfish.

1.2 Alternatives Considered

1.2.1 Alternative 1: No Action. Under this alternative, the following (current) definitions of ABC and overfishing would remain in place:

Acceptable biological catch is a seasonally determined catch or range of catches that may differ from MSY for biological reasons. It may be lower or higher than MSY in some years for species with fluctuating recruitments. Given suitable biological justification by the Plan Team and/or Scientific and Statistical Committee, the ABC may be set anywhere between zero and the current biomass less the threshold value. The ABC may be modified to incorporate safety factors and risk assessment due to uncertainty. Lacking other biological justification, the ABC is defined as the exploitation rate multiplied by the size of the biomass for the relevant time period. The ABC is defined as zero when the stock is at or below its threshold.

Threshold is the minimum size of a stock that allows sufficient recruitment so that the stock can eventually reach a level that produces MSY. Implicit in this definition are rebuilding schedules. They have not been specified since the selection of a schedule is a part of the OY determination process. Interest instead is on the identification of a stock level below which the ability to rebuild is uncertain. The estimate given should reflect use of the best scientific information available. Whenever possible, upper and lower bounds should be given for the estimate.

Overfishing is defined as a maximum allowable fishing mortality rate. For any stock or stock complex under management, the maximum allowable mortality rate will be set at the level corresponding to maximum sustainable yield (F_{MSY}) for all biomass levels in excess of the level corresponding to maximum sustainable yield (B_{MSY}). For lower biomass levels, the maximum allowable fishing mortality rate will vary linearly with biomass, starting from a value of zero at the origin and increasing to a value of F_{MSY} at B_{MSY}, consistent with other applicable laws. If data are insufficient to calculate F_{MSY} or B_{MSY}, the maximum allowable fishing mortality rate will be set equal to the following (in order of preference):

1) the value that results in the biomass-per-recruit ratio (measured in terms of spawning biomass) falling to 30 percent of its pristine value;
2) the value that results in the biomass-per-recruit ratio (measured in terms of exploitable biomass) falling to 30 percent of its pristine value; or
3) the natural mortality rate (M).

If data are insufficient to estimate any of the above, the TAC shall not exceed the average catch taken since 1977.

1.2.2 Alternative 2: Proposed Revision. The revision proposed is to strike the existing FMP language defining "threshold" and replace the existing FMP language defining ABC and overfishing with the following (the proposed ABC definition--except for the last sentence--is taken directly from the 50 CFR 600.310(e)):

Acceptable Biological Catch is a preliminary description of the acceptable harvest (or range of harvests) for a given stock or stock complex. Its derivation focuses on the status and dynamics of the stock, environmental conditions, other ecological factors, and prevailing technological characteristics of the fishery. The fishing mortality rate used to calculate ABC is capped as described under "overfishing" below.

Overfishing is defined as any amount of fishing in excess of a prescribed maximum allowable rate. This maximum allowable rate is prescribed through a set of six tiers which are listed below in descending order of preference, corresponding to descending order of information availability. The SSC will have responsibility for determining whether a given item of information is "reliable" for the purpose of this definition, and may use either objective or subjective criteria in making such determinations. The Ssc shall recommend its determination to the Council, who may then recommend the SSC's determination to the Secretary for final approval. For tier (1), a "pdf" refers to a probability density function (see Appendix A). For tiers (1-3), the coefficient α is set at a default value of 0.05, with the understanding that the SSC may establish a different value for a specific stock or stock complex as merited by the best available scientific information. Figure 1 provides a hypothetical illustration of the behavior of tiers (1-3). For tiers (2-4), a designation of the form "$F_{X\%}$" refers to the F associated with an equilibrium level of spawning per recruit (SPR) equal to $X\%$ of the equilibrium level of spawning per recruit in the absence of any fishing. If reliable information sufficient to characterize the entire maturity schedule of a species is not available, the SSC may choose to view SPR calculations based on a knife-edge maturity assumption as reliable. For tier (3), the term $B_{40\%}$ refers to the long-term average biomass that would be expected under average recruitment and $F=F_{40\%}$.

1) *Information available: Reliable point estimates of B and B_{MSY} and reliable pdf of F_{MSY}.*
   - 1a) *Stock status:* $B/B_{MSY} > 1$
     - $F_{OFL} = \mu_A$, the arithmetic mean of the pdf (see Appendix A)
     - $F_{ABC} \leq \mu_H$, the harmonic mean of the pdf (see Appendix A)
   - 1b) *Stock status:* $\alpha < B/B_{MSY} \leq 1$
     - $F_{OFL} = \mu_A \times (B/B_{MSY} - \alpha)/(1 - \alpha)$
     - $F_{ABC} \leq \mu_H \times (B/B_{MSY} - \alpha)/(1 - \alpha)$
   - 1c) *Stock status:* $B/B_{MSY} \leq \alpha$
     - $F_{OFL} = 0$
     - $F_{ABC} = 0$
2) *Information available: Reliable point estimates of B, B_{MSY}, F_{MSY}, $F_{30\%}$,* and $F_{40\%}$.
   - 2a) *Stock status:* $B/B_{MSY} > 1$
     - $F_{OFL} = F_{MSY} \times (F_{30\%}/F_{40\%})$
     - $F_{ABC} \leq F_{MSY}$
   - 2b) *Stock status:* $\alpha < B/B_{MSY} \leq 1$
     - $F_{OFL} = F_{MSY} \times (F_{30\%}/F_{40\%}) \times (B/B_{MSY} - \alpha)/(1 - \alpha)$
     - $F_{ABC} \leq F_{MSY} \times (B/B_{MSY} - \alpha)/(1 - \alpha)$

2c) *Stock status:* $B/B_{MSY} \leq \alpha$

$F_{OFL} = 0$

$F_{ABC} = 0$

3) *Information available: Reliable point estimates of* B, $B_{40\%}$, $F_{30\%}$, and $F_{40\%}$.

3a) *Stock status:* $B/B_{40\%} > 1$

$F_{OFL} = F_{30\%}$

$F_{ABC} \leq F_{40\%}$

3b) *Stock status:* $\alpha < B/B_{40\%} \leq 1$

$F_{OFL} = F_{30\%} \times (B/B_{40\%} - \alpha)/(1 - \alpha)$

$F_{ABC} \leq F_{40\%} \times (B/B_{40\%} - \alpha)/(1 - \alpha)$

3c) *Stock status:* $B/B_{40\%} \leq \alpha$

$F_{OFL} = 0$

$F_{ABC} = 0$

4) *Information available: Reliable point estimates of* B, $F_{30\%}$, and $F_{40\%}$.

$F_{OFL} = F_{30\%}$

$F_{ABC} \leq F_{40\%}$

5) *Information available: Reliable point estimates of B and natural mortality rate M.*

$F_{OFL} = M$

$F_{ABC} \leq 0.75 \times M$

6) *Information available: Reliable catch history from 1978 through 1995.*

OFL = the average catch from 1978 through 1995, unless an alternative value is established by the SSC on the basis of the best available scientific information

ABC ≤ 0.75 × OFL

1.2.3 Summary of Similarities and Differences Between the Alternatives. The major similarities and differences between the alternatives are summarized below in order of "tier," using the numbering given in Alternative 2:

<u>All Tiers</u>

Similarities: F_{OFL} is a maximum allowable fishing mortality rate involving F_{MSY}, $F_{30\%}$, M, or the average catch. F_{OFL} varies linearly with biomass under some circumstances.

Differences: Alternative 2 provides a buffer between ABC and OFL; Alternative 1 does not. Alternative 2 gives the SSC specific responsibility for determining reliability of estimates; Alternative 1 does not.

<u>Tier (1)</u>

Similarities: For healthy stocks (1a), F_{OFL} and the upper limit on F_{ABC} are both independent of biomass level. For moderately depleted stocks (1b), F_{OFL} varies linearly with biomass level.

Differences: Alternative 2 takes the degree of uncertainty surrounding F_{MSY} into consideration; Alternative 1 does not. For moderately depleted stocks (1b), Alternative 2 forces the upper limit on F_{ABC} to vary linearly with biomass level; Alternative 1 does not. For severely depleted stocks (1c), Alternative 2 sets both F_{OFL} and F_{ABC} equal to zero; Alternative 1 does not.

Tier (2)

Similarities: For healthy stocks (2a), F_{OFL} and the upper limit on F_{ABC} are both independent of biomass level. For moderately depleted stocks (2b), F_{OFL} varies linearly with biomass level.

Differences: For healthy stocks (2a), Alternative 2 sets F_{OFL} higher than F_{MSY}; Alternative 1 does not. For moderately depleted stocks (2b), Alternative 2 forces the upper limit on F_{ABC} to vary linearly with biomass level; Alternative 1 does not. For severely depleted stocks (2c), Alternative 2 sets both F_{OFL} and F_{ABC} equal to zero; Alternative 1 does not.

Tier (3)

Similarities: For healthy stocks (3a), F_{OFL} is set at $F_{30\%}$, independent of biomass level.

Differences: For healthy stocks (3a), Alternative 2 caps F_{ABC} at the $F_{40\%}$ level; Alternative 1 does not. For moderately depleted stocks (3b), Alternative 2 forces both F_{OFL} and the upper limit on F_{ABC} to vary linearly with biomass level; Alternative 1 does not. For severely depleted stocks (3c), Alternative 2 sets both F_{OFL} and F_{ABC} equal to zero; Alternative 1 does not.

Tier (4)

Similarities: F_{OFL} is set at $F_{30\%}$.

Differences: Alternative 2 caps F_{ABC} at the $F_{40\%}$ level; Alternative 1 does not.

Tier (5)

Similarities: F_{OFL} is set equal to M.

Differences: Alternative 2 caps F_{ABC} at 75 percent of M; Alternative 1 does not.

Tier (6)

Similarities: F_{OFL} is set equal to average catch, at least as a default value.

Differences: Alternative 2 fixes the terminal year of the time series used to compute average catch at 1995; Alternative 1 does not. Alternative 2 allows the default OFL value to be adjusted in special cases on the basis of the best available scientific information; Alternative 1 does not. Alternative 2 caps ABC at 75 percent of OFL; Alternative 1 does not.

2.0 NEPA REQUIREMENTS: ENVIRONMENTAL IMPACTS OF THE ALTERNATIVES

An environmental assessment (EA) is required by the National Environmental Policy Act of 1969 (NEPA) to determine whether the action considered will result in significant impact on the human environment. The environmental analysis in the EA provides the basis for this determination and must analyze the intensity or severity of the impact of an action and the significance of an action with respect to society as a whole, the affected region and interests, and the locality. If the action is determined not to be significant based on an analysis of relevant considerations, the EA and resulting finding of no significant impact (FONSI) would be the final environmental documents required by NEPA. An environmental impact study (EIS) must be prepared for major Federal actions significantly affecting the human environment.

An EA must include a brief discussion of the need for the proposal, the alternatives considered, the environmental impacts of the proposed action and the alternatives, and a list of document preparers. The purpose and alternatives were discussed in Sections 1.1 and 1.2, and the list of preparers is in Section 8. This section contains the discussion of the environmental impacts of the alternatives including impacts on threatened and endangered species and marine mammals.

2.1 Environmental Impacts of the Alternatives

The environmental impacts generally associated with fishery management actions are effects resulting from 1) harvest of fish stocks which may result in changes in food availability to predators, changes in the abundance and population structure of target fish stocks, and changes in community structure; 2) changes in the physical and biological structure of the benthic environment as a result of fishing practices (e.g., effects of gear use and fish processing discards); and 3) entanglement/entrapment of non-target organisms in active or inactive fishing gear. A summary of the effects of the 1996 groundfish total allowable catch amounts on the biological environment and associated impacts on marine mammals, seabirds, and other threatened or endangered species are discussed in the final environmental assessment for the 1996 groundfish total allowable catch specifications.

2.1.1 Alternative 1: No Action. Because this alternative simply preserves the status quo, no significant environmental impacts are anticipated.

2.1.2 Alternative 2: Proposed Revision.

In terms of ABC, the definition contained in Alternative 2 can be viewed as a restricted version of the status quo. That is, nothing in the proposed redefinition of ABC is disallowed under the current definition. Therefore, in the sense that the Council currently has the ability to follow the restrictions on ABC contained in Alternative 2, the environmental impacts of adopting this alternative may be minimal (i.e., the Council might choose, even under Alternative 1, to impose *voluntarily* the same restrictions on ABC that would be *required* under Alternative 2). However, because the current definition of ABC is essentially open-ended except in cases where an estimate of F_{MSY} is available, there are insufficient built-in safeguards against imprudent harvest rates. By instituting such safeguards, Alternative 2 is expected to generate *positive* environmental impacts relative to the status quo by reducing the chance of setting allowable catches too high. This is accomplished primarily by placing an upper limit on the fishing mortality rate used to calculate ABC.

For example, when the amount of uncertainty associated with an estimate of F_{MSY} can be determined (tier [1]), Alternative 2 prescribes a cap on F_{ABC} based on the risk-averse optimization presented in Appendix B. However, it should be emphasized that, since the data necessary to ascertain this amount of uncertainty are not currently available, the main short-term effect of tier (1) may simply be to indicate the intended direction of future groundfish management in the North Pacific. Had Alternative 2 been in place when the 1996 harvest specifications were established, for example, *none* of the 1996 ABCs would have been determined under this tier. Even the more modest information requirement of an F_{MSY} point estimate (tier [2]) is presently satisfied for only two species (eastern Bering Sea pollock and GOA Pacific ocean perch, Table 1).

When information is more limited (tiers [3-4]), Alternative 2 caps F_{ABC} at the $F_{40\%}$ level, following the recommendation of Clark (1993) and Mace (1994). Justification for the $F_{40\%}$ strategy was presented in the December 1995 SSC minutes. In brief, the best available scientific information indicates that $F_{40\%}$ represents a reasonable upper bound on safe harvest rates when no reliable estimate of F_{MSY} is available. It is slightly more conservative than the $F_{35\%}$ strategy which has been used to set many ABCs for North Pacific groundfish in recent years, and seems to be especially appropriate when variability in recruitment is high or when instances of low recruitment tend to occur in groups (i.e., when a weak year class is more likely to be followed by another weak year class than by a strong year class). As Clark states, "The year-to-year variability of yield is hardly affected by the target level of spawning biomass per recruit, but the frequency of episodes of low spawning biomass -- if defined as less than 20 percent of the unfished level -- may be reduced substantially by fishing at $F_{40\%}$ rather than $F_{35\%}$, even though there is only a small difference in average spawning biomass between $F_{35\%}$ and $F_{40\%}$." The SSC concluded that there was "general agreement" between itself and the Plan Teams to the effect that $F_{40\%}$ is "a desirable harvest rate for consideration in the evolution of conservative harvest rate policies." Approximately one-fifth of the 1996 ABCs were based on the $F_{40\%}$ strategy. Had Alternative 2 been in place when the 1996 harvest specifications were established, probably about one-half of all ABCs would have been set at the $F_{40\%}$ level (Table 1).

When information is extremely limited (tier [5]), Alternative 2 caps F_{ABC} at a level somewhat (specifically, 25 percent) below the natural mortality rate M, following the recommendation of Deriso (1982) and Thompson (1993). Deriso showed that M often exceeds F_{MSY}, and Thompson showed that M can exceed even $F_{30\%}$. According to the formulae presented by Thompson, capping the harvest rate at 75 percent of M should generally keep F below $F_{30\%}$. Had the proposed definition been in place when the 1996 harvest specifications were established, probably about one-fourth of all ABCs would have been set at the F=0.75M level (Table 1).

