NOAA Technical Memorandum NOS NGS-11

APPLICABILITY OF ARRAY ALGEBRA

Rockville, Md.
February 1978

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NOAA geodetic publications

<u>Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys</u>. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1974, reprinted 1975, 1976, 12 p. (PB265442). National specifications and tables show the closures required and tolerances permitted for first-, second-, and third-order geodetic control surveys.

<u>Specifications to Support Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys</u>. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1975, reprinted 1976, 30 p. (PB261037). This publication provides the rationale behind the original publications, "Classification, Standards of Accuracy, ...".

(Continued at end of publication)

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Richard A. Snay

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APPLICABILITY OF ARRAY ALGEBRA

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ABSTRACT. Matrix equations associated with gridded data reduce to equivalent array equations, with proper assumptions for the mathematical model and the weight matrix. Array equations involve significantly fewer computations and less computer storage than the equivalent matrix equations.

INTRODUCTION

Consider a rectangular grid in the xy-plane as pictured in figure 1. At each point of the grid (x_i, y_j) a number z_{ij} is associated that can be considered as the observed value of an unknown function $z = f(x,y)$, that is,

$$z_{ij} = f(x_i, y_j) \text{ for } 1 \leq i \leq 4 \text{ and } 1 \leq j \leq 3.$$

One may think of the x and y values as spatial positions and the z values as heights or gravity anomalies or distortions of a photograph. Alternatively, x may represent temperature, y pressure, and z refractive index. Whatever the physical interpretation, two fundamental problems exist, namely, that of estimation and prediction.

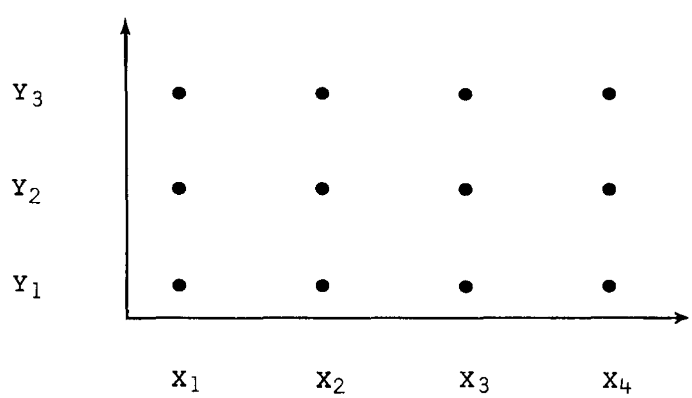

Figure 1.--Gridded data.

In estimation one wishes to approximate the unknown function $z = f(x,y)$ with a suitable model, say

$$z = \sum_{p=0}^{m} \sum_{q=0}^{n} c_{pq} x^p y^q , \qquad (1)$$

by estimating the parameters c_{pq} for $0 \leq p \leq m$ and $0 \leq q \leq n$. In prediction the problem is to determine the values of $z = f(x,y)$ for values of x and y where z is not observed. In general, there is no advantage computationally in having the observations gridded; however, with certain mathematical models and with certain patterns for the weight matrix associated with the observations significant computational savings are realized. These same conditions on the model and the weight matrix allow also for the estimation or prediction problem to be solved with less computer storage when the data are gridded. An example is given in the next two sections of the paper.

The advantages of using gridded data have led to the development of a notational system entitled array algebra. Introduced by the photogrammetrist, Urho Rauhala, this algebra provides a language, with specific grammatical rules, in which computations with gridded data can be easily expressed. The fundamentals of array algebra are contained in a following section. A more complete presentation can be found in the dissertation of Rauhala (1974).

AN EXAMPLE IN ESTIMATION

Consider a situation where observations are given as depicted in figure 1. One wishes to approximate the function $z = f(x,y)$ by a polynomial of the form

$$z = \sum_{p=0}^{2} \sum_{q=0}^{1} c_{pq} x^p y^q .$$

Estimating the coefficients of this polynomial by the technique of least-squares leads to the following set of observation equations:

$$\underset{(12\mathrm{x}6)}{A} \quad \underset{(6\mathrm{x}1)}{X} \quad = \quad \underset{(12\mathrm{x}1)}{Z} \quad + \quad \underset{(12\mathrm{x}1)}{V} \tag{2a}$$

