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Motion Compensation for Shipborne Radars and Lidars

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CONTENTS

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ABSTRACT. Three radars and a lidar were shipborne during the RICO experiment. Their data require correction of the measured Doppler velocities for instrument motion as well as determination of the spatial position above the Earth to which each datum pertains. This report gives a thorough analysis of how to use the data to calculate the required quantities for the RICO experiment. The analysis is general enough to apply to all ship- and aircraft-borne sensors.

1. INTRODUCTION

The NOAA/K cloud and precipitation scanning radar, the University of Miami (Univ. Miami) X-band and W-band cloud radars, the NOAA Mini-MOPA (Master Oscillator Power Amplifier) scanning Doppler lidar, and the NOAA/ETL flux instruments, which include a sonic anemometer, were onboard the Research Vessel (R/V) *Seward Johnson* during the Rain in Cumulus over the Ocean (RICO) experiment. The RICO data from those instruments were obtained from an area northwest of the island of Barbuda during January 2005. The trade-wind swell caused substantial motion of the R/V *Seward Johnson*. Data from all of those instruments need to be corrected to compensate for the ship's motion.

In Section 2, the ship's and the Earth's coordinate systems are defined, as are the angular rate of rotation of the ship and the transformation of coordinates between the ship's and the Earth's coordinate systems. The most essential equation for motion compensation is the relationship between the velocities at any two points on the ship; that relationship is given in Section 2. Section 3 shows how to calculate the radars' and lidar's radial direction in the Earth's coordinate system and how to calculate the antenna's radial velocity for correction of the Doppler velocity. Each datum from radars and the lidar must be associated with the spatial position from which the backscattering occurred. Section 4 shows how to calculate those spatial positions in latitude, longitude, and height. Section 5 compares the correction for motion using the ship's versus the lidar's motion-detection systems. The formulation in Section 5 enables comparison of data from the two motion-detection systems. Such a comparison can determine the accuracy of the systems. The ship's system is called the Position and Orientation System for Marine Vessels, abbreviated POS MV. Section 6 shows how either the ship's or lidar's motion-detection system can be used to correct data from any of the radars or the lidar. Section 6 also shows how to correct the sonic anemometer's data using the ship's motion-detection system. In Section 7, the equations for motion compensation are derived in celestial, Earth, and ship's reference frames. Section 7 thereby connects the rigid body equations to the standard meteorological equations; in particular, the absence of Coriolis force in the ship's reference frame is explained. The survey of the positions of the instruments onboard the ship is essential to motion compensation. The surveyed positions are given in the Appendix.

Motion correction equations in this report were derived at sea without reference to published literature. Subsequent comparison with previous publications (e.g., Edson *et al.*, 1998; Schulz *et al.*, 2005) shows variations of the formulation that arise because of different definitions of coordinate systems, angular rates, and Euler angles. Much confusion could result unless such definitions are clearly stated. A particular distinction is made by Edson *et al.* (1998) between gyro-stabilized systems and strapped-down systems. The lidar's motion compensation system is the strapped-down type. The POS MV system onboard the R/V *Seward Johnson* is the strapped-down type. The angular rates of body motion form a vector. For the RICO experiment, the components of the angular rate vector are given by the lidar's system in the lidar's coordinate system and by the ship's POS MV system in the ship's coordinate system. That is unlike the formulations stated by Edson *et al.* (1998) and Schulz *et al.* (2005); those formulations use components of the angular rate vector in Earth-fixed coordinates.

Airborne Doppler radars require correction for aircraft motion. There is substantial literature on that topic: Bosart et al. (2002), Heymsfield et al (1996), Lee et al. (1994, 2003), Testud et al. (1995). Shipborne phased-array Doppler radars have used electronic phasing of the antenna and ship-motion measurements to retrieve wind profiles (Law et al., 2002). None of those studies considers the difference between the velocity of the phase center of the radar antenna and the velocity at the location where it is measured by an inertial navigation unit. That velocity difference is one essential aspect of the present report and of the studies by Edson *et al.* (1998) and Schulz *et al.* (2005).

The POS MV system allows output of velocities, accelerations, angular rates, and orientation angles for any three points on the ship. For the RICO experiment, those three points were chosen to be the phase center of the NOAA/K radar antenna, the Univ. Miami X-band radar antenna, and the red accelerometer box within the lidar's sea container. Details are given in the Appendix.

