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Megatrend Solutions in Physical Geodesy

Arne Bjerhammar

Rockville, Maryland
February 1986

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MEGATREND SOLUTIONS IN PHYSICAL GEODESY

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ABSTRACT. A technique for renormalization of integral equations is used for obtaining very robust solutions. The number of multiplications used for inverting the integral equations can be reduced dramatically and mostly only weighted means will be needed. Theoretical proportional gain in computer time might be as much as 10^5 for the most favorable cases when using 1,000 unknowns. (Practical gains will be considerably less.) Solutions have been obtained with increased accuracy compared with the classical technique of integral equations. Surface elements could be of arbitrary size, but the method is optimal for a global approach with equal area elements. The solutions were found strictly invariant with respect to the depth of the embedded sphere when using simpler models.

INTRODUCTION

Classical geodesy has had to face a very difficult mathematical problem, namely the free boundary value problem. The most widely used technique was based on an application of resolvents for a strictly spherical boundary surface. The actual formulas requiring integration over the whole Earth were given by Stokes for the disturbing potential and by Vening Meinesz for vertical deflection.

Molodensky presented integral equations that could give the solutions for a nonspherical Earth. The existence as well as uniqueness of a solution of the free boundary value problem was analyzed rigorously by Hörmander (1976). This study assumed a continuous gravity field. The existence of a solution was proved

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This study was performed during a 6-month stay in 1984 when the author was a Senior Visiting Scientist at the National Geodetic Survey, under the auspices of the Committee on Geodesy, National Research Council, National Academy of Sciences, Washington, DC.

for rather smooth surfaces (Hölder class $H^{2+\epsilon}$). Hörmander also gave the foundations for constructing a solution.

Gravity data are available only at discrete points on the surface of the Earth and, therefore, we have an infinite number of singularities. Existence and uniqueness of a solution are ascertained by using six assumptions:

1. The solution satisfies all given data.
2. The solution generates all missing data.
3. The solution is harmonic down to a fully embedded sphere (or equivalent).
4. The number of unknowns equals the number of observations.
5. A radius vector intersects the surface only once.
6. Uniqueness for the overdetermined case is obtained in the least squares sense for a Gauss-Markov model (excluding assumptions 1 and 4).

This kind of solution is described in Bjerhammar (1962, 1964). The validity is obvious for the case with only a finite number of observations.

A somewhat academic question is the validity of the solution if the number of observations becomes infinite. The following are some mathematical theorems of interest:

1. Walsh (1929) proved uniform convergence for harmonic continuation down to an internal sphere. There were no restrictions on the external surface, but the applied compact set had to be connected and at a finite distance from the surface of the Earth.

2. Keldych and Lavrentieff (1937) extended the proof to the case when the compact set goes down to the surface of the Earth (unstable points excluded).

3. Deny (1949) proved that extension to an unconnected complement is valid (multibody problem), but harmonic polynomials cannot be used.

Goldberger (1962) made an extension of the Gauss-Markov model to the case where the residuals are used for additional "prediction." He called his technique "best linear unbiased prediction in generalized linear regression models."

Krarup (1969) and Moritz (1972) independently developed an equivalent theory. It was called "least squares collocation" by these authors, and applied mostly to the "discrete free boundary value." The theory of stochastic processes was now also incorporated.

Geodetic models of this type have often been designed without the trend function. This approach gives a minumum norm solution (not least squares solution) without any degrees of freedom if the covariance function is not estimable. Confidence intervals for estimated variances are then unlimited (infinite). The minimum norm solution has strongly influenced geodetic literature.

The classical deterministic methods of Stokes and Vening Meinesz benefited from a simple structure of the solution. No systems of equations were needed. However, there were some obvious limitations affecting these integral methods:

1. Vening Meinesz' formula had a singularity at zero.
2. No predictions were involved.
3. Nonspherical surfaces were excluded.

The more recent Molodensky approach required solutions of integral equations with a theoretically infinite number of unknowns. The existence of the solution was open to question.

The discrete approach by Bjerhammar, Krarup, and Moritz seemed attractive for local solutions where the number of unknowns can be kept small. However, a minimization of an L_2-norm on the internal sphere will be strictly meaningful only for a global approach. Unfortunately, the global approach leads to unacceptable mathematical complications, and very little has been done with least squares collocation in a global mode.

Present "megatrends" in geodesy (Bossler 1984) justify the application of global methods that can handle very large data sets without excessive use of computer time. Strictly discrete techniques are still needed.

The following study will display a robust technique where global solutions of increased accuracy can be obtained with very large savings in computer time.

The largest system to be solved simultaneously for n unknowns is:

Collocation and reflexive prediction: n eqs.; multiplications: $\simeq n^3/6$

Proposed technique: Mostly weighted means with only one unknown.

The computer time mostly increases with the third power of the number of unknowns. In reality much less savings are obtained because global systems will always contain many nondiagonal terms that are different from zero but still without any significance. It should finally be emphasized that the proposed technique is not directly suitable for local applications, but can be used after slight modifications.

1. RENORMALIZATIONS OF INTEGRAL EQUATIONS IN GEODESY

Modern geodesy benefits from using very large data sets. Computer solutions are required when dealing with several million observations. Furthermore, the solutions are often obtained from integral equations that formally postulate an infinite number of observations. These integral equations mostly have no zero elements and, therefore, the "band technique" is not directly applicable. It will be shown here how some typical geodetic problems can be solved with great simplification and improved accuracy. The mathematical procedure uses a renormalization of the integral equations; similar procedures have been applied mostly in quantum field theory (Ingraham 1980). We choose, as an example, the free boundary value problem for the linear case, but the application is not restricted to this kind of problem.