In terms of preventing overfishing, the definition contained in Alternative 2 is also expected to generate positive environmental impacts relative to the status quo by imposing additional safeguards under those conditions where they are most needed. Although Alternative 2 relaxes the current overfishing definition slightly for healthy stocks, it is more restrictive than the current definition for stocks that have fallen significantly below their target levels of abundance (Figure 1).

In terms of the need for action outlined in Section 1.1, Alternative 2 addresses the specific ODRP and SSC concerns as follows:

1) *F_{OFL} should vary directly with biomass when the latter is between B_{thr} and B_{pre}.* The proposed definition satisfies this concern in tiers (1-3) by establishing a linear scale for F_{OFL} when biomass is between B_{thr} and B_{pre} and by setting B_{pre} equal to either B_{MSY} (tiers [1-2]) or $B_{40\%}$ (tier [3]). This concern is not satisfied in tiers (4-6) because it is impossible to identify an appropriate precautionary biomass level when basic biological information is largely or entirely lacking.

2) *For healthy stocks, F_{OFL} should exceed F_{MSY}.* The proposed definition satisfies this concern in tiers (1-2) by setting a buffer based either on the ratio between the arithmetic and harmonic means of the pdf (tier [1]) or on the ratio between $F_{30\%}$ and $F_{40\%}$ (tier [2]). This concern is not satisfied in tiers (3-6) because it is impossible to ensure that any particular F is greater than F_{MSY} if F_{MSY} cannot be estimated.

3) *The authority for determining reliability of information should be specified.* The proposed definition satisfies this concern by vesting within the SSC final authority for determining reliability of information.

4) *Ambiguity should be eliminated in any text relating SPR to exploitable biomass.* The proposed definition satisfies this concern by eliminating the previous definition's text relating SPR to exploitable biomass.

5) *F_{ABC} should be reduced when $B<B_{MSY}$.* The proposed definition satisfies this concern in tiers (1-2) by establishing a linear scale for F_{ABC} when biomass is between B_{thr} and B_{MSY}. This concern is not satisfied in tiers (3-6) because it is impossible to measure biomass relative to B_{MSY} when B_{MSY} cannot be estimated.

6) *More caution should be required when less information is available.* The proposed definition satisfies this concern in tier (1) by setting the F_{ABC}/F_{OFL} ratio equal to the ratio between the harmonic and arithmetic means, a quantity which tends to decrease as the coefficient of variation (a measure of uncertainty or lack of information) increases. This concern is not necessarily satisfied in tiers (2-6) taken individually (e.g., comparing F_{OFL}s for two different stocks under tier [4]), because these tiers are designed to group stocks together on the basis of similarity of available information, making it difficult to distinguish between levels of uncertainty for stocks managed within any one of these tiers. Neither is this concern necessarily satisfied in tiers (2-6) taken sequentially (e.g., comparing F_{OFL}s calculated for the same stock under tiers [5] and [6]), because it is difficult to ensure that an F computed under any given tier is lower than the F that would have been computed under a more information-intensive tier if the requisite information is lacking (which it is, by definition).

7) *F_{OFL} should exceed F_{ABC}.* The proposed definition satisfies this concern by providing an explicit buffer between F_{OFL} and F_{ABC}.

8) *OFL should remain constant over time when catch history is the only information available.* The proposed definition satisfies this concern in tier (6) by terminating the catch time series in 1995 (i.e., the endpoint of the catch time series would be fixed at 1995, not set at the current year as in the status quo). This concern is not relevant to tiers (1-5).

2.2 Impacts on Endangered, Threatened, or Candidate Species

Listed and candidate species under the Endangered Species Act ESA) that may be present in the GOA and BSAI include:

Endangered

Northern right whale	*Balaena glacialis*
Sei whale	*Balaenoptera borealis*
Blue whale	*Balaenoptera musculus*
Fin whale	*Balaenoptera physalus*
Humpback whale	*Megaptera novaeangliae*
Sperm whale	*Physeter macrocephalus*
Snake River sockeye salmon	*Oncorhynchus nerka*
Snake River fall chinook salmon	*Oncorhynchus tshawytscha*
Short-tailed albatross	*Diomedea albatrus*

Threatened

Steller sea lion	*Eumetopias jubatus*
Snake River spring and summer chinook salmon	*Oncorhynchus tshawytscha*

Spectacled eider *Somateria fischeri*

The impact of BSAI and GOA groundfish fisheries on Steller sea lions was addressed in a formal consultation on April 19, 1991, and in various informal consultations since then. NMFS has determined that the groundfish fisheries are not likely to affect Steller sea lions in a way or to an extent not already considered in these consultations.

An informal consultation conducted on effects of the GOA and BSAI groundfish fisheries concluded that the continued operation of these fisheries would not adversely affect listed species of salmon as long as current observer coverage levels continued and salmon bycatch was monitored on a weekly basis. Consultation must be reinitiated if chinook salmon bycatch exceeds 40,000 fish in the GOA or 55,000 in the BSAI or sockeye salmon bycatch exceeds 200 fish in the BSAI or 100 fish in the GOA.

Endangered, threatened, proposed, and candidate species of seabirds that may be found within the regions of the GOA and BSAI where the groundfish fisheries operate, and potential impacts of the groundfish fisheries on these species are discussed in the EA prepared for the TAC specifications. The U.S. Fish and Wildlife Service (USFWS), in the informal consultation on the 1995 specifications and subsequent actions consistent with the biological opinions, concluded that groundfish operations are likely to result in an unquantified level of mortality to short-tailed albatrosses, a listed species, but will not jeopardize the continued existence of the population. The take level was not expected to exceed that authorized in the USFWS consultation conducted on the implementation of the Marine Mammal Exemption Program (1988).

Neither Alternative 1 (No Action) nor Alternative 2 (Proposed Revision) is anticipated to affect threatened, endangered, or candidate species in a way or to an extent not already considered in the above-mentioned consultations.

2.3 Impacts on Marine Mammals

Marine mammals not listed under the Endangered Species Act that may be present in the GOA and BSAI include cetaceans [minke whale (*Balaenoptera acutorostrata*), killer whale (*Orcinus orca*), Dall's porpoise (*Phocoenoides dalli*), harbor porpoise (*Phocoena phocoena*), Pacific white-sided dolphin (*Lagenorhynchus obliquidens*), and the beaked whales (e.g., *Berardius bairdii* and *Mesoplodon* spp.)] as well as pinnipeds [e.g., northern fur seals (*Callorhinus ursinus*) and Pacific harbor seals (*Phoca vitulina*)] and the sea otter (*Enhydra lutris*).

Relative to the status quo, neither Alternative 1 (No Action) nor Alternative 2 (Proposed Revision) is anticipated to have an adverse impact on any marine mammal species.

2.4 Coastal Zone Management Act

Implementation of either alternative would be conducted in a manner consistent, to the maximum extent practicable, with the Alaska Coastal Management Program within the meaning of Section 30(c)(1) of the Coastal Zone Management Act of 1972 and its implementing regulations.

3.0 ECONOMIC ANALYSIS OF THE ALTERNATIVES

3.1 Economic and Socioeconomic Impacts of the Alternatives

This section provides information about the economic and socioeconomic impacts of the alternatives including identification of the individuals or groups that may be affected by the action, the nature of these impacts, quantification of the economic impacts if possible, and discussion of the tradeoffs between qualitative and quantitative benefits and costs.

3.1.1 Alternative 1: No Action. Because this alternative simply preserves the status quo, no significant economic or socioeconomic impacts are anticipated.

3.1.2 Alternative 2: Proposed Revision. As noted in Section 2.1.2 above, the definition of ABC contained in Alternative 2 can be viewed as a restricted version of the status quo. That is, nothing in the proposed redefinition of ABC is disallowed under the current definition. Therefore, in the sense that the Council currently has the ability to follow the restrictions on ABC contained in Alternative 2, the economic and socioeconomic impacts of adopting this alternative may be minimal (i.e., the Council might choose, even under Alternative 1, to impose *voluntarily* the same restrictions on ABC that would be *required* under Alternative 2).

Nevertheless, had Alternative 2 been in place when the 1996 groundfish specifications were put into place, it appears that some short-term economic impacts would have been experienced by the fishing industry. From Table 1, for example, it appears that 1996 ABCs for most flatfish stocks would have decreased on the order of 15-20 percent and that 1996 ABCs for BSAI rockfish stocks other than Pacific ocean perch would have decreased on the order of 25 percent (assuming that ABC is roughly proportional to F_{ABC}). However, changes in TAC would in many cases have been less extreme, since TAC was already well below ABC for many species. Table 2 shows the relative amounts by which 1996 TAC differed with respect to 1996 ABC, the relative amounts by which 1996 ABC would have been reduced had Alternative 2 been in place (assuming that ABC is roughly proportional to F_{ABC}), and the relative amounts by which 1996 TAC would have been reduced had Alternative 2 been in place. Note that the relative differences between actual 1996 ABC and actual 1996 TAC were already *greater* than the relative reductions in ABC that would probably have been required under Alternative 2 for all GOA flatfish except rex sole, GOA "other" slope rockfish, all BSAI flatfish, BSAI sablefish, and BSAI squid. This leaves only GOA rex sole (with a reduction in 1996 TAC of about 1-5 percent); GOA sablefish (with a reduction in 1996 TAC of about 8 percent); GOA shortraker/rougheye rockfish (with a reduction in 1996 TAC of about 9 percent); and BSAI rockfish other than Pacific ocean perch (with reductions in 1996 TAC of about 15 percent) as requiring modification in the final TAC had Alternative 2 been in place during the 1996 specification process.

One area in which economic impacts are particularly difficult to evaluate is the management of certain species on the basis of average catch. In 1996, average catch was the basis for ABC in the cases of BSAI squid and "other species," and the basis for OFL in the case of BSAI squid. The 1996 OFL for BSAI "other species" was based on an $F=M$ strategy, giving a buffer of well over 100,000 t for that species complex. Also, the Council has given some consideration to the possibility of removing dusky rockfish from the GOA pelagic shelf rockfish complex (of which dusky rockfish is the dominant species). If dusky rockfish were removed from the pelagic shelf rockfish complex, it is possible that the remaining species in that complex would start to be managed on the basis of average catch. In general, it is difficult to argue strongly in favor of past catch history as implying very much about what future catches ought to be, except to note that if average catch is the only information available for a particular species it is difficult to imagine what else might be used to determine an appropriate harvest level. Of course, the use of average catch to set future harvest levels for a species should typically send a strong signal that more research is needed into the biology of that species.

The "average catch" rule might also come into play in the case of a new target groundfish species, that is, a groundfish species for which a target fishery first develops after 1995. Here, the proposed definition might work in either of two ways:

1) If the new target species is currently listed as a member of the "other species" complex, it already has a catch history (though only as a bycatch species). A target fishery for that species could develop under the existing TAC and OFL for the complex until such time as the species is split out into its own category by FMP amendment. If it is split out into its own category, it would be managed under the highest tier for which it qualifies (like any other species). If it qualifies for management only under tier (6), ABC and OFL would be based on the species' average catch prior to 1996 (i.e., when it was taken as bycatch only) unless an alternative OFL value is established by the SSC on the basis of the best available scientific information.

2) If the new target species is *not* currently listed as a member of the "other species" complex, it does not have a catch history. Because it would not automatically come under Federal management, an unconstrained target fishery for the species could develop until such time as the appropriate FMP is amended to include it as a member of the "other species" complex or as its own category. Lacking a catch history, however, it could not be given its own ABC or OFL under tier (6), meaning that it would have to qualify for management under some other tier in order to receive its own ABC or OFL.

While some short-term negative economic impacts may result from adoption of Alternative 2, it should be remembered that the measures incorporated into this alternative were developed with long-term optimization explicitly in mind as required by 50 CFR 600.310. This means that increases in long-term benefits are expected eventually to outweigh any short-term losses, assuming that long-term average yield (or something like it) is a reasonable measure of long-term benefits and that the discount rate is sufficiently low. On the other hand, it should be noted that different measures of long-term benefits or a sufficiently high discount rate could lead to different conclusions. The specific assumptions or conditions under which the net economic impacts of short-term costs might outweigh those of the expected long-term benefits have not been determined.

3.2 Administrative, Enforcement, and Information Costs

No additional administrative, enforcement, or information costs are expected under either alternative. Moreover, because Alternative 2 would require the maintenance of a reasonable buffer between ABC and OFL, its adoption is expected to make administration of the fishery management system easier and to reduce the average amount of unharvested TAC, because it is easier to achieve a target harvest amount if the goal is to come as close to the target as possible than if the goal is to come as close as possible without going over.

4.0 SUMMARY AND CONCLUSIONS

Reviews by the ODRP and SSC have indicated that the definitions of ABC and overfishing contained in the BSAI and GOA Groundfish FMPs could and should be improved. Suggestions for improvement include the following: A) greater imprecision in parameter estimates should correspond to more conservative fishing mortality rates; B) for a stock below its target abundance level, fishing mortality rates should vary directly with biomass and ultimately fall to zero should the stock become critically depleted; and C) a buffer should be maintained between ABC and the overfishing level.

This plan amendment proposal contains two alternatives: Alternative 1 (No Action) maintains the current definitions, and Alternative 2 (Proposed Revision) modifies the current definitions in response to the suggestions made by the ODRP and SSC. The differences between the two alternatives can perhaps best be illustrated by considering the case in which a point estimate of F_{MSY} is available together with a reliable description of the amount of uncertainty surrounding that estimate. Under the current definitions, F_{ABC} and F_{OFL} are both set equal to the point estimate of F_{MSY}, regardless of the level of uncertainty associated with that estimate. Under the proposed revision, the ratio between F_{ABC} and F_{OFL} varies inversely with the level of uncertainty (i.e., the greater the uncertainty in the estimate of F_{MSY}, the lower F_{ABC} is in relation to F_{OFL}).

Even in cases where reliable descriptions of the level of uncertainty associated with a point estimate of F_{MSY} are not available, the proposed revision maintains an appropriate buffer between F_{ABC} and F_{OFL}. Also, whenever a target abundance level can be reasonably identified, the proposed revision reduces fishing mortality rates as stock size falls below that target level. The current definitions do neither of these.

Because the proposed revision institutes new safeguards against overly aggressive harvest rates, particularly under conditions of high uncertainty or low stock size, the revision is expected to result in positive environmental impacts, if any, but is not likely to significantly effect the quality of the human environment. The revision would also be expected to result in positive long-term economic impacts in those cases where the objective of optimizing long-term average yield on a species-by-species basis is a suitable proxy for maximizing long-term economic impacts. However, it is possible that negative economic impacts could be generated in the short term for a few fisheries, particularly Bering Sea and Aleutian Islands fisheries targeting on rockfish species other than Pacific ocean perch, where total allowable catch might be reduced by about 15 percent.

The assumptions or conditions under which the net economic impacts of such short-term costs might outweigh those of the expected long-term benefits have not been determined.

4.1 Conclusions or Finding of No Significant Impact

Neither of the alternatives is likely to affect significantly the quality of the human environment, and the preparation of an environmental impact statement for the proposed action is not required by Section 102(2)(C) of NEPA or its implementing regulations.

[signature]	JAN 9 1997
	Date

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Appendix A:
Nontechnical Definitions of Statistical Terms

Probability density function (pdf): A description of the probability associated with different values of a variable. For example, in a coin flip the probability of tossing "heads" is 50 percent and the probability of tossing "tails" is 50 percent. As another example, in tossing a six-sided die, the probability of tossing a "1" is 16.667 percent and the probability of tossing something other than a "1" is 83.333 percent. The probabilities in a pdf must always sum to 100 percent.