The description of the data recorded by the POS MV system is given in Corcoran and Pronk (2003). The user of data must rely on that description for substitution of POS MV data into the equations of this report. The Appendix of this report refers to data at the reference point, and at Sensor 1 and Sensor 2. Those data are from data groups described in Corcoran and Pronk (2003). The data Group 1 is for the reference point; data Group 102 is for Sensor 1; data Group 103 is for Sensor 2. Angular rates and accelerations for all three data groups are given by Corcoran and Pronk (2003) as "longitudinal, transverse, and down" components, which we assume means the ship's coordinate system given in the next section. Corcoran and Pronk (2003) give the velocity components of Group 1 as north, east, and down, which is the Earth's coordinate system described in the next section. Corcoran and Pronk (2003) give the velocity components of Groups 102 and 103 as "along track, across track, and down". Those velocities are explained in Section 2.1. In particular, Tucker (2005) shows that the "down" velocity component recorded in Group 102 is, in fact, positive when the velocity at Sensor 1 is upward; that is also the case for Sensor 2 recorded as Group 103. Therefore, there appears to be a mistake in the report by Corcoran and Pronk (2003). The "down velocity" components must be multiplied by -1 to produce a right-handed coordinate system. Tucker (2005) shows that the "down" component of velocity recorded in Group 1 is correctly positive for downward velocity.

2. THE SHIP'S AND EARTH'S COORDINATE SYSTEM AND MOTION DATA

2.1 Definition of Coordinate Systems

A vector is a quantity that is independent of coordinate systems, and as such, it is denoted without a superscript. Arbitrary vector $\mathbf{U}$ is an example. The same vector $\mathbf{U}$ with its components obtained on the ship's, or Earth's, or lidar's coordinate system is denoted by $\mathbf{U}^S$, or $\mathbf{U}^E$, or $\mathbf{U}^L$, respectively. In data Group 1, the POS MV system outputs velocities in the Earth's coordinate system, but outputs accelerations and angular rates in the ship's coordinate system. In the notation used below, the POS MV outputs include $\mathbf{v}_S^E$, $\mathbf{a}_S^S$, and $\mathbf{\Omega}_S^S$, where the subscript S denotes a quantity measured by the ship's system. Note that vectors are denoted by bold type. Axes 'forward', 'starboard' (i.e., 'right of forward'), and 'downward' constitute, in that order, an orthogonal right-handed coordinate system. It is called the ship's coordinate system. Unit vectors aligned along the axes in the positive sense are denoted by

$$\begin{array}{cc} \text{forward} & \widehat{\mathbf{x}}^S \\ \text{starboard} & \widehat{\mathbf{y}}^S \\ \text{downward} & \widehat{\mathbf{z}}^S \end{array} \tag{1}$$

The caret denotes that a vector has unit magnitude.

The angle about direction forward is 'roll' ϕ, about direction starboard is 'pitch' θ, about direction downward is 'heading' ψ. These angles are known as Euler angles. These angles do not constitute a vector; as such they are not components in any coordinate system. The angular rates are $\frac{d\phi}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, which have units of radians per second. An angular rate vector can be formed by the ordered triple

$$\mathbf{\Omega}^S = \begin{pmatrix} \widehat{\mathbf{x}}^S \cdot \mathbf{\Omega} \\ \widehat{\mathbf{y}}^S \cdot \mathbf{\Omega} \\ \widehat{\mathbf{z}}^S \cdot \mathbf{\Omega} \end{pmatrix} = \begin{pmatrix} \Omega_x^S \\ \Omega_y^S \\ \Omega_z^S \end{pmatrix} = \begin{pmatrix} \frac{d\phi}{dt} \\ \frac{d\theta}{dt} \\ \frac{d\psi}{dt} \end{pmatrix} . \tag{2}$$

The dot product in (2) using the ships' unit vectors $\left(\widehat{\mathbf{x}}^S, \widehat{\mathbf{y}}^S, \widehat{\mathbf{z}}^S\right)$ produces the components of the vector in the ship's coordinate system; hence the superscript S, and the subscripts x, y, and z.