Let Δg^* be a gravity anomaly on a sphere with radius r_o. Furthermore, $r_o \Delta g^*$ is considered harmonic. Then the gravity anomaly can be computed for points outside the sphere according to Poisson's integral

$$\Delta g_j = (4\pi)^{-1} (s^2 - s^4) \iint_\Omega \Delta g^* d^{-3} d\Omega \tag{1.01}$$

where

$$d^2 = 1 + s^2 - 2s \cos \omega$$

$$s = r_o/r_j \text{ and } r_j > r_o .$$

The gravity anomaly at a fixed point P_j is denoted Δg_j and Ω is the unit sphere. The geocentric distance of this point is r_j and the geocentric angle between P_j and the moving point on Ω is ω. A formally more rigorous geodetic approach requires omission of the two first-order Legendre polynomials (Bjerhammar 1962, 1964).

Gravity anomalies are known only at discrete points and, therefore, eq. (1.01) cannot be applied directly. For a discrete application, we postulate an equal area approach and obtain the limiting value

$$\lim_{i \to \infty} [(4\pi)^{-1}(s^2-s^4)\iint_{\Omega} d^{-3}d\Omega] \; [\sum_i \Delta g_i^* d_{ji}^{-3}/\sum_i d_{ji}^{-3}] = \Delta g_j \tag{1.02}$$

where we easily obtain

$$(4\pi)^{-1}(s^2-s^4)\iint_{\Omega} d^{-3}d\Omega = s^2 .$$

The notation d_{ji} is used in the discrete mode instead of d, and Δg_i^* instead of Δg^* (Bjerhammar 1970, Svensson 1983).

Clearly, we have a predictor for Δg_j <u>in space and on the given sphere</u>

$$\Delta\hat{g}_j = s^2 \sum_i \Delta g_i^* d_{ji}^{-3}/\sum_i d_{ji}^{-3} . \tag{1.03}$$

Svensson (1983) found this yielded uniform convergence to the correct value, extraordinary economy, uniform prediction errors, and uniform estimates for the propagation of observation errors.

The discrete formula defines a prediction of all missing points on the sphere with the outcome space

$$\Delta g^*_{min} < \hat{\Delta g} < g^*_{max} \ . \tag{1.04}$$

For further details, see Bjerhammar (1970), Katsambalos (1980) and Sünkel (1980). The trivial case with s = 1 is simply the weighted mean which has been explored many times in geodesy and elsewhere when making predictions.

The study by Katsambalos (1980) indicated no significant gain in accuracy when using the more complicated "least squares collocation." A lack of error estimates was considered as an objection against this "inversion-free prediction." This is not a major issue because the prediction errors can be estimated simply by applying "autoprediction." See section 3 for further details concerning standard deviations.

Sünkel (1980) noted that prediction between the given points shows a tendency of "step effects." These comments refer to the problem of predicting on the sphere and are not applicable in the following application.

In this approach, we consider solving the integral equation (1.01) for the case where discrete gravity data are given on the nonspherical surface of the Earth. The gravity anomaly Δg^* on the internal sphere (fully embedded) will be an unknown quantity. Our solution will be obtained with the discrete renormalization defined by eq. (1.03).

Our system of integral equations (1.01) is now replaced by a system of linear equations

$$\Delta g = C \ \Delta g^* \tag{1.05}$$

where the elements of the matrix C are defined by (for n observations)

$$C_{ji} = s_j^2 d_{ji}^{-3} \Big/ \sum_{i=1}^{n} d_{ji}^{-3} \qquad \text{("robust base function")} \tag{1.06}$$

Here C_{ji} represents the elements of the base function. It is anticipated that the observations are given with "equal spacing" on the external surface with Δg^*-values on the corresponding verticals. For $C_{ji}/C_{ii} < 10^{-5}$ $(j\neq i)$, we are justified in using the solution (smallest possible depth of the internal sphere is preferred)

$$\Delta g_j^* = \Delta g_j s_j^{-2} . \tag{1.07}$$

For $C_{ji}/C_{ii} > 10^{-5}$ $(j\neq i)$, we use instead

$$\Delta g^* = D^{-1}\Delta g + D^{-1}(\Delta g - C\, D^{-1}\Delta g) \tag{1.07a}$$

where $D_{ji} = C_{jj}$ for $i = j$ and $D_{ji} = 0$ for $i \neq j$. The solution requires only an inversion of a diagonal matrix. We transform the last expression to obtain

$$\Delta g^* = (2I - D^{-1}C)D^{-1}\,\Delta g \quad . \tag{1.07b}$$

Even this solution is directly accessible. Further improvements are hardly justified in a global approach. The residuals V are then obtained

$$V = \Delta g - C\,\Delta g^* . \tag{1.08}$$

If $V^TV/n < \varepsilon^2$, where ε is the observation error, then no further improvements are needed. For convergence conditions see section 6.

Some filtering is included in this approach. If additional unused observations are available, then these can be used for a determination of variances. See section 3 for further details.

For a depth to the internal sphere of h and a grid distance of L we have

$$(C_{j,(j+1)}/C_{jj}) \cong h^3L^{-3} \quad (<10^{-8} \text{ for } h=1 \text{ km and } L=500 \text{ km}).$$

At a grid distance of 1° this ratio will be about 10^{-5}.

2. PREDICTIONS

Predictions of Δg on the surface of the Earth and in space are obtained from eq. (1.05). The geoidal height is computed from the discretized Stokes formula

$$N_q = \sum_i (1/\gamma_q) s\, r_o (1+2/d - 3d - 5s \cos \omega_{qi} - 3s \cos \omega_{qi}\, \ell n\, u)\, \Delta g_i^*/n$$

$$d^2 = 1 + s^2 - 2s \cos \omega_{qi} \qquad s = r_o/r_q \tag{2.01}$$

$$u = ((d+1)^2 - s^2)/4 \qquad n = \text{number of unknowns}$$

$$\gamma_q = \text{normal value of gravity at } q$$

Vertical deflections are obtained from the discretized Vening Meinesz' formula

$$\left\{ {\xi_q \atop \eta_q} \right\} = \left\{ \sum_i F_{qi}\, \Delta g_i^*/n \right\} \tag{2.02}$$

$$F_{qi} = \gamma_q^{-1} s^2 \sin \omega_{qi} (8 - 2d^{-3} - [3(d+1)^2/2du] + 3\ell n\, u) \left\{ {\cos \alpha_{qi} \atop \sin \alpha_{qi}} \right\} \tag{2.03}$$

where α_{qi} is the azimuth from q to i, ξ vertical deflection north, and η vertical deflection east. Predictions are made for a point q. We note that our earlier renormalization gives a solution for Δg^* which is not restricted to isolated "Dirac points" on the internal sphere. For a selected surface element ΔS, we obtain

$$r_o^{-2} (4\pi)^{-1} \iint_{\Delta S} \Delta g_i^*\, dS = \Delta g_i^*/n$$

when postulating an equal area approach and constant Δg_i^* inside the selected surface element. For arbitrary spacing, see section 5.