Arithmetic mean: For a random variable *X*, the arithmetic mean is the sum of the possible values of *X* weighted by the respective probabilities of those values. For example, consider a game of chance based on a coin flip, where the random variable *X* denotes the prize associated with the game. The player gets $72 if he or she tosses "heads" and $24 if he or she tosses "tails." The arithmetic mean prize for this game is

$$(50\% \times \$72) + (50\% \times \$24) = \$48.$$

As another example, consider a game of chance based on the toss of a six-sided die, where again the random variable *X* denotes the prize associated with the game. The player gets $72 if he or she tosses a "1" and $24 if he or she tosses anything else. The arithmetic mean prize associated with this game is

$$(16.667\% \times \$72) + (83.333\% \times \$24) = \$32.$$

Harmonic mean: Unfortunately, when written out in words, the definition of harmonic mean is a little complicated, but here goes (hopefully, the examples which follow will make things clearer): For a random variable *X*, the harmonic mean is the reciprocal of the sum of the reciprocals of the possible values of *X* weighted by the respective probabilities of those values. For example, consider the game of chance based on a coin flip described under "arithmetic mean" above. The harmonic mean prize associated with this game is

$$\frac{1}{\frac{50\%}{\$72} + \frac{50\%}{\$24}} = \$36.$$

As another example, consider the game of chance based on the toss of a six-sided die described under "arithmetic mean" above. The harmonic mean prize associated with this game is

$$\frac{1}{\frac{16.667\%}{\$72} + \frac{83.333\%}{\$24}} = \$27.$$

Note that the harmonic mean is less than the arithmetic mean in both of these examples (\$36 versus \$48 for the coin flip and \$27 versus \$32 for the die toss). For all practical purposes, this relationship always holds (i.e., the harmonic mean is always less than the arithmetic mean). Thus, if the random variable X represents a fishing mortality rate, the harmonic mean is a more conservative (i.e., lower) rate than the arithmetic mean.

Appendix B:
Risk-Averse Optimal Harvesting in a Biomass Dynamic Model

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Table 1: Summary of impacts on 1996 ABC and OFL fishing mortality rates had Alternative 2 been in place (see footnotes).

Gulf of Alaska

	ABC Fishing Mortality Rate			OFL Fishing Mortality Rate		
Species	1996 Actual [1]	Alternative 2 [2]	%Change [3]	1996 Actual [1]	Alternative 2 [2]	%Change [3]
Walleye pollock	FABC=0.30	same	0	F30%=0.50	same	0
Pacific cod	F40%=0.40	same	0	F30%=0.57	same	0
Deepwater flatfish	F35%=0.125	F40%=? [4]	-? [4]	F30%=0.146	same	0
Rex sole	F35%=0.125	F40%=? [4]	-? [4]	F30%=0.146	same	0
Shallow water flatfish	F35%=0.149	F40%=? [4]	-? [4]	F30%=0.175	same	0
Flathead sole	F35%=0.145	F40%=? [4]	-? [4]	F30%=0.159	same	0
Arrowtooth flounder	F35%=0.125	F40%=? [4]	-? [4]	F30%=0.146	same	0
Sablefish	F35%(adj.)=0.112	F40%=0.103	-8	F30%=0.153	same	0
Pacific ocean perch	F44%(adj.)=0.052	same	0	FMSY(adj.)=0.065	FMSY(adj.)=0.082	+26
Shortraker	F=M=0.03	M x 0.75 = 0.023	-25	F=M=0.03	same	0
Rougheye	F=M=0.025	same (?) [5]	0	F30%=0.046	same	0
Other slope rockfish	F=M=0.05	same (?) [5]	0	F30%=0.08	same	0
Northern rockfish	F=M=0.06	same (?) [5]	0	F30%=0.113	same	0
Pelagic shelf rockfish	F=M=0.09	same (?) [5]	0	F30%=0.151	same	0
Demersal shelf rockfish	F=M=0.02	same (?) [5]	0	F30%=0.04	same	0
Thornyhead rockfish	F40%=0.059	same	0	F30%=0.09	same	0
Atka mackerel	F=M/2=0.15	same	0	F30%=0.45	same	0

Bering Sea and Aleutians

	ABC Fishing Mortality Rate			OFL Fishing Mortality Rate		
Species	1996 Actual [1]	Alternative 2 [2]	%Change [3]	1996 Actual [1]	Alternative 2 [2]	%Change [3]
EBS Walleye pollock	F40%=0.30	same	0	FMSY=0.38	FMSY(adj.)=0.46	+21
AI Walleye pollock	F40%=0.34	same	0	F30%=0.45	same	0
Bogoslof Walleye pollock	F40%/2=0.11	same	0	FMSY(adj.)=0.11	FMSY(adj.)=0.17	+54
Pacific cod	F40%=0.30	F40%=0.30	0	F30%=0.43	same	0
Yellowfin sole	F35%=0.13	F40%=0.11	-15	F30%=0.16	same	0
Greenland turbot	F40%(adj.)=0.184	same	0	F30%=0.37	same	0
Arrowtooth flounder	F35%=0.27	F40%=0.22	-19	F30%=0.34	same	0
Rock sole	F35%=0.18	F40%=0.15	-17	F30%=0.22	same	0
Flathead sole	F35%=0.19	F40%=0.16	-16	F30%=0.23	same	0
Other flatfish	F35%=0.17	F40%=0.14	-18	F30%=0.20	same	0
Sablefish	F35%(adj.)=0.112	F40%=0.103	-8	F30%=0.15	same	0
EBS True POP	F44%=0.06	same	0	F30%=0.096	same	0
EBS Other red rockfish	F=M=0.05	M x 0.75 = 0.038	-25	F=M=0.05	same	0
AI True POP	F44%=0.06	same	0	F30%=0.096	same	0
AI Sharpchin/northern	F=M=0.06	M x 0.75 = 0.045	-25	F=M=0.06	same	0
AI Shortraker/rougheye	F=M=0.03	M x 0.75 = 0.023	-25	F=M=0.03	same	0
EBS Other rockfish	F=M=0.07	M x 0.75 = 0.053	-25	F=M=0.07	same	0
AI Other rockfish	F=M=0.07	M x 0.75 = 0.053	-25	F=M=0.07	same	0
Atka mackerel	F40%=0.49	same	0	F30%=0.75	same	0
Squid	Fave=?	Fave x 0.75 = ?	-25	Fave=?	same	0
Other species	Fave=?	same	0	F=M=0.20	same	0

Notes:

1) "1996 Actual" lists the fishing mortality rate corresponding to the ABC or OFL approved by the Council for 1996.
Rates bearing the suffix "(adj.)" have been adjusted by the ratio of current biomass to target biomass.

2) If Alternative 2 would have required a reduction in the 1996 rate, this column lists the maximum rate that would have been allowed.
If Alternative 2 would *not* have required a reduction in the 1996 rate, a listing of "same" is given.
Rates bearing the suffix "(adj.)" have been adjusted by the ratio of current biomass to target biomass. However, adjustments that might have been required as a result of biomass falling below B40% are not shown, as estimates of B40% are generally unavailable.

3) "%Change" lists the percentage change between "1996 Actual" and "Alternative 2."

4) Estimates of F40% for GOA flatfish are not available. If BSAI flatfish rates are used as a proxy, F40% is 15-19% less than F35%.

5) Estimates of F40% for some GOA rockfish are not available. However, it appears likely that F40% would be greater than '96 F(ABC).

Table 2: Estimated net impacts on 1996 TAC had Alternative 2 been in place (see footnotes).

Gulf of Alaska Species	Actual 1996 ABC	Actual 1996 TAC	[1] Actual % Difference (ABC:TAC)	[2] % Reduction in ABC Required by Alternative 2	[3] % Reduction in TAC Required by Alternative 2
Walleye pollock	54810	54810	0	0	0
Pacific cod	65000	65000	0	0	0
Deepwater flatfish	14590	11080	24	15-19	0
Rex sole	11210	9690	14	15-19	1-5
Shallow water flatfish	52270	9740	81	15-19	0
Flathead sole	28790	18630	35	15-19	0
Arrowtooth flounder	198130	35000	82	15-19	0
Sablefish	17080	17080	0	8	8
Pacific ocean perch	8060	6959	14	0	0
Shortraker/rougheye	1910	1910	0	9	9
Other slope rockfish	7110	2020	72	0	0
Northern rockfish	5270	5270	0	0	0
Pelagic shelf rockfish	5190	5190	0	0	0
Demersal shelf rockfish	950	950	0	0	0
Thornyhead	1560	1248	20	0	0
Atka mackerel	3240	3240	0	0	0

Bering Sea and Aleutians Species	Actual 1996 ABC	Actual 1996 TAC	[1] Actual % Difference (ABC:TAC)	[2] % Reduction in ABC Required by Alternative 2	[3] % Reduction in TAC Required by Alternative 2
EBS Walleye pollock	1190000	1190000	0	0	0
AI Walleye pollock	35600	35600	0	0	0
Bogoslof Walleye pollock	121000	1000	99	0	0
Pacific cod	305000	270000	11	0	0
Yellowfin sole	278000	200000	28	15	0
Greenland turbot	10300	7000	32	0	0
Arrowtooth flounder	129000	9000	93	19	0
Rock sole	361000	70000	81	17	0
Flathead sole	116000	30000	74	16	0
Other flatfish	102000	35000	66	18	0
Sablefish	2500	2300	8	8	0
EBS True POP	1800	1800	0	0	0
EBS Other red rockfish	1400	1260	10	25	15
AI True POP	12100	12100	0	0	0
AI Sharpchin/northern	5810	5229	10	25	15
AI Shortraker/rougheye	1250	1125	10	25	15
EBS Other rockfish	497	447	10	25	15
AI Other rockfish	952	857	10	25	15
Atka mackerel	116000	106157	8	0	0
Squid	3000	1000	67	25	0
Other species	27600	20125	27	0	0

Notes:

1) This column gives the percentage by which actual 1996 ABC was higher than actual 1996 TAC.
2) This column gives the percentage by which actual 1996 ABC would have been reduced had Alternative 2 been in place. Listings do not include any adjustments that might have occurred as a result of biomass falling below B40%. Required reductions for GOA flatfish ABCs are assumed to be in the 15-19% range by analogy with the BSAI flatfish species. Required reductions for some GOA rockfish (see Table 1) are assumed to be 0 on the basis of the large buffer between 1996 F(ABC) and F30% (i.e., F40% is assumed to be higher than 1996 F(ABC) and therefore not constraining).
3) This column gives the percentage by which actual 1996 TAC would have been reduced had Alternative 2 been in place. Species for which Alternative 2 would likely have had a noticeable impact on TAC in 1996 are shaded.

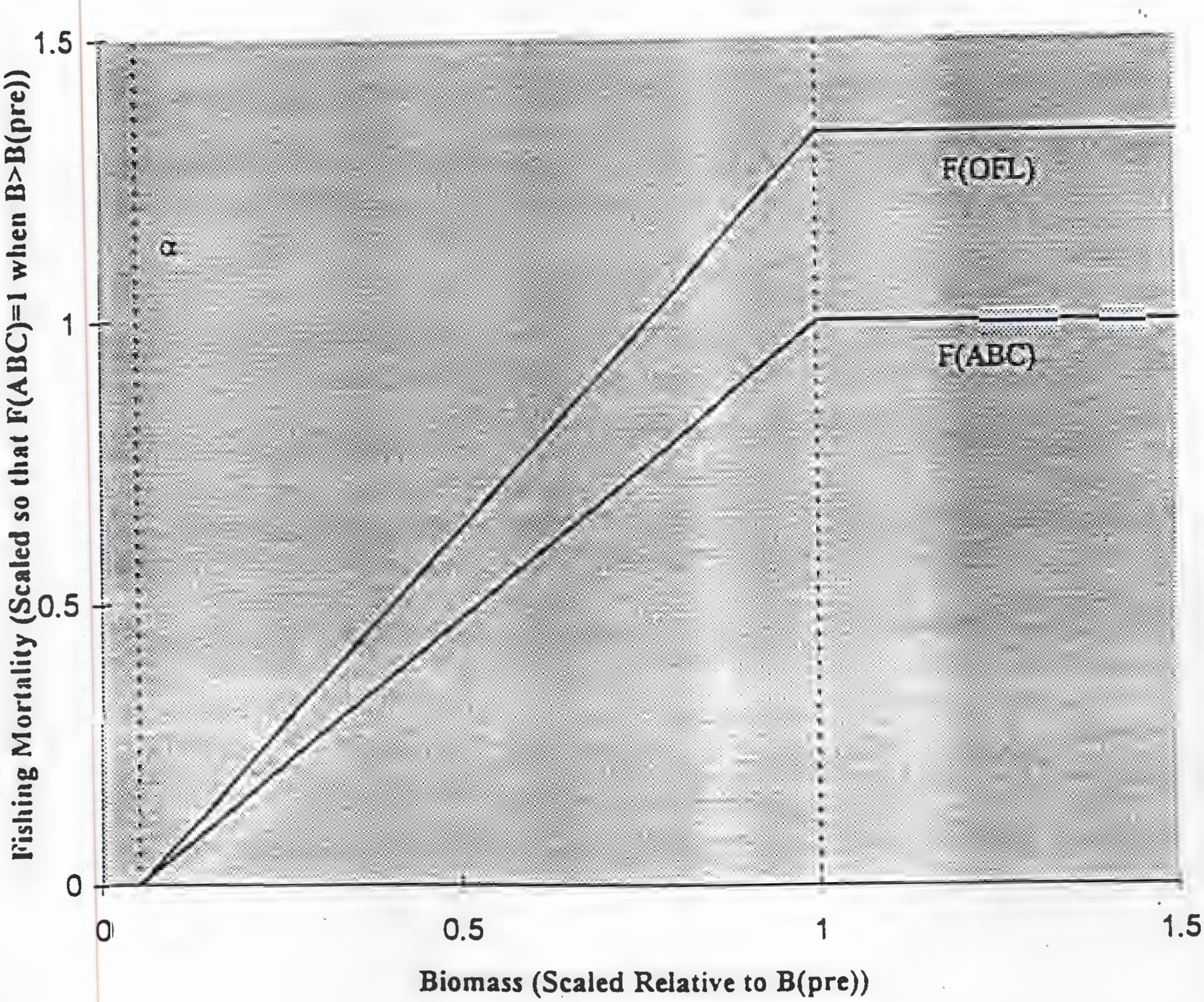

Figure 1. Hypothetical example illustrating relationship of F(OFL) to F(ABC) as a function of biomass.

Appendix B:
Risk-Averse Optimal Harvesting in a Biomass Dynamic Model

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SUMMARY

Considerable interest has been expressed in the fishery science literature toward finding an objectively risk averse long-term management strategy that takes account of both measurement error and process error, factors which affect estimates of present stock size and projections of future stock sizes under alternative harvest strategies. The present paper takes for its underlying model of stock dynamics a stochastic differential equation deriving from the deterministic Gompertz growth function. Process error in this model is formally lognormal. This property, combined with an assumption of lognormal measurement error, renders the (log transformed) model amenable to estimation via the Kalman filter, which can be interpreted as a Bayesian method of updating stock size estimates. The Kalman filter defines a likelihood function which, given prior distributions on certain model parameters, can then be used to obtain Bayesian estimates of those parameters from their respective posterior distributions. A theory of relative risk aversion is presented, describing the relationship between risk and moments of the applicable probability density function (pdf). From this theory, the harvest rate that maximizes the expected value of the logarithm of stationary yield (a formally risk averse harvest objective) is shown to be equal to the harmonic mean of the pdf of the harvest rate that maximizes sustainable yield in the deterministic case. This result is shown to be somewhat sensitive to changes in the level of relative risk aversion, but potentially quite robust to changes in model structure.

Key words: Fisheries; Harvest strategy; Risk aversion; Decision theory; Bayesian statistics; Kalman filter; Gompertz growth; Stochastic differential equations; Harmonic mean.