Consider a position vector $\mathbf{r}^S(AB)$ from point A on the ship to point B on the ship, where the components of $\mathbf{r}^S$ are in the ship's coordinate system. Given that the ship can be treated as a rigid body, the velocity of point B relative to point A is the cross product $\mathbf{\Omega}^S \times \mathbf{r}^S(AB)$. The cross product is defined by

$$\mathbf{\Omega} \times \mathbf{r} = \begin{pmatrix} \widehat{\mathbf{x}} \cdot \mathbf{\Omega} \times \mathbf{r} \\ \widehat{\mathbf{y}} \cdot \mathbf{\Omega} \times \mathbf{r} \\ \widehat{\mathbf{z}} \cdot \mathbf{\Omega} \times \mathbf{r} \end{pmatrix} = \begin{pmatrix} \Omega_y r_z - \Omega_z r_y \\ \Omega_z r_x - \Omega_x r_z \\ \Omega_x r_y - \Omega_y r_x \end{pmatrix} , \tag{3}$$

wherein the absence of a superscript denotes that (3) is independent of the coordinate system. Note that the argument (AB) of the vector $\mathbf{r}^S(AB)$ is omitted in (3) for clarity. The cross product (3) is always performed in the ship's coordinate system because $\mathbf{r}^S(AB)$ is a constant in that coordinate system and $\mathbf{\Omega}^S$ is measured in that coordinate system.

The Earth's coordinate system is north, east, down in that order; it is a right-handed coordinate system. Thus, analogous to (1) the unit vectors of the Earth's coordinate system aligned along the axes in the positive sense are denoted by

$$\begin{array}{cc} \text{northward} & \widehat{\mathbf{x}}^E \\ \text{eastward} & \widehat{\mathbf{y}}^E \\ \text{downward} & \widehat{\mathbf{z}}^E \end{array} \tag{4}$$

The motion-detection systems report velocities relative to the solid Earth with the components of that velocity in the Earth's coordinate system. Denote such velocities with superscript E, e.g. $\mathbf{V}^E$. Let the velocity of point A on the ship be denoted by $\mathbf{V}^E(A)$ and the velocity at point B on the ship be denoted by $\mathbf{V}^E(B)$. Given that the ship is a rigid body, the relationship between those two velocities is

$$\mathbf{V}^E(A) = \mathbf{V}^E(B) + \left(\mathbf{\Omega}^S \times \mathbf{r}^S(AB)\right)^E \quad . \tag{5}$$

This relationship between velocities is central to motion compensation of radar and lidar data and can also be used to correct data from the tower-mounted sonic anemometer. As the notation in (5) implies, the vector $\mathbf{\Omega}^S \times \mathbf{r}^S(AB)$ is determined in the ship's coordinate system and must have its components determined in the Earth's coordinate system before it can be added within (5) to the velocity vector in Earth's coordinate system. How to calculate $\left(\mathbf{\Omega}^S \times \mathbf{r}^S(AB)\right)^E$ is the topic of Sections 2.2 and 2.3.

Tucker (2005) uses rigid-body equations (as in this report) to compare POS MV data of Group 1 with that of Group 102. Data in Group 102 contains the along track component of velocity, V_{along}, and across track component of velocity, V_{across}. Tucker (2005) demonstrates that V_{along} and V_{across}, are related to the north and east components of velocity of the position designated as Sensor in the Appendix by

$$\mathbf{V}^E \cdot \widehat{\mathbf{x}}^E = \cos(\Theta)\sqrt{V_{\text{along}}^2 + V_{\text{across}}^2} \tag{6}$$

$$\mathbf{V}^E \cdot \widehat{\mathbf{y}}^E = \sin(\Theta)\sqrt{V_{\text{along}}^2 + V_{\text{across}}^2} \quad , \tag{7}$$

where Θ is the track angle. Tucker (2005) defines track angle as the angle from north to the tangent to the loci of latitude and longitude data written in Group 102. The track angle is positive for track velocity oriented east of north. The algorithm (6)-(7) is not what was intended for Group 102 data. Note that Tucker (2005) shows that the "down velocity" component recorded in Group 102 is, in fact, positive when the velocity at Sensor 1 is upward; that is likely also the case for Sensor 2 recorded as Group 103. Therefore, there is also a sign error in the report by Corcoran and Pronk (2003). Tucker (2005) shows that the "down velocity" component recorded in Group 1 is correctly positive for downward velocity.