3. STANDARD DEVIATIONS

We assume that a subset of the observations has been excluded from the previous analysis. Then we can make use of our solution of Δg^* for a computation of predicted Δg-values in the subset of unused data.

Then we have

$$V = \Delta g - C\ \Delta g^* \tag{3.01}$$

with variance

$$\sigma^2 = V^T V/n \tag{3.02}$$

where n is the number of residuals. Knowing the variance of the observations, we can compute the standard deviation of any wanted quantity. For traditional least squares estimates, see section 4. Equation (3.02) gives an unbiased estimator of the variance.

4. LEAST SQUARES SOLUTION

First we consider a system without overdeterminations

$$C\ \Delta g^* = \Delta g\ . \tag{4.01}$$

If $n = m$ we obtain for C full rank

$$\Delta g^* = C^{-1}\Delta g. \tag{4.02}$$

The least squares solution is (for $n > m$)

$$\Delta g^* = (C^T PC)^{-1} C^T P\ \Delta g \tag{4.03}$$

$$\hat{\sigma}^2 = (\Delta g - C\Delta g^*)^T\ P(\Delta g - C\Delta g^*)/(n-m) \tag{4.04}$$

where P is the weight matrix and $\hat{\sigma}^2$ is the estimated variance.

This kind of solution can be used in combination with MINQUE (Minimum Norm Quadratic Unbiased Estimation) procedures for a computation of unbiased estimates of the weight matrix. Formally mixed data can then be considered. The loss of stability could be a serious problem for such an approach. If gravimetric data are mixed with altimetric data, then the embedded sphere should be replaced by an embedded ellipsoid of smallest possible depth, in order to obtain maximal diagonal dominance.

5. ARBITRARY SPACING

The previous sections hold for the case with "equal spacing." This concept is somewhat intricate for operations on a sphere. A practical procedure has been presented by Rapp (1972) who used equal latitute differences and variable longitude differences to obtain equal area blocks. This is probably the most convenient approach. If the observations are not given with such spacing we can easily compute the gravity anomaly inside each equal area block. Several procedures can be considered:

1. Arithmetic means
2. Weighted means
3. Eq. (1.03) for inversion-free prediction
4. Least squares collocation

An alternative approach is to introduce unequal area surface elements of size $\Delta\phi\Delta\lambda$. The formulas (2.01) through (2.03) should then be modified and $\Delta g_i^*/n$ replaced by

$$\Delta g_i^* p_i / 4\pi \tag{5.01}$$

where

$p_i = \Delta\phi_i \, \Delta\lambda_i \cos\phi_i$
$\Delta\phi_i$ = latitude span of the i-th element (radians)
$\Delta\lambda_i$ = longitude span of the i-th element (radians)
ϕ_i = latitude of the center of the i-th element

This technique will always impair the diagonal stability of the system.

6. CONVERGENCE

A sufficient condition for convergence in an iterative approach is found if all diagonal elements are larger than the sum of all remaining elements in the same equation

$$2|C_{jj}| > \sum_{i}^{n} |C_{ji}| \qquad 1 \le j \le n \cdot \tag{6.01}$$

Similar conditions for convergence are also formally valid for the Dirac approach. Convergence can always be expected for models where the depth to the internal sphere is smaller than half the minimum grid distance.

7. THE ROBUST BASE FUNCTION COMPARED WITH THE OLD BASE FUNCTION

The principal difference between the old and the robust base functions has its origin in the leading term. We have the following alternatives.

Dirac approach:

$$K_{ji} = \frac{(r_j^2 - r_o^2) r_o^2}{r_j^4} \cdot \frac{1}{d_{ji}^3} = \frac{s^2 - s^4}{d_{ji}^3} \tag{7.01}$$

Robust approach:

$$C_{ji} = \frac{r_o^2}{r_j^2 \sum_i (1/d_{ji}^3)} \cdot \frac{1}{d_{ji}^3} = \frac{s^2}{\sum_i (1/d_{ji}^3)} \cdot \frac{1}{d_{ji}^3} \quad \text{(renormalized kernel)} \tag{7.02}$$

$$\lim_{r_j \to \infty} K_{jj} = 0 \qquad \lim_{r_j \to \infty} C_{jj} = 0$$

$$\lim_{r_j \to r_o} K_{jj} = \infty \qquad \lim_{r_j \to r_o} C_{jj} = 1$$

The difference between the two predictors is most clearly seen if $r_j = r_o$. Then the leading term will be infinite in the Poisson kernel and all predictions on the external surface (except for the observation points) will be equal to zero. The robust approach is remarkable because it gives meaningful predictions between the given points on the sphere. The original Poisson formula cannot predict missing points on the given sphere!

8. GLOBAL MODELS

We start with a trivial model. Let Δg be the observed gravity anomaly at two given points on an external spherical surface. The predicted gravity anomaly ($\Delta\hat{g}_j$) for any point on the external surface in the joint "great circle," according to the Dirac approach (Bjerhammar 1976), is then

$$\Delta\hat{g}_j = \frac{(1+s^2-2s\cos x)^{-3/2}+(1+s^2-2s\cos(\omega-x))^{-3/2}}{(1-s)^{-3}+(1+s^2-2s\cos\omega)^{-3/2}}\,\Delta g \qquad (8.01)$$

where

$$d_{12} = d_{21},\ d_{11} = d_{22},\ \text{and}\ \Delta g_1 = \Delta g_2 = \Delta g\ .$$

x is the geocentric angle between the first given point and the prediction point, ω-x the geocentric angle between the second given point and the prediction point, and ω the geocentric angle between the two given points. Furthermore, r_o is the radius of the internal sphere, r_j the radius of the external sphere, and $s = r_o/r_j$.