1. Introduction

Considerable interest has been expressed in the fishery science literature toward finding an optimal long-term management strategy that takes account of the various sources of uncertainty inherent in fishery stock assessment, as evidenced in volumes edited by Smith, Hunt, and Rivard (1993) and Kruse et al. (1993) as well as in recent papers by Cordue and Francis (1994); Megrey, Hollowed, and Baldwin (1994); McAllister et al. (1994); Rosenberg and Restrepo (1994); Walters and Ludwig (1994); Frederick and Peterman (1995); Ianelli and Heifetz (1995); and Adkison and Peterman (1996). These studies typically make use, in some form or combination, of Bayesian statistics, decision theory, or risk analysis. The present paper is an attempt to expand on the already considerable foundation laid in the above (and other) studies. The objective is to put together an integrated approach to the problems posed by uncertainty and risk in fishery management, one which covers the areas of stock dynamics, parameter estimation, and harvest optimization, all within a formally risk-averse framework. In other words, the objective is to provide a start-to-finish, practical example of how fisheries can be managed in an optimal, risk-averse fashion.

A convenient place to start in developing such an approach is with a biomass dynamic model (Hilborn and Walters, 1992), also called a "production" or "surplus production" model. Biomass dynamic models are among the simpler tools used in fishery stock assessment, but may perform as well as or better than more complicated models in some cases, and are basically the assessment scientist's only choice when age data are unavailable (Hilborn and Walters, 1992). In continuous-time form, a biomass dynamic model describes the time derivative of stock biomass as some function of stock biomass itself. This function typically takes the form of one of the classical growth curves, such as that of Gompertz (1825), Verhulst (1838), or Richards (1959). The biomass dynamic models corresponding to the Gompertz, Verhulst, and Richards curves were developed by Fox (1970), Graham (1935) and Schaefer (1954), and Pella and Tomlinson (1969), respectively. It may be noted that the Gompertz-Fox and Verhulst-Graham-Schaefer models are special cases of the Richards-Pella-Tomlinson model. Recent developments in the use of biomass dynamic models are described by Polachek et al. (1993); Hoenig, Warren, and Stocker (1994); Prager (1994); and Walters (1995).

The approach outlined in this paper will take the Gompertz-Fox model as an appropriate deterministic metaphor for stock dynamics, then generalize the Gompertz growth function into stochastic differential equation form as described by Capocelli and Ricciardi (1974). Parameter estimation and derivation of the risk-averse optimal fishing mortality rate will be based on a Bayesian methodology (e.g., Berger 1985, Lee 1989), applying the principles of decision theory to posterior distributions of model parameters. The likelihood function will be generated by a Kalman filter approach (e.g., Harvey 1990), which itself can be interpreted as a Bayesian methodology (Harrison and Stevens, 1971, 1976; Meinhold and Singpurwalla, 1983). Previous applications of the Kalman filter in the fisheries literature have been made by Mendelssohn (1988), Collie and Walters (1991), Sullivan (1992), Gudmundsson (1994 and 1995), and Walters and Parma (1996), and very general treatments of the method have been presented in a fisheries context by Pella (1993) and Schnute (1994).

For the purpose of providing a practical example, the approach will be applied to the eastern Bering Sea stock of flathead sole (*Hippoglossoides elassodon*), a lightly exploited stock which has been assessed by a standardized trawl survey annually since 1982 (Walters and Wilderbuer 1995).

The outline of the paper is as follows:

2. Statistical Terminology and Notation

Some notational conventions will be helpful to note early on: 1) Single capital Roman letters will refer to logarithms of their lower-case counterparts, except when used in an acronym (e.g., Y will refer to the logarithm of yield y except when it appears in an acronym such as *MSY*, the abbreviation for "maximum sustainable yield"). 2) The symbols μ and σ will refer to the mean and standard deviation of a normal distribution. 3) A "prime" symbol will designate a parameter of a prior distribution, while the absence of a "prime" symbol will designate a parameter of a posterior distribution (e.g., μ'_X will represent the prior mean of X, while μ_X will represent the posterior mean of X). 4) For coefficients that are functions of time (t), the limit as t goes to infinity will be indicated by the absence of a time argument (e.g., μ_X will denote the limit of $\mu_X(t)$ as t approaches infinity). 5) The symbol $g_X(X)$ will be used to designate the probability density function (pdf) of X. A complete list of symbols is given in Table 1 (note that this table applies primarily to use of symbols in the main text; certain Greek characters have different usages in Attachment 1).

Because the normal and lognormal distributions play such an important role in the remainder of the paper, a brief review of their functional forms is in order. If the variable X is normally distributed, that is, if it has a probability density function (pdf) of the form

$$g_X(X) = \sqrt{\frac{1}{2\pi}}\left(\frac{1}{\sigma_X}\right)\exp\left(-\left(\frac{1}{2}\right)\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right), \tag{2.1}$$

where μ_X and σ_X represent the mean and standard deviation of X, respectively, then the variable $x=e^X$ is distributed lognormally with pdf

$$g_x(x) = \sqrt{\frac{1}{2\pi}}\left(\frac{1}{\sigma_X x}\right)\exp\left(-\left(\frac{1}{2}\right)\left(\frac{\ln(x)-\mu_X}{\sigma_X}\right)^2\right). \tag{2.2}$$

If $g_x(x)$ represents a lognormal pdf of the variable x, the jth moment about zero is given by

$$\int_0^\infty x^j g_x(x)dx = \exp\left(j\mu_X + \frac{j^2\sigma_X^2}{2}\right). \tag{2.3}$$

Note that j need not be restricted to integer values.

The jth root of the jth moment of a pdf is known as the "mean of order j" (Mitrinovic, Pecaric, and Fink; 1993), and will be denoted here by

$$m_x(j) = \left(\int_0^\infty x^j g_x(x)dx\right)^{1/j}. \tag{2.4}$$

If the coefficients defining $g_x(x)$ are time variant, the jth-order mean may be written $m_x(t,j)$.

A well-known characteristic of the function $m_x(j)$ is that it is monotone increasing with respect to j, regardless of the form of $g_x(x)$, provided that $g_x(x)=0$ for all $x\leq 0$ (Mitrinovic et al. 1993). Important special cases correspond to $j=1$ (the arithmetic mean), the limit as j approaches 0 (the geometric mean), and $j=-1$ (the harmonic mean).

For the lognormal distribution, the jth-order mean is given by

$$m_x(j) = \exp\left(\mu_X + \frac{j\sigma_X^2}{2}\right). \tag{2.5}$$

3. Model Development

3.1 *Deterministic Dynamics*

Define some basic model parameters as follows: a growth rate a, a prevailing fishing mortality rate f, and a carrying capacity k (f is called the "prevailing" fishing mortality rate to emphasize its role as a determinant of the historic trajectory of stock size and to distinguish it from the normative or "target" fishing mortality rate ϕ which will be introduced in Section 4). The respective logarithms of these parameters will be denoted A, F, and K. The time derivative of stock size x in the Gompertz growth function can then be written

$$\frac{dx}{dt} = ax(K - \ln(x)) - fx. \tag{3.1}$$

Equilibrium stock size b is given by

$$b = ke^{-f/a}, \tag{3.2}$$

which simplifies the time derivative to

$$\frac{dx}{dt} = ax(B - \ln(x)), \tag{3.3}$$

where B=ln(b).

The time derivative of log stock size X is simply the linear relationship

$$\frac{dX}{dt} = a(B - X). \tag{3.4}$$

The parameter a thus represents: 1) the per-capita growth rate of x at x=b/e, or 2) the per-capita growth rate of X at X=$B/2$.

Given an initial stock size x_0, the population trajectory is described by

$$x(t) = b\left(\frac{x_0}{b}\right)^{e^{-at}}, \tag{3.5}$$

and for initial log stock size X_0 the trajectory of log stock size is described by

$$X(t) = e^{-at}X_0 + (1 - e^{-at})B. \tag{3.6}$$

Yield y is given by the simple relationship $y=fx$. As shown by Fox (1970, using a slightly different parametrization), *MSY* is obtained by fishing at a rate equal to a, which results in an equilibrium stock size of k/e or an equilibrium log stock size of K-1.

3.2 *Stochastic Dynamics (Process Error)*

To introduce a stochastic component into the deterministic model presented above, it is convenient to begin with the well-known Ornstein-Uhlenbeck process of mathematical physics (Uhlenbeck and Ornstein 1930, Ricciardi 1977), which can be written for arbitrary parameters a and B and arbitrary variable X as

$$\frac{dX}{dt} = a(B - X) + s\epsilon(t), \tag{3.7}$$

where ε is a standard white noise process and s is a scale parameter describing the intensity of the noise. Note that Equation (3.7) is identical to Equation (3.4) except for the specific interpretation of a, B, and X in Equation (3.4) and the fact that Equation (3.7) includes the term $s\varepsilon(t)$ on the right-hand side (RHS).

The transition pdf of the Ornstein-Uhlenbeck process is normal with parameters

$$\mu'_X(t) = e^{-at}X_0 + (1 - e^{-at})B \tag{3.8}$$

and

$$\sigma_{PX}(t) = s\sqrt{\frac{1 - e^{-2at}}{2a}}, \tag{3.9}$$

where X_0 is the (known) value of X at time t=0. The subscript "PX" (rather than just "X") is used in Equation (3.9) to indicate that this is the variability due to *process* error only.

Using the Stratonovich interpretation of stochastic differential equations (e.g., Ricciardi 1977), Equation (3.7) can be transformed into a stochastic version of Equation (3.3) by the chain rule of ordinary calculus, giving

$$\frac{dx}{dt} = ax(B - \ln(x)) + sx\epsilon(t). \tag{3.10}$$

The transition pdf of x is then lognormal (Capocelli and Ricciardi 1974) with parameters given by Equations (3.8) and (3.9).

Suppose instead that the value of X_0 is not known, but is assumed to follow a lognormal distribution with parameters μ_0 and σ_0. In this case, the coefficients of the transition pdf are

$$\mu'_X(t) = e^{-at}\mu_0 + (1 - e^{-at})B \tag{3.11}$$

and

$$\sigma'_X(t) = \sqrt{\sigma_{PX}(t)^2 + e^{-2at}\sigma_0^{\,2}}, \tag{3.12}$$

where $\sigma_{PX}(t)$ is given by Equation (3.9).

In the limit as t goes to infinity, the above equations (either (3.8-3.9) or (3.11-3.12)) reduce to the coefficients of the stationary distribution, namely

$$\mu'_X = B \tag{3.13}$$

and

$$\sigma'_X = \frac{s}{\sqrt{2a}}. \tag{3.14}$$

Assume that the conditional transition distribution of log yield Y is normal with parameters $F+X$ and σ_{PY}. Note that this is not the same as substituting $K-f/a$ for B in Equation (3.12) and then substituting y for fx in the resulting expression, which would lead to a two-dimensional (and thus considerably less tractable) stochastic differential equation. Instead, the simpler assumption is made that error in the harvest process $y=fx$ affects y but not x.

Given this assumption, the marginal transition distribution of yield y is lognormal (i.e., Y is normal) with coefficients

$$\mu'_Y(t) = F + \mu'_X(t) \tag{3.15}$$

and

$$\sigma'_Y(t) = \sqrt{\sigma'_X(t)^2 + \sigma_{PY}^2}. \tag{3.16}$$

3.3 *Measurement Error*

Suppose the following: A stock of size (biomass) x is distributed over a range of area r. The stock's size has been estimated $n+1$ times by a survey, specifically at times t_i, $i=0,1,2,\ldots,n$. Each survey consists of a large number of observations, each of which in turn measures, on a per-unit-area basis, the segment of the population contained in some sampling site or quadrat (e.g., the portion of the seabed swept by a single haul in a trawl survey). Survey observations may be biased (either upward or downward) by a "catchability coefficient" q.

For example, a standardized trawl survey has been used to assess groundfish stocks of the eastern Bering Sea annually since 1982 (Walters and Wilderbuer 1995). The survey includes sampling stations distributed throughout an area of approximately 4.634 million ha, a figure which is typically used as a proxy for the area inhabited by the flathead sole stock. The survey is typically viewed as unbiased for this stock. Thus, for the flathead sole example, one might set r=4.634 million ha, n=13 (through 1995), and q=1.

It will be assumed here that the observations generated by a given survey represent a random draw from some pdf with mean z (in units of kg/ha), where Z=ln(z) is a consistent estimator of log stock size plus a constant. Given an assumption about the form of the pdf, Z can be estimated by the method of maximum likelihood (a straightforward extension of the method suggested by Kappenman 1994). Then, if the survey sample size is large enough, the asymptotic normality of maximum likelihood estimates can be invoked, meaning that Z_i is normal with expected value $X(t_i)+Q-R$, where Q=ln(q) adjusts the survey estimate for bias and R=ln(r) adjusts the survey estimate to reflect the stock's abundance over its entire range rather than just its density per unit area. The standard deviation of Z_i, σ_{MX_i}, represents the variability in the survey estimate due to measurement error (the subscript label "MX" stands for "measurement error in X").

The following estimates were obtained by this method for the flathead sole survey time series (Figure 1):

i:	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Z:	1.57	1.70	1.99	1.76	2.07	2.13	2.56	2.42	2.61	2.45	2.71	2.53	2.71	2.55
σ_{MX}:	.212	.077	.117	.085	.146	.116	.150	.108	.130	.093	.148	.066	.072	.122

In similar fashion, true yield y_i is not known, but rather is measured by an estimate w_i (here, in units of metric tons). The logarithm of this estimate, W_i, may be viewed as normal with parameters $Y(t_i)$ and σ_{MY_i}. Unfortunately, an empirical estimate of σ_{MY} is not available for the flathead sole fishery, but the following point estimates of W can be identified:

i:	0	1	2	3	4	5	6	7	8	9	10	11	12	13
W:	8.41	8.56	8.40	8.64	8.56	8.28	8.82	8.19	9.92	9.66	9.56	9.52	9.70	9.20

It should be noted that use of annual catch data in the present model constitutes an approximation, since the model technically requires an estimate of the *instantaneous rate* of catch at the time of the survey. This approximation should not pose any serious difficulty so long as the distribution (as opposed to the magnitude) of harvests within the year does not have a major impact on stock dynamics.

3.4 *The Kalman Filter*

The model described above is ideally suited to estimation via the Kalman filter. In the present context, the Kalman filter consists of iteratively reweighting the coefficients of the transition pdf of X. Let the time difference between each successive pair of surveys be given by $\tau_i = t_i - t_{i-1}$, $i=1,2,\ldots,n$. For the flathead sole example, $\tau_i=1$ for all i. Then, the prior and posterior estimates of σ_X are given recursively for observations $i=1,2,\ldots,n$ by

$$\sigma'_{X_i} = \sqrt{e^{-2\alpha\tau_i}\sigma_{X_{i-1}}^{\ 2} + \sigma_{PX_i}^{\ 2}} \tag{3.17}$$

and

$$\sigma_{X_i} = \sqrt{\frac{1}{\frac{1}{\sigma'^{\,2}_{X_i}} + \frac{1}{\sigma_{MX_i}^{\ 2}} + \frac{1}{\sigma_{PY_i}^{\ 2} + \sigma_{MY_i}^{\ 2}}}}\,, \tag{3.18}$$

where

$$\sigma_{X_0} = \sqrt{\frac{1}{\frac{1}{\sigma_{MX_0}^{\ 2}} + \frac{1}{\sigma_{PY_0}^{\ 2} + \sigma_{MY_0}^{\ 2}}}}\,, \tag{3.19}$$

that is, where it is assumed that the estimate of the standard deviation of X prior to the first observation ($i=0$) is infinite.