2.2 The Euler-Angle Rotation Matrix

The coordinate transformation from the Earth's coordinate system to that of the ship is needed. The transformation matrix is obtained from the product of three rotation matrixes. Transformation matrixes are denoted by bold type. Begin with a Cartesian coordinate system aligned with the Earth's coordinate system (north, east, down). Rotate that coordinate system until it coincides with the ship's coordinate system as follows. First, rotate that coordinate system about its heading axis (i.e., down axis) by ψ; denote the rotation matrix by $\mathbf{C}$.

$$\mathbf{C} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Second, rotate about the new pitch axis by θ; denote the rotation matrix by $\mathbf{B}$.

$$\mathbf{B} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} .$$

Last, rotate about the new roll axis by ϕ; denote the rotation matrix by $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} .$$

The coordinate transformation matrix is the product $\mathbf{ABC}$ as follows:

$$\mathbf{Q} \equiv \mathbf{ABC} = \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & \cos\theta\sin\phi \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi & \cos\theta\cos\phi \end{pmatrix} .$$

The inverse of this matrix is its transpose because it is an orthogonal transformation.

$$\mathbf{Q}^{-1} = \mathbf{Q}^T = \begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} .$$

At each time step, new values of the angles ϕ, θ, ψ are determined by integration of the angular rate vector $\mathbf{\Omega}$. At each time step, the above matrix $\mathbf{Q}$ must be computed from the ϕ, θ, ψ. The definitions above apply to both the lidar's and the ship's coordinate systems.

2.3 How to Determine $\mathbf{\Omega} \times \mathbf{r}$ in the Earth's Coordinate System

The above definition of the Euler angles ϕ, θ, ψ gives the transformation between components of a vector in the ship's coordinate system to the Earth's coordinate system. Let $\mathbf{U}^E$ denote any vector when its components are in the Earth's coordinate system, and let $\mathbf{U}^S$ be that same vector in the ship's coordinate system. Then,

$$\mathbf{U}^S = \mathbf{Q}\mathbf{U}^E$$
$$\mathbf{U}^E = \mathbf{Q}^{-1}\mathbf{U}^S \quad .$$

For use in (5) the required computation at each time step is to calculate the components of $\mathbf{\Omega}^S \times \mathbf{r}^S$, then transform that vector to the Earth's coordinate system at which point it is denoted by $(\boldsymbol{\Omega} \times \mathbf{r})^E$. Since (3) is $\boldsymbol{\Omega}^S \times \mathbf{r}^S = (\boldsymbol{\Omega} \times \mathbf{r})^S$ is in the ship's coordinate system, we obtain $(\boldsymbol{\Omega} \times \mathbf{r})^E$ from

$$(\boldsymbol{\Omega} \times \mathbf{r})^E = \mathbf{Q}^{-1} (\boldsymbol{\Omega} \times \mathbf{r})^S \quad . \tag{8}$$

The above also applies to the lidar's coordinate system for which case superscript S is replaced by L.

3. HOW TO DETERMINE THE RADAR'S RADIAL DIRECTION IN THE EARTH'S COORDINATE SYSTEM AND CALCULATE THE ANTENNA'S RADIAL VELOCITY

First, define the radar radial unit vector in the ship's coordinate system. Assume that the radar's measurement of azimuth φ is level with the main deck, zero degrees azimuth is forward, and the azimuth is positive if the rotation is from forward toward starboard. Assume that the radar's measurement of elevation ε is positive for upward tilt of the antenna from the plane containing the main deck. Then the unit vector pointing outward from the radar's antenna in the ship's coordinate system is

$$\widehat{\mathbf{p}}^S = \begin{pmatrix} \widehat{\mathbf{x}}^S \cdot \widehat{\mathbf{p}}^S \\ \widehat{\mathbf{y}}^S \cdot \widehat{\mathbf{p}}^S \\ \widehat{\mathbf{z}}^S \cdot \widehat{\mathbf{p}}^S \end{pmatrix} = \begin{pmatrix} p_x^S \\ p_y^S \\ p_z^S \end{pmatrix} = \begin{pmatrix} \cos\varphi \cos\varepsilon \\ \sin\varphi \cos\varepsilon \\ -\sin\varepsilon \end{pmatrix} \quad . \tag{9}$$

Verify that $\widehat{\mathbf{p}}^S$ is a unit vector as follows:

$$\widehat{\mathbf{p}}^S \cdot \widehat{\mathbf{p}}^S = \left(\cos^2\varphi + \sin^2\varphi\right) \cos^2\varepsilon + \sin^2\varepsilon = \cos^2\varepsilon + \sin^2\varepsilon = 1 \quad .$$