We note that the predictions are not invariant with respect to s.

The predictions depend strongly on the choice of radius of the internal sphere, as shown in table 1 (page 14). The corresponding prediction by the "robust base function" is in an unfiltered rigorous solution.

Δg^*-determination (no Legendre polynomials excluded)

$$d_{11}^{-3}\Delta g_1^* + d_{12}^{-3}\Delta g_2^* = \Delta g_1 \, (d_{11}^{-3} + d_{12}^{-3})s^{-2} \tag{8.02}$$

$$d_{21}^{-3}\Delta g_1^* + d_{22}^{-3}\Delta g_2^* = \Delta g_2 \, (d_{21}^{-3} + d_{22}^{-3}) \, s^{-2} \tag{8.03}$$

where

$$d_{12} = d_{21} \; , \; d_{11} = d_{22} \text{ and } \Delta g_1 = \Delta g_2 = \Delta g \; .$$

Thus

$$\Delta g_1^* = \Delta g_2^* = s^{-2}\Delta g = r_j^2 \, r_o^{-2}\Delta g \; . \tag{8.04}$$

with the prediction for any point on the external surface

$$\Delta\hat{g} = r_o^2/r_j^2 \, \Delta g^* = \Delta g \, . \tag{8.05}$$

Thus we have proved that this prediction is strictly invariant with respect to the radius of the internal sphere (for this simple model).

The invariance is lost to some degree if the external surface is nonspherical or if the gravity anomaly is not constant. However, the predictions are almost "invariant" with respect to the radius of the external sphere, as demonstrated in the following example.

Models with extremely "tricky" data (like the Molodensky model) are expected to give much better results with a Dirac approach than the robust approach. Models with "mixed" data can be considered when using an "embedded ellipsoid" with a depth of about 100 m.

Table 1.--Two identical observations of Δg are given on a spherical external surface. The separation between the given points is 5°. Predictions $\Delta\hat{g}$ are made at 0.5° ($\Delta\omega$) equidistance between the given points.

Geocentric angle between the closest given point and the prediction point	Dirac approach s = 0.999	Dirac approach s = 0.9	Robust approach s = 0.999	Robust approach s = 0.9
$\omega = 0$	$\Delta\hat{g}_1 = 1.000000\ \Delta g$	$1.0000\ \Delta g$	$1.000\ \Delta g$	$1.000\ \Delta g$
$\omega = 0.5°$	$\Delta\hat{g}_2 = 0.001480\ \Delta g$	$1.0332\ \Delta g$	$1.000\ \Delta g$	$1.000\ \Delta g$
$\omega = 1.0°$	$\Delta\hat{g}_3 = 0.000190\ \Delta g$	$1.0567\ \Delta g$	$1.000\ \Delta g$	$1.000\ \Delta g$
$\omega = 1.5°$	$\Delta\hat{g}_4 = 0.000060\ \Delta g$	$1.0718\ \Delta g$	$1.000\ \Delta g$	$1.000\ \Delta g$
$\omega = 2.0°$	$\Delta\hat{g}_5 = 0.000030\ \Delta g$	$1.0801\ \Delta g$	$1.000\ \Delta g$	$1.000\ \Delta g$
$\omega = 2.5°$	$\Delta\hat{g}_6 = 0.000024\ \Delta g$	$1.0827\ \Delta g$	$1.000\ \Delta g$	$1.000\ \Delta g$

For the Dirac approach in table 1 we used $r_o = sr_j$. Here the corresponding robust approach is strictly invariant with respect to the radius of the internal sphere.

A test model was studied with point estimates of the gravity anomaly for the center of each individual surface element. A numerical solution was made with harmonic coefficients from GEM 10B. Gravity anomalies were given in an equal area 5° x 5° grid. Predictions were made of Δg and N at approximately 900 points outside the given points. The number of unknowns was 1,654. The following prediction errors (rms) were obtained.

	COVA	Dirac s=0.90	Dirac s=0.95	Dirac s=0.97	Robust s=0.95	Robust s=0.99	Robust s=0.999
rms N m	± 10.2	±14.954	±3.81	±9.87	±1.96	±2.09	±1.96
rms Δg mgal	± 13.0	±11.38	±6.09	±3.95	±7.05	±6.64	±6.48

COVA solutions were made according to model 4 of Tscherning-Rapp (1974) (four iterations). Optimum value for the Dirac approach is expected for $s = (r_j - b/2)/r_j = 0.95$, where b is the grid distance.