The prior estimates of the standard deviations of Z and W are given by

$$\sigma'_{Z_i} = \sqrt{\sigma'^{\,2}_{X_i} + \sigma_{MX_i}^{\ 2}} \tag{3.20}$$

and

$$\sigma'_{W_i} = \sqrt{\sigma'^{\,2}_{X_i} + \sigma_{PY_i}^{\ 2} + \sigma_{MY_i}^{\ 2}}\,, \tag{3.21}$$

with correlation coefficient

$$\rho_{ZW_i} = \frac{\sigma'^{\,2}_{X_i}}{\sigma'_{Z_i}\sigma'_{W_i}}\,. \tag{3.22}$$

The prior and posterior estimates of μ_X are given recursively for observations i=1,2,...,n by

$$\mu'_{X_i} = e^{-a\tau_i}\mu_{X_{i-1}} + (1 - e^{-a\tau_i})B \tag{3.23}$$

and

$$\mu_{X_i} = \sigma_{X_i}^{2}\left(\frac{\mu'_{X_i}}{\sigma'^{2}_{X_i}} + \frac{Z_i - Q + R}{\sigma_{MX_i}^{2}} + \frac{W_i - F}{\sigma_{PY_i}^{2} + \sigma_{MY_i}^{2}}\right), \tag{3.24}$$

where

$$\mu_{X_0} = \sigma_{X_0}^{2}\left(\frac{Z_0 - Q + R}{\sigma_{MX_0}^{2}} + \frac{W_0 - F}{\sigma_{PY_0}^{2} + \sigma_{MY_0}^{2}}\right). \tag{3.25}$$

The prior means of Z and W are given, respectively, by

$$\mu'_{Z_j} = \mu'_{X_j} + Q - R \tag{3.26}$$

and

$$\mu'_{W_j} = \mu'_{X_j} + F. \tag{3.27}$$

3.5 *Likelihood Function*

Given Equations (3.20-3.22) and (3.26-3.27), the log likelihood is

$$-n\ln(2\pi) - \left(\sum_{i=1}^{n} \ln(\sigma'_{Z_i}) + \ln(\sigma'_{W_i}) + \left(\frac{1}{2}\right)\ln(1 - \rho_{ZW_i}^{2}) + \frac{\left(\frac{Z_i - \mu'_{Z_i}}{\sigma'_{Z_i}}\right)^{2} - 2\rho_{ZW_i}\left(\frac{Z_i - \mu'_{Z_i}}{\sigma'_{Z_i}}\right)\left(\frac{W_i - \mu'_{W_i}}{\sigma'_{W_i}}\right) + \left(\frac{W_i - \mu'_{W_i}}{\sigma'_{W_i}}\right)^{2}}{2(1 - \rho_{ZW_i}^{2})}\right). \tag{3.28}$$

In Expression (3.28), the vectors σ'_Z, σ'_W, and ρ are all functions of parameters a and s and vectors σ_{MX}, σ_{PY}, and σ_{MY}. The vectors μ'_Z and μ'_W are all functions of the same, plus parameters f, k, q, and r.

The topic of parameter estimation will be considered in the following sections. For now, suffice it to say that at least some parameters, for example a and s, could potentially be estimated by the method of maximum likelihood, that is, by maximizing Expression (3.28). However, an interior maximum to the likelihood surface does not always exist in this model. That is, there is always a maximum to the likelihood surface at a=0 with s positive and another at s=0 with a positive, but there is not always a maximum with both a and s positive. In addition to this practical difficulty, the maximum likelihood estimates (MLEs) suffer from a lack of any clear relationship to alternative levels of risk aversion (discussed in the following section), and they ignore prior knowledge about the relative believability of alternative parameter values. To address these shortcomings, a Bayesian estimation methodology is set forth below.

4. A Theory of Relative Risk Aversion (RRA)

For some quantity which can be thought of as a proxy for nominal income, such as yield y in the case of a fishery, Pratt (1964) defined relative risk aversion RRA as

$$RRA = -y\left(\frac{\frac{d^2l}{dy^2}}{\frac{dl}{dy}}\right), \tag{4.1}$$

where l is the "loss" function which, when multiplied by a negative constant, describes the level of well-being or "utility" associated with a given level of y. The loss function may be an attribute of an individual, a group, or a society. The scale of l is arbitrary.

A convenient choice for l is the following:

$$l(y) = \frac{1 - y^j}{j}. \tag{4.2}$$

When j=1 Equation (4.2) gives a linear loss function, and in the limit as j approaches zero Equation (4.2) gives a logarithmic loss function. Using the definition of relative risk aversion given in Equation (4.1), Equation (4.2) implies a constant (i.e., y-independent) level of relative risk aversion, namely RRA=1-j. Thus, j=1 corresponds to the "risk neutral" approach where RRA=0, and j=0 corresponds to a distinctly risk averse approach where RRA=1.

In Bayesian decision theory, the objective is to minimize risk, where risk is defined as expected loss. For example, let l be given by Equation (4.2) and write y as a function of target fishing mortality ϕ and some uncertain parameter θ with pdf $g_\theta(\theta)$. Then, the objective is to choose ϕ so as to minimize

$$EL(\phi) = \int_{-\infty}^{\infty} l(y(\phi,\theta))g_\theta(\theta)d\theta = \frac{1 - m_y(j)^j}{j}. \tag{4.3}$$

Differentiating Equation (4.3) with respect to ϕ and setting the result equal to zero gives

$$\frac{dEL}{d\phi} = -m_y(j)^{j-1}\left(\frac{dm_y(j)}{d\phi}\right) = 0. \tag{4.4}$$

Note that the value of ϕ which *minimizes* the expected loss (the Bayes optimum) is simply the value of ϕ which *maximizes* the jth-order mean of y, where j is equal to 1 minus the chosen level of relative risk aversion. For example, if RRA=0 (i.e., j=1) the Bayes optimum is the value of ϕ which maximizes the *arithmetic* mean of y, while if RRA=1 (i.e., j=0) the Bayes optimum is the value of ϕ which maximizes the *geometric* mean of y.

Note also that ϕ is a special type of parameter in that its value can be *chosen*, that is, ϕ is a decision variable. Other parameters, however, may best be thought of as "states of nature," not readily subject to manipulation. Parameters such as a, f, k, q, r, and s in the present model would be examples. In general, such parameters cannot be estimated within the framework outlined above. For instance, future yield y could be written as a function of carrying capacity k as well as target fishing mortality ϕ. However, the derivative of $m_y(j)$ with respect to k is a positive constant, so no solution to Equation (4.4) would exist.

However, it is possible to modify the framework slightly so as to enable states of nature to be estimated in a manner very analogous to decision variables. To begin with, let stock size x be written as a function of some uncertain state of nature θ and let l be redefined as follows:

$$l(\theta,\hat{\theta}) = \left(\frac{x(\theta)^j - x(\hat{\theta})^j}{j}\right)^2, \tag{4.5}$$

where $\hat{\theta}$ is an estimate of θ. Since RRA was shown to be equal to 1-j when RRA and l were defined by Equations (4.1) and (4.2), respectively, assume for the present that RRA can again be equated with 1-j when l is defined by Equation (4.5), even though the definition given by Equation (4.1) is not applicable to the "state of nature" case.

The expected loss is given by

$$EL(\hat{\theta}) = \int_{-\infty}^{\infty} l(\theta,\hat{\theta})g_\theta(\theta)d\theta = \frac{m_x(2j)^{2j} - 2m_x(j)^j x(\hat{\theta})^j + x(\hat{\theta})^{2j}}{j^2}. \tag{4.6}$$

Differentiating Equation (4.6) with respect to $\hat{\theta}$ and setting the result equal to zero gives

$$\frac{dEL}{d\theta} = -2x(\theta)^{j-1}\left(\frac{dx}{d\theta}\right)\left(\frac{m_x(j)^j - x(\theta)^j}{j}\right) = 0\,. \tag{4.7}$$

That is, the value of θ which minimizes the expected loss is simply the value which sets $x=m_x(j)$. (If the derivative of x with respect to θ can be set equal to zero, this will also be a minimum or maximum.)

In summary, the approaches to choosing a value for a decision variable and for a state of nature are as follow: For a decision variable, choose the value that maximizes $m_y(j)$. For a state of nature, choose the value that sets $x=m_x(j)$. Even though the definition of relative risk aversion given by Equation (4.1) is not meaningful for the loss function defined by Equation (4.5), the fact that the solutions to Equations (4.4) and (4.7) are so similar suggests that the relationship RRA=1-j derived from the combination of Equations (4.1) and (4.2) is also a reasonable measure of relative risk aversion when l is defined by Equation (4.5), at least for the case where dy/dx is always positive (as it is here).

For the remainder of this paper, results will focus primarily on a risk averse approach corresponding to an *RRA* value of unity.

5. Parameter Estimation

5.1 *Overview*

The parameters in this model are the vectors σ_{MX}, σ_{PY}, and σ_{MY} and the scalars ϕ, a, f, k, q, r, and s. Note that the *prevailing* fishing mortality rate f may in general be different than the *optimal* rate ϕ. Estimation of ϕ will follow the method for decision variables described in the preceding section.

It will be assumed that the vector σ_{MX} is known, which in practice means viewing an independent estimate of σ_{MX} (such as the one presented in Section 3.3 for the flathead sole example) as certain. Ideally, an independent estimate of the vector σ_{MY} would be available as well, but in practice it often is not, as is the case with the flathead sole example. If σ_{MY} is unknown, it is impossible to estimate the vector σ_{PY}, since the two terms never appear separately. For the flathead sole example, then, an *ad hoc* value of

$$\sqrt{\sigma_{PY_i}^{\ 2} + \sigma_{MY_i}^{\ 2}} = 0.5 \tag{5.1}$$

will be assumed for all i. This value is high enough that it has the effect of downweighting the importance of the catch time series, thereby letting the model focus on tracking the survey abundance time series. Such an approach seems fairly reasonable for the flathead sole example, since the complexities of the eastern Bering Sea groundfish fishery and its management are such that anything close to a time-invariant proportionality between catch and biomass for this stock is probably not a very appropriate assumption.

To facilitate estimation of the remaining parameters, define two new parameters which, given A, Q, and R, prescribe a one-to-one mapping into F and K:

$$C = F - Q + R \tag{5.2}$$

and

$$D = K - e^{F-A} + Q - R. \tag{5.3}$$

Note that $B=D-Q+R$. The important things abound C and D (or, alternatively, c and d) is that their MLEs can be computed in closed form (Attachment 1) and that these MLEs are independent of both Q and R. Thus, it is convenient to reduce the dimensionality of the model by setting parameters c and d at their MLEs, conditional on the other parameters (i.e., c and d become explicit functions of the other parameters). Walters and Ludwig (1994) call this an "approximate Bayesian" procedure.

This leaves parameters a, q, r, and s to be estimated. These four parameters are of two distinct types in terms of their estimability. Parameters a and s appear separately in the terms making up the likelihood (i.e., they are not formally confounded, though they may be correlated) and their values directly influence the likelihood even when c and d are set at their respective MLEs. Parameters q and r, on the other hand, *are* formally confounded (specifically, they always appear in the form r/q), and their values have no influence on the likelihood when c and d are set at their respective MLEs. Thus, it is natural to consider estimation of a and s separately from estimation of q and r.

Parameters a and s will be estimated by applying the method for states of nature described in the preceding section, where the computation of $m_x(j)$ will involve integrating across the joint posterior distribution of a and s. The joint posterior distribution, in turn, is obtained by assuming a joint prior distribution for a and s, then multiplying by the likelihood (from Expression (3.28)). A convenient form for a joint prior distribution is the bivariate lognormal:

$$g'_{a,s}(a|s) = \left(\frac{1}{2\pi\sigma'_A\sigma'_S a s}\right)\sqrt{\frac{1}{1-\rho'^2_{AS}}}\exp\left(-\frac{\left(\frac{\ln(a)-\mu'_A}{\sigma'_A}\right)^2+\left(\frac{\ln(s)-\mu'_S}{\sigma'_S}\right)^2}{2(1-\rho'^2_{AS})}+\frac{\rho'_{AS}\left(\frac{\ln(a)-\mu'_A}{\sigma'_A}\right)\left(\frac{\ln(s)-\mu'_S}{\sigma'_S}\right)}{1-\rho'^2_{AS}}\right), \tag{5.4}$$

where μ'_A, μ'_S, σ'_A, and σ'_S represent the prior means and standard deviations of the marginal distributions of A and S, respectively, and ρ'_{AS} is the prior correlation between A and S. For the flathead sole example, the parameter values $\mu'_A = -1.96$, $\mu'_S = -0.99$, $\sigma'_A = \sigma'_S = 0.833$, and

$\rho'_{AS} = 0$ will be chosen. These give $m_a(1)$=0.2 (the point estimate of the natural mortality rate for flathead sole, Walters and Wilderbuer 1995) and a coefficient of variation (CV) equal to unity for the marginal prior pdfs of a and s as well as for the stationary distribution of x when Equation (3.14) is evaluated at the means of the respective marginal priors. (It should be noted that although the bivariate lognormal will be used in the flathead sole example, the estimation methodology presented below does not depend on the prior following this functional form.)

Estimation of parameters q and r will follow basically the same scheme, except that the joint posterior pdf is identical to the joint prior pdf because the value of the likelihood is invariant with respect to these two parameters. A bivariate lognormal pdf will be assumed, with parameters μ_Q=-0.01961, μ_R=17.632, σ_Q=σ_R=0.198, and ρ_{QR}=0. These parameter values set $m_q(1)$ and $m_r(1)$ equal to the respective point estimates of 1.0 and 4.634 million given in Section 3.3, and give a CV of 0.20 for the marginal pdfs of both q and r.

The following three subsections treat, in turn, estimation of the optimal fishing mortality rate ϕ, the parameters a and s, and the parameters q and r. In each subsection, the estimation process is divided into two cases: The first is based on the relevant jth-order mean when parameter values are known with certainty (i.e., process error only) and the second is based on the relevant jth-order mean when parameter uncertainty is incorporated.

5.2 *Optimal Fishing Mortality ϕ*

5.2.1 *Parameter Values Certain*

Using the loss function defined by Equation (4.2), the expected loss is given by Equation (4.3), where $m_y(t,j)$ is defined by Equations (3.15-3.16) with target fishing morality rate ϕ substituted for historic fishing mortality rate f:

$$\begin{aligned} m_y(t,j) &= \phi \exp\left(\mu'_X(t) + \frac{j\sigma'_Y(t)^2}{2} \right) \\ &= \phi \exp\left(e^{-at}\mu_0 + (1 - e^{-at})\left(\ln(k) - \frac{\phi}{a} \right) + \frac{j\sigma'_Y(t)^2}{2} \right). \end{aligned} \tag{5.5}$$

In the limit as t approaches infinity, Equation (5.5) becomes

$$m_y(j) = \phi \exp\left(\ln(k) - \frac{\phi}{a} + \left(\frac{j}{2}\right)\left(\frac{s^2}{2a} + \sigma_{PY}^2\right)\right). \tag{5.6}$$

Differentiating $m_y(t,j)$ with respect to ϕ and setting the resulting expression equal to zero gives the solution for the fishing mortality rate at maximum expected utility (MEU):

$$\phi_{MEU}(t) = \frac{a}{1 - e^{-at}}. \tag{5.7}$$

Stated another way, the degree of relative risk aversion does not impact the choice of exploitation rates when model parameters are known with certainty *regardless of the level of process error*, and as t approaches infinity, the solution collapses to $\phi_{MEU}=\phi_{MST}=a$.

