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A Model Comparison in Vertical Crustal Motion Estimation Using Leveling Data

Günter W. Hein

Rockville, MD
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A MODEL COMPARISON IN VERTICAL CRUSTAL MOTION ESTIMATION USING LEVELING DATA

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ABSTRACT. A new generalized linear regression model, also called "mixed" model, is presented for the computation of zero-epoch heights and height velocities using releveling observations. The method is compared with the single point velocity model having a multiquadric interpolator for the velocity surface.

Tests based on simulated error-free and noisy data show slight advantages for the mixed model. The multiquadric technique provides a good interpolation method in the deterministic sense when certain specific precautions are considered. Error statistics of multiquadrics using a stochastic approach through least-squares should be interpreted with extreme care.

The appendix contains geophysical and geological information useful for defining the trend part in the mixed model. Simple numerical investigations in the Houston-Galveston area lead to the conclusion that it might be possible to predict the subsidence of bench marks due to groundwater withdrawal with a standard deviation of 2 to 3 mm/yr.

1. INTRODUCTION

The new adjustment of the North American Vertical Datum (Holdahl 1984) requires that (theoretically) all data have to refer to one common epoch in order to avoid distortions in the results due to possible bench mark motions. To hold costs within reasonable limits it may be preferable to find a method or algorithm to select

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from the data those reobservations with only white noise, so that it can be assumed that no systematic errors due to movements have taken place in the considered area. Thus, all observations can be viewed as static.

There are, however, some areas where rapid changes in elevation due to seismic or human engineering activities (groundwater table changes, etc.) occur. Those areas have to be adjusted dynamically either before the adjustment in order to obtain observations reduced to a common epoch or, when excluding those data from the readjustment, to adjust them separately afterwards.

For that purpose a variety of dynamical models has been developed; for a review see, e.g., Gubler (1984), Holdahl (1978). This paper presents a new generalized linear regression model for determining zero-epoch heights and height velocities from releveling data. It includes an interpolation of the velocity surface and takes advantage of the signal-to-noise ratio in the data. To assess the new method, it is compared with the linear single point velocity model using multiquadrics for the velocity surface fitting.

Chapter 2 contains a review of the linear single point velocity method and discusses the surface interpolation. This is followed by the presentation of the new generalized linear regression model in chapter 3. Test computations and corresponding results are detailed in chapter 4. Finally, the conclusions section outlines the results of the study.

The appendix reports the results of some numerical investigations on the deterministic trend determination of vertical movements due to water withdrawal. As an example, geodetic and geological data are used from the Houston-Galveston subsidence area.

2. THE LINEAR SINGLE POINT VELOCITY MODEL

2.1 Observation Equations and Solution

Each observed height difference Δh_{ij} between marks P_j and P_i at time t can be expressed by

$$\Delta h_{ij,t} = H_{j,t_o} - H_{i,t_o} + \dot{H}_j(t-t_o) - \dot{H}_i(t-t_o) + n_{ij} \qquad (2\text{-}1)$$

where H_{i,t_0}, H_{j,t_0} are the heights of P_i and P_j at the reference epoch t_o, $\dot{H}_i$, $\dot{H}_j$ are the corresponding linear vertical point velocities, and n_{ij} is the observational noise. In classical least squares adjustment n_{ij} is considered to be the negative residual. Observation eq. (2-1) holds under the assumptions that

(i) no motion took place during the time span of the observation itself, and t is a representative value of this span,

(ii) the motion can be modeled linearly,

(iii) the gravity variations in the area considered are so small that the orthometric height difference Δh_{ij} can be associated in a one-to-one correspondence with geopotential differences ΔW_{ij},

$$\Delta W_{ij} = \int_{P_i}^{P_j} g \, dn \doteq \sum_{P_i}^{P_j} g \, \delta n \qquad (2\text{-}2)$$

where δn are the height increments and g is the gravity along the observation line, and

(iv) the secular variation of gravity for the time interval $(t-t_o)$ is negligible so that its influence on the observation is smaller than the noise level. (A secular variation of a gravity difference of $\delta(\Delta g) \geq 50$ µgal can cause an error in height of $\delta(\Delta h) \geq 0.2$ mm; for numerical values see Kistermann and Hein (1979).)

If (ii) is not fulfilled, the observation equation (2-1) can be extended by adding acceleration terms

$$\tfrac{1}{2} [\ddot{H}_j(t-t_o)^2 - \ddot{H}_i(t-t_o)^2]. \qquad (2\text{-}3)$$

If (iii) and/or (iv) are not fulfilled, Δh_{ij} has to be replaced by $\Delta W_{ij} \equiv C_{ij}$. (See eq. (2-2).) Consequently, the unknowns in (2-1) are then geopotential numbers C and corresponding velocities $\dot{C}$. A check of these assumptions is essential in areas with large changes in the Earth's crust, e.g., due to mining, oil and gas withdrawal, groundwater table changes, and other factors.

Let p be the number of observations and u the number of parameters.

The solution of an overdetermined $p > u$ linear system of equations such as of (2-1) is obtained by minimizing

$$\underline{n}^T \underline{C}_{nn}^{-1} \underline{n} = \min \tag{2-4}$$

where $\underline{C}_{nn}$ is the variance-covariance matrix of observational noise, usually defined in the form of a diagonal matrix for uncorrelated observations, $\underline{C}_{nn} = \sigma_{ij}^2 \underline{I}$, $\sigma_{ij}^2 = \mathrm{var}(\Delta h_{ij,t})$. $\underline{I}$ is the $p \times p$ identity matrix, and p is the number of observations.

Thus, one gets the estimate of the unknown vector of heights $\hat{H}_{i,t_o}$, $\hat{H}_{j,t_o}$ and the linear velocities $\hat{\dot{H}}$ by

$$\hat{\underline{x}} = (\underline{A}^T \underline{C}_{nn}^{-1} \underline{A})^{-1} \underline{A}^T \underline{C}_{nn}^{-1} \underline{l} \quad , \tag{2-5}$$

where $\hat{\underline{x}}$ is the $u \times 1$ vector consisting of the two $(u/2)\times 1$ subvectors $\hat{\underline{x}}_1$, $\hat{\underline{x}}_2$,

$$\hat{\underline{x}} = [\hat{\underline{x}}_1^T | \hat{\underline{x}}_2^T]^T = [\hat{\underline{H}}_{t_o}^T | \hat{\dot{\underline{H}}}^T]^T \quad . \tag{2-6}$$

$\underline{A}$ is the $p \times u$ design matrix of known coefficients,

$$\underline{A} = [\underline{A}_1 | \underline{T}\ \underline{A}_1] \tag{2-7}$$

where the $p \times (u/2)$ matrix $\underline{A}_1$ is defined as in the usual static adjustment of vertical networks. $\underline{T}$ is a $p \times p$ diagonal matrix defined by

$$\underline{T} = \mathrm{diag}(t - t_o) \qquad t = \{t_1, t_2, \ldots, t_p\}$$

$\underline{l}$ is the $p \times 1$ observation vector,

$$\underline{l} = (\Delta h_{ij,t}) \quad .$$

In order to remove the defect $d = 2$ of $(\underline{A}^T \underline{C}_{nn}^{-1} \underline{A})$, one has to fix the datum of the heights and velocities by holding as a minimum one of each fixed. For convenience, this is normally done at one point P_o, e.g.,

$$H(P_0) = \text{constant}$$
$$\dot{H}(P_0) = \text{constant}. \tag{2-8}$$

Additionally, it is required that any bench mark P_i, P_j considered is observed at least in two epochs. Otherwise singularities appear and the inverse of $(\underline{A}^T \underline{C}_{nn}^{-1} \underline{A})$ does not exist.

The error statistics of eq. (2-5) are given by the variance-covariance matrix of the unknowns,

$$\underline{C}_{xx} = \hat{\sigma}_0^2 (\underline{A}^T \underline{C}_{nn}^{-1} \underline{A})^{-1} \tag{2-9}$$

with the a posteriori variance factor of unit weight

$$\hat{\sigma}_0^2 = (\underline{n}^T \underline{C}_{nn}^{-1} \underline{n})/(p-u). \tag{2-10}$$

Obviously, the results from eq. (2-9) are dependent on the introduced datum, eq. (2-8).

2.2 Representation of the Velocity Surface by Multiquadrics

In order to fit a two-dimensional surface to the computed vertical velocities, any appropriate interpolation method can be used in principle. (See, e.g., Hein and Lenze 1979.) In practical applications two-dimensional polynomials (Vaníček and Christodoulidis 1974), orthogonal polynomials (Vaníček et al. 1979), and multiquadrics (Hardy 1978, Holdahl and Hardy 1979) have been previously chosen. The latter one is the method currently applied in crustal movement studies at NGS, see e.g., Holdahl (1982).

The unknown velocity $\hat{\dot{H}}$ at a prediction point P_α is determined by

$$\hat{\dot{H}}_\alpha = \underline{f}^T{}_{\alpha\beta} \underline{k}_\beta \qquad \text{for} \quad \beta \leq u/2 \tag{2-11}$$

where P_β are the so-called nodal points. $\underline{f}_{\alpha\beta}$ is a known $\beta \times 1$ vector defined later in eq. (2-17). $\underline{k}_\beta = \underline{k}(P_\beta)$ is the $\beta \times 1$ vector of unknown multiquadric coefficients. Inserting eq. (2-11) in the basic observation eq. (2-1) results in

$$\Delta h_{ij,t} = H_{j,t_o} - H_{i,t_o} + (t-t_o)(\underline{f}^T_{j\beta} - \underline{f}^T_{i\beta})\underline{k}_\beta + n_{ij} \quad . \tag{2-12}$$

The unknowns in eq. (2-12) are now the heights at reference time t_o and the multi-quadric coefficients $\underline{k}_\beta$ defining the velocity surface. Thus $\hat{\underline{x}}$ in eq. (2-6) has to be substituted by

$$\hat{\underline{x}} = [\hat{\underline{x}}_1^T \,|\, \hat{\underline{x}}_2^T]^T = [\hat{\underline{H}}^T_{t_o} \,|\, \underline{k}^T_\beta]^T \tag{2-13}$$

and $\underline{A}$ in eq. (2-7) by

$$\underline{A} = [\underline{A}_1 \,|\, \underline{T}\ \underline{A}_1 \underline{F}_\beta] \tag{2-14a}$$

where

$$\underline{F}_\beta = \begin{bmatrix} \underline{f}^T_{1\beta} \\ \underline{f}^T_{2\beta} \\ \vdots \\ \underline{f}^T_{\alpha\beta} \end{bmatrix} . \tag{2-14b}$$

Since $\beta \le u/2$ the dimension of $\hat{\underline{x}}$ can now be smaller than u. The same holds for the number of columns of $\underline{A}$ in eq. (2-7).