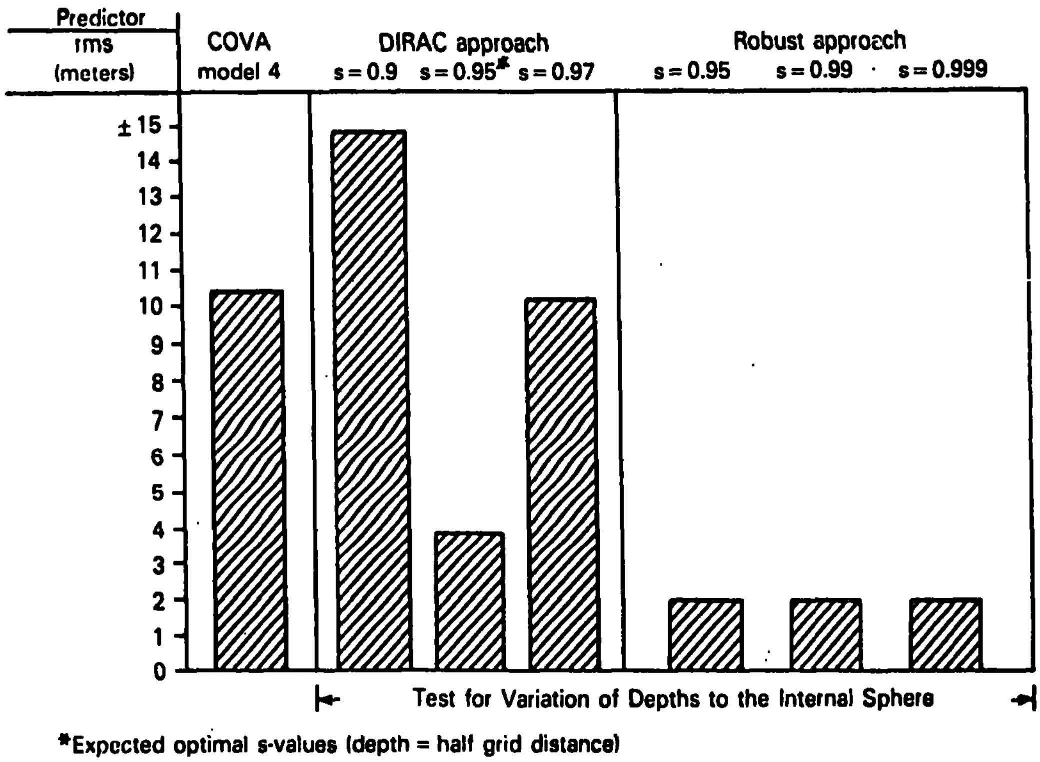

Figure 1.--Global predictions of geoidal heights from an equal area model with 1,654 points.

The COVA model is a least squares collocation solution with prescribed parameters, according to Rapp and Tscherning (1974). The Dirac approach is based on Bjerhammar (1964) and (1976). The robust approach is according to eq. (1.07a). The rms values have been computed from the prediction errors with respect to theoretical values in the spherical harmonic expansion. All available harmonics were included. The errors of the robust approach are about the same size as the errors introduced when representing the surface element by its point estimate at the center of the element. All studies are based on point estimates. Considerable improvement can be expected from estimates with mean values (for surface elements).

All solutions were made with single precision. The COVA solution benefits from the favorable choice of degree variances which is suppose to correspond closely to the 'true' harmonic coefficients in GEM 10B. The lower accuracy might be explained by the poor condition number, lack of isotropy, and homogeneity. Other explanations cannot be excluded. A better covariance function might be found. (See fig. 1.)

The Dirac approach is useful in a Gauss-Markov model with overdeterminations. Least squares estimates of the variance are then obtained.

Robust solutions were computed according to eq. (1.07b). The solutions were almost invariant with respect to the choice of s-values for the selected range (contrary to the Dirac approach). Computational costs for the robust computation of Δg^* were almost negligible. We note that the robust Δg^* is not a Dirac quantity but varies smoothly on the internal sphere.

The test model was based on a data set supplied by Prof. Richard H. Rapp of Ohio State University. The results for the robust approach are mapped in figure 2.

9. LOCAL MODELS

The robust approach seems directly justified for global models, and optimal application can be expected for an equal area approach, according to Rapp (1972).

If a global solution is available, then there is still an interest in finding improved local solutions. For this purpose, the difference between the observed and the global gravity anomaly can be computed.

$$\Delta g_{local} = \Delta g_{observed} - \Delta g_{global} = \Delta g_o$$

where the global gravity anomaly is predicted locally by using eq. (1.05).

The local solution is difficult to obtain because rugged topography can complicate the computations. The most extensive study of various solutions was made by Katsambalos (1981), who used a model according to Molodensky with the following parameters:

Region size: 0.40° x 0.40°: Predictions for 10,000 m above the ground
Grid interval: 1'
Inclination of the cone: 10.543°
Number of anomalies: 576 (Green method: 57,600)