5.2.2 *Parameter Values Uncertain*

When the parameters a, f, k, q, r, s, and σ_{PY} are uncertain, Equation (5.6) changes as shown below:

$$m_y(j) = \phi\left(\int_0^\infty \cdots \int_0^\infty \exp\left(j\left(\ln(k) - \frac{\phi}{a}\right) + \frac{j^2 \sigma_Y^2}{2}\right) g_{\theta_1,\ldots,\theta_7}(\theta_1, \ldots, \theta_7)\, d\theta_1 \ldots d\theta_7\right)^{1/j}, \tag{5.8}$$

where each of the θ_i corresponds to one of the uncertain parameters (σ_{PY} is treated here as a scalar for notational convenience, though in general it could be viewed as a vector) and $g_{\theta_1,\ldots,\theta_7}(\theta_1, \ldots, \theta_7)$ represents the joint pdf of those parameters .

For the special case where j approaches 0, Equation (5.8) simplifies to

$$\begin{aligned} m_y(0) &= \phi \exp\left(\int_0^\infty \int_0^\infty \left(\ln(k) - \frac{\phi}{a}\right) g_{a,k}(a,k)\, da\, dk\right) \\ &= \phi\, m_k(0) \exp\left(-\frac{\phi}{m_a(-1)}\right), \end{aligned} \tag{5.9}$$

where $g_{a,k}(a,k)$ is the joint marginal pdf of a and k.

Differentiating Equation (5.9) with respect to ϕ and solving for zero gives the harvest rate that maximizes expected log stationary yield (*MELSY*), a harvest strategy suggested by Thompson (1992b):

$$\phi_{MELSY} = m_a(-1). \tag{5.10}$$

That is, the risk-averse (*RRA*=1) long-term optimal harvest rate is simply the harmonic mean of the marginal posterior pdf of a. For the flathead sole example, $m_a(-1)$=0.108. By way of comparison, $m_a(1)$=0.150. The marginal prior and posterior pdfs of a for the flathead sole example are compared in Figure 2. Although the two distributions appear roughly similar, the CV of the posterior is actually 43% smaller than that of the prior.

5.3 *Growth Rate "a" and Process Error Scale "s"*

5.3.1 *Parameter Values Certain*

From the transition pdf of x, the jth-order mean of x can be written

$$m_x(t,j) = \exp\left(e^{-at}\mu_{x_0} + (1-e^{-at})\left(\ln(k) - \frac{\phi}{a}\right) + \left(\frac{j}{2}\right)\left(e^{-2at}\sigma_{x_0}^{\ 2} + (1-e^{-2at})\left(\frac{s^2}{2a}\right)\right)\right). \qquad (5.11)$$

In the limit as t approaches infinity, Equation (5.11) reduces to

$$m_x(j) = k\exp\left(-\frac{\phi}{a} + \frac{js^2}{4a}\right). \qquad (5.12)$$

5.3.2 *Parameter Values Uncertain*

When the parameters a and s are uncertain, Equation (5.12) changes to

$$m_x(j) = k\left(\int_0^\infty \int_0^\infty \exp\left(-\frac{j\phi}{a} + \frac{j^2 s^2}{2a}\right) g_{a,s}(a,s)\,da\,ds \right)^{1/j}, \qquad (5.13)$$

which, in the special case where j approaches 0, becomes

$$\begin{aligned} m_x(0) &= k\exp\left(\int_0^\infty \left(-\frac{\phi}{a}\right) g_a(a)\,da \right) \\ &= k\exp\left(-\frac{\phi}{m_a(-1)}\right). \end{aligned} \qquad (5.14)$$

Thus, the optimal estimator of a (for this limiting case) is $m_a(-1)$, which, as noted earlier, has a value of 0.108 in the flathead sole example. Because $m_a(-1)$ turns out to be both the optimal estimator of a and the risk-averse optimal fishing mortality rate, the deterministic result equating a with the optimal fishing mortality rate is preserved in the stochastic case.

An optimal estimator of s is not so obvious, since s does not appear in Equation (5.14). However, a reasonable choice can be initiated by noting that the φth-order mean of $m_x(j)$ can be written

$$m_{m_x(j)}(\varphi) = k\left(\int_0^\infty \int_0^\infty \exp\left(-\frac{\varphi\phi}{a} + \frac{\varphi j s^2}{2a}\right) g_{a,s}(a,s)\,da\,ds \right)^{1/\varphi}, \qquad (5.15)$$

which, in the limit as φ approaches zero, becomes

$$\begin{aligned} m_{m_s(j)}(0) &= k\exp\left(\int_0^\infty\int_0^\infty\left(-\frac{\phi}{a}+\frac{js^2}{2a}\right)g_{a,s}(a,s)\,da\,ds\right) \\ &= k\exp\left(-\frac{\phi}{m_a(-1)}+\frac{jm_s(2)^2}{2m_a(-1)}\right). \end{aligned} \tag{5.16}$$

Thus, since the geometric mean of $m_x(j)$ is given by setting $s=m_s(2)$ for arbitrary j, it makes sense to set $s=m_s(2)$ for the special case where j=0. In the flathead sole example, $m_s(2)$=0.128.

5.4 *Catchability "q" and Range "r"*

5.4.1 *Parameter Values Certain*

Noting that

$$\mu_{X_t} = \mu_{Z_t} - \ln(q) + \ln(r) \tag{5.17}$$

and

$$k = d\left(\frac{r}{q}\right)\exp\left(\frac{cq}{ar}\right), \tag{5.18}$$

and defining

$$\eta(t,j) = \exp\left(e^{-at}\mu_{Z_0} - (1-e^{-at})\left(\frac{\phi}{a}\right) + \left(\frac{j}{2}\right)\left(e^{-2at}\sigma_{X_0}{}^2 + (1-e^{-2at})\left(\frac{s^2}{2a}\right)\right)\right), \tag{5.19}$$

it is possible to rewrite Equation (5.11) as

$$m_x(t,j) = \left(\frac{r}{q}\right)\exp\left((1-e^{-at})\left(\ln(d)+\frac{cq}{ar}\right)\right)\eta(t,j). \tag{5.20}$$

Importantly, nothing in $\eta(t,j)$ depends on either q or r. The fact that neither q nor r enters into the calculation of σ_X can be confirmed from Equations (3.9) and (3.17-3.19), and the independence of μ_Z from q and r may be established by substituting the expression for μ_X shown in Attachment 1 into Equation (5.17).

In the limit as t approaches infinity, Equation (5.20) becomes

$$m_x(j) = d\left(\frac{r}{q}\right)\exp\left(\frac{cq}{ar} - \frac{\phi}{a} + \frac{js^2}{4a}\right). \tag{5.21}$$

5.4.2 *Parameter Values Uncertain*

When q and r are uncertain, Equation (5.21) changes as shown below:

$$m_x(j) = d\left(\int_0^\infty\int_0^\infty\left(\frac{r}{q}\right)^j\exp\left(\frac{jcq}{ar} - \frac{j\phi}{a} + \frac{j^2s^2}{4a}\right)g_{q,r}(q,r)\,dq\,dr\right)^{1/j}, \tag{5.22}$$

which, in the special case where j approaches 0, is simply

$$m_x(0) = d\left(\frac{m_r(0)}{m_q(0)}\right)\exp\left(\frac{cm_q(1)}{am_r(-1)} - \frac{\phi}{a}\right). \tag{5.23}$$

Note that Equation (5.23) can also be obtained by evaluating Equation (5.18) at the point $(q,r)=(m_q(1),m_r(-1))$ and then multiplying by the terms $m_q(1)/m_q(0)$, $m_r(0)/m_r(-1)$, and $e^{-\phi/a}$. In general, the value of k implied by Equation (5.23) may be less than or greater than the value of k given by evaluating Equation (5.18) at the point $(q,r)=(m_q(1),m_r(1))$. In the flathead sole example, for instance, the two quantities are virtually the same. That is, the risk-averse optimal estimates of q and r are approximately equal to the arithmetic means of the respective marginal pdfs. This approximation will arise whenever q and r follow lognormal distributions with the same value of σ, so long as μ_R is sufficiently large. The value of k under the risk-averse optimal parameter estimates is approximately 942,000 t. The corresponding estimate of $m_x(0,0)$, that is, the geometric mean estimate of biomass at the time of the most recent survey, is approximately 640,000 t.

6. Discussion

6.1 *Robustness of the Harmonic Mean Strategy with Respect to the Level of RRA*

From Equation (5.10), it is evident that there is a one-to-one mapping between the harmonic mean (order j=-1) of a and an *RRA* value of 1. Given this fact, it seems reasonable to consider how (or whether) other jth-order means of a map into values of *RRA*, thus shedding some light on the robustness of the harmonic mean strategy with respect to the level of *RRA*. For example, might the arithmetic mean of a (i.e., the mean of order j=1) describe the optimal fishing mortality rate under an *RRA* value of 0? To begin to answer this question, first substitute Equation (5.6) into Equation (4.2) to obtain the expected loss conditional on a and ϕ (and some other parameters, too, but a and ϕ are the important ones in this context):

$$EL_{a\phi}(a,\phi) = \frac{1 - \left(\phi k \exp\left(-\frac{\phi}{a} + \left(\frac{1-RRA}{2}\right)\left(\frac{s^2}{2a} + \sigma_{PY}^2\right)\right)\right)^{1-RRA}}{1-RRA}. \tag{6.1}$$

To add an element of parametric uncertainty to the problem, suppose that a follows an inverse Gaussian distribution (Tweedie, 1957) with arithmetic mean κ_a and shape parameter λ_a, that is:

$$g_a(a) = \sqrt{\frac{\lambda_a}{2\pi a^3}} \exp\left(-\left(\frac{\lambda_a}{2a}\right)\left(\frac{a-\kappa_a}{\kappa_a}\right)^2\right). \tag{6.2}$$

Notation will be simplified by setting $a=\kappa_a$ in the definition of σ'_Y, by defining the CV of a, and by scaling ϕ relative to κ_a as follows:

$$\sigma'_Y = \sqrt{\frac{s^2}{2\kappa_a} + \sigma_{PY}^2}, \quad CV_a = \sqrt{\frac{\kappa_a}{\lambda_a}}, \quad \nu = \frac{\phi}{\kappa_a}. \tag{6.3}$$

Next, define the following pair of functions:

$$\xi(\nu) = \sqrt{2(1-RRA)\left(\nu - \left(\frac{1-RRA}{2}\right)\sigma'^2_Y\right)CV_a^2 + 1} \tag{6.4}$$

and

$$\zeta(\nu) = \left(\nu\, CV_a^2\right)^{1-RRA} \exp\left(-\frac{\xi(\nu)-1}{CV_a^2}\right). \tag{6.5}$$

Then, integrating the product of Equations (6.1) and (6.2) with respect to a gives

$$EL_\nu(\nu) = \left(\frac{1}{1-RRA}\right)\left(1 - \frac{(\lambda_a k)^{1-RRA}\zeta(\nu)}{\xi(\nu)}\right). \tag{6.6}$$

Differentiating Equation (6.6) with respect to ν gives

$$\frac{dEL_v}{dv} = (\lambda_a k)^{1-RRA}\zeta(v)\left(\frac{vCV_a^2\,\xi(v)^{-3} + v\xi(v)^{-2} - \xi(v)^{-1}}{v}\right). \tag{6.7}$$

The value of v that sets Equation (6.7) equal to zero is the risk-minimizing solution. To express that solution in terms of a jth-order mean of a, first note that the jth-order mean of the inverse Gaussian distribution can be scaled relative to κ_a (the first-order mean) as follows:

$$u(CV_a,j) = \left(\exp\left(\frac{1}{CV_a^2}\right)\sqrt{\frac{2}{\pi CV_a^2}}\,\mathrm{BesselK}_{|j-1/2|}\left(\frac{1}{CV_a^2}\right)\right)^{1/j}, \tag{6.8}$$

where $\mathrm{BesselK}_j(a)$ is a "modified Bessel function of the second kind" (Watson, 1944) with order j and argument a. When j is an integer, Equation (6.8) can be expressed in simpler terms (Tweedie, 1957). Important special cases include those corresponding to j=1 (where u=1) and j=-1 (where $u=1/(1+CV_a^2)$).

The problem, then, is this: Suppose that v were set equal to u given some chosen value of j. What would the level of RRA have to be in order for this choice of j to be optimal? The answer is a function of σ'_Y and CV_a, as shown below:

$$RRA(\sigma'_Y, CV_a, j) = 1 - \left(\frac{u(CV_a,j)}{\sigma'_Y}\right)\left(1 - \sqrt{1 - \left(\frac{\sigma'_Y}{u(CV_a,j)CV_a}\right)^2\left(u(CV_a,j)CV_a^2 - 1\right) - \left(\frac{\sigma'_Y}{CV_a}\right)^2\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{CV_a^2}{u(CV_a,j)}}\right)}\right). \tag{6.9}$$

When j=-1, Equation (6.9) reduces to RRA=1, confirming the solution provided by Equation (5.10). However, it turns out that -1 is the only value of j that maps into a unique value of RRA. All other values of j map into a range of RRA values, depending on the values of σ'_Y and CV_a. To get a handle on the behavior of this solution, it is helpful to note that there is a critical value of σ'_Y above which Equation (6.9) becomes complex, and that Equation (6.9) is monotone decreasing with respect to σ'_Y for all values below this critical point (except at j=-1, where the solution is independent of σ'_Y). In other words, RRA is maximized when σ'_Y is minimized, and minimized when σ'_Y is maximized.

In contrast to its behavior with respect to σ'_Y, Equation (6.9) remains real for all values of CV_a. Thus, the behavior of Equation (6.9) can be bracketed by taking limits as σ'_Y approaches 0 and the critical value and as CV_a approaches 0 and infinity. For the case where σ'_Y is maximized and CV_a is minimized, the RHS of Equation (6.9) reduces simply to $-j$. For the case where both σ'_Y and CV_a are maximized, the RHS of Equation (6.9) reduces to

$$2\left(\sqrt{\frac{1}{\pi}}\,\Gamma\left(\frac{1}{2}-j\right)\right)^{-1/j} . \tag{6.10}$$

For the case where σ'_Y and CV_a are both minimized, the RHS of Equation (6.9) reduces simply to $(1\text{-}j)/2$. Finally, for the case where σ'_Y is minimized and CV_a is maximized, the RHS of Equation (6.9) reduces to

$$\frac{1}{2}+\left(\sqrt{\frac{1}{\pi}}\,\Gamma\left(\frac{1}{2}-j\right)\right)^{-1/j} . \tag{6.11}$$

The behavior of these four limits is illustrated in Figure 3. As shown in the figure, the range of *RRA* values corresponding to a given choice of j can be fairly broad. For example, setting ϕ equal to the arithmetic mean of a (i.e., setting j=1) can correspond to an *RRA* value anywhere between -1 and 1/2. For another example, the figure indicates that in order to guarantee a positive *RRA* value, j would have to be set no higher than zero. Viewed from the other direction, any *RRA* value less than 1/2 corresponds to an infinite range of j values.

In general, then, it does not appear that very much can be concluded about the level of *RRA* under which a harvest strategy defined by any particular jth-order mean of a is optimal, with the exception of the special relationship that exists between *RRA*=1 and j=-1. Conversely, the harmonic mean strategy does not appear to be robust to large changes in *RRA*.

6.2 *Robustness of the Harmonic Mean Strategy with Respect to Model Structure*

The derivation culminating in Equation (5.10) is, of course, conditional on the model structure presented in Sections 3.1 and 3.2. As noted in the Introduction, biomass dynamic models such as the one used here are among the simpler analytical tools used in fishery stock assessment. In particular, age structure is not considered in biomass dynamic models, and the processes of recruitment, natural mortality, and individual growth are subsumed in an unspecified way into the overall population growth function. One way to test the robustness of the harmonic mean strategy, then, would be to see how well it compares with the true optimal harvest rate in a model with a more complex structure, specifically an age-structured or "dynamic pool" model. A convenient example is the simple dynamic pool model presented by Thompson (1992a), where the particular functional forms chosen for the stock-recruitment and individual growth processes permit a closed-form solution for ϕ_{MST}. This model would seem to be a reasonably comparable age-structured alternative to the Gompertz-Fox model, as the relationship $x_{MST}/k=1/e$ found in the latter obtains as a limiting case in the former.