The datum defect of the normal equation matrix remains the same as before. Recall from eq. (2-8) that one height and one velocity have to be fixed. The nodal points P_β can be, in principle, located arbitrarily in the considered area. The simplest choice is to put them at the same location as the bench marks. However, the solvability of the normal equations requires certain considerations (Holdahl and Hardy 1979). Nodal points can be situated anywhere, but are best situated in places where velocity information can be inferred from the observations. A theoretical advantage, at least of the model, eq. (2-12), is the fact that the number β of nodal points can be smaller than the number of discrete velocity unknowns considered in eq. (2-1),

$$\dim \underline{x}_2(\underline{k}_\beta) \le \dim \underline{x}_2(\hat{\dot{\underline{H}}}). \tag{2-15}$$

The velocity at any point can be predicted by eq. (2-11) after having performed the adjustment. The standard deviation of the predicted velocity can be found using covariance propagation for the function, eq. (2-11)

$$\hat{\sigma}_{\dot{H}_\alpha} = \hat{\sigma}_0 (\underline{f}^T_{\alpha\beta} \underline{Q}_{\beta\beta} \underline{f}_{\alpha\beta})^{0.5} . \tag{2-16}$$

$\hat{\sigma}_0$ can be taken from eq.(2-10). $\underline{Q}_{\beta\beta}$ is the covariance matrix of determined multiquadric coefficients $\underline{k}_\beta$, the lower right submatrix of $(\underline{A}^T \underline{C}^{-1}_{nn} \underline{A})^{-1}$.

For the definition of

$$\underline{f}^T_{\alpha\beta} = [f_{\alpha\beta_1} f_{\alpha\beta_2} \ldots f_{\alpha\beta_n}]$$

the recommended quadric was chosen to be a hyperboloid,

$$f_{\alpha\beta_k} = [(x_{\beta_k} - x_\alpha)^2 + (y_{\beta_k} - y_\alpha)^2 + D^2]^{0.5} \tag{2-17}$$

where D is some kind of smoothing constant which has to be empirically defined. (x_β, y_β) are the horizontal coordinates of nodal points P_β, $\beta = (1,2,\ldots) \leq u/2$, and (x_α, y_α) the corresponding coordinates of the prediction point P_α .

It is obvious that the determination of reference epoch heights and velocities as well as the interpolation of the latter can be done in two steps. Savings in computer time can be achieved only in the case that dim $\underline{x}_2(\underline{k}_\beta)$ is considerably smaller than dim $\underline{x}_2(\dot{\underline{H}})$,

$$\dim \underline{x}_2(k_\beta) \ll \underline{x}_2(\hat{\underline{H}}) . \tag{2-18}$$

Multiquadric analysis was previously used by the author as the interpolation method (in a second <u>separate</u> step <u>after</u> the adjustment) for mapping height changes in the F.R. Germany (DGK-Arbeitskreis 1979) and in the Rhenish Massif (Mälzer et al. 1983).

2.3 Representation of the Velocity Surface by Collocation

Instead of using eq. (2-11) for the multiquadric representation of velocities $\hat{\dot{H}}$, least squares collocation also can be applied to the adjusted quantities $\hat{\dot{\underline{H}}}$ as pseudo-observations in the form

$$\dot{\underline{H}} = \underline{B}\underline{X} + \underline{s} + \tilde{\underline{n}} \tag{2-19}$$

where $\underline{B}\underline{X}$ is a trend function describing the large scale motion (such as plate tilting) of the area under consideration, e.g., using a first order polynomial approximation for describing an inclined plane,

$$f_{plane} = a_0 + a_1 x + a_2 y \quad .$$

Thus, $\underline{B}$ and $\underline{X}$ are defined by

$$\begin{aligned} \underline{B} &= (1 \quad x \quad y)_\alpha \\ \underline{X} &= (a_0 \; a_1 \; y_2)^T . \end{aligned} \tag{2-20}$$

The signal $\underline{s}$ is considered to represent the regional crustal movements and $\tilde{n}$ is random individual motion at certain points. The solution for $\underline{X}$ and $\underline{s}$ is derived by considering the hybrid minimum condition

$$\underline{s}^T \underline{C}_{\underline{ss}}^{-1} \underline{s} + \tilde{\underline{n}}^T \underline{C}_{\tilde{n}\tilde{n}}^{-1} \tilde{\underline{n}} = \min \tag{2-21}$$

where $\underline{C}_{ss}$ and $\underline{C}_{\tilde{n}\tilde{n}}$ are the corresponding covariance matrices of $\underline{s}$ and $\tilde{\underline{n}}$. Thus, we get as the solution (see e.g., Moritz 1980; p. 116)

$$\hat{\underline{X}} = (\underline{B}^T \bar{\underline{C}}^{-1} \underline{B})^{-1} \underline{B}^T \bar{\underline{C}}^{-1} \dot{\underline{H}} \tag{2-22}$$

$$\hat{\underline{s}} = \hat{\underline{s}}(P_\alpha) = \underline{C}_{\alpha\beta}^T \bar{\underline{C}}^{-1} (\dot{\underline{H}} - \underline{B}\hat{\underline{X}}) \tag{2-23}$$

where

$$\underline{\bar{C}} = \underline{C}_{ss} + \underline{C}_{\tilde{n}\tilde{n}} \quad . \tag{2-24}$$

A possible choice for $\underline{C}_{\tilde{n}\tilde{n}}$ is the lower right submatrix of $(\underline{A}^T\underline{C}_{nn}\underline{A})^{-1}$ resulting from the foregoing estimation of single point velocities in eq. (2-1). $\underline{C}_{ss}$ can be found from the adjusted discrete velocities $\dot{H}$ by determining an empirical covariance function (for details see chapter 3).

A slightly different model where a collocation-type representation of the velocity was used in one computational step was given by Hein and Kistermann (1981). The observation equation reads

$$\Delta h_{ij,t} = H_{j,t_0} - H_{i,t_0} + (t-t_0)(\dot{H}_j-\dot{H}_i) + (t-t_0)(s_j-s_i) + n_{ij} \quad . \tag{2-25}$$

There collocation was used in the form

$$\underline{l} = \underline{B}\underline{X} + \underline{R}\underline{s} + \underline{n} \tag{2-26}$$

where

$$\underline{B} = [\underline{A}_1 | \underline{T}\underline{A}_1]$$

$$\underline{X} = (H_{j,t_0}, H_{i,t_0}, \ldots, \dot{H}_j, \dot{H}_i, \ldots)^T \tag{2-27}$$

$$\underline{R} = \underline{T}\,\underline{A}_1$$

and $\underline{A}_1$ and $\underline{T}$ are matrices defined earlier in eq. (2-7). In this approach the velocities $\dot{H}$ included in the deterministic part of eq. (2-26) are considered to be regional changes of recent crustal movements and the signal is interpreted as the local field of nontectonic influences.

For the detailed solution of eq. (2-26), see section 3.1 of Hein and Kistermann (1981).

2.4 Multiquadrics Versus Collocation for Velocity Surface Representation

In the following, some remarks will be made about the two methods discussed above with respect to the problem of velocity surface fitting.

Expression (2-11) can be formally considered as the basic equation for both methods when writing it in the form

$$\hat{\dot{H}}_\alpha = \underline{f}^T_{\alpha\beta}\,\underline{k}_\beta = \underline{f}^T_{\alpha\beta}\,\underline{C}^{-1}_{\beta\beta}\,\underline{\dot{H}}_\beta .$$

This corresponds to a (cross covariance-) prediction estimator. ($\underline{C}^{-1}_{\beta\beta}$ is then an autocovariance matrix of given centered velocities $\dot{H}_\beta$.) In the multiquadric method, however, no inversion is necessary since $\underline{k}_\beta$ is obtained by solving the linear symmetric system of equation

$$\underline{C}_{\beta\beta}\,\underline{k}_\beta = \underline{\dot{H}}_\beta$$

where $\underline{C}_{\beta\beta}$ is explicitly defined by (2-14b) and (2-17)

$$\underline{C}_{\beta\beta} := \underline{F}_\beta .$$

The (arbitrary) choice of the special function and the "smoothing" constant D defining $\underline{f}^T_{\alpha\beta}$ for multiquadrics have similarities to those of the covariance function and the signal-to-noise ratio in collocation. However, neither a convincing stochastic nor geophysical interpretation can be given in multiquadrics for the selection of nodal points, the use of any specific quadric, or the value of D. Although only collocation requires beforehand some type of trend elimination in $\underline{\dot{H}}$, it is well known that the results of multiquadrics improve considerably when applying a similar procedure (Schut 1975). Therefore, as long as both methods are considered as pure interpolation methods for the velocity surface, and we are dealing with adjusted quantities (at the same locations) without stochastic signal properties, it may be concluded that the two methods yield meaningful predictions and, hence, corroborate previous studies (Hein and Lenze 1979; Wolf 1981).

A question, however, arises if eq. (2-18) is present in the one-step adjustment of linear single point velocities in eq. (2-12). This means that the number of nodal points is smaller than the number of possible discrete velocity unknowns. How good is the approximation then? It should be stressed here that the excellent performance of the multiquadric method in the comparative study of Hein and Lenze (1979) was obtained on the basis that all data point locations served as nodal points.

An answer to this question will be given later in this report.

3. A GENERALIZED LINEAR REGRESSION MODEL OR MIXED MODEL FOR DETERMINING VERTICAL MOVEMENTS FROM LEVELING DATA

3.1 Observation Equations and Solution

Let us start again with the linear observation equation (2-1)

$$\Delta h_{ij,t} = H_{j,t_0} - H_{i,t_0} + (t-t_0)\,(\dot{H}_j-\dot{H}_i) + n_{ij} \tag{3-1}$$

and assign the different terms in it to a model of the form

$$\underline{l} = \underline{A}_1\underline{X} + \underline{R}\underline{s} + \underline{n} \tag{3-2}$$

where

- $\underline{l}$ is the p×1 vector of observed height differences at time t, $\underline{l} = (\Delta h_{ij,t})$
- $\underline{X}$ is the u_1×1 vector of unknown heights at the reference epoch t_0, $\underline{X} = (H_{i,t_0},\ H_{j,t_0},\ \ldots)$
- $\underline{R}$ is the p×u_2 matrix of known coefficients, $\underline{R} = \underline{T}\,\underline{A}_1$, where $\underline{T} = \mathrm{diag}(t-t_0)$, $t = (t_1,t_2,\ldots,t_p)$

and $\underline{s}$ is the unknown u_2×1 vector of height velocities, $\underline{s} = (\dot{\underline{H}})$.

The matrix $\underline{A}_1$ is defined as in an ordinary adjustment of leveling networks. In section 2.1 the solution for an overdetermined system of linear observation eqs. (2-1) or (3-1) was obtained by considering the unknown vector $\underline{H}_{t_0}$ of heights at reference epoch t_0 and the vertical velocity vector $\dot{\underline{H}}$ as fixed effects in a

Gauss-Markov model

$$\underline{l} = [\underline{A}_1 \underline{R}] \begin{bmatrix} \underline{X} \\ \underline{s} \end{bmatrix} + \underline{n} \tag{3-3}$$

with

$$\begin{aligned} E(\underline{l}) &= [\underline{A}_1 \underline{R}] \begin{bmatrix} \underline{X} \\ \underline{s} \end{bmatrix} \\ E(\underline{n}) &= 0 \\ E(\underline{n}\underline{n})^T &= \underline{C}_{nn} = \text{var}(n) \end{aligned} \tag{3-4}$$

where the symbol E stands for the expectation operator.

Let us now assume that in contrast to eq. (3-3) not only $\underline{n}$ but also $\underline{s} = \dot{\underline{H}}$ is a random vector in eq. (3-2), so that

$$\begin{aligned} E(\underline{l}) &= \underline{A}_1 \underline{X} + \underline{R}\underline{\mu} \\ E(\underline{s}) &= \underline{\mu} = 0 \\ E(\underline{n}) &= 0 \quad . \end{aligned} \tag{3-5}$$

Model (3-2) in conjunction with (3-5) is known in statistical literature as a mixed model (Harville 1976) or generalized linear regression model (Goldberger 1962). If $\underline{s}$ is an $\infty \times 1$ vector of infinite random components and $\underline{R}$ its corresponding $p \times \infty$ matrix (p is the number of observations), then model (3-2) is called (least squares) collocation in physical geodesy (see e.g., Rummel 1976). In order to avoid any confusion with the pure representation or interpolation of the (adjusted) velocities by collocation as discussed in 2.3, we will use the term "mixed model" for the general adjustment model (3-1), (3-2), and (3-5).