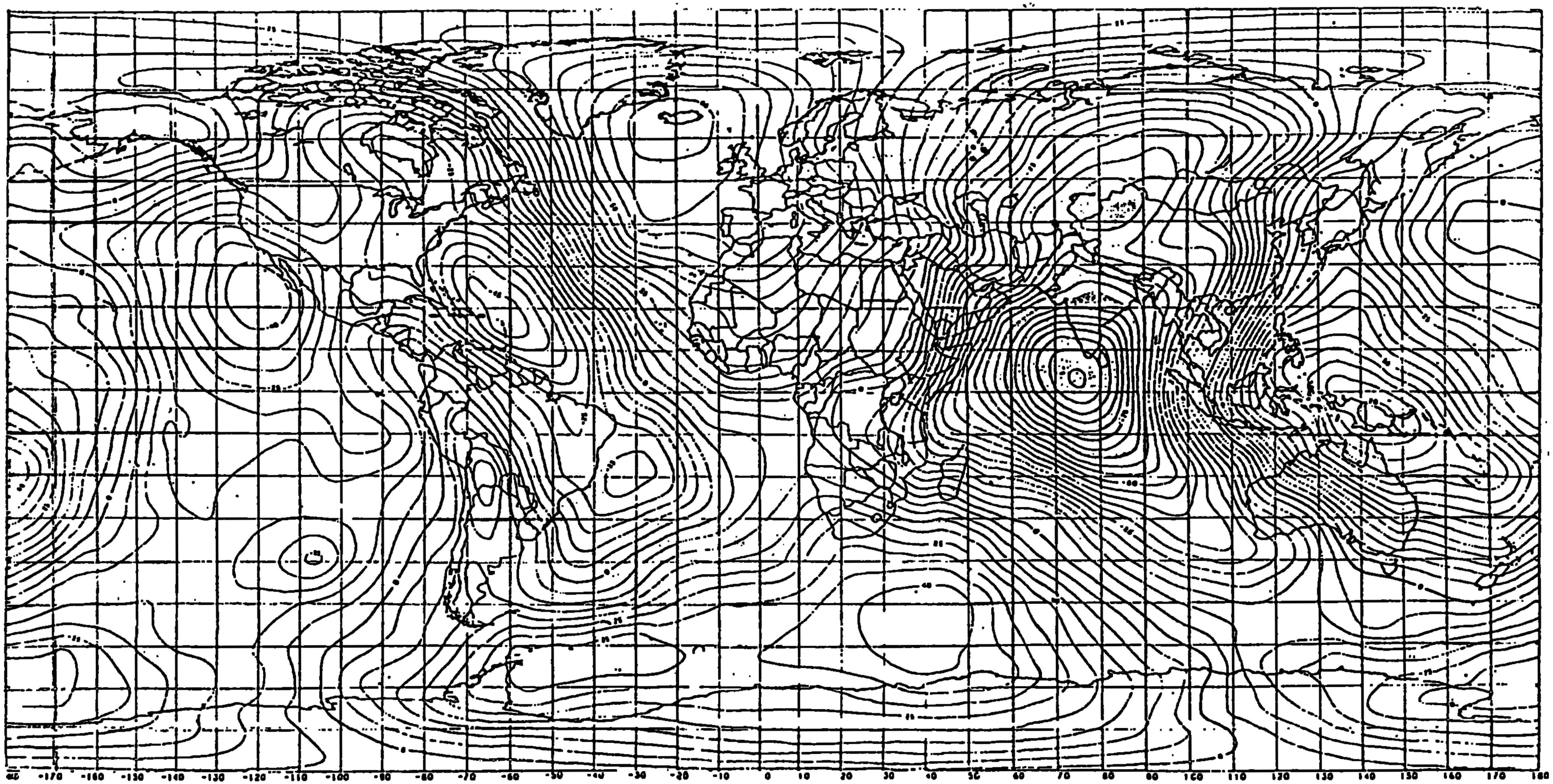

Figure 2.--Robust solution made from 5° x 5°, equal area gravity anomalies (1,654 unknowns on a microcomputer). Robust approach with s = 0.999. Reference ellipsoid GRS 1980.

Method	Radial gravity (rms)	Horizontal gravity (rms)
Green	±3.19 mgal	±0.98
Bjerhammar-Dirac	0.37	0.14
Classical	9.36	2.93
Krarup-Moritz[2]	(19.45)?	(2.77)?

Katsambalos (1981) presented no results from predictions at the physical surface but recommended here (for low altitudes) the Krarup-Moritz approach. S. Stocki (private communication) extended the model study to surface predictions and obtained the following results when using surface elements of constant latitude and longitude, with the pole at the top of the mountain.

Gravity prediction:

Depth of sphere (m)	Bjerhammar-Dirac (mgal)	Krarup-Moritz (COVA model 4) (mgal)
10	±9.04	± 87
375	7.27	180
750	5.82	---
1,200	----	506
1,500	3.75	overflow
3,000	2.30	overflow
6,000	43.00	overflow

Vertical deflection:

Depth (m)	Bjerhammar-Dirac (arc sec)	Krarup-Moritz (COVA model 4) (arc sec)
10	± 1.17	± 402
375	0.97	961
750	0.79	----
1,200	----	4,446
1,500	0.56	overflow
3,000	0.38	overflow
6,000	5.91	overflow

[2] The actual figures were not disclosed for this model and grid interval. The values given here represent a compilation from a larger number of rms values. Figures for the Krarup-Moritz model were obtained by extrapolation from solutions with greater grid intervals. Applied covariance function is according to Tscherning. Katsambalos (1981: p. 84) concluded, concerning the Bjerhammar-Dirac approach, "It is quite remarkable that even over the edge of the area at 5 km altitude, the errors are no more than 9%. More remarkable is the fact that directly above the model, at 5 km, 10 km and 20 km altitude, the errors are less than 3%, as opposed to almost 25% from the Green approach. ...at 100 km the errors from the Dirac approach are almost four times smaller than the Green approach."

For a nonsingular matrix K, the condition number κ was computed from the relation

$$\kappa = \| K \|^{\infty} \; \| K^{-1} \|^{\infty}$$

where $\| K \|^{\infty}$ represents the infinity norm. The associated condition numbers were computed by S. Stocki.

Condition number: (A smaller condition number gives a more stable solution).

Depth (m)	Bjerhammar-Dirac	Krarup-Moritz (COVA model 4)
10	11,437	36,232,433
375	2,137	111,324,562
750	1,930	-----
1,200	---	1,434,135,264
1,500	3,721	-----
3,000	25,628	overflow
6,000	12,739,160	overflow

All collocation computations were made with the covariance function of Tscherning and Rapp (1974) for the Krarup-Moritz approach.

The condition number is a good measure of the stability of the solution. For a depth of 375 m, the condition number was about 5,300 times smaller when using the Dirac technique instead of the COVA model.

Tscherning (1983) introduced a number of modifications to the covariance function and stated: "We should be able to get results as good as these obtained using the Dirac ... approach." He also showed a series of results with comparable quality, after deleting all degrees below 20 of the kernel. There might be a corresponding improvement for the same omission in the Dirac approach.

An evaluation of the results from Katsambalos (1981), Stocki, and Tscherning (1983) for the Krarup-Moritz method is very difficult. They all excluded the balancing of the gravity field. The expectation of the gravity anomaly should