or

$$\begin{bmatrix} 1 & y_1 & x_1 & x_1y_1 & x_1^2 & x_1^2y_1 \\ 1 & y_2 & x_1 & x_1y_2 & x_1^2 & x_1^2y_2 \\ 1 & y_3 & x_1 & x_1y_3 & x_1^2 & x_1^2y_3 \\ 1 & y_1 & x_2 & x_2y_1 & x_2^2 & x_2^2y_1 \\ 1 & y_2 & x_2 & x_2y_2 & x_2^2 & x_2^2y_2 \\ 1 & y_3 & x_2 & x_2y_3 & x_2^2 & x_2^2y_3 \\ 1 & y_1 & x_3 & x_3y_1 & x_3^2 & x_3^2y_1 \\ 1 & y_2 & x_3 & x_3y_2 & x_3^2 & x_3^2y_2 \\ 1 & y_3 & x_3 & x_3y_3 & x_3^2 & x_3^2y_3 \\ 1 & y_1 & x_4 & x_4y_1 & x_4^2 & x_4^2y_1 \\ 1 & y_2 & x_4 & x_4y_2 & x_4^2 & x_4^2y_2 \\ 1 & y_3 & x_4 & x_4y_3 & x_4^2 & x_4^2y_3 \end{bmatrix} \begin{bmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \\ c_{20} \\ c_{21} \end{bmatrix} = \begin{bmatrix} z_{11} \\ z_{12} \\ z_{13} \\ z_{21} \\ z_{22} \\ z_{23} \\ z_{31} \\ z_{32} \\ z_{33} \\ z_{41} \\ z_{42} \\ z_{43} \end{bmatrix} + \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{31} \\ v_{32} \\ v_{33} \\ v_{41} \\ v_{42} \\ v_{43} \end{bmatrix} . \tag{2b}$$

Given two matrices, $\underset{(m \times n)}{G}$ and $\underset{(r \times s)}{H}$, the concept of the Kronecker or direct product of G and H is denoted by $G \otimes H$, and defined by the equation

$$G \otimes H = \begin{bmatrix} g_{11}H & g_{12}H & \cdots & g_{1n}H \\ g_{21}H & g_{22}H & \cdots & g_{2n}H \\ \vdots & \vdots & & \vdots \\ g_{m1}H & g_{m2}H & \cdots & g_{mn}H \end{bmatrix} \tag{3}$$

where $g_{ij}H$ denotes the matrix obtained by multiplying each entry of H by the number g_{ij}. Thus $G \otimes H$ is a matrix of order $(mr \times ns)$.

Because of the gridded data and the way the observations and unknowns have been ordered, eq. (2a) can be rewritten as

$$\underset{(4x3)}{(A_x} \otimes \underset{(3x2)}{A_y)} \quad \underset{(6x1)}{X} = \underset{(12x1)}{Z} + \underset{(12x1)}{V} \tag{4}$$

where

$$\underset{(12x6)}{A} = \underset{(4x3)}{(A_x} \otimes \underset{(3x2)}{A_y)} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & y_1 \\ 1 & y_2 \\ 1 & y_3 \end{bmatrix} . \tag{5}$$

If P denotes the weight matrix associated with the observations, then the least-squares estimator of the matrix X is given by the familiar equation

$$\underset{(6x1)}{X} = (\underset{(6x12)}{A^T} \underset{(12x12)}{P} \underset{(12x6)}{A})^{-1} \underset{(6x12)}{A^T} \underset{(12x12)}{P} \underset{(12x1)}{Z} . \tag{6}$$

Assume the existence of matrices P_x and P_y such that

$$\underset{(12x12)}{P} = \underset{(4x4)}{P_x} \otimes \underset{(3x3)}{P_y} . \tag{7}$$

Such is the case when $P = s^2 I_{12}$, where I_{12} denotes the identity matrix of order twelve and s is a given constant. In this case eq. (7) is satisfied with $P_x = sI_4$ and $P_y = sI_3$. Under the assumption of eq. (7), one can write eq.(6) as

$$\underset{(6x1)}{X} = \left[(\underset{(4x3)}{A_x} \otimes \underset{(3x2)}{A_y})^T (\underset{(4x4)}{P_x} \otimes \underset{(3x3)}{P_y}) (\underset{(4x3)}{A_x} \otimes \underset{(3x2)}{A_y})\right]^{-1} (\underset{(4x3)}{A_x} \otimes \underset{(3x2)}{A_y})^T (\underset{(4x4)}{P_x} \otimes \underset{(3x3)}{P_y}) \underset{(12x1)}{Z} . \tag{8}$$

Examining the definition of the Kronecker product, one finds by straight forward algebraic manipulations that

$$\underset{(m\times n)}{(G} \otimes \underset{(r\times s)}{H)}^{T} = \underset{(n\times m)}{G^{T}} \otimes \underset{(s\times r)}{H^{T}} \tag{9}$$

and

$$(\underset{(m\times n)}{G_1} \otimes \underset{(r\times s)}{G_2'})\;(\underset{(n\times p)}{H_1} \otimes \underset{(s\times t)}{H_2}) = \underset{(m\times n)(n\times p)}{G_1 H_1} \otimes \underset{(r\times s)(s\times t)}{G_2 H_2} \quad . \tag{10}$$

From eq. (10) and the uniqueness of the matrix inverse, it follows that

$$(\underset{(m\times m)}{G} \otimes \underset{(n\times n)}{H})^{-1} = \underset{(m\times m)}{G^{-1}} \otimes \underset{(n\times n)}{H^{-1}} \quad . \tag{11}$$