We rewrite the mixed model (3-2) in the form

$$\underline{l} = \underline{A}_1 \underline{X} + \underline{R}(\underline{O}_s + \underline{s}) + \underline{n} \tag{3-6}$$

where the null vector $\underline{0}_s$ represents a vector of pseudo-observations of the random vector $\underline{s}$. Equation (3-6) is a so-called Gauss-Helmert model having the stochastic properties

$$\operatorname{cov}\begin{bmatrix} \underline{l} \\ \underline{0}_s \end{bmatrix} = \begin{bmatrix} \underline{C}_{nn} & \underline{0} \\ \underline{0} & \underline{C}_{ss} \end{bmatrix} . \tag{3-7}$$

Substituting in eq. (3-6)

$$\underline{0}_s + \underline{s} = \underline{s}' \tag{3-8}$$

and adding these equations for the pseudo-observations $\underline{0}_s$ to the original system of observation equations, we get the general Gauss-Markov model

$$\begin{aligned} \underline{l} &= \underline{A}_1\underline{X} + \underline{R}\underline{s}' + \underline{n} \\ \underline{0}_s &= \underline{s}' - \underline{s} \end{aligned} \qquad \operatorname{cov}\begin{bmatrix} \underline{l} \\ \underline{0}_s \end{bmatrix} = \begin{bmatrix} \underline{C}_{nn} & \underline{0} \\ \underline{0} & \underline{C}_{ss} \end{bmatrix} \tag{3-9}$$

where s' can now formally be considered a fixed effect with $\operatorname{var}(s') = 0$, or in short form

$$\tilde{\underline{l}} = \tilde{\underline{A}}\,\tilde{\underline{X}} - \tilde{\underline{v}} \ , \qquad \underline{C}_{\tilde{l}\tilde{l}} \tag{3-10}$$

with

$$\tilde{\underline{A}}_1 = \begin{bmatrix} \underline{A}_1 & \underline{R} \\ -\underline{0} & \underline{I} \end{bmatrix}$$

$$\tilde{\underline{X}} = \begin{bmatrix} \underline{X} \\ \underline{s}' \end{bmatrix}$$

$$\tilde{\underline{v}} = \begin{bmatrix} -\underline{n} \\ \underline{s} \end{bmatrix}$$

$$\underline{C}_{\tilde{l}\tilde{l}} = \begin{bmatrix} \underline{C}_{nn} & \underline{0} \\ \underline{0} & \underline{C}_{ss} \end{bmatrix}.$$

Using the least squares minimum condition

$$\tilde{\underline{v}}^T \underline{C}_{\tilde{l}\tilde{l}}^{-1} \tilde{\underline{v}} = \underline{n}^T \underline{C}_{nn}^{-1} \underline{n} + \underline{s}^T \underline{C}_{ss}^{-1} \underline{s} = \min \tag{3-11}$$

we derive the estimators for $\underline{X}$ and $\underline{s}$

$$\hat{\underline{X}} = (\underline{A}_1^T \bar{\underline{C}}^{-1} \underline{A}_1)^{-1} \underline{A}_1^T \bar{\underline{C}}^{-1} \underline{l} \tag{3-12}$$

$$\hat{\underline{s}} = \underline{C}_{ss} \underline{R}^T \bar{\underline{C}}^{-1} (\underline{l} - \underline{A}_1 \hat{\underline{X}}) \tag{3-13}$$

where

$$\bar{\underline{C}} = \underline{C}_{nn} + \underline{R}\, \underline{C}_{ss}\, \underline{R}^T . \tag{3-14}$$

Inserting in eq. (3-13) the appropriate cross-covariance matrix $\underline{C}_{ts}$ instead of $\underline{C}_{ss}$ allows the user to also predict signals $\underline{t}$ at stations different from those where the observations $\underline{l}$ are given.

The error statistics are given by

$$\underline{C}_{\hat{X}\hat{X}} = (\underline{A}_1^T \bar{\underline{C}}^{-1} \underline{A}_1)^{-1} \tag{3-15}$$

$$\underline{C}_{\hat{s}\hat{s}} = \underline{C}_{ss} - \underline{C}_{st}\, \underline{R}^T \bar{\underline{C}}^{-1} [\underline{I} - \underline{A}_1 (\underline{A}_1^T \bar{\underline{C}}^{-1} \underline{A}_1)^{-1} \underline{A}_1^T \bar{\underline{C}}^{-1}] \underline{R}\, \underline{C}_{ts} . \tag{3-16}$$

For the detailed derivation the reader is referred to Moritz (1972) and Wolf (1977).

3.2 Discussion of the Estimation Process

(1) The application of the hybrid minimum norm (3-11) implies only one datum defect with respect to the heights at reference epoch t_o. The height velocities $\dot{H}_j, \dot{H}_i$ are constrained by the second term in eq. (3-11). Therefore it is sufficient

to fix one height H_{i,t_o}.

(2) The mixed model above is a general algorithm able to predict velocities $\dot{H}$ at any point in the area under consideration and height changes in time.

If the vector $\underline{l} = (\Delta h_{ij,t})$ consists only of observations belonging to one epoch, then $\underline{s} = \dot{\underline{H}}$ results in

$$\hat{\underline{s}} = (t-t_o)\, \underline{C}_{ss}\, \underline{A}_1^T \bar{\underline{C}}^{-1}(\underline{l}-\underline{A}_1\hat{\underline{X}})$$

$$\hat{\underline{s}} = (t-t_o)\, \underline{C}_{ss}\, (\underline{A}_1^T \bar{\underline{C}}^{-1}\underline{l} - \underline{A}_1^T \bar{\underline{C}}^{-1}\underline{A}_1\hat{\underline{X}}) = \underline{0} \qquad (3\text{-}17)$$

which simply expresses the fact that no velocity information is inherent in the data.

If a bench mark is observed at only one epoch, then the corresponding column of $\underline{R}$ is a multiple of the columns of $\underline{A}_1$ so that the corresponding row of $\underline{R}^T\bar{\underline{C}}^{-1}(\underline{l}-\underline{A}_1\hat{\underline{X}})$ in eq. (3-13) vanishes, and therefore no contribution of that observation point is made to the estimate of $\underline{s}$.

(3) If the point P(t), where the signal $t = \dot{H}$ has to be predicted, is far from the data points P(s), then

$$\begin{aligned} \underline{C}_{st} &\rightarrow \underline{0} \\ \hat{\underline{t}} &\rightarrow \underline{0} \\ \underline{C}_{\hat{t}\hat{t}} &\rightarrow \underline{C}_{ss} \end{aligned} \qquad (3\text{-}18)$$

which is a reasonable result for an extrapolation.

(4) Whereas in model (2-1), using the minimum condition (2-4), the adjusted velocities $\hat{\underline{x}}_2 = \hat{\dot{\underline{H}}}$ refer to an (arbitrary) point with fixed velocity (2-6), the datum for $\hat{\underline{s}} = \dot{\underline{H}}$ in the mixed model (See eq. 3.1) is derived from the data themselves. The reference frame is defined by minimizing the (weighted) sum of squares of velocities. (See eq. 3-11.) Therefore the signals $\underline{s}$ can be considered in some way as "inner velocities" similar to the inner or free adjustment theory (Meissl 1969).

In order to clarify this point, the reader is reminded that leveling observations are relative measurements without any information about the absolute height and motion of the network. The following phenomena can occur to the area covered by the network:

(i) Changes in translation (height) and rotation of the whole area with respect to an inertial frame without changing the geometric configuration of the network. An example is the motion of a tectonic plate on which the leveling network is situated.

(ii) Homogeneous deformation, changes in the volume of the area which are constant in amount and direction.

(iii) Inhomogeneous deformation as a function of the location of the points.

(iv) Single point movements, irregular in appearance and limited to very local phenomena, as e.g., subsidence of one bench mark due to (random) unknown causes in the neighborhood (local soil swelling, nearby engineering projects, etc.).

Whereas the information about (ii) to (iv) can be extracted from the data, an assumption has to be made about (i) or some information coming from another source has to enter the model. (See (2-8).) However, in most practical applications the information about an absolute velocity is not available or, at least, is uncertain. In addition, the covariance matrix (2-9) and consequently, the error statistics are dependent on the introduced fixed velocity. Why then not choose a reference frame for the vertical velocities $\dot{H}$ which is based only on the data and inner adjustment constraints (shown in the following) similar to the work of Blaha (1971) and Meissl (1969)? The mixed model proposed in section 3.1 exactly follows this philosophy and tries to find the value of the velocities based on the "optimal" reference frame defined by the data, since their absolute values are nonestimable on the basis of the considered observations. (See also Papo and Perelmuter 1983.)

If, in addition, some information about the velocity of one or several points in the network is available, the results of the mixed models, the signals $\hat{\underline{s}} = \dot{\underline{H}}$, can be transformed to the new reference frame using Meissl (1969), Blaha (1971: p. 76) or Baarda's S-transformations. (See, e.g., Baarda 1973.)

Denote by $\tilde{\dot{H}}$ the velocities in the new reference frame and by $\hat{\dot{H}}$ the adjusted velocities referring to the inner coordinate system. Then, the transformation can be expressed by

$$\underline{\tilde{\dot{H}}} = \underline{\hat{\dot{H}}} + \underline{G}\,\underline{dt} = [\underline{I}\ \underline{G}]\begin{vmatrix} \underline{\hat{\dot{H}}} \\ \underline{dt} \end{vmatrix} \tag{3-19}$$

where the vector of differential velocity shift $\underline{dt}$ is defined by

$$\underline{dt} = \underline{F}\,\underline{\tilde{\dot{H}}} \tag{3-20}$$

where

$$\underline{F} = (\underline{G}^T\underline{\bar{G}})^{-1}\underline{\bar{G}}^T \quad . \tag{3-21}$$

$\underline{G}$ is Helmert's transformation vector of dimension $u_2 \times 1$, where u_2 is the number of velocities,

$$\underline{G}^T = [1\ 1\ \ldots\ 1] \quad . \tag{3-22}$$

Thus, eq. (3-19) finally becomes

$$\underline{\tilde{\dot{H}}} = [\underline{I} + \underline{G}(\underline{G}^T\underline{\bar{G}})^{-1}\underline{\bar{G}}^T]\ \underline{\hat{\dot{H}}} \tag{3-23}$$

with

$$\underline{\bar{G}} = \underline{C}_{ss}^{-1}\ \underline{G} \quad . \tag{3-24}$$

When introducing only one point with a fixed velocity $\tilde{\dot{H}}_p$, eq. (3-23) simplifies to

$$\underline{\tilde{\dot{H}}} = \underline{\hat{\dot{H}}} + \underline{G}(\hat{\dot{H}}_p - \tilde{\dot{H}}_p) \tag{3-25}$$

which means that all the adjusted velocities $\underline{\hat{\dot{H}}}$ have to be changed by one constant. The new variance-covariance matrix $\underline{\tilde{C}}_{ss}$ of $\underline{\tilde{\dot{H}}}$ is given by Blaha (1971: 80).

$$\tilde{\underline{C}}_{ss} = \underline{C}_{ss} + \underline{G}\,\underline{F}\,\underline{C}_{ss} + \underline{C}_{ss}\,\underline{G}\,\underline{F} - GF\,C_{ss}\,GF. \tag{3-26}$$

Thus, the inner constraint applied on the height velocities $\dot{\underline{H}}$ is

$$\underline{G}^T \underline{C}_{\hat{s}\hat{s}}^{-1} \hat{\dot{\underline{H}}} = 0. \tag{3-27}$$

In conclusion, the mixed model has the advantage of computing velocities from the data themselves referring to the average datum defined by eq. (3-11). In a second step the vector of velocities and its covariance matrix can be transformed to any arbitrary datum by use of eqs. (3-23) and (3-26). If more than one velocity is fixed, the transformation (3-19) will yield the answer whether or not these assumptions are confirmed by the data.