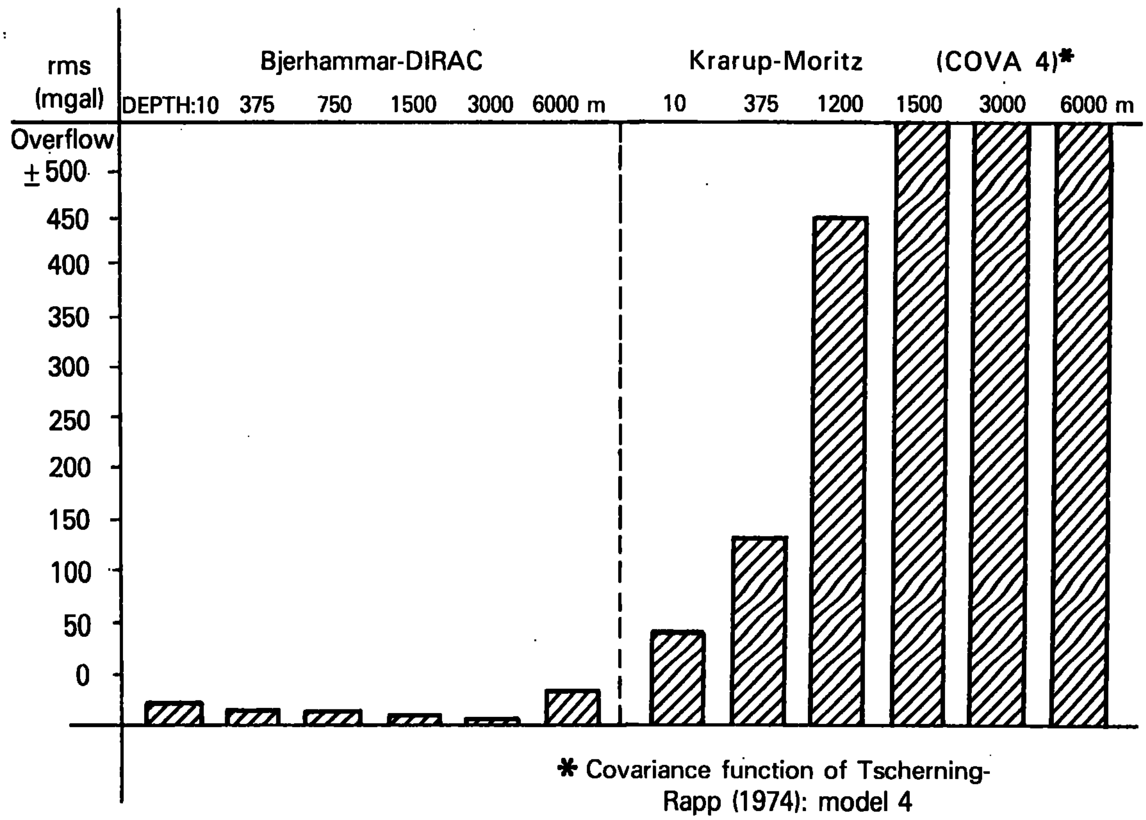

Figure 3.--Molodensky mountain study (local model). Predictions of gravity on the surface of the Earth. (Investigator: S. Stocki. See also Tscherning (1983).

be equal to zero in the least squares collocation. This condition is not satisfied in these studies of the Molodensky mountain model. The solutions presented are minimum norm solutions which are not necessarily unbiased. However, the unbiased approach which operates on the residuals seems to give worse results for the applied covariance function. Further investigations are needed.

The best choice of a covariance function for a local model is somewhat unclear. In fact, the covariance function is not estimable for the actual geodetic model. Some empirical procedures can perhaps give the wanted information.

10. STRICTLY INVERSION-FREE PREDICTION

From eq. (1.03) we obtain the following limiting value for the robust approach

$$\lim_{r_o \to r_j} \Delta g^* = \Delta g$$

This means that we have an inversion-free approach on an external spherical surface:

Δg-predictions: eq. (1.03) with $s = 1$.

N-predictions: Classical Stokes' formula.

ξ,η-predictions: Classical Vening Meinesz' formula.

The gravity anomaly prediction is made by the formula

$$\Delta g_j = s^2 \sum_i \Delta g_i^* d_{ji}^{-3} p_i \Big/ \sum_i d_{ji}^{-3} p_i \qquad (10.1)$$

where

$(4\pi)^{-1}(s^2 - s^4) \iint_\Omega d^{-3} d\Omega = s^2$, and p_i is given by eq. 5.01 (but is mostly unity.)

For this kind of approach see Bjerhammar (1970).

Svensson (1983) used an equivalent approach for prediction of geoidal heights.

$$N_j = r_o \gamma^{-1} \sum_i \Delta g_i S(\omega)_i p_i \left[(4\pi)^{-1} \iint_\Omega d^{-1}\, d\Omega\right] \Big/ \sum_i d_{ji}^{-1} p_i \qquad (10.2)$$

where $S(\omega)$ is Stokes' function and $(4\pi)^{-1} \iint_\Omega d^{-1} d\Omega = s$. This prediction formula avoids a previous prediction of the gravity anomalies on the external surface. However, the corresponding prediction formula for the vertical deflection is less satisfactory. (Correct values of p_i are needed in eqs. (10.2) and (10.3).)

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}_j = (4\pi\gamma)^{-1} \quad \sum_i \Delta g_i \; V(\omega)_i \; \begin{bmatrix}\cos\\ \sin \alpha_i\end{bmatrix} p_i \iint_\Omega d^{-2} d\Omega \;\; / \; \sum_i d_{ji}^{-2} p_i \tag{10.3}$$

where $V(\omega)$ represents the Vening Meinesz function and

$$(4\pi)^{-1} \iint_\Omega d^{-2} d\Omega = \tfrac{1}{2}\, s \log \frac{r_j + r_o}{r_j - r_o}.$$

The simplicity of these procedures is obvious. Some comments are justified.

1. The geoidal height is computed with the weight kernel for Stokes' function given by d_{ji}^{-1} . The Stokes function itself cannot be used because it goes from positive numbers to negative numbers. The selected weighting function avoids the associated singularity.
2. The vertical deflection is computed with the weight kernel for Vening Meinesz' formula given by d_{ji}^{-2} . There is a well-known singularity for $r_j = r_o$.

This technique uniquely defines the predictions on the external surface and in space for gravity anomalies and geoidal heights. Vertical deflections are more difficult to handle because of the singularity for $r_j = r_o$. The singularity problem is avoided by using the robust approach with an internal sphere at a small depth.