Hence, eq. (8) is equivalent to

$$\underset{(6\times 1)}{X} = \left\{ \left[\underset{(3\times 4)(4\times 4)(4\times 3)}{(A_x^T P_x A_x)^{-1}} \underset{(3\times 4)(4\times 4)}{A_x^T P_x} \right] \otimes \left[\underset{(2\times 3)(3\times 3)(3\times 2)}{(A_y^T P_y A_y)^{-1}} \underset{(2\times 3)(3\times 3)}{A_y^T P_y} \right] \right\} \underset{(12\times 1)}{Z} \quad . \tag{12}$$

In going from eq. (6) to eq. (12), the numerical value of the solution has not changed, but the number of numerical operations to compute X has decreased. Observe that eq. (6) requires an inversion of a 6×6 matrix, while in eq. (12) one inverts two smaller matrices of dimensions 3×3 and 2×2. As the number of operations to invert an nxn matrix is of the order n^3, one obtains the following ratio between the two methods

$$\frac{\text{operations in (12)}}{\text{operations in (6)}} = \frac{2^3 k + 3^3 k + r_1}{6^3 k + r_2} = \frac{35\ k + r_1}{216\ k + r_2} \quad ,$$

where r_i represents the number of noninversion operations in the respective methods and k is the factor of proportionality.

The numerical savings are even more dramatic. Observe the standard matrix product $\underset{(r\times s)}{G}\ \underset{(s\times t)}{H}$ requires r × s × t multiplications. Assuming that the P matrix is full, then just to form the matrix $\underset{(6\times 12)(12\times 12)(12\times 6)}{A^T\ P\ A}$ requires 1296 multiplications (D = P A requires 12 x 12 x 6 = 864 multiplications and A^T D requires 6 x 12 x 6 = 432 multiplications). On the other hand, it only takes 84 multiplications to form $A_x^T P_x A_x$ and 30 multiplications to form $A_y^T P_y A_y$. Once

$N_x = A_x^T P_x A_x$ and $N_y = A_y^T P_y A_y$ are computed and inverted, still further savings in the number of operations are achieved by calculating $N_x^{-1} A_x^T P_x$ and $N_y^{-1} A_y^T P_y$ as opposed to $N^{-1} A^T P$, where $N = A^T P A$. If either P or A is sparse, then many operations can be avoided when using either eq. (6) or eq. (12). However, eq. (12) is still computationally advantageous.

Equation (12) is of the form

$$\underset{(6x1)}{X} = (\underset{(3x4)}{B_x} \otimes \underset{(2x3)}{B_y}) \quad \underset{(12x1)}{Z} \tag{13}$$

where

$$B_x = (A_x^T P_x A_x)^{-1} A_x^T P_x$$

and

$$B_y = (A_y^T P_y A_y)^{-1} A_y^T P_y \quad .$$

In eq. (13) the implied sequence of operations is to obtain first the matrix $B = B_x \otimes B_y$, and then to obtain the product B Z.

With the use of the array algebra described in the next section, the operation $B_x \otimes B_y$ can be circumvented, resulting in still greater computational efficiency.

ARRAY ALGEBRA

In the sense that a matrix is a vector whose components are vectors, a k-array is a vector whose components are (k-1)-arrays. Hence, a 1-array is a vector, a 2-array is a matrix, a 3-array is a vector of matrices, and so forth. A k-array is denoted as $\underset{(n_1 x n_2 x \ldots x n_k)}{\overline{Z}}$, and its elements are denoted as $\overline{z}_{i_1 i_2 \ldots i_k}$ for $1 \leq i_j \leq n_j$ and $1 \leq j \leq k$. The notion of an array is familiar in computer science as a storage structure for a k-dimensional grid of values.

Given a k-array $\underset{(n_1 x n_2 x \ldots x n_k)}{\bar{Z}}$ and a set of matrices $\left\{\underset{(m_1 x n_1)}{G_1}, \underset{(m_2 x n_2)}{G_2}, \ldots, \underset{(m_k x n_k)}{G_k}\right\}$, define the k-array

$$\bar{X} = < G_1, G_2, \ldots, G_k > \bar{Z} \tag{14}$$

by the equations

$$\bar{x}_{i_1 i_2 \ldots i_k} = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \ldots \sum_{j_k=1}^{n_k} g(1)_{i_1 j_1} g(2)_{i_2 j_2} \ldots g(k)_{i_k j_k} \bar{z}_{j_1 j_2 \ldots j_k} \tag{15}$$

for $1 \leq i_r \leq m_r$ and $1 \leq r \leq k$. Here $g(r)_{ij}$ represents the i, j element of G_r. The operation of eq. (14) is called R-matrix multiplication in Rauhala (1974).