(5) The most inconvenient part (and possibly the only drawback) of the mixed model is the definition of the velocity covariance matrix. $\underline{C}_{ss}$ can be computed using an analytical positive covariance function which is only dependent on the distance between data points, after postulating homogeneity, isotropy, and ergodicity for the sample of random quantities $\underline{s}_i = \dot{\underline{H}}_i$. For a deeper understanding of the subject the reader is referred to the theory of stochastic processes (e.g., Papoulis 1965).

An empirical covariance function can be found from preliminary adjustments of $\dot{\underline{H}}$ (or any other source where height velocities are determined) by

$$\mathrm{cov}(\tau_e) = \frac{1}{m} \Sigma\, (\dot{H}-\bar{\dot{H}})_{r_i}\, (\dot{H}-\bar{\dot{H}})_{r_i+\tau} \tag{3-28}$$

where r_i is the location of the points with known velocity $s_i = \dot{H}_i$, and m is the number of these points (or products) in zone e with distance interval τ_e. $\bar{\dot{H}}$ is a mean value of $\dot{H}$ in the considered area for balancing the velocities, $E(\dot{H}-\bar{\dot{H}}) = 0$. Such a trend elimination is necessary for generating quantities representing a (pseudo-) stochastic process.

In order to handle the step function (3-28) simply, it can be approximated by a (positive) analytical function, as, e.g., Hirvonen's function

$$C(r) = \frac{C_0}{1 + (r/\xi)^2} \tag{3-29}$$

where $C_0(r{=}0)$ is the variance of balanced height velocities, $r = [(x_i - x_j)^2 + (y_i - y_j)^2]^{0.5}$ is the distance between the data points and ξ the so-called correlation length ("Halbwertsbreite").

Other functions are also suited for the approximation of (3-28). Gauss' function is another example. If a least squares adjustment is used for determining C_0 and ξ in (3-29), then the noise in the data should cancel out,

$$C_{ss}(0) = C_0(0) + C_{nn}(0). \tag{3-30}$$

(See Mikhail 1976: 399.) Following Moritz (1980: 169) we can characterize each covariance function by means of three essential parameters:

- C_0 the variance of the covariance function, $C(r)$ for $r{=}0$,
- ξ the correlation length (argument for which the covariance function has the value $C(\xi) = C_0/2$),
- κ the curvature parameter $\kappa = k\xi^2/C_0$ (k is the curvature of the covariance function at $r{=}0$)

For Hirvonen's function (3-29) the corresponding quantity is

$$\kappa = 2 \ln 2. \tag{3-31}$$

Once the three parameters mentioned above are known, eq. (3-29) is uniquely defined, and the covariance matrix $\underline{C}_{ss}$ can be derived from eq. (3-29) by

$$\underline{C}_{ss} = \begin{bmatrix} C_0 & C(r_{12}) & C(r_{13}) & \dots \\ & C_0 & C(r_{23}) & \dots \\ (\text{sym.}) & & C_0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{3-32}$$

Assume that the covariance matrix of $\underline{n}$ in eq. (3-9) is known,

$$\underline{C}_{nn} = {}_1k_o^2\underline{Q}_{nn} \qquad (3\text{-}33)$$

where ${}_1k_o^2$ is the corresponding variance of unit weight, whereas the covariance matrix $\underline{C}_{ss}$ of the pseudo-observations $\underline{0}_s$ in eq. (3-9) is only known up to an unknown factor

$$\underline{C}_{0_s0_s} = {}_1k_o^2\underline{Q}_{ss} = e\underline{k}_{ss} \,. \qquad (3\text{-}34)$$

Then the final value e can be found by variance estimation for the group of pseudo-observations (Schwintzer 1984) using the estimation given by Förstner (1979) which is based on the relation

$$E(\hat{\underline{s}}^T\underline{Q}_{ss}^{-1}\hat{\underline{s}}) = \sigma_o^2 r_s \qquad (3\text{-}35)$$

where

$$r_s = u_2 - \mathrm{tr}(\underline{Q}_{ss}^{-1}\underline{Q}_{\hat{s}\hat{s}}) \qquad (3\text{-}36)$$

u_2 is the number of signals. The symbol "tr" stands for trace.

From a first adjustment using an approximate value e_o for e in eq. (3-34), we get the a posteriori estimator for $\hat{e}_o$ using only $\hat{s}$ by

$$k_o^2(\hat{s})_1 = \left[\frac{\hat{\underline{s}}^T\underline{Q}_{ss}^{-1}\hat{\underline{s}}}{r_s} \right]_o . \qquad (3\text{-}37)$$

In the next iteration we use an improved value for e,

$$e_1 = \frac{k_o^2(\hat{s})_1}{{}_1k_o^2}\, e_o \qquad (3\text{-}38)$$

so that finally the whole iteration process can be described by the following two recursive formulas:

$$k_0^2(\hat{s})_{\nu+1} = \left[\frac{\hat{\underline{s}}^T \underline{Q}_{ss}^{-1} \hat{\underline{s}}}{{}_1k_0^2} \right]_\nu \qquad \nu = \{1,2,\ldots\} \tag{3-39}$$

and

$$e_{\nu+1} = \frac{k_0^2(\hat{s})_{\nu+1}}{{}_1k_0^2}\, e_\nu . \tag{3-40}$$

The iteration will be finished if

$$| k_0^2(\hat{s}) - {}_1k_0^2 | < \varepsilon \tag{3-41}$$

where ε is a given tolerance. Test computations indicate that three to five iterations are sufficient for the desired convergence.

(6) These additional remarks refer to the assumption, (3-5), $E(s) = E(\dot{\underline{H}}) = 0$ in the mixed model: Let us consider as an illustration the leveling network in figure 1 which was observed twice, once at time t_1 and again at t_2. The observed height differences are

$$\Delta h_{ij,t_1} = \Delta h_{ij,t_2} \qquad ij = \{12,13,24,34\}$$

$$\Delta h_{01,t_2} = \Delta h_{01,t_1} + 2\text{ mm}$$

and the assumed $\sigma_{\Delta h}$ is considered to be of the same order as the change in Δh_{01} between observation epochs t_1 and t_2. The linear single point velocity based on the minimum condition (2-4) in section 2.1 yields the following estimates for $\dot{H}_i$, $i = \{1,\ldots,4\}$, with $\dot{H}_0 = 0$ and $(t_2-t_1) = 1$ yr,

$$\dot{H}_i = 2\text{ mm/yr},$$

whereas the mixed model where no velocity has to be fixed determines

$$\dot{H}_i = 0 \text{ mm/yr}, \qquad i = \{0,1,\ldots,4\}$$

and interprets the difference in Δh_{01} between the different observation epochs as measurement error. Isn't that answer based only on the data and the relative weighting of signal and noise at least as good as the other one mentioned above?

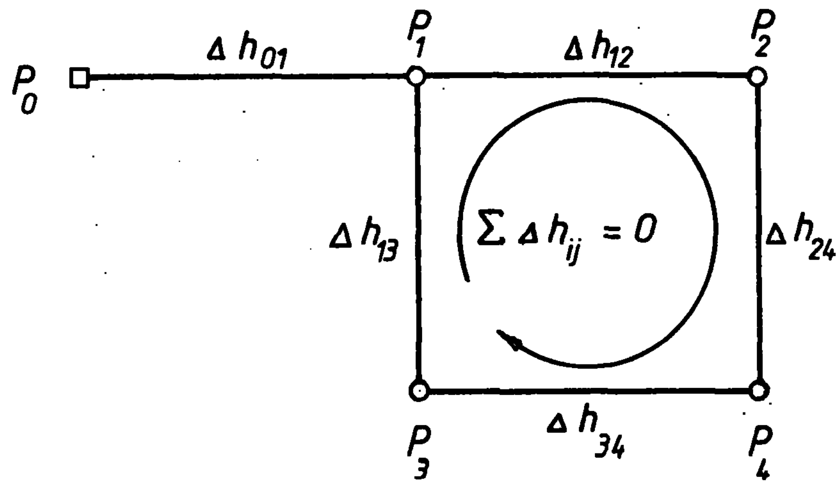


Figure 1.--Leveling network observed at t_1, t_2.

It is obvious that the error estimates in the model above will give some indications of the uncertainty of the computed parameters. However, in no case is the situation as clear as here, and the least squares adjustment tends to produce smearing effects. On the other hand, where do we have sufficient information on the absolute velocity value of a point? Therefore, in conclusion, the assumption $E(s) = E(\dot{H}) = 0$ is a useful working hypothesis which leads us to a data-based reference frame in recent crustal movement research.

(7) If geophysical information about the cause of the uplift or subsidence of an area is available, it can be used to describe the deterministic part $\underline{A}_1\underline{X}$ of the mixed model (3-2). Any kind of linear regression coefficients (See, e.g., Fahlbusch et al. 1980; Koch 1983) can be solved for simultaneously in the mixed model. For example, it is well known from hydrology in the Houston-Galveston area where subsidence due to groundwater withdrawal is observed, that the thickness of the aquifer, changes in the groundwater table, and the coefficients of compressibility of the soil can be used to a certain extent to describe the surface movements.

Therefore, observation eq. (3-1) can be extended by replacing H_i, H_j, e.g., by (here outlined only for H_j)

$$H_{j,t_0} \rightarrow H_{j,t_0} + a_j\delta_1 + a_j^2\delta_2 + b_j\delta_3 + b_j^2\delta_4 + c_j\delta_5 \tag{3-42}$$

where δ_i, $i = \{1,\ldots,5\}$ are unknown regression coefficients to be determined in $\underline{X}$ together with the vector of zero-epoch heights, and a_j could be the thickness of the aquifer, b_j, the groundwater table change, and c_j, the coefficient of compressibility at P_j.

Applying such additional terms in eq. (3-1) has the nice effect that $E(\underline{s}+\underline{n}) = 0$ $E(\underline{s}) = \underline{0}$ in eq. (3-2).

(8) Sequential analysis of vertical velocities is possible by using the stepwise collocation algorithm (Moritz 1980: 144). Special attention has to be given the datum problem when adding new observation points at each step (See also Papo and Perelmuter 1984). Formulas (3-19) to (3-26) have to be considered to refer the unknowns to the new datum.