Similar to (8), the radar radial unit vector in the Earth's coordinate system is

$$\widehat{\mathbf{p}}^E = \mathbf{Q}^{-1} \widehat{\mathbf{p}}^S \quad . \tag{10}$$

Assume the convention that motion toward the radar antenna is negative and motion away from the radar antenna is positive. The radial velocity correction in Earth coordinates is

$$\widehat{\mathbf{p}}^E \cdot \mathbf{v}^E \quad , \tag{11}$$

where $\mathbf{v}^E$ is the velocity at the phase center of the radar's antenna; of course, $\mathbf{v}^E$ is in the Earth's coordinate system. This correction must be added to (not subtracted from) the radar's measurement of radial velocity.

To compute (11) for the Univ. Miami radars we must first determine $\mathbf{v}^E$ from the Group 103 or Group 1 velocity data. From Group 103 data, the along-track and across-track velocity components are substituted into (6-7) and the track angle can be calculated from Group 103 data, as described by Tucker (2005). That calculation gives the north and east components of velocity v_x^E and v_y^E. Finally, the Group 103 "down velocity" is also positive for upward motion, as described in the Introduction. Hence, the Group 103 "down velocity" must be multiplied by -1 to produce v_z^E. Alternatively, to use Group 1 data we must use (5) as described in Section 6.

It is a good approximation that the Univ. Miami radars are pointed straight up relative to the main deck. Therefore,

$$\widehat{\mathbf{p}}^S = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad . \tag{12}$$

The correction to the Univ. Miami radar's Doppler velocity is the negative of the ship's velocity at the location of the radar antenna; namely,

$$\widehat{\mathbf{p}}^E \cdot \mathbf{v}^E = \left(\mathbf{Q}^{-1}\widehat{\mathbf{p}}^S\right) \cdot \mathbf{v}^E =$$

$$\begin{aligned} &(-\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi)\, v^E_{north} \\ &+ (\cos\psi\sin\phi - \cos\phi\sin\theta\sin\psi)\, v^E_{east} \\ &+ (-\cos\theta\cos\phi)\, v^E_{down} \quad . \end{aligned} \tag{13}$$

Here, we used the fact that the Earth's coordinate system is north, east, down in that order. Also used was the following matrix multiplication:

$$\widehat{\mathbf{p}}^E = \mathbf{Q}^{-1}\widehat{\mathbf{p}}^S =$$

$$\begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi \\ \cos\psi\sin\phi - \cos\phi\sin\theta\sin\psi \\ -\cos\theta\cos\phi \end{pmatrix} \quad . \tag{14}$$

More generally, for $\widehat{\mathbf{p}}^S$ given by (9) such that

$$\widehat{\mathbf{p}}^E = \mathbf{Q}^{-1}\widehat{\mathbf{p}}^S =$$

$$\begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} \cos\varphi\cos\varepsilon \\ \sin\varphi\cos\varepsilon \\ -\sin\varepsilon \end{pmatrix} =$$

$$\begin{pmatrix} \begin{bmatrix} \cos\theta\cos\varepsilon\cos\psi\cos\varphi - (\sin\varepsilon)(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi) \\ + (\cos\varepsilon\sin\varphi)(-\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi) \end{bmatrix} \\ \begin{bmatrix} \cos\theta\cos\varepsilon\cos\varphi\sin\psi + (\cos\varepsilon\sin\varphi)(\cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi) \\ -(\sin\varepsilon)(-\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi) \end{bmatrix} \\ [-\cos\theta\cos\phi\sin\varepsilon - \sin\theta\cos\varepsilon\cos\varphi + \cos\theta\cos\varepsilon\sin\phi\sin\varphi] \end{pmatrix} \quad . \tag{15}$$

Therefore,

$$\widehat{\mathbf{p}}^E \cdot \mathbf{v}^E =$$

$$\begin{aligned} &\begin{bmatrix} \cos\theta\cos\varepsilon\cos\psi\cos\varphi - (\sin\varepsilon)(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi) \\ + (\cos\varepsilon\sin\varphi)(-\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi) \end{bmatrix} v^E_{north} \\ &+ \begin{bmatrix} \cos\theta\cos\varepsilon\cos\varphi\sin\psi + (\cos\varepsilon\sin\varphi)(\cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi) \\ -(\sin\varepsilon)(-\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi) \end{bmatrix} v^E_{east} \\ &+ [-\cos\theta\cos\phi\sin\varepsilon - \sin\theta\cos\varepsilon\cos\varphi + \cos\theta\cos\varepsilon\sin\phi\sin\varphi]\, v^E_{down} \quad . \end{aligned} \tag{16}$$

This latter expression is needed to correct the Doppler velocity measured by the NOAA/K radar. The required data for (16) are the angles φ, ε, ϕ, θ, ψ, and velocity $\mathbf{v}^E$ from the Group 1 data. Those data were recorded with the NOAA/K radar data.