In eq. (15) the second subscript of $g(r)_{ij}$ agrees with the r^{th} index of the array element. Thus, if $\bar{Z}$ is a 1-array, then $< G > \bar{Z}$ is the regular product of a matrix and a vector. If $\bar{Z}$ is a 2-array, then $< G_1, G_2 > \bar{Z}$ is the matrix product $G_1 \bar{Z} G_2^T$. Such a reformulation of eq. (14) into standard matrix notation is not directly possible if $\bar{Z}$ is a k-array for a value of k greater than 2. However, a natural transformation from an array to a vector exists which sets up an equivalence between equations in the form of eq. (14) and certain matrix equations. Specifically, the transformation takes an array $\underset{(n_1 x n_2 x \ldots x n_k)}{\bar{Z}}$ to a vector $\underset{(n)}{Z}$ where n is the product $n_1 n_2 \ldots n_k$. The transformation is defined by the equation $z_j = \bar{z}_{i_1 i_2 \ldots i_k}$ where $j = (i_1 - 1) n_2 n_3 \ldots n_k + (i_2 - 1) n_3 n_4 \ldots n_k + \ldots + (i_{k-1} - 1) n_k + i_k$. For example, the 3-array $\underset{(2x2x2)}{\bar{Z}}$ transforms into the vector

$$\underset{(8)}{Z} = \begin{bmatrix} z_{111} \\ z_{112} \\ z_{121} \\ z_{122} \\ z_{211} \\ z_{212} \\ z_{221} \\ z_{222} \end{bmatrix} .$$

Computers actually store arrays as vectors either according to this transformation or a similar transformation that increments the indices to the left first.

As is demonstrated by Blaha (1977a), this transformation, or more precisely its inverse, allows one to transform a matrix equation of the type

$$\underset{(m \times 1)}{X} = (\underset{(m_1 \times n_1)}{G_1} \otimes \underset{(m_2 \times n_2)}{G_2} \otimes \cdots \otimes \underset{(m_k \times n_k)}{G_k}) \underset{(n \times 1)}{Z} , \tag{16}$$

where $m = m_1 m_2 \ldots m_k$ and $n = n_1 n_2 \ldots n_k$, into the equivalent array equation

$$\underset{(m_1 \times m_2 \times \ldots \times m_k)}{\overline{X}} = < \underset{(m_1 \times n_1)}{G_1}, \underset{(m_2 \times n_2)}{G_2}, \ldots, \underset{(m_k \times n_k)}{G_k} > \underset{(n_1 \times n_2 \times \ldots \times n_k)}{\overline{Z}} . \tag{17}$$

From the examples of the previous section, let

$$\underset{(3x2)}{\overline{X}} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \\ C_{20} & C_{21} \\ & \end{bmatrix} \quad \text{and} \quad \underset{(4x3)}{\overline{Z}} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{41} & Z_{42} & Z_{43} \end{bmatrix} ,$$

then eq. (13) becomes

$$\underset{(3\text{x}2)}{\overline{X}} = \underset{(3\text{x}4)}{<B_x}, \underset{(2\text{x}3)}{B_y} > \underset{(4\text{x}3)}{\overline{Z}} .$$

Note that the expression $G_1 \otimes G_2 \otimes \ldots \otimes G_k$ used in eq. (16) is well defined since the operator $\otimes$ is associative, i.e.,

$$(G \otimes H) \otimes J = G \otimes (H \otimes J).$$

As a result of the equivalence of eqs. (16) and (17), and the property

$$(G_1 \otimes G_2 \otimes \ldots \otimes G_k)\ (H_1 \otimes H_2 \otimes \ldots \otimes H_k) = G_1H_1 \otimes G_2H_2 \otimes \ldots \otimes G_kH_k, \quad (18)$$

it follows that

$$\overline{X} = < G_1H_1, G_2H_2, \ldots, G_kH_k > \overline{Z}$$

is equivalent to

$$\overline{X} = < G_1, G_2, \ldots, G_k > \left[< H_1, H_2, \ldots, H_k > \overline{Z} \right] .$$

In particular, the equation

$$\overline{X} = < H_1, H_2, H_3 > \overline{Z} \quad (19)$$

can be written as

$$\overline{X} = < H_1, I, I > \left[< I, H_2, I > \left[< I, I, H_3 > \overline{Z} \right] \right] . \quad (20)$$

The reason for expanding eq. (19), as in eq. (20), is to emphasize a method of computing $\overline{X}$. When G_i is the identity matrix for all but one value of i, say $G_r \neq I$, then the evaluation of

$$\overline{X} = < G_1, G_2, \ldots, G_k > \overline{Z}$$

reduces from eq. (15) to the formula

$$\overline{x}_{i_1 \ldots i_r \ldots i_k} = \sum_{j_r=1}^{n_r} g(r)_{i_r j_r} \overline{z}_{i_1 \ldots j_r \ldots i_k}$$

for $1 \leq i_h \leq m_h$ and $1 \leq h \leq k$.

The numerical efficiency of the array algebra eq. (17) over the equivalent Kronecker product eq. (16) can be illustrated by counting the number of multiplications required to calculate each expression. This comparison is made here between

$$\underset{(mn\times 1)}{X} = \underset{(m\times r)}{(G_1} \otimes \underset{(n\times s)}{G_2)} \ \underset{(rs\times 1)}{Z} \tag{21}$$

and

$$\underset{(m\times n)}{\overline{X}} = \langle \underset{(m\times r)}{G_1}, \underset{(n\times s)}{G_2} \rangle \underset{(r\times s)}{\overline{Z}} \ . \tag{22}$$

In eq. (21) the formation of $\underset{(mn\times rs)}{G} = \underset{(m\times r)}{G_1} \otimes \underset{(n\times s)}{G_2}$ requires mnrs multiplications and the product $\underset{(mn\times rs)}{G} \ \underset{(rs\times 1)}{Z}$ requires another mnrs multiplications for a total of 2 mnrs multiplications.