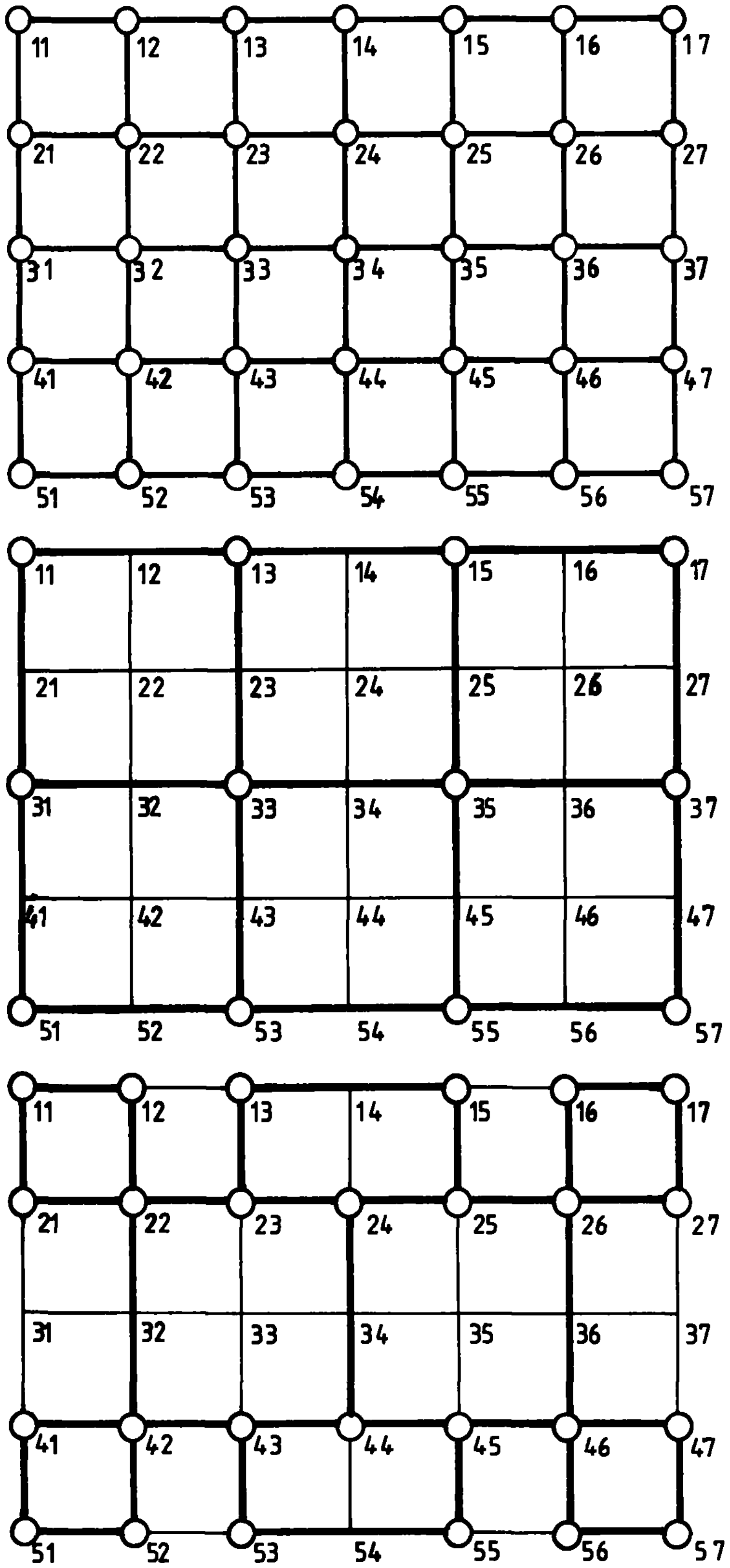
(9) The mixed model takes advantage of the signal-to-noise ratio and avoids misinterpretations. The more the noise is increased, the more the velocity surface is smoothed.

4. TEST COMPUTATIONS

To assess the two adjustment models discussed in chapters 2 and 3, two FORTRAN programs were written and several test computations were carried out. The following sections describe the generation of the test data, the different test runs, and their results.

4.1 Simulation of Test Data

For the test computations a simulated leveling net consisting of a regular grid with 35 junction points (bench marks) was chosen (fig. 1). The distance between two bench marks is considered to be 1 km. The network was "observed" three times,



Figures 1 to 3.--Design of leveling network observed at first ($t_1 = 1981.5$), second ($t_2 = 1982.5$), and third epoch ($t_3 = 1984.5$). The distance between two neighboring bench marks is considered to be 1 km.

$$t_1 = 1981.5$$
$$t_2 = 1982.5$$
$$t_3 = 1984.5$$

each time considering a different observation scheme (see figs. 1-3). At the first epoch the whole network was observed, carrying out $p_1 = 58$ observations, the second time only six loops were releveled with $p_2 = 17$ observations, and the third time $p_3 = 33$ leveling lines were surveyed. The three observation designs were selected in such a way that bench marks no. 14, 32, 34, 36, and 54 were only visited at $t_1 = 1981.5$. Consequently, the generated observation data contain no velocity information at those points and interpolation has to be applied in the models. This can correspond to practice when bench marks are destroyed, for example, and presents a good opportunity to compare the interpolation properties of the methods described above.

The isolines of assumed uplift and subsidence in the area of test example 1 are shown in figure 4. The corresponding values at bench mark locations are given in figure 5. The difference between the two velocity surfaces provides some idea of the extent to which the assumed original surface was recovered by the discrete bench mark locations. It is one of the basic problems in geodesy that the continuum always has to be represented by a discrete number of points without knowing the positions which are suited to get the best approximation.

Besides a simulation program which can automatically generate the error-free observation set from the matrix of "true" height velocities, a subroutine could also be called for the computation of normally distributed random numbers.

All heights referring to $t_1 = 1981.5$ were taken to be zero, so that the first observation set consists of zero's if an error-free set is desired, or otherwise just of the observational noise.

Whereas the velocity surface of test example 1 is more or less smooth having positive values in the upper left corner and negative values in the lower right one, test example 2 (fig. 6) represents an irregular surface with numbers of fast changing sign.

The covariance function and its characteristics were computed using (3-28) and a least squares adjustment for the approximation of the empirical function by

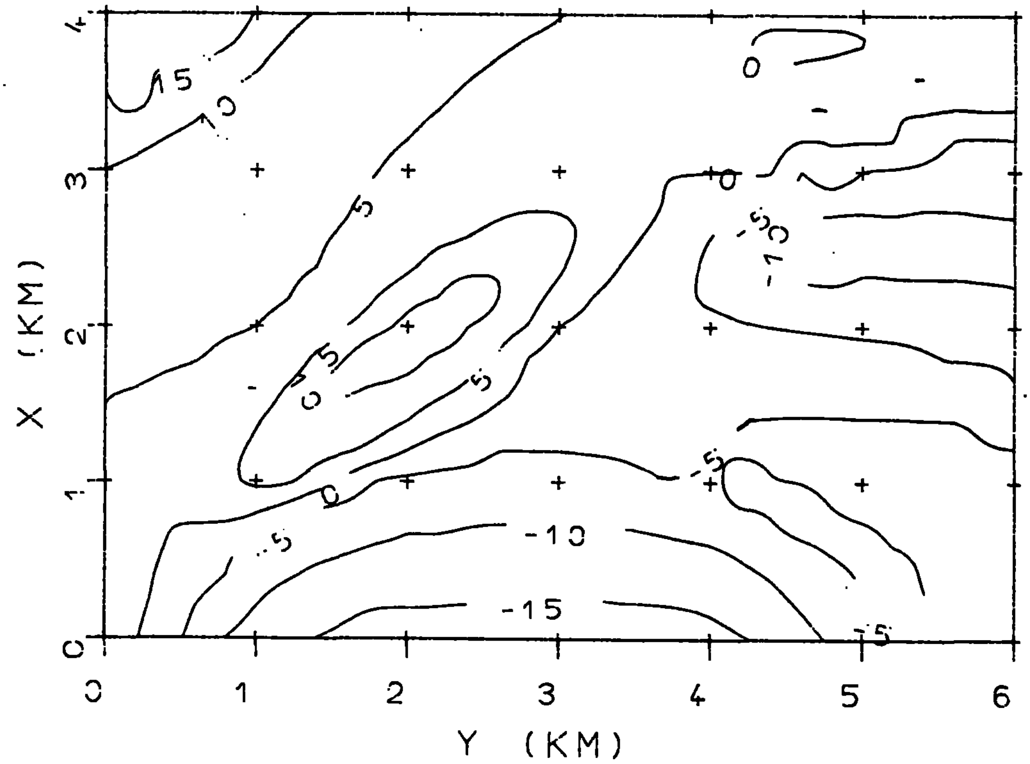


Figure 4.--Simulated isolines of uplift and subsidence (test example 1) in the area covered by the leveling network. Unit of isolines in mm/yr.

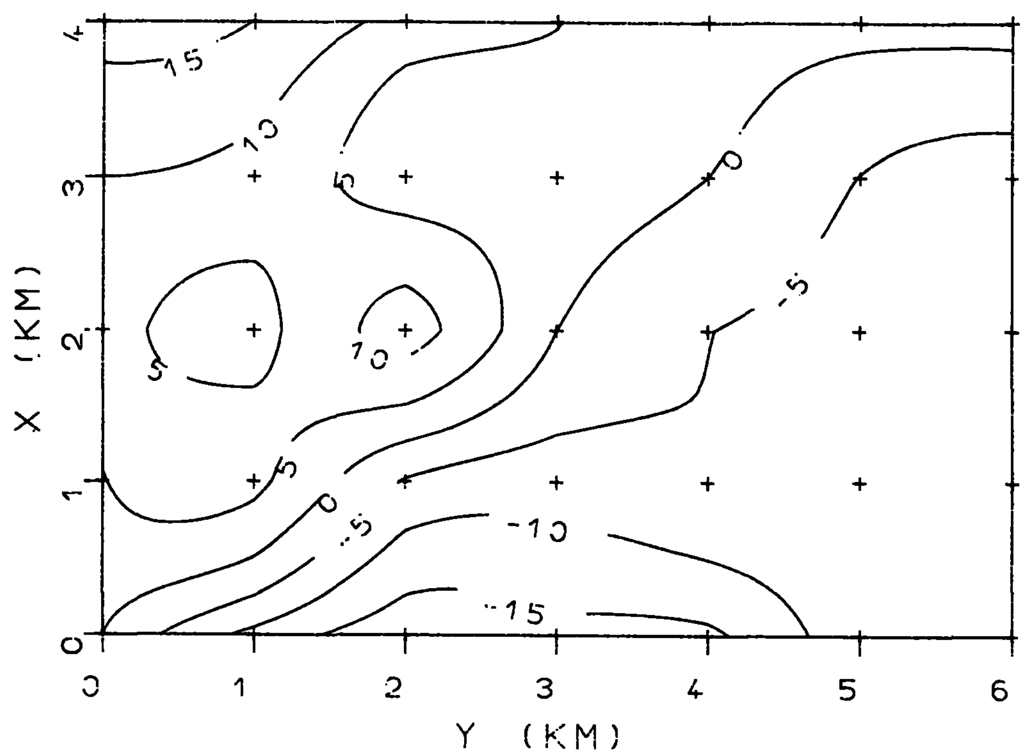


Figure 5.--Height velocities recovered by the location of the bench marks. The differences between the two velocity surfaces represent the loss of information due to the discretization of the problem. Unit of isolines in mm/yr. The height differences used in the adjustments are obtained by digitization at bench mark locations and, therefore, reflect the situation in figure 5.