For the simple case in which process error is absent and all parameter values are known except for the degree of compensation p in the stock-recruitment relationship (Shepherd, 1982), a closed-form solution for ϕ_{MELSY} can also be obtained (Thompson, 1992b) if the level of uncertainty surrounding p can be described by a beta distribution:

$$g_p(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}, \tag{6.12}$$

where α and β are positive constants. (Note that in the derivation given by Thompson (1992b), the uncertain parameter was the *complement* of the compensation parameter, i.e., 1-p. To correspond with the derivation given in the 1992 paper, the constants α and β used in the present paper would need to be reversed.)

The solutions presented by Thompson (1992a and 1992b) scale the optimal fishing mortality rates relative to the instantaneous rate of natural mortality. The same convention will be observed in this section. Thus, the reader should remain aware that expressions for ϕ given in this section will be in relative terms, in contrast to the rest of the paper where ϕ is given in absolute terms. The risk-minimizing solution given by Thompson (1992b) can be reparametrized as

$$\phi_{MELSY} = \frac{-(3\phi_{max}+1)\beta - 2(\alpha-1) + \sqrt{((3\phi_{max}+1)\beta+2(\alpha-1))^2 + 8\phi_{max}(\phi_{max}-1)\beta(\alpha-1)}}{2\beta(\phi_{max}-1)}, \tag{6.13}$$

where ϕ_{max} is the fishing mortality rate that maximizes the level of yield per recruit.

Meanwhile, the certainty-equivalent solution given by Thompson (1992a) can be reparametrized as

$$\phi_{MSY} = \frac{-\phi_{max}(1-p) - 3 + p + \sqrt{((\phi_{max}+1)p - (3\phi_{max}+1))^2 - 4(\phi_{max}^2-1)p}}{2(\phi_{max}-1)(1-p)} - 1. \tag{6.14}$$

Note that the harmonic mean of Equation (6.12) is given by

$$m_p(-1) = \frac{\alpha-1}{\alpha+\beta-1} \tag{6.15}$$

and the CV is given by

$$CV_p = \sqrt{\frac{\beta}{\alpha(\alpha+\beta+1)}}. \tag{6.16}$$

In the beta distribution, any given value for the harmonic mean determines a maximum possible CV, as described below:

$$MCV_p = \frac{\sqrt{1 - m_p(-1)}}{1 + \sqrt{2m_p(-1)}}. \tag{6.17}$$

Dividing Equation (6.16) by Equation (6.17) gives a relative CV, RCV_p, which ranges between 0 and 1. The constants α and β can be expressed entirely in terms of $m_p(-1)$ and RCV_p, meaning that Equations (6.12-6.14) can be expressed in these terms as well.

Interestingly, the value of Equation (6.14) evaluated at $p=m_p(-1)$ is the same as the solution given by Equation (6.13). In other words, a fishery manager who acted to minimize risk in the face of uncertainty regarding p would harvest at the same rate as a fishery manager who acted as though the harmonic mean of p were the only possible value.

Thus, the optimal harvest rates in both models (the biomass dynamic model presented here and the simple dynamic pool model presented in the 1992 papers) are obtained by setting the key uncertain parameters (population growth rate a in the biomass dynamic model and stock-recruitment compensation p in the dynamic pool model) equal to their respective harmonic means. One important difference, however, is that the key uncertain parameter in the biomass dynamic model (a) is formally identical with the deterministic value of ϕ_{MST}, whereas the key uncertain parameter in the simple dynamic pool model (p) is related to the deterministic value of ϕ_{MST} in a comparatively complicated way, as indicated by Equation (6.14). Solving Equation (6.14) for p and defining h as the ratio of ϕ_{MST} to ϕ_{max} (with a range of 0 to 1) enables p to be written as

$$p(h) = 1 - \frac{2(1-h)}{((\phi_{max} - 1)h + 2)(h\phi_{max} + 1)}, \tag{6.18}$$

with derivative

$$\frac{dp}{dh} = 2\left(\frac{(2-h)h(\phi_{max} - 1)\phi_{max} + (3\phi_{max} + 1)}{((\phi_{max} - 1)h + 2)^2(h\phi_{max} + 1)^2}\right). \tag{6.19}$$

The pdf of h is then given by substituting Equations (6.12), (6.18), and (6.19) into

$$g_h(h) = g_p(p(h))\frac{dp}{dh}. \tag{6.20}$$

Note that if Equation (6.18) were linear and homogeneous, the harmonic mean of p would be proportional to the harmonic mean of h and the strategy described by Equation (5.10) would hold here as well. However, since Equation (6.18) is neither linear nor homogeneous, the strategy described by Equation (5.10) will necessarily be biased to some extent if applied to the simple dynamic pool model. Some idea of the range of possible bias is given by Figure 4, which considers ϕ_{max} values of 2, 4, and 6 (again, expressed relative to the natural mortality rate); $m_p(-1)$ values ranging from 0.3 to 0.9; and RCV_p values ranging from 0.3 to 0.9 as well. The axis labeled "harmonic mean" corresponds to $m_p(-1)$ and the axis labeled "coefficient of variation" corresponds to RCV_p. The figure was constructed by solving (numerically) for the harmonic mean of Equation (6.20) and computing the relative amount ("bias") by which this value exceeded the true risk-minimizing solution given by Equation (6.13). Note that the amount of bias varies directly with ϕ_{max}, $m_p(-1)$, and RCV_p, but that in none of the cases considered does the bias reach 10%. In fact, for the vast majority of parameter combinations considered, the amount of bias imposed by harvesting at the harmonic mean of Equation (6.20) rather than the true solution is less than 2%. Thus, this comparison indicates that the harmonic mean strategy may be reasonably robust to changes in model structure.

7. Conclusions

In summary, the following conclusions may be drawn from the above:

1) In the field of fishery stock assessment, it is possible to model both process and measurement error simultaneously in a formal, rigorous fashion, meaning that the assessment scientist does not necessarily have to choose between "pure process error" and "pure measurement error" models.

2) The existence of lognormal process error in population dynamics, often assumed on an ad hoc basis, can actually be derived in the case of the model presented here, though the relationship between error magnitude and stock size is more complex than generally assumed.

3) The Kalman filter provides a straightforward means of addressing the time-series nature of at least some of the estimation problems typically encountered in fishery stock assessment.

4) It should be possible, at least in some models, to estimate the level of process error internally.

5) In terms of computational overhead, the cost of viewing the catchability coefficient and similar quantities as uncertain parameters rather than as known constants may be minimal.

6) Even when information regarding the age structure of the stock is not available or is ignored, a time series of trawl survey stock size estimates may provide sufficient information to achieve a substantial reduction in the CV of the MSY fishing mortality rate (i.e., comparing the CV of the posterior pdf to that of the prior pdf).

7) The jth-order means of the pdfs of stationary yield and stock size are related in a straightforward and heuristic way to alternative levels of relative risk aversion. For example, a risk-averse optimal fishing mortality rate can be defined as that which maximizes the geometric mean of stationary yield, equivalent to the MELSY (maximum expected log stationary yield) strategy.

8) In the model presented here, the harmonic mean of the posterior pdf of the Gompertz growth parameter is the MELSY solution.

9) Because the Gompertz growth parameter is identical to the MSY fishing mortality rate in the deterministic case, the preceding result suggests the hypothesis that the harmonic mean of the posterior pdf of the MSY fishing mortality rate may be a good proxy for the risk-averse

optimum *in general* (i.e., not just for the model presented here). Comparisons with a simple dynamic pool model indicate that the harmonic mean strategy may be a good approximation of the true MELSY harvest rate under a wide range of circumstances.

10) However, the optimality of the harmonic mean strategy is somewhat sensitive to the level of relative risk aversion used. Other levels of *RRA* do not map uniquely into *j*th-order means of the MSY fishing mortality rate in the model presented here.

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Table 1. List of symbols used in main text.

Roman characters		Greek characters	
a	Gompertz growth parameter	α	1st parameter of the beta distribution
b	equilibrium stock size	β	2nd parameter of the beta distribution
c	1st of 2 surrogate parameters for f and k		
d	2nd of 2 surrogate parameters for f and k		
e	Napier's constant (2.718...)	ε	Gaussian white noise process
f	historic fishing mortality rate	ϕ	target fishing mortality rate
g	any probability density function		
h	ratio of ϕ_{MST} to ϕ_{max}	η	a function used to simplify $m_x(t,j)$
i	a counter, used to index time		
j	a counter, used to index order of mean	φ	a counter, used to index order of mean
k	carrying capacity	κ	inverse Gaussian arithmetic mean
l	loss function	λ	inverse Gaussian shape parameter
m	mean	μ	normal arithmetic mean
n	number of observations in the series		
p	stock-recruitment compensation	π	Pi (3.141...)
q	catchability coefficient	θ	arbitrary parameter
r	size of the stock's range (area)	ρ	correlation coefficient
s	process error scale	σ	normal standard deviation
t	time	τ	time difference between 2 points in series
u	std. inverse Gaussian jth-order mean		
v	ratio of target fishing mortality to $m_a(1)$		
w	observed yield		
x	stock size	ξ	a function used to simplify expected loss
y	true yield		
z	survey estimate of stock size	ζ	a function used to simplify expected loss

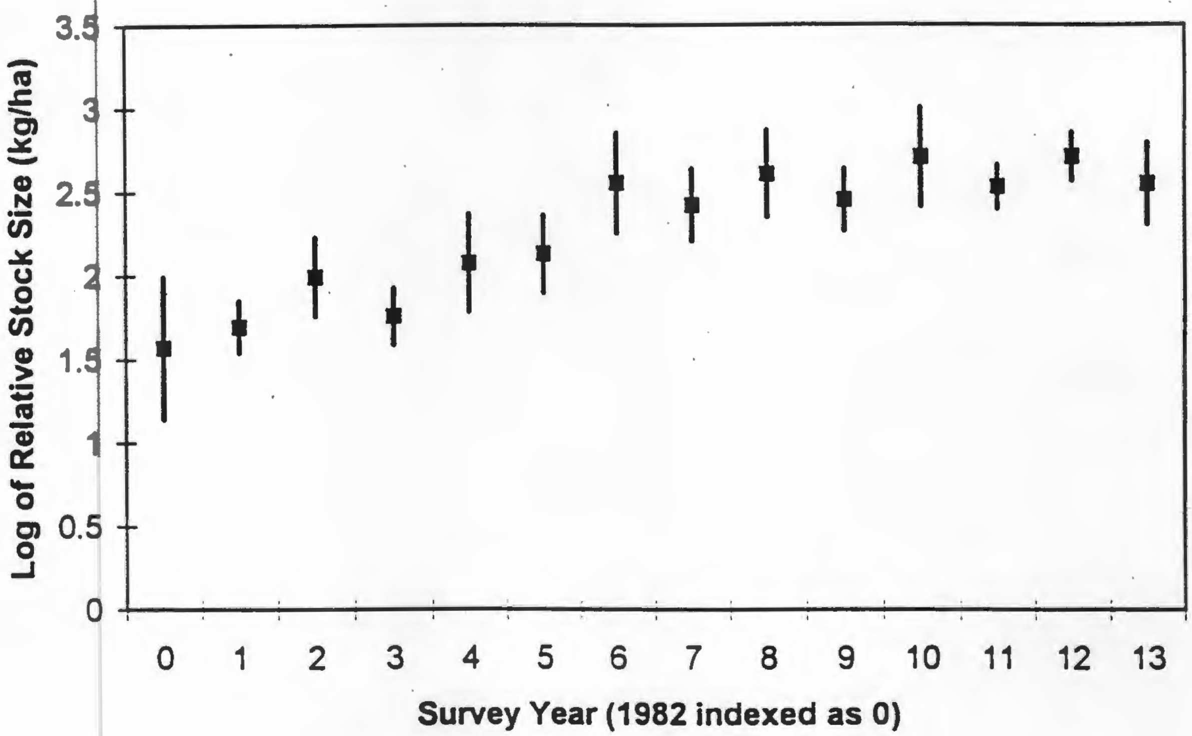

Figure 1. Trawl survey estimates of relative flathead sole abundance on a log scale, plus or minus two standard deviations.

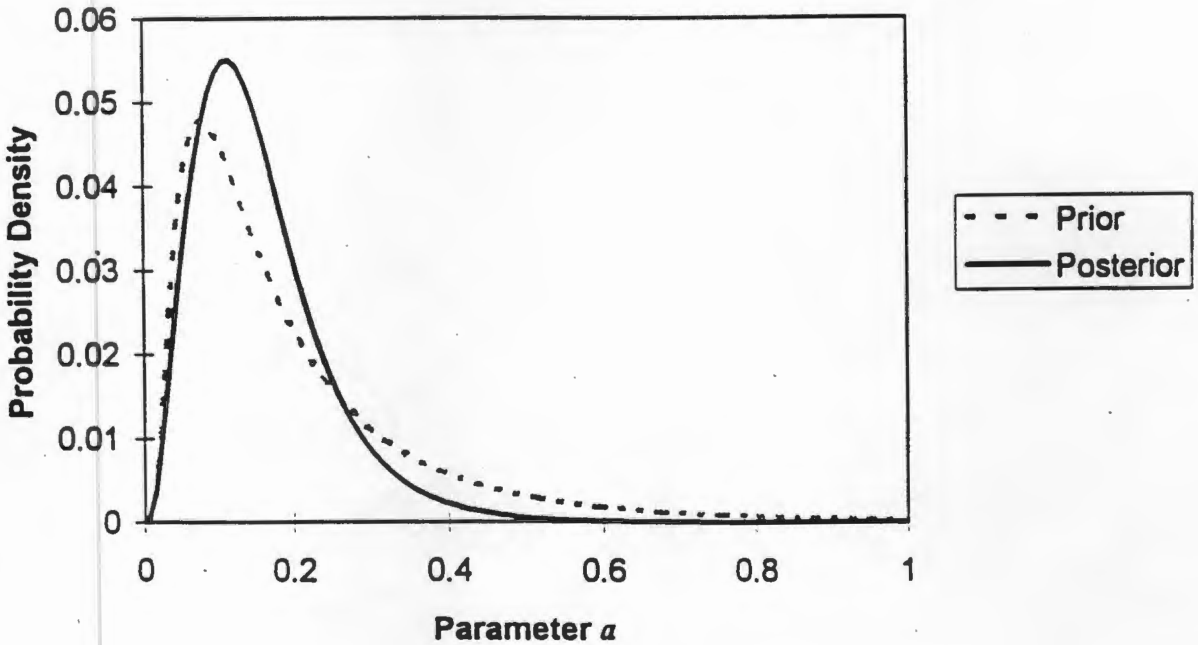

Figure 2. Prior and posterior probability density functions for parameter a in the flathead sole example.