Alternatively, to compute $\overline{X}$ by eq. (22) one considers the equation

$$\underset{(m\times n)}{\overline{X}} = \langle \underset{(m\times r)}{G_1}, \underset{(n\times n)}{I} \rangle \left[\langle \underset{(r\times r)}{I}, \underset{(n\times s)}{G_2} \rangle \underset{(r\times s)}{\overline{Z}} \right] .$$

Let $\overline{W} = \langle I, G_2 \rangle \overline{Z}$. Since $\overline{w}_{ij} = \sum_{k=1}^{s} g(2)_{jk}\overline{z}_{ik}$ requires s multiplications and $\overline{W}$ has rn elements, it requires srn multiplications to compute $\overline{W}$. Now $\overline{X} = \langle G_1, I \rangle \overline{W}$. Since $\overline{x}_{ij} = \sum_{k=1}^{r} g(1)_{ik}\overline{w}_{kj}$ requires r multiplications and $\overline{X}$ has mn elements, an additional rmn multiplications are required to compute $\langle G_1, I \rangle \overline{W}$ for a total of srn + rmn multiplications to evaluate eq. (22). If, for example, m = n = r = s = 10, then eq. (21) requires 20,000 multiplications, and eq. (22) requires 2000 multiplications. Note, as a result of the interchangeability of the order of summation in eq. (15), one may write eq. (22) equivalently as

$$\underset{(m\times n)}{\overline{X}} = \langle \underset{(m\times m)}{I}, \underset{(n\times s)}{G_2} \rangle \left[\langle \underset{(m\times r)}{G_1}, \underset{(s\times s)}{I} \rangle \underset{(r\times s)}{\overline{Z}} \right] ,$$

which requires mrs + smn multiplications to obtain $\bar{X}$; i.e., the number of multiplications in eq. (22) depends on the order of summation.

Returning to earlier considerations, recall that the data existed in a two-dimensional grid and satisfied some unknown function $z = f(x,y)$ that is modeled by the polynomial

$$z = \sum_{p=0}^{2} \sum_{q=0}^{1} c_{pq} x^p y^q \quad .$$

Because of the gridded data and the model, the observation eq. (4) may be written in array notation as

$$< A_x, A_y > \bar{X} = \bar{Z} + \bar{V} \quad .$$

With additional assumptions on the weight matrix P, the least-squares estimate of $\bar{X}$ is obtained by

$$\underset{(3x2)}{\bar{X}} = < \underset{(3x4)}{(A_x^T P_x A_x)^{-1} A_x^T P_x} , \underset{(2x3)}{(A_y^T P_y A_y)^{-1} A_y^T P_y} > \underset{(4x3)}{\bar{Z}} \quad .$$

The motivation for writing the solution in this manner was simply for the computational efficiency of the method. However, as previously mentioned, there is an additional advantage gained with the use of array algebra in terms of storage savings. Observe if $\underset{(mnxrs)}{B} = \underset{(mxr)}{B_x} \otimes \underset{(nxs)}{B_y}$, then B requires mnrs storage locations, whereas B_x and B_y together require only mr + ns storage locations. Furthermore, in making the computation $\bar{X} = < B_x, B_y > \bar{Z}$ as

$$\bar{X} = < B_x, I > \left[< I, B_y > \bar{Z} \right] ,$$

only one of B_x or B_y has to be directly accessible at a given time.

LIMITATIONS ON THE APPLICABILITY OF ARRAY ALGEBRA

In addition to the requirement of working with gridded data, the limitations to the use of array algebra involve the mathematical model and the weight matrix. Examination of these limitations follows.

In order to factor the design matrix A as $A_u \otimes A_v$ and express the observations in an array $\overline{Z}$, it is necessary to have gridded data. However, there is some flexibility in the nature of the grid. First, there is no restriction placed on the spacing of the grid. A grid such as pictured in figure 2 is permissible. Second, a grid based on any parameterization is allowable. Consider a grid in polar coordinates as is illustrated in figure 3,

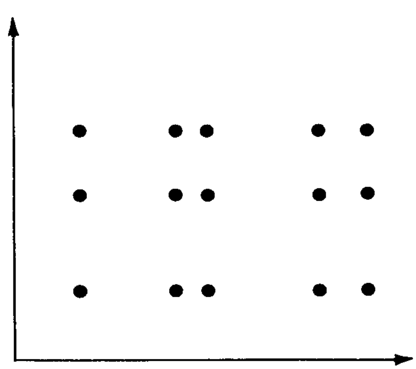

Figure 2 --Irregularly spaced grid.