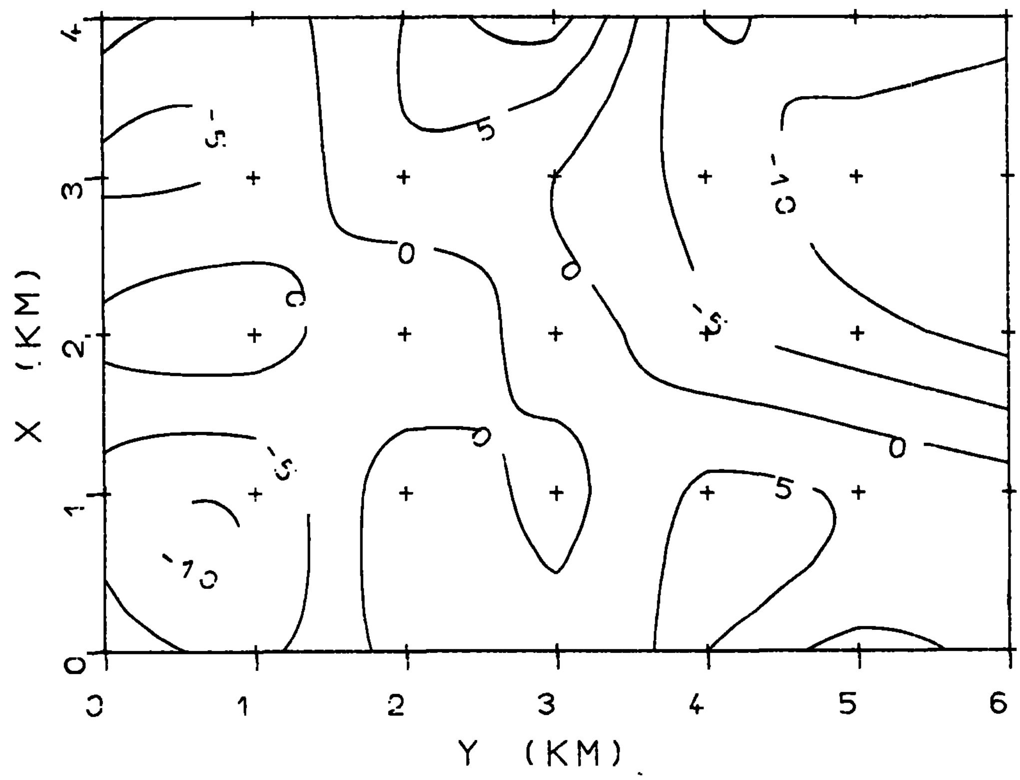


Figure 6.--Height velocity surface of test example 2.
Unit of isolines in mm/yr.

Table 1.--Characteristics of covariance functions of centered height velocities of the two test examples

	Test example 1	Test example 2
Mean value $\dot{H}$ of height velocities	- 0.76 mm/yr	- 1.73 mm/yr
Variance C_0 of centered height velocities $(\underline{\dot{H}}-\bar{\dot{H}})$	74.97 $(mm/yr)^2$	36.76 $(mm/yr)^2$
Correlation length ξ (3-30)	0.9028 km	0.4809 km
Curvature parameter κ (3-31)	2.000	2.000

Hirvonen's function, eq. (3-29). The results are summarized in table 1.

For all test computations the reference epoch t_0 = 1981.5 was chosen.

4.2 Results of Test Runs

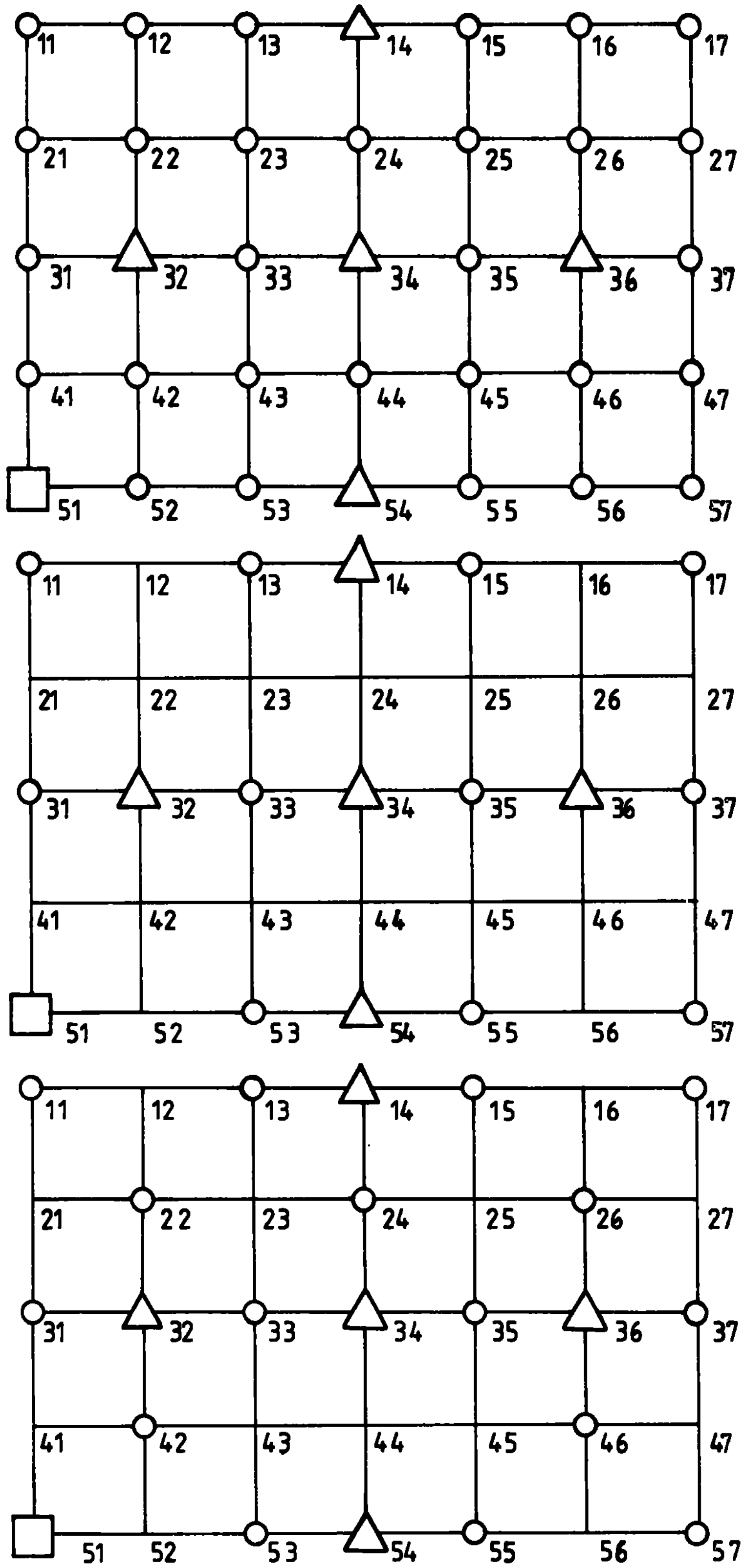
First, some test runs were carried out assuming "error-free data" and the linear single point velocity model using the multiquadric surface representation (2-12), in the following discussion abbreviated by MQUA. The use of "true data" was mainly done to understand and evaluate the behavior of the model when varying the location and the number of nodal points. Three different arrangements of the nodal points were chosen. In the first one the location of the points (nod = 34) coincides with all of the bench marks with velocity information (fig. 7). The height H_{51,t_0} at reference epoch t_0 and the velocity $\dot{H}_{51,t_0}$ were fixed. Recall that in order to avoid a singularity of the normal equation matrix, eq. (2-15) must hold. Figures 8 and 9 show two other configurations with reduced numbers of nodal points (nod = 11 and nod = 17, resp.). There was no evidence to put the nodal points at a certain depth, since to date no convincing geophysical explanation of the nodal points is available. Another common practice of putting the nodal points on the surface peaks and valleys in order to get the best possible approximation was not used herein since in those computations one never knows the exact answer beforehand (velocity surface). Incidentally, notice that the velocity surface used in test example 1 is approximately a tilted plane; hence no a priori indication for the location of nodal points is obvious.

For the so-called smoothing constant D in multiquadrics (2-17) Hardy (1978) suggests (with respect to a plane) the value

$$D = 0.665\ l^2 \tag{4-1}$$

where l is the rectangular nodal grid spacing, or equivalently, the mean l for irregularity spaced points. Holdahl (1984, private communication) uses in his NGS studies

$$D = \left(\frac{V*W}{n}\right) * 0.4283, \tag{4-2}$$



Figures 7 to 9.--Location of nodal points (O) used for multiquadrics. Bench mark no. 51 (□) serves as fixed station with height velocity $\dot{H} = 0$. At bench marks marked with Δ no velocity information is inherent in the data.

where V and W are the side lengths of the rectangular area under study and n is the number of nodal points.

The following error statistics were used in the comparison of the results:

(1) Mean of differences between true height H_{i,t_0} at epoch t_0 and the computed height $\hat{H}_{i,t_0}$ at t_0

$$m(H_{t_0}) = \frac{1}{n} \sum_{i=1}^{n} (H_{i,t_0} - \hat{H}_{i,t_0}) \tag{4-3}$$

(n is the number of samples in eqs. (4-5) to (4-10).

(2) Mean of differences between true height velocity $\dot{H}$ and computed height velocity $\hat{\dot{H}}$

$$m(\dot{H}) = \frac{1}{n} \sum_{i=1}^{n} (\dot{H}_i - \hat{\dot{H}}_i) . \tag{4-4}$$

(3) RMS of true errors in zero-epoch height determination

$$\varepsilon(H_{t_0}) = (\sum_{i=1}^{n} (H_{i,t_0} - H_{i,t_0})^2/n)^{0.5} . \tag{4-5}$$

(4) RMS of true errors in height velocity estimation

$$\varepsilon(\dot{H}) = (\sum_{i=1}^{n} (\dot{H}_i - \hat{\dot{H}}_i)^2/n)^{0.5} . \tag{4-6}$$

(5) Mean estimated standard deviation of zero-epoch heights

$$\bar{\sigma}(H_{t_0}) = \frac{1}{n} \sum_{i=1}^{n} \hat{\sigma}_{H_{i,t_0}} . \tag{4-7}$$

(6) Mean estimated standard deviation of height velocities

$$\bar{\sigma}(\dot{H}) = \frac{1}{n} \sum_{i=1}^{n} \hat{\sigma}_{\dot{H}_i} . \tag{4-8}$$

Table 2.--Results using error-free data of test example 1 for MQUA with varying number of nodal points (nod) and varying smoothing constant D (interpolation points 14, 32, 34, 36, 54 not included)

MODEL: MQUA			$m(H_{t_0})$	$m(\dot{H})$	$\varepsilon(H_{t_0})$	$\varepsilon(\dot{H})$	$\bar{\sigma}(H_{t_0})$	$\bar{\sigma}(\dot{H})$
nod	D	see Fig.	mm	mm/yr	mm	mm/yr	mm	mm/yr
29	0	7	0.00	0.00	0.00	0.00	0.00	0.00
29	0.43	7	0.00	0.00	0.00	0.00	0.00	0.00
17	0	8	2.08	2.51	2.87	3.79	4.34	2.44
17	0.61	8	2.40	3.03	2.95	4.06	3.93	2.25
11	0	9	0.05	0.42	3.56	3.06	6.17	3.36
11	0.86	9	-0.69	-0.54	3.71	3.14	6.32	3.49

Table 3.--Results using noisy data of test example 1 (standard deviation of an observed height difference of 1 km $\sigma_{ij} = \pm$ 2.0 mm, mean = 0.0) for MQUA with varying number of nodal points (nod) and varying smoothing constant D (interpolation points 14, 32, 34, 36, 54 not included)