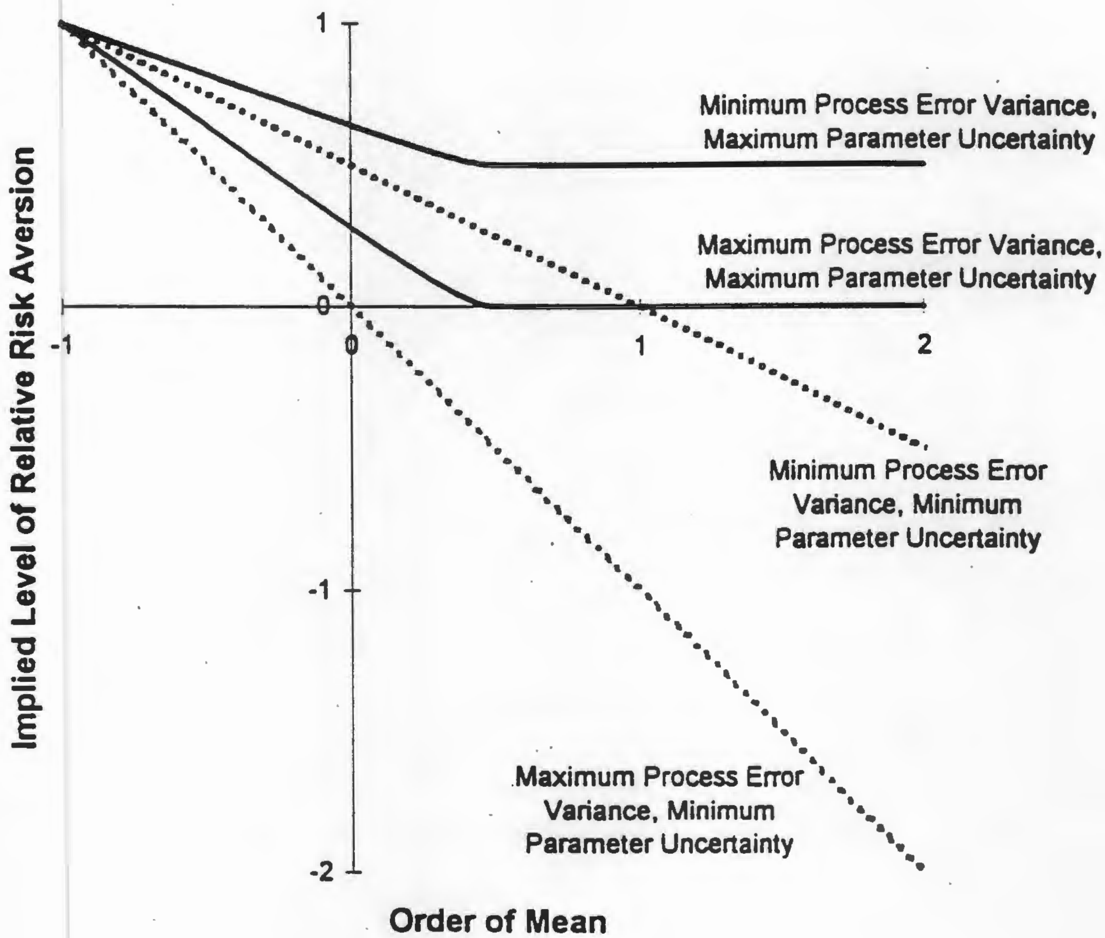

Figure 3. Limits on relative risk aversion as a function of the order of the mean.

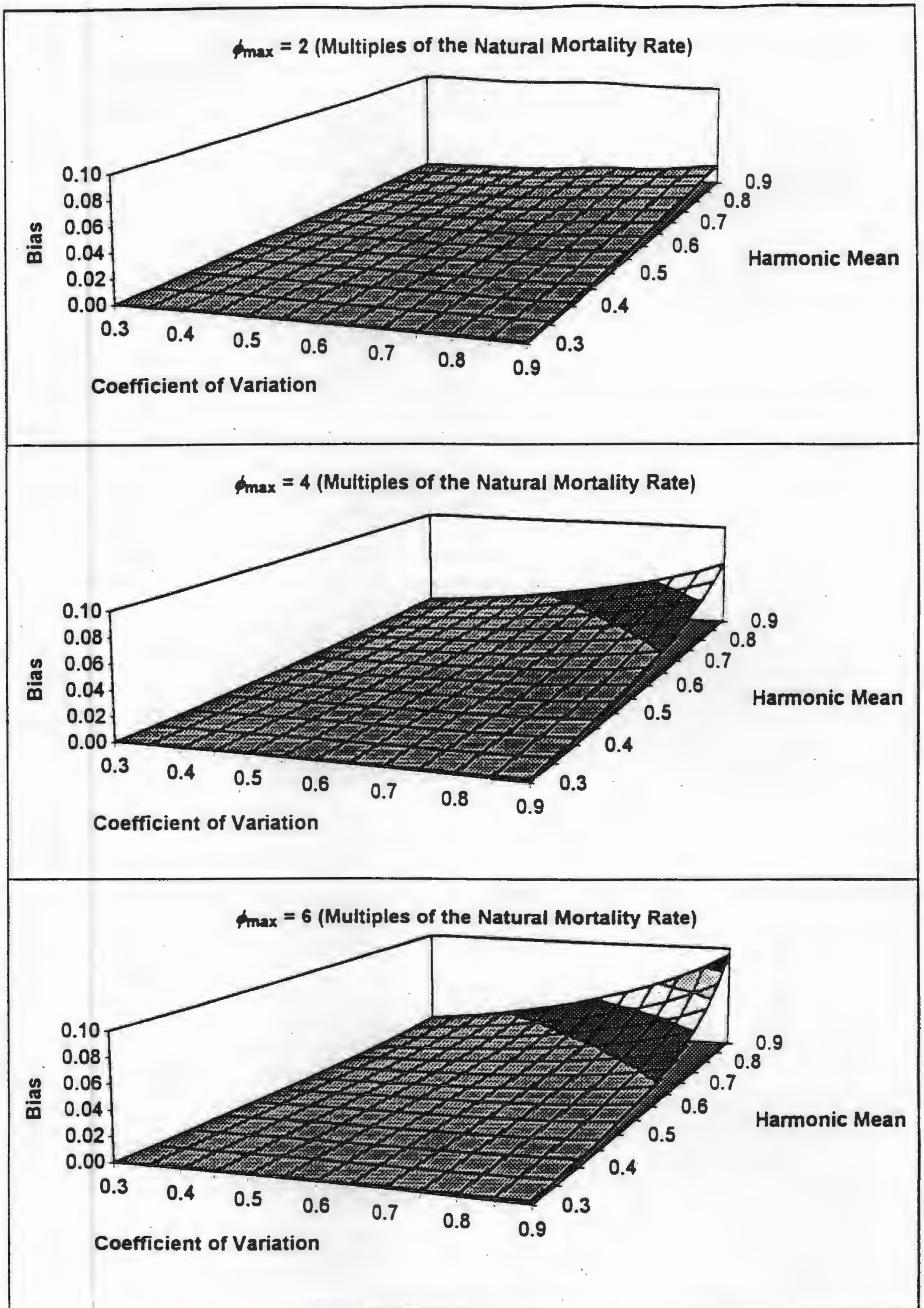

Figure 4. Bias imposed by using the harmonic mean of the MSY fishing mortality rate as a proxy for the risk-averse optimum. Each contour represents an additional 2% bias, starting from 0-2% for the broadest contour.

Attachment 1: Maximum Likelihood Estimation of Parameters *C* and *D*

Define some coefficients:

$$\alpha_0 := \left(\frac{\sigma X_0}{\sigma MX_0}\right)^2 \qquad \alpha_i := e^{-\sigma \cdot t_i} \cdot \left(\frac{\sigma X_i}{\sigma' X_i}\right)^2 \cdot \alpha_{i-1} + \left(\frac{\sigma X_i}{\sigma MX_i}\right)^2$$

$$\beta_0 := \left(\frac{\sigma X_0}{\sigma MX_0}\right)^2 \cdot Z_0 \qquad \beta_i := e^{-\sigma \cdot t_i} \cdot \left(\frac{\sigma X_i}{\sigma' X_i}\right)^2 \cdot \beta_{i-1} + \left(\frac{\sigma X_i}{\sigma MX_i}\right)^2 \cdot Z_i$$

$$\chi_0 := \frac{\left(\sigma X_0\right)^2}{\left(\sigma PY_0\right)^2 + \left(\sigma MY_0\right)^2} \qquad \chi_i := e^{-\sigma \cdot t_i} \cdot \left(\frac{\sigma X_i}{\sigma' X_i}\right)^2 \cdot \chi_{i-1} + \frac{\left(\sigma X_i\right)^2}{\left(\sigma PY_i\right)^2 + \left(\sigma MY_i\right)^2}$$

$$\delta_0 := \left[\frac{\left(\sigma X_0\right)^2}{\left(\sigma PY_0\right)^2 + \left(\sigma MY_0\right)^2}\right] \cdot W_0 \qquad \delta_i := e^{-\sigma \cdot t_i} \cdot \left(\frac{\sigma X_i}{\sigma' X_i}\right)^2 \cdot \delta_{i-1} + \left[\frac{\left(\sigma X_i\right)^2}{\left(\sigma PY_i\right)^2 + \left(\sigma MY_i\right)^2}\right] \cdot W_i$$

Define some more coefficients which are linear combinations of the above:

$$\upsilon_i := e^{-\sigma \cdot t_i} \cdot \left(\beta_{i-1} + \delta_{i-1}\right) \qquad \varpi_i := e^{-\sigma \cdot t_i} \cdot \chi_{i-1} \qquad \omega_i := 1 - e^{-\sigma \cdot t_i} \cdot \left(\alpha_{i-1} + \chi_{i-1}\right)$$

The above coefficients enable the prior means of *X* to be written as linear functions of *C* and *D*:

$$\mu' X_i = \upsilon_i - \varpi_i \cdot C + \omega_i \cdot D - Q + R$$

and the posterior means as linear functions of *C*, *D*, *Z*, and *W*:

$$\mu X_i = \left(\sigma X_i\right)^2 \cdot \left[\frac{\upsilon_i - \varpi_i \cdot C + \omega_i \cdot D}{\left(\sigma' X_i\right)^2} + \frac{Z_i}{\left(\sigma MX_i\right)^2} + \frac{W_i - C}{\left(\sigma PY_i\right)^2 + \left(\sigma MY_i\right)^2}\right] - Q + R$$

Define partial derivatives of posterior means (of *X*) with respect to *C* and *D*:

$$\delta\mu\delta C_0 := -\frac{\left(\sigma X_0\right)^2}{\left(\sigma PY_0\right)^2 + \left(\sigma MY_0\right)^2} \qquad \delta\mu\delta C_i := e^{-\sigma \cdot t_i} \cdot \left(\frac{\sigma X_i}{\sigma' X_i}\right)^2 \cdot \delta\mu\delta C_{i-1} - \frac{\left(\sigma X_i\right)^2}{\left(\sigma PY_i\right)^2 + \left(\sigma MY_i\right)^2}$$

$$\delta\mu\delta D_0 := 0 \qquad \delta\mu\delta D_i := \left(\frac{\sigma X_i}{\sigma' X_i}\right)^2 \cdot \left[e^{-\sigma \cdot t_i} \cdot \left(\delta\mu\delta D_{i-1} - 1\right) + 1\right]$$

Define a pair of vectors:

$$\gamma := \begin{bmatrix} -\displaystyle\sum_{i=1}^{n} \frac{\left[\left(\sigma' W_i\right)^2 \cdot e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} \cdot \varpi_i + \left(\sigma' Z_i\right)^2 \cdot \left(e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} + 1\right) \cdot \left(\varpi_i - 1\right) \ldots \atop + \left(\sigma' X_i\right)^2 \cdot \left[\left(e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} + 1\right) \cdot \varpi_i + e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} \cdot \left(\varpi_i - 1\right)\right]\right]}{\left(\sigma' Z_i\right)^2 \cdot \left(\sigma' W_i\right)^2 - \left(\sigma' X_i\right)^4} \\ \displaystyle\sum_{i=1}^{n} \frac{\left[\left(\sigma' W_i\right)^2 \cdot \left[e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} \cdot \left(Z_i - \upsilon_i\right)\right] + \left(\sigma' Z_i\right)^2 \cdot \left(e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} + 1\right) \cdot \left(W_i - \upsilon_i\right) \ldots \atop + \left(\sigma' X_i\right)^2 \cdot \left[\left(e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} + 1\right) \cdot \left(Z_i - \upsilon_i\right) + e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} \cdot \left(W_i - \upsilon_i\right)\right]\right]}{\left(\sigma' Z_i\right)^2 \cdot \left(\sigma' W_i\right)^2 - \left(\sigma' X_i\right)^4} \\ -\displaystyle\sum_{i=1}^{n} \frac{\left[\left(\sigma' W_i\right)^2 \cdot e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} - \left(\sigma' X_i\right)^2 \cdot \left(2 \cdot e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} + 1\right) + \left(\sigma' Z_i\right)^2 \cdot \left(e^{-\alpha\cdot\tau_t} \cdot \delta\mu\delta C_{i-1} + 1\right)\right] \cdot \omega_i}{\left(\sigma' Z_i\right)^2 \cdot \left(\sigma' W_i\right)^2 - \left(\sigma' X_i\right)^4} \end{bmatrix}$$

$$\eta := \begin{bmatrix} \displaystyle\sum_{i=1}^{n} \frac{\left[e^{-\alpha\cdot\tau_t} \cdot \left(\delta\mu\delta D_{i-1} - 1\right) + 1\right] \cdot \left[\left(\sigma' W_i\right)^2 - 2 \cdot \left(\sigma' X_i\right)^2 + \left(\sigma' Z_i\right)^2\right] \cdot \omega_i}{\left(\sigma' Z_i\right)^2 \cdot \left(\sigma' W_i\right)^2 - \left(\sigma' X_i\right)^4} \\ \displaystyle\sum_{i=1}^{n} \frac{\left[e^{-\alpha\cdot\tau_t} \cdot \left(\delta\mu\delta D_{i-1} - 1\right) + 1\right] \cdot \left[\left[\left(\sigma' W_i\right)^2 - \left(\sigma' X_i\right)^2\right] \cdot \left(Z_i - \upsilon_i\right) + \left[\left(\sigma' Z_i\right)^2 - \left(\sigma' X_i\right)^2\right] \cdot \left(W_i - \upsilon_i\right)\right]}{\left(\sigma' Z_i\right)^2 \cdot \left(\sigma' W_i\right)^2 - \left(\sigma' X_i\right)^4} \\ \displaystyle\sum_{i=1}^{n} \frac{\left[e^{-\alpha\cdot\tau_t} \cdot \left(\delta\mu\delta D_{i-1} - 1\right) + 1\right] \cdot \left[\left[\left(\sigma' W_i\right)^2 - \left(\sigma' X_i\right)^2\right] \cdot \varpi_i + \left[\left(\sigma' Z_i\right)^2 - \left(\sigma' X_i\right)^2\right] \cdot \left(\varpi_i - 1\right)\right]}{\left(\sigma' Z_i\right)^2 \cdot \left(\sigma' W_i\right)^2 - \left(\sigma' X_i\right)^4} \end{bmatrix}$$

Solve for the maximum likelihood estimates of *C* and *D* simultaneously:

$$C := \frac{\gamma_1 \cdot \eta_0 + \gamma_2 \cdot \eta_1}{\gamma_0 \cdot \eta_0 - \gamma_2 \cdot \eta_2} \qquad D := \frac{\gamma_0 \cdot \eta_1 + \gamma_1 \cdot \eta_2}{\gamma_0 \cdot \eta_0 - \gamma_2 \cdot \eta_2}$$

Note that the maximum likelihood estimates of *C* and *D* can also be written as linear functions of each other:

$$C := \frac{\gamma_1 + \gamma_2 \cdot D}{\gamma_0} \qquad D := \frac{\eta_1 + \eta_2 \cdot C}{\eta_0}$$

Solve for *F* and *K* as functions of *A*, *C*, *D*, and *Q*:

$$F := C + Q - R \qquad K := D + e^{C + Q - R - A} - Q + R$$