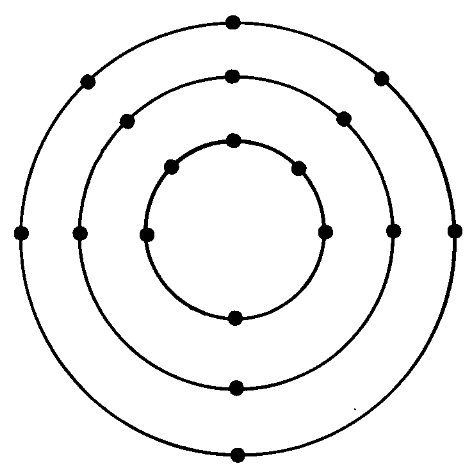

Figure 3 --Grid in polar coordinates.

or a grid on the sphere in spherical coordinates. Finally, the restriction to two-dimensional grids in this paper is merely for simplicity of presentation. The results stated for two-dimensional grids can be generalized to grids of dimension three and higher. In general, the greater the dimension of the grid, the greater the computational efficiency of the multilinear system over the equivalent monolinear system.

Focusing now on the limitations of the model, observe that the ability to transform the observation equations

$$(A_u \otimes A_v)\ X = Z + V$$

into an array equation

$$< A_u, A_v > \overline{X} = \overline{Z} + \overline{V}$$

requires some symmetry of pattern in the vector of parameters X. A condition that insures this symmetry is a model of the form

$$z = \sum_{i\varepsilon I} \sum_{j\varepsilon J} c_{ij} f_i(u) h_j(v) \quad , \tag{23}$$

where I and J are indexing sets and f_i and h_j are real valued functions for $i\varepsilon I$ and $j\varepsilon J$.

To give an example, other than a polynomial, for which the model of eq. (23) is appropriate, consider the model

$$z = \sum_{r=0}^{m} \sum_{s=0}^{n} a_{rs} u^r \cos(sv) + \sum_{r=0}^{m} \sum_{s=1}^{n} b_{rs} u^r \sin(sv) \quad .$$

Here,

$$f_i(u) = u^i \qquad \text{for } i = 0,1,2,\ldots,m$$

and

$$h_j(v) = \begin{cases} \cos(jv) & \text{for } j = 0,1,2,\ldots,n \\ \sin\Big((j-n)v\Big) & \text{for } j = n+1,\ n+2,\ldots,2n \end{cases} \quad .$$

The condition of eq. (23) is sufficient, but not necessary; i.e., some models do not satisfy this condition but are solvable by array algebra. For example, the simple polynomial $z = c_{00} + c_{10}u + c_{11}uv$ is not of the form specified by eq. (23). However, Rauhala (1974, p. 114) and Jancaitis and Magee (1977) demonstrated how to solve such a model in array algebra by using the model

$$z = c_{00} + C_{10}u + c_{11}uv + c_{01}v \tag{24a}$$

with the constraint

$$c_{01} = 0 \ . \tag{24b}$$

Briefly, given the observation equations

$$A\ X = Z + V \tag{25a}$$

with the constraints

$$D\ X = W \quad , \tag{25b}$$

the least-squares estimate of X is given by the equation (Uotila, 1967, p. 63)

$$X = N^{-1} \left[A^T PZ + D^T (DN^{-1} D^T)^{-1} \ (W - DN^{-1} A^T PZ) \right] \tag{26}$$

where

$$N = A^T PA \quad .$$

If the constraint matrix D of eq. (25b) satisfies the condition $D = D_u \otimes D_v$, then eqs. (25a) and (25b) can be expressed in array algebra as

$$< A_u \ , A_v > \overline{X} = \overline{Z} + \overline{V} \tag{27a}$$

and

$$< D_u , D_v > \overline{X} = \overline{W} \quad . \tag{27b}$$

The solution is

$$\overline{X} = < N_u^{-1}, N_v^{-1} > \Big[< A_u^T P_u, A_v^T P_v > \overline{Z}$$

$$+ < D_u^T \left(D_u N_u^{-1} D_u^T\right)^{-1}, D_v^T \left(D_v N_v^{-1} D_v^T\right)^{-1} > \overline{K} \Big]$$

where

$$\overline{K} = \overline{W} - < D_u N_u^{-1} A_u^T P_u, D_v N_v^{-1} A_v^T P_v > \overline{Z}$$

$$N_u = A_u^T P_u A_u$$

$$N_v = A_v^T P_v A_v \quad .$$

In the example of eqs. (24a) and (24b), let

$$\underset{(2x2)}{\overline{X}} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} \quad , \quad \underset{(1x4)}{D} = \underset{(1x2)}{D_u} \otimes \underset{(1x2)}{D_v} = (1,0) \otimes (0,1), \text{ and } \underset{(1x1)}{\overline{W}} = 0 \quad ,$$

then the constraint $c_{01} = 0$ has the form $< D_u, D_v > \overline{X} = \overline{W}$.

The restrictions on the weight matrix P rest on the condition that P must factorize as $P_u \otimes P_v$. Given a two-dimensional grid $\left\{(u_i, v_j) : 1 \le i \le m \text{ and } 1 \le j \le n\right\}$ with observations $z_{ij} = f(u_i, v_j)$, Jancaitis and Magee (1977) point out that, under the assumption of independent observations, a factorization, $P = P_u \otimes P_v$, is possible if there exist numbers $r_1, r_2, \ldots, r_m$ associated with the rows of the grid and numbers $k_1, k_2, \ldots, k_n$ associated with the columns of the grid such that the observation z_{ij} has weight $p_{ij} = r_i k_j$.