MODEL: MQUA			$m(H_{t_0})$	$m(\dot{H})$	$\varepsilon(H_{t_0})$	$\varepsilon(\dot{H})$	$\bar{\sigma}(H_{t_0})$	$\bar{\sigma}(H)$
nod	D	see Fig.	mm	mm/yr	mm	mm/yr	mm	mm/yr
29	0	7	2.07	0.96	2.68	1.73	2.14	1.44
29	0.43	7	2.07	0.96	2.68	1.73	2.14	1.44
17	0	8	4.53	4.19	5.03	5.34	4.74	2.59
17	0.61	8	4.71	4.68	5.16	5.66	4.38	2.45
11	0	9	3.06	1.57	4.60	3.82	6.58	3.44
11	0.86	9	2.06	0.27	4.07	3.50	6.63	3.54

Table 4.--Results at interpolation points using error-free data of test example 1 for MQUA with varying number of nodal points (nod) and varying smoothing constant D

MODEL: MQUA			$m(H_{t_0})$	$m(\dot{H})$	$\varepsilon(H_{t_0})$	$\varepsilon(\dot{H})$	$\bar{\sigma}(H_{t_0})$	$\bar{\sigma}(\dot{H})$
nod	D	see Fig.	mm	mm/yr	mm	mm/yr	mm	mm/yr
29	0	7	0.00	-0.02	0.00	0.13	0.00	0.00
29	0.43	7	0.00	-0.02	0.00	0.59	0.00	0.00
17	0	8	2.20	2.47	2.35	2.72	4.36	2.20
17	0.61	8	2.58	3.40	2.24	3.64	3.95	2.12
11	0	9	0.06	0.10	1.00	0.99	6.28	3.12
11	0.86	9	-0.68	-0.65	1.42	1.02	6.42	3.34

Table 5.--Results of interpolation points using noisy data (standard deviation of an observed height difference of 1 km σ_{ij} = ± 2.0 mm, mean = 0.0) of test example 1 for MQUA with varying number of nodal points and varying smoothing factor D

MODEL: MQUA			$m(H_{t_0})$	$m(\dot{H})$	$\varepsilon(H_{t_0})$	$\varepsilon(\dot{H})$	$\bar{\sigma}(H_{t_0})$	$\bar{\sigma}(\dot{H})$
nod	D	see Fig.	mm	mm/yr	mm	mm/yr	mm	mm/yr
29	0	7	1.71	1.20	2.27	1.55	2.11	1.31
29	0.43	7	1.71	0.18	2.27	3.47	2.11	1.43
17	0	8	4.20	4.50	4.41	4.64	4.71	2.32
17	0.61	8	4.45	5.47	4.66	5.51	4.35	2.29
11	0	9	2.70	1.56	3.06	1.85	6.61	3.17
11	0.86	9	1.69	0.49	2.35	0.82	6.65	3.37

Table 6.--Comparison of results of the single point velocity model using multiquadrics (MQUA) with nod = 29 nodal points and smoothing constant D = 0 and the mixed model approach (MIX). Numbers in parentheses refer to the results at interpolation points.

(* ... different noise generator
σ_{ij}.. standard deviation of an observed height difference Δh_{ij} of 1 km)

Model	DATA (e=error-free n=noisy)	Example No.	$m(H_{t_0})$ mm	$m(\dot{H})$ mm/yr	$\varepsilon(H_{t_0})$ mm	$\varepsilon(\dot{H})$ mm/yr	$\bar{\sigma}(H_{t_0})$ mm	$\bar{\sigma}(\dot{H})$ mm/yr
MQUA	e	1	0.00 (0.00)	0.00 (-0.02)	0.00 (0.00)	0.00 (0.13)	0.00 (0.00)	0.00 0.00
MIX	e	1	0.02 (0.02)	0.01 (-0.59)	0.03 (0.02)	0.01 (1.63)	0.11 (0.10)	3.34 (7.10)
MQUA	n ; σ_{ij}=2 mm	1	2.09 (1.71)	0.96 (1.20)	2.68 (2.27)	1.73 (1.55)	2.14 (2.11)	1.44 (1.31)
MIX	n ; σ_{ij}=2 mm	1	2.00 (1.63)	0.88 (0.74)	2.58 (2.24)	1.65 (2.57)	2.29 (2.26)	3.41 (7.17)
MQUA	n ; σ_{ij}=8 mm	1	-11.11 (-11.77)	-2.11 (-1.03)	13.57 (13.51)	5.43 (4.54)	9.49 (9.36)	6.41 (5.85)
MIX	n ; σ_{ij}=8 mm	1	-11.37 (-12.42)	-2.31 (-2.20)	13.42 (13.82)	4.42 (4.40)	8.70 (8.65)	4.81 (7.50)
MQUA	n*; σ_{ij}=2 mm	1	0.78 (-0.53)	2.97 (2.47)	1.97 (1.43)	3.29 (2.63)	2.07 (2.04)	1.40 (1.28)
MIX	n*; σ_{ij}=2 mm	1	0.70 (-0.60)	2.81 (2.08)	1.86 (2.92)	3.15 (1.47)	2.29 (2.26)	3.47 (7.17)
MQUA	n ; σ_{ij}=1.5mm	2	-2.08 (-2.21)	0.19 (4.28)	2.54 (2.53)	1.02 (6.82)	1.78 (1.75)	1.21 (1.10)
MIX	n ; σ_{ij}=1.5mm	2	-2.13 (-2.26)	-0.43 (4.43)	2.64 (2.60)	1.05 (7.31)	1.71 (1.69)	1.85 (5.84)

Tables 2 to 5 summarize the results of computations with the single velocity model in connection with a multiquadric representation of the velocity surface, eq. (2-11), abbreviated by MQUA, as mentioned above. Error-free and noisy data were simulated assuming varying numbers of nodal points as well as different smoothing constants D.

Table 6 compares the results of MQUA and the mixed model (MIX) described in chapter 3.

4.3 Discussion of the Results

From tables 2 to 5 one can conclude that MQUA gives good results only if the model considers one nodal point at every bench mark with known velocity information. Any reduction in the number of nodal points leads to a considerable loss in accuracy in determining the zero-epoch heights and the height velocities. However, the loss in accuracy is not a function of the number of nodal points (see tables 2 to 5). Therefore, there might be an optimal configuration of these points smaller in the number than the discrete velocity unknowns in the model which could represent the velocity surface with the desired accuracy. However, this is not known in advance by the user. In addition, irregular location of the nodal points requires some solvability analysis of the model beforehand (Holdahl and Hardy 1979). The test computations with test example 1 (fig. 5) where the velocity surface is approximately a tilted plane have clearly shown that in such cases any attempt to establish the location of nodal points correlated with surface peaks and valleys fails. Due to the absence of any theoretical or heuristic directive to find the optimal location of the nodal points, each velocity unknown in the model should be replaced by a nodal point. In that case, no substantial saving of computer time is achieved in solving for both the nodal points and the discrete unknowns. Hence, the approximation and the interpolation of the velocity surface can be separated from the adjustment itself and done in a second step.

The use of previously recommended smoothing constants (see eqs. (4-1) and (4-2)) introduced no improvement in the final results. Since just the opposite happened, it seems to be preferable to avoid the smoothing constant completely, setting $D = 0$.

Tables 4 and 5 corroborate the good interpolation properties of multiquadrics and confirm early results presented by Hein and Lenze (1979) where nodal points were situated at each station. However, although MQUA may be an excellent deterministic interpolation method, the meaning of the computed error estimates is not always reliable as will be discussed later. They should be interpreted with extreme caution. The pure interpolation of the velocity surface carried out by multiquadrics is better than those of MIX at stations where the surface presents a nearly linear behavior.

From the results obtained by using noisy data we can deduce that the estimated zero-epoch heights and velocities are disturbed by the same order of magnitude as the standard deviation of the noise.

The results of the mixed model (MIX) are only slightly better (closer to the true values) than those of MQUA. (See table 6.) The great advantage of MIX, however, is that its error estimates are more reliable. In figure 10 a typical example of discrepancies between true and computed values is shown (results of last two lines in table 6). The stations are marked where the computed standard deviations are three times larger than the estimated velocities. Whereas the 3σ-error statistics of MQUA indicate that all estimated velocities are reliable, those of MIX consider eight values as bad. A comparison with the (printed) true errors at the corresponding locations shows that six out of eight reflect the true picture. Based on that type of analysis of several test computations one can state that the error estimates from MQUA are not reliable. This seems to be a serious drawback, since in practical applications the results have to be assessed through the computed error estimates.

Further research regarding the recovery of the autocovariance function of the height velocities should be done; whereas the variance can be estimated in the mixed model, the correlation length must be found by other means. However, any source providing information about the changes in height in the considered area can be used. There is hope that geophysical information about the type of motion can define the correlation length properly.

5. CONCLUSIONS

The linear single point velocity model used in connection with multiquadrics for the velocity surface representation provides good results only if:

TRUE VELOCITY ERRORS - MODEL MQUA

-0.71	$\boxed{-1.04}$	-1.60	$\boxed{12.38}$	-1.21	0.31	-0.78
-0.76	-0.60	-1.15	$\boxed{-0.88}$	-1.00	-0.40	-1.07
$\boxed{1.99}$	7.26	2.65	4.37	0.37	-2.74	-2.02
-0.51	-0.52	-0.09	0.35	0.14	-0.62	0.02
0.00	-1.07	-0.49	0.15	-0.29	-0.78	0.27

TRUE VELOCITY ERRORS - MODEL MIX

-0.64	$\boxed{-1.00}$	-1.55	$\boxed{13.79}$	-1.33	0.10	-1.03
-0.74	-0.58	-1.11	-0.89	-1.11	-0.60	-1.35
$\boxed{2.05}$	$\boxed{5.52}$	2.29	$\boxed{4.01}$	0.02	$\boxed{-4.45}$	-2.68
-0.54	-0.54	-0.02	$\boxed{0.41}$	0.25	-0.55	0.11
0.00	-1.09	-0.37	$\boxed{3.27}$	0.14	$\boxed{-0.69}$	0.40

Figure 10.--Typical example (see table 6, last two lines) of discrepancies between true velocities and computed ones (upper matrix: MQUA, lower matrix: MIX). Marked boxes refer to stations where the standard deviation is three times larger than the computed velocities.

The matrices are printed in the same arrangement as the leveling net (see fig. 1).

(i) if nodal points are considered at every bench mark with velocity information, and

(ii) the smoothing constant D is set to zero.

There is no need, in principle, to perform the adjustment and the velocity surface interpolation in one step, since no saving in computer time can be achieved (the number of nodal points is equal to the number of discrete velocity unknowns). Multiquadrics is mainly a deterministic interpolation method without any stochastic model. A serious drawback of the linear single point velocity model (MQUA) is the fact that the error statistics seem to be unreliable, and no assessment of the quality of the determined heights at epoch zero and the velocity unknowns can be made using the adjustment results. The model further requires the fixing of one velocity and one height as datum. Consequently, the derived parameters are dependent on this absolute constraint, a fact that should always be kept in mind.

The generalized linear regression or mixed model takes the datum of the velocities from the data via the hybrid minimum norm. This is more pleasing to the type of data considered. Since leveling observations are relative by their very nature, only relative heights and velocities can be estimated. The results of the model MIX are only slightly better than those of MQUA. However, the error statistics are much more useful for assessing the estimated unknowns properly. Since a stochastic model is involved, misinterpretations can be avoided and the signal-to-noise ratio is properly considered.