There exists software to produce images of NOAA/K radar's reflectivity, Doppler velocity, and other measured parameters as functions of the radar's elevation angle. Those images are distorted by ship motion. However, if the elevation above the Earth's horizon is calculated and used, then the images will be corrected. Similar to (9), the unit vector pointing outward from the radar's antenna in the Earth's coordinate system is

$$\widehat{\mathbf{p}}^E = \begin{pmatrix} \widehat{\mathbf{x}}^E \cdot \widehat{\mathbf{p}}^E \\ \widehat{\mathbf{y}}^E \cdot \widehat{\mathbf{p}}^E \\ \widehat{\mathbf{z}}^E \cdot \widehat{\mathbf{p}}^E \end{pmatrix} = \begin{pmatrix} p_x^E \\ p_y^E \\ p_z^E \end{pmatrix} = \begin{pmatrix} \cos\varphi^E \cos\varepsilon^E \\ \sin\varphi^E \cos\varepsilon^E \\ -\sin\varepsilon^E \end{pmatrix} . \tag{17}$$

Recall that both the ship's and the Earth's coordinate systems have their z axes positive in the downward direction. Therefore, ε^E is elevation angle that is positive upward from the horizon. Also, φ^E is azimuth in radians eastward from north. From (15), the value of p_z^E is already known to be

$$p_z^E = -\cos\theta\cos\phi\sin\varepsilon - \sin\theta\cos\varepsilon\cos\varphi + \cos\theta\cos\varepsilon\sin\phi\sin\varphi.$$

Consequently, from (17) an elevation angle above the Earth's horizon is

$$\varepsilon^E = \arcsin\left(-p_z^E\right). \tag{18}$$

The arcsin function outputs in the range $\pi/2$ to $-\pi/2$. Hence, the elevation angle, ε^E from (18), is elevation above the horizon and does not exceed $\pi/2$ (90 degrees for pointing to zenith); it is negative when the antenna points below the horizon. Note that ε^E depends on radar azimuth φ and elevation ε and on the ship's pitch θ and roll ϕ, but not on the ship's heading ψ. Unlike scans obtained with the radar on solid ground, the azimuth φ^E varies with the ship's motion. Consider the following special case. The radar onboard the R/V *Seward Johnson* was usually pointed to starboard; thus, let the radar azimuth be 90 degrees, $\varphi = \pi/2$, then $p_z^E = \cos\theta\left(-\cos\phi\sin\varepsilon + \cos\varepsilon\sin\phi\right) = \cos\theta\sin\left(\phi - \varepsilon\right)$. The reason for the proportionality to $\cos\theta$ is clear from Section 2.1. Namely, roll is defined about the axis of the ship in its pitched position. If the ship's pitch is zero, $\theta = 0$, such that $\cos\theta = 1$, then $p_z^E = -\cos\phi\sin\varepsilon + \cos\varepsilon\sin\phi\sin\varphi$. Both cases considered together, that is, $\theta = 0$ and $\varphi = \pi/2$, gives $p_z^E = \sin\left(\phi - \varepsilon\right)$ such that $\varepsilon^E = \varepsilon - \phi$, as expected.

Using the ships' angular orientation data, $\widehat{\mathbf{p}}^E$ is known from (15) and ε^E is known from (18). Therefore, we can use (17) to solve for radar azimuth in Earth coordinates, i.e., solve for φ^E, using $p_x^E = \cos\varphi^E\cos\varepsilon^E$ and $p_y^E = \sin\varphi^E\cos\varepsilon^E$. The algorithm is: $\varphi^E = ATAN2(p_y^E, p_x^E)$ which gives φ^E in radians from -π to π, multiply by $(180/\pi)$ to obtain degrees from radians, and if $\varphi^E < 0$, then $\varphi^E = \varphi^E + 360$. Now, φ^E is azimuth of the radar beam which varies, like ship's heading ψ, from 0 degrees at north, 90 degrees at east, 180 degrees at south, 270 degrees at west, and returning to north from west, it approaches 360 degrees. Of course, φ^E varies with time. In summary,

$$\begin{aligned} \varphi^E &= (180/\pi)\, ATAN2(p_y^E, p_x^E) \\ \text{if } \varphi^E < 0, &\text{ then } \varphi^E = \varphi^E + 360 \end{aligned} \tag{19}$$

When the NOAA/K radar's antenna sweeps (at fixed azimuth) from the horizon on the starboard side to beyond radar's vertical (thereby pointing up and toward the port side of the ship), the NOAA/K angle encoder for elevation angle varies beyond 90 degrees. Recall that ε^E from (18) does not exceed $\pi/2$ (90 degrees). Recall our objective of correcting the images of radar data using the existing software that plots data as functions of the radar encoder's elevation angle. For that objective, we calculate a new elevation angle, ε^R, that varies from 0 at the starboard horizon to $\pi/2$ at zenith, and to π at the port-side horizon, and is negative below the starboard horizon. If $\varphi^E < \psi$ or $\varphi^E > \psi + 180$, then the radar beam points to port of the vertical plane that contains the ship's forward axis and contains the radial vector from the Earth's center that intersects the ship's forward axis. Therefore, if $\varphi^E < \psi$ or $\varphi^E > \psi + 180$, then $\varepsilon^R = \pi - \varepsilon^E$, otherwise, $\varepsilon^R = \varepsilon^E$. Then ε^R is the elevation angle that corrects the radar images for ship motion, although ε^E may be used if the radar's encoder elevation is less than about 80 degrees because in that case the radar is almost certainly pointing to starboard.

4. HOW TO CALCULATE THE SPATIAL POSITION OF EACH DATUM OF RADAR AND LIDAR MEASUREMENT

The radius of the Earth is $R_{Earth} = 6378$ km. Note that the GPS geoid is about 45 m above sea level in the RICO study area; that corresponds to an altitude of the water line of the ship of $z = -45$ m. The NOAA/K radar and the lidar measure each datum within their averaging volume at their recorded range r_{ange}, elevation ε, and azimuth φ. The Univ. Miami radars record only range because their elevation is fixed at perpendicular to the ship's deck. The position vector of each datum relative to the antenna in the Earth's coordinate system is the product $\left(\widehat{\mathbf{p}}^E\right) r_{ange}$. See (10) above for the calculation of $\widehat{\mathbf{p}}^E$. The ship's POS MV system gives latitude, longitude and altitude of the center of the radars' antennas and of the lidar's accelerometer box as functions of time: $(lat\,(t)\,, lon\,(t)\,, z\,(t))$. The unit of $lat\,(t)$ and $lon\,(t)$ is decimal degrees, and the unit of $z\,(t)$ is meters. Here, increments of latitude and longitude are calculated in the small angle approximation, e.g., $\sin\left(r_{ange}/R_{Earth}\right) \simeq r_{ange}/R_{Earth}$. The increment of latitude in degrees associated with the north component of $\left(\widehat{\mathbf{p}}^E\right) r_{ange}$ is

$$\Delta lat = \frac{180}{\pi R_{Earth}} \left(\widehat{p}^E_{north}\right) r_{ange} \quad . \tag{20}$$

The increment of longitude associated with the east component of $\left(\widehat{\mathbf{p}}^E\right) r_{ange}$ is

$$\Delta lon = \frac{180}{\pi R_{Earth} \cos\left(lat\right)} \left(\widehat{p}^E_{east}\right) r_{ange} \quad . \tag{21}$$

Altitude written to file by POS MV increases upward despite the fact that the POS MV convention for vectors is (Northward, Eastward, Downward). The negative of the increment of altitude is

$$\Delta z = \left(\widehat{p}^E_{down}\right) r_{ange} \quad . \tag{22}$$

Finally, the latitude, longitude, and altitude of each datum of radars and lidar is

$$lat_{datum} = lat\,(t) + \Delta lat \tag{23}$$

$$lon_{datum} = lon\,(t) + \Delta lon \tag{24}$$

$$z_{datum} = z\,(t) - \Delta z \quad . \tag{25}$$

The height H above the sea surface for each datum can be obtained to within about 0.2 m. In Appendix A the following is given: The main deck is 1