Here,

$$\underset{(m\times m)}{P_u} \otimes \underset{(n\times n)}{P_v} = \begin{bmatrix} r_1 & & & \bigcirc \\ & r_2 & & \\ & & \ddots & \\ \bigcirc & & & r_m \end{bmatrix} , \otimes \begin{bmatrix} k_1 & & & \bigcirc \\ & k_2 & & \\ & & \ddots & \\ \bigcirc & & & k_n \end{bmatrix} .$$

Another situation that yields a suitable weight matrix occurs when the covariance between two observations $z_{ij} = f(u_i, v_j)$ and $z_{hk} = f(u_h, v_k)$ satisfies the equation

$$\mathrm{Cov}(ij,hk) = C^2 \exp\left[- a^2 (u_i - u_h)^2 - b^2 (v_j - v_k)^2\right]$$

for fixed parameters C, a, and b. In this case the observations can be ordered such that the covariance matrix Σ is of the form

$$\Sigma = \Sigma_u \otimes \Sigma_v \quad .$$

Here, the i,h element of Σ_u is $C \exp\left[- a^2 (u_i - u_h)^2\right]$,

and the j,k element of Σ_v is $C \exp\left[- b^2 (v_j - v_k)^2\right]$.

Consequently,

$$P = (\Sigma_u \otimes \Sigma_v)^{-1} = \Sigma_u^{-1} \otimes \Sigma_v^{-1} \quad .$$

USE OF SEQUENTIAL TECHNIQUE TO MODIFY THE GRID

Sometimes the nature of the data might be such that one or more observations exist in addition to those of the grid, or conversely, one or more observations might be missing from a complete grid. The second situation can occur when observations have been purposely eliminated because of blunders made in obtaining or processing the data. Such irregularities in the grid can be handled with array algebra using the technique of sequential adjustments.

The situation of adding one set of observations to another leads to the following matrix of observation equations

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{28}$$

with associated weight matrix

$$\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} .$$

Let

$$X^* = N_1^{-1} A_1^T P_1 Z_1 \quad \text{where} \quad N_1 = A_1^T P_1 A_1 .$$

Then, as demonstrated in Uotila (1973), the solution to eq. (28) is given by

$$X = X^* + N_1^{-1} A_2^T \left(A_2 N_1^{-1} A_2^T + P_2^{-1} \right)^{-1} \left(Z_2 - A_2 X^* \right) \tag{29}$$

when N_1 has full rank.

Suppose that the observations corresponding to the equation

$$A_1 X = Z_1 + V_1$$

are such that

$$A_1 = A_{1u} \otimes A_{1v} \qquad \text{and} \qquad P_1 = P_{1u} \otimes P_{1v} \quad ; \tag{30}$$

that is, these observations are gridded and X, Z_1, and V_1 can be transformed into the 2-arrays, $\overline{X}$, $\overline{Z}_1$, and $\overline{V}_1$.

Let

$$\overline{X}^* = < \left(A_{1u}^T P_{1u} A_{1u} \right)^{-1} A_{1u}^T P_{1u} \; , \; \left(A_{1v}^T P_{1v} A_{1v} \right)^{-1} A_{1v}^T P_{1v} > \overline{Z}_1$$

and let X* denote the vector formed from the 2-array $\overline{X}^*$. Further, let

$$N_1^{-1} = \left(A_{1u}^T P_{1u} A_{1u}\right)^{-1} \otimes \left(A_{1v}^T P_{1v} A_{1v}\right)^{-1} .$$

Then the solution to eq. (28) can be obtained by substituting these values for X* and N_1^{-1} into eq. (29). It is seen that the number of rows in the matrix A_2 is a critical factor in determining whether this technique is computationally more efficient than solving eq. (28) completely without the use of array algebra and the sequential process.

In the case of having an incomplete grid of observations, construct a set of phony observations together with made-up weights to complete the grid. Let $A_1 X = Z_1 + V_1$ comprise the entire set of real and phony observations with associated weight matrix P_1, and let $A_2 X = Z_2 + V_2$ comprise the set of phony observations with associated weight matrix P_2. Then the least-squares estimate of X as determined by the real observations only is equivalent to the least-squares estimate for the observation equations

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

with associated weight matrix

$$\begin{pmatrix} P_1 & 0 \\ 0 & -P_2 \end{pmatrix} .$$

Hence, the solution can be obtained using the sequential technique by replacing P_2 in eq. (29) with $-P_2$. With the assumptions of eq. (30), one can make use of the array algebra in the first step of the sequential process.