Thus, multiquadrics has its place where a pure deterministic approach is desired. When measurements with unavoidable noise are present one should not mix such a deterministic method with stochastic considerations within a least squares adjustment model.

In particular, if the signal-to-noise ratio approaches the value one, misinterpretations of the results of model (2-1) with multiquadric representation of the velocity surface are possible. (See the simple example in chapter 3.2, paragraph (6).)

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APPENDIX.--SOME SIMPLE NUMERICAL INVESTIGATIONS ON THE DETERMINISTIC TREND DETERMINATION OF VERTICAL MOVEMENTS DUE TO WATER WTHDRAWAL. EXAMPLE: HOUSTON-GALVESTON AREA.

In section 3.2, paragraph (7), it was suggested that geological and geophysical information on the cause of height changes can be used to describe the deterministic part $\underline{A}_1\underline{X}$ of the mixed model (3-2). Some simple computations should demonstrate this in the Houston-Galveston area (fig. A 1) where mainly withdrawal of water has caused a pattern of subsidence. A specially designed geodetic leveling network monitors the large subsidence measured since 1906.

The most recent surveys were carried out in 1978 (Balazs 1980) and 1983 (Zilkoski 1984). The groundwater withdrawal is monitored by the U.S. Geological Survey in cooperation with the Texas Department of Water Resources and the Harris-Galveston Coastal Subsidence District (see, e.g., Gabrysch 1982; Strause and Ranzau 1983). It has resulted in water-level declines of as much as 76 m in the Chicot aquifer and as much as 91 m in wells completed in the Evangeline aquifer. The center of regional subsidence is the Pasadena area, where more than 3.0 m of subsidence have ocurred since 1906.

Due to limited data available in digital form, a simple regression was chosen to describe the height changes,

$$H_{i,t_2} - H_{i,t_1} = \delta_0 + a_{1,i}\delta_1 + a_{2,i}\delta_2 + b_{1,i}\delta_3 + b_{2,i}\delta_4 \qquad \text{(A-1)}$$

where

$H_{i,t_2} - H_{i,t_1}$ is the difference in height of bench mark P_i in the time interval $t_2 - t_1$,

$\delta_0,\ldots,\delta_4$ are unknown regression coefficients,

$a_{1,i}$, $a_{2,i}$ are the water level changes in wells of the Evangeline and Chicot aquifer respectively, in the considered time interval at bench mark P_i ,

$b_{1,i}$, $b_{2,i}$ are the thicknesses of clay of the Evangeline and of the Chicot aquifer respectively, at bench mark P_i .

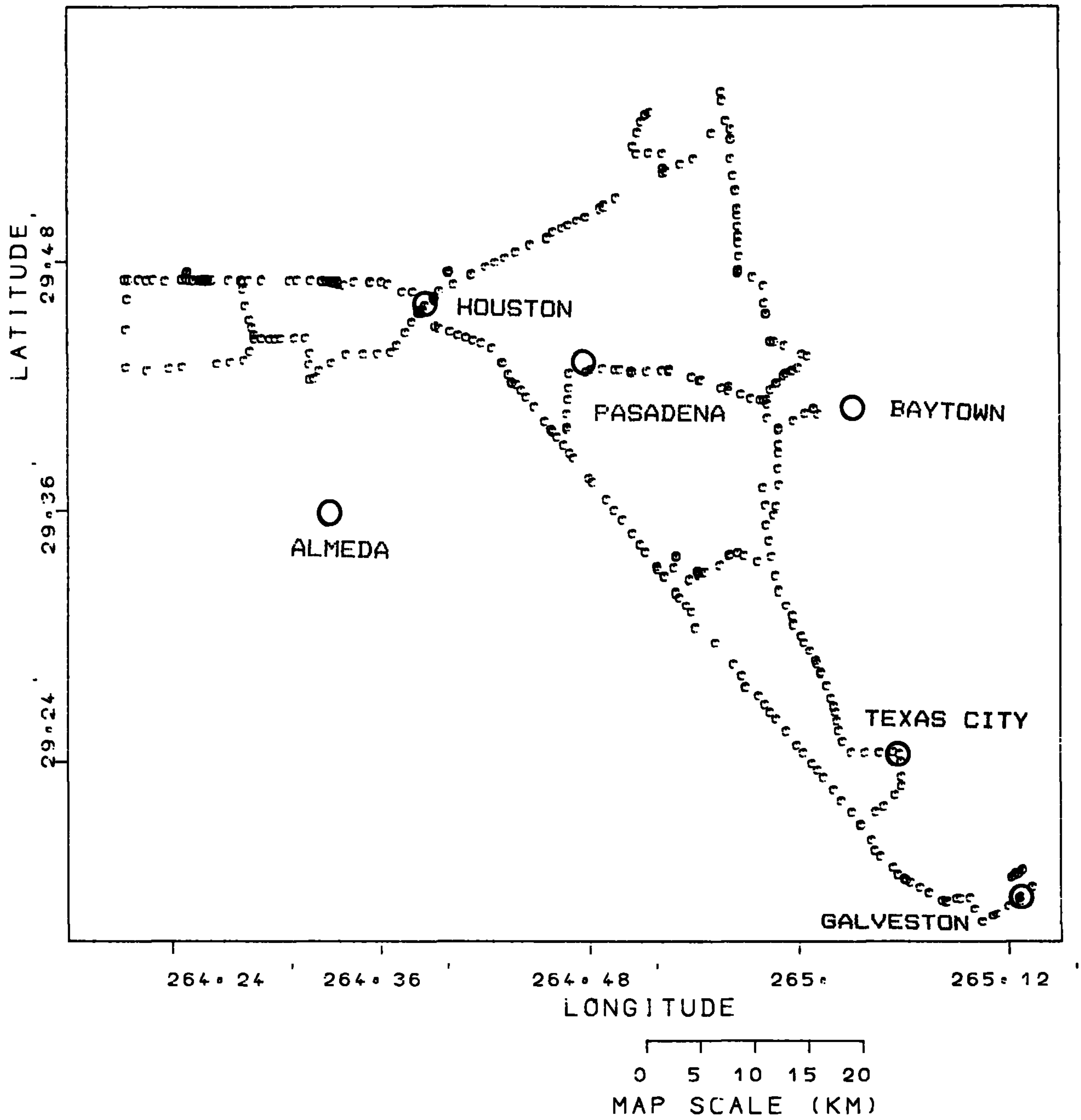

Figure A 1. Leveling stations (denoted by tiny circles) in the Houston-Galveston, Texas, Area.

Table A 1.--Results of the regression analysis in the Houston-Galveston area

```
STEP-WISE MULTIPLE REGRESSION.....  HOUSTON

NUMBER OF OBSERVATIONS  383
NUMBER OF VARIABLES       5
NUMBER OF SELECTIONS      3

CONSTANT TO LIMIT VARIABLES  0.00000

VARIABLE      MEAN       STANDARD
   NO.                   DEVIATION
    1     -98.72298     86.51490  HEIGHT DIFFERENCE IN MM 1978-1983
    2      31.43969     46.87114  WATER LEVEL CHANGE IN EVANG. AQUIFER
    3      26.42768     37.53087  WATER LEVEL CHANGE IN CHICOT AQUIFER
    4    1603.43473    433.80165  CLAY THICKNESS OF EVANGELINE AQUIFER
    5     366.30444     66.98117  CLAY THICKNESS OF CHICOT     AQUIFER

                                  (VARIABLES 2-5 IN FEET)


CORRELATION MATRIX

ROW  1
    1.00000     0.87798     0.79705     0.57644     0.14083

ROW  2
    0.87798     1.00000     0.84304     0.67573     0.15221

ROW  3
    0.79705     0.84304     1.00000     0.33213    -0.27418

ROW  4
    0.57644     0.67573     0.33213     1.00000     0.56863

ROW  5
    0.14083     0.15221    -0.27418     0.56863     1.00000


STEP  4

VARIABLE ENTERED..... 4

SUM OF SQUARES REDUCED IN THIS STEP....         32.101
PROPORTION REDUCED IN THIS STEP........          0.000

CUMULATIVE SUM OF SQUARES REDUCED......    2287137.379
CUMULATIVE PROPORTION REDUCED..........          0.800  OF  2859204.318

FOR  4 VARIABLES ENTERED
  MULTIPLE CORRELATION COEFFICIENT...     0.894
       (ADJUSTED FOR D.F.)...........     0.893
  F-VALUE FOR ANALYSIS OF VARIANCE...   377.813
  STANDARD ERROR OF ESTIMATE.........    38.903
       (ADJUSTED FOR D.F.)...........    39.056

  VARIABLE       REGRESSION      STD. ERROR OF      COMPUTED
   NUMBER        COEFFICIENT      REG. COEFF.        T-VALUE
     2             0.79745         0.13800            5.779
     3             1.12276         0.15507            7.240
     5             0.26499         0.04923            5.382
     4             0.00121         0.00832            0.146
  INTERCEPT     -252.47518
```

As mentioned above, the results of the last two relevelings (1978 and 1983) were used, thus $t_2 - t_1 = 5$ yr . The water level changes in the Evangeline and Chicot aquifers were digitized in a grid of approximately 4×4 km^2 from figure 3 and figure 1, respectively, in the report of Strause and Ranzau (1983). Thereby a difference of some months in the time interval with respect to that used for the releveling results had to be accepted. The thickness of clay in the Evangeline and Chicot aquifers were digitized from figures 36 and 37 of Gabrysch (1982), using a grid of approximately 6×6 km^2. The final values $a_{1,i}$, $a_{2,i}$ and $b_{1,i}$, $b_{2,i}$ at bench mark locations were computed by a simple weighted average procedure using the corresponding values at the grid points and weighting it with reciprocal distance. To assess the significance of introduced parameters in the model (A-1), a stepwise multiple regression procedure was chosen.

The results of the regression study can be summarized as follows. Variable no. 4, the thickness of clay $b_{1,i}$ in the Evangeline aquifer does not contribute to the regression at all. The determined coefficient δ_3 is insignificant. There is also only a small improvement in terms of the correlation coefficient from 0.884 to 0.894 when taking into account variable no. 5, the thickness of clay $b_{2,i}$ in the Chicot aquifer. Thus, accepting a loss of accuracy of 2 mm in the standard error of estimate, only the water level changes in the two aquifers have to be considered.

The results using all variables in the regression are outlined in table A 1.

Conclusions

Using a simple regression analysis it is possible to determine the subsidence by groundwater level changes in the Houston-Galveston area in the time interval 1978 to 1983 with a standard deviation of 3-4 cm, which corresponds to about 6-8 mm/yr. This is an interesting result in spite of more sophisticated models, like three-dimensional finite elements analysis, which are currently under research by geologists and geophysists in that area.

Considering the facts that:

(i) the data had to be digitized from small scale maps,

(ii) the water well locations were not available at the time of this study, and

(iii) the corresponding time intervals of considered height changes and geological parameters do not coincide exactly,

it can be concluded that using better data, the water well locations, and the regression trend model equipped with a stochastic signal part as suggested by the mixed model (3-2), it might be possible to predict the subsidence in such areas with an accuracy of ± 2 mm/yr.

The application of such a procedure allows for separate treatment of subsidence areas in the project to readjust the North American Vertical Datum (NAVD). Thus, the heights in those areas can be reduced to a common epoch for final inclusion in the NAVD using the above approach.

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