There is an important theorem given by Svensson (1983) for predictions of type (1.03) or equivalent. The external surface is postulated to be a homeomorphic differentiable surface, embedded in space, by radial projection, to a sphere. In particular, latitude and longitude can be used for coordinates on this surface. The proof shows that there is uniform convergence to the correct value for predictions on the sphere and in space when using an equal area approach and other specified grids. For a nonspherical surface, the predictions on the surface (and in space) will still be uniformly convergent to the correct value, but the necessary procedures are not directly available.

A user of the Svensson approach for Stokes' as well as Vening-Meinesz' formulas has to observe that every summation must include the whole Earth. This is also the case for eq. (1.01) when making predictions in space.

Predictions on the external surface can of course be made with truncation if it is only a question of gravity anomalies.

Svensson found that predictors of second power instead of the third power are not uniformly convergent to the correct value for predictions on the surface. (See eq. (10.3).)

The inversion-free predictors give no direct estimates of the prediction errors. Methods for estimating the prediction errors are given in section 11.

11. AUTOPREDICTION

Autoprediction is defined as a technique employing a predictor on a set of given observations for the prediction of a selected observation from the subset of remaining observations. The rms value of these predictions s_A is a measure of the quality of the predictor.

Katsambalos (1980) asked for estimates of the standard deviation of a prediction by the inversion-free predictors. This is of special interest for evaluation of the given predictions.

Least squares collocation gives direct measures of the prediction errors, but these estimates utilize a priori information which is crucial for the final results. The inversion-free predictors have no corresponding measures in the most simple application. However, autoprediction gives a direct measure of the prediction error in the following way.

The inversion-free prediction (on the external surface) is defined by the predictor

$$\hat{y}_j = \sum_i y_i d_{ji}^{-v} p_i \;/\; \sum_i d_{ji}^{-v} p_i \qquad (11.1)$$

where $\hat{y}_j$ is the predicted value, y_i an observation, d_{ji} the distance between the prediction point and the observation point, p_i the weight, and v a positive scalar (mostly an integer).

Now we consider a case where the predictions as well as the observations are available for all given points. Then we have the "prediction error" e

$$e_j = y_j - \hat{y}_j \tag{11.2}$$

For the predictions, we compute the following rms value

$$s_A = \sqrt{\sum_j e_j^2 / n} \tag{11.3}$$

where n is the number of observations. Here s_A is an estimate of the prediction error from our predictor. The computer time for estimating s_A can be reduced by using a sampling technique.

Autoprediction is also a useful tool for comparing different predictors. The most simple application is a study of the influence of truncation. For example, it can be found that the prediction of the gravity anomaly yields mostly the same prediction error if we use only the 10 closest observations instead of all gravity anomalies. It has also been found that no improvement is obtained by using the more sophisticated least squares collocation (Katsambalos 1980).

Predictions of the gravity anomaly on the external surface can be made with various values of v. If $p_i = 1$, then we have:

1. v = 0. The prediction is the arithmetic mean.
2. v > 0. The prediction is a weighted mean.

Svensson (1983) excluded v = 2, because this approach is not uniformly convergent to the true value. (See the formula for vertical deflection in eq. (11.3) above.) However, v = 2 has exceptional merits for local predictions of gravity anomalies in large data sets. The time-consuming square roots can be avoided. If v = 2 is combined with severe truncation, then the technique can be fully justified. This can be verified by autoprediction on the actual data set.

Filtering is obtained in the inversion-free approach by the following transformation

$$d_{ji}^{-v} \rightarrow d_{ji}^{-v} + \delta \qquad \delta > 0$$

where δ is conveniently studied by using the autoprediction technique. Predictions without filtering show a slight step effect in the fully inversion-free approach (Sünkel 1980, 1981). This step effect can be eliminated by using appropriate filtering.

Estimates of the prediction errors from autoprediction represent a pessimistic approach, because the distance to the closest observations will always be less for any prediction point inside the given set of observations. For example, prediction inside a rectangular grid can be made with a minimum distance of L/2, but autopredictions are made with a minimum distance of L, if the grid distance is L.

Predictions on the external surface are fully justified according to eq. (11.1) for the following cases:

Gravity anomalies from a given set of gravity anomalies.
Geoidal heights from a given set of geoidal heights, e.g., altimetric heights.

12. CONCLUDING REMARKS

The classical, free boundary value problem has been given very simple solutions. These solutions are closely related to inversion-free solutions presented earlier which were made without an embedded sphere. However, harmonicity down to an internal sphere is used to justify a suitable renormalization of the original integral equations. Extremely small depths to the embedded surface can then be used to obtain extreme diagonal dominance for global models.

Predictions will still be almost invariant with respect to the depth of this embedded surface as long as the depth is at least 10 times smaller than the grid distance. The robust approach combines the simplicity of the classical integral methods with the advantage of the discrete predictors. It can be used for reduction of data down to a sphere (or ellipsoid) in a spherical harmonic expansion or as a final presentation. If a more accurate solution is required, then the Dirac approach is a promising alternative when operating on the residuals. Least squares collocation is another alternative for cases with known covariance functions.

Estimates of variances are somewhat controversial for a problem where we have a primary integral equation and an infinite number of singularities. The inversion-free prediction uses a technique for autoprediction when estimating the prediction errors.

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APPENDIX.--ROBUST PREDICTOR

The mathematical justification for the robust predictor is concisely given below. If $r\Delta g$ is considered harmonic, then we can write in a rigorous way

$$\Delta g_j = \frac{(4\pi)^{-1} s^2 \iint\limits_{\Omega} \Delta g^* \, d^{-3} \, d\Omega}{(4\pi)^{-1} \iint\limits_{\Omega} d^{-3} \, d\Omega} \qquad \text{"Renormalized Poisson integral"}$$

For an equal area approach we obtain

$$\lim_{i \to \infty} \left(s^2 \sum_i \Delta g^* d_{ji}^{-3}\right) / \left(\sum_i d_{ji}^{-3}\right) = \Delta g_j$$

The renormalized integral formula has given a discrete formula where the discretization errors balance in the numerator and the denominator.

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