AN EXAMPLE IN PREDICTION

Up to this point only applications of array algebra in least-squares estimation with matrices of full rank have been discussed. Actually, array algebra can be applied in any situation that involves matrix equations pertaining to gridded data (provided the model and the weight matrix are suitable). For instance, the equations of this presentation could be translated into the language of generalized inverses of matrices whenever appropriate (Rauhala, 1974). Also, working equations for prediction and filtering are given in Blaha (1977b). The transformation from monolinear notation to array algebra for a problem in prediction is presented here in the form of an example.

Consider a two-dimensional grid of ground positions, all of whose heights have been observed at the times $t_1, t_2, \ldots, t_r$.
Let

$$z_{ijk} = f(x_i, y_j, t_k) \text{ for } 1 \leq i \leq m,\ 1 \leq j \leq n, \text{ and } 1 \leq k \leq r$$

represent the observed heights, and let s equal the product mnr. Assume further that the covariance between two observations is given by

$$\begin{aligned} \mathrm{Cov}(ijk,uvw) = C_x C_y C_t \ \exp \Big[& - a^2 (x_i - x_u)^2 \\ & - b^2 (y_j - y_v)^2 - c^2 (t_k - t_w)^2 \Big] \quad . \end{aligned}$$

Then the observations can be ordered such that the covariance matrix is of the form

$$\underset{(s\times s)}{\Sigma} = \underset{(m\times m)}{\Sigma_x} \otimes \underset{(n\times n)}{\Sigma_y} \otimes \underset{(r\times r)}{\Sigma_t} \quad .$$

The predicted value $\hat{z}$ of the height z at an arbitrary point (x,y) and at an arbitrary time t that minimizes the variance of the error, $e = \hat{z} - z$, is given (Liebelt, 1967, p. 138) by the equation

$$\underset{(1\times 1)}{\hat{z}} = \underset{(1\times s)(s\times s)(s\times 1)}{\Lambda^T \Sigma^{-1} Z} \qquad (31)$$

where

$$Z = \begin{bmatrix} z_{111} \\ z_{112} \\ \cdot \\ \cdot \\ \cdot \\ z_{11r} \\ z_{121} \\ z_{122} \\ \cdot \\ \cdot \\ \cdot \\ z_{mnr} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_{111} \\ \lambda_{112} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_{121} \\ \lambda_{122} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_{mnr} \end{bmatrix} .$$

Here,

$$\lambda_{ijk} = C_x C_y C_t \exp\left[- a^2 (x_i - x)^2 - b^2 (y_j - y)^2 - c^2 (t_k - t)^2 \right] .$$

Thus,

$$\Lambda = \Lambda_x \otimes \Lambda_y \otimes \Lambda_t = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_m \end{bmatrix} \otimes \begin{bmatrix} \eta_1 \\ \eta_2 \\ \cdot \\ \cdot \\ \cdot \\ \eta_n \end{bmatrix} \otimes \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \cdot \\ \cdot \\ \cdot \\ \zeta_r \end{bmatrix}$$

where

$$\xi_i = C_x \exp\left[- a^2 (x_i - x)^2 \right] \qquad \text{for } 1 \le i \le m,$$

$$\eta_j = C_y \exp\left[- b^2 (y_j - y)^2 \right] \qquad \text{for } 1 \le j \le n, \text{ and}$$

$$\zeta_k = C_t \exp\left[- c^2 (t_k - t)^2 \right] \qquad \text{for } 1 \le k \le r \quad .$$

Hence,

$$\hat{z} = \left(\Lambda_x \otimes \Lambda_y \otimes \Lambda_t\right)^T \left(\Sigma_x \otimes \Sigma_y \otimes \Sigma_t\right)^{-1} Z$$

$$= \left(\Lambda_x^T \Sigma_x^{-1} \otimes \Lambda_y^T \Sigma_y^{-1} \otimes \Lambda_t^T \Sigma_t^{-1}\right) Z$$

or equivalently,

$$\underset{(1x1)}{\hat{z}} = < \underset{(1xm)}{\Lambda_x^T \Sigma_x^{-1}}, \underset{(1xn)}{\Lambda_y^T \Sigma_y^{-1}}, \underset{(1xr)}{\Lambda_t^T \Sigma_t^{-1}} > \underset{(mxnxr)}{\overline{Z}} \tag{32}$$

where $\underset{(mxnxr)}{\overline{Z}}$ is the 3-array formed from the elements of $\underset{(sx1)}{Z}$.

Equation (32) is the formulation in array algebra to be compared against the monolinear formulation of eq. (31). Note, if $m = n = r = 10$, then eq. (32) involves the inversion of three 10×10 matrices, whereas eq. (31) would involve the inversion of a $1{,}000 \times 1{,}000$ matrix.

CONCLUDING REMARKS

In addition to the advantages noted herein, Kratky (1976) and Woltring (1977) point out that the array algebra formulation improves the numerical accuracy of the solution over the equivalent monolinear formulation. This is especially significant because polynomial models have a tendency to be ill-conditioned. Moreover, array algebra can often be used in combination with other techniques employed with matrix equations, such as orthogonal triangularization (Woltring, 1977). Fields in which array algebra has found application include satellite imagery (Kratky, 1976), terrain modeling (Jancaitis and Magee, 1977), and photogrammetry (Woltring, 1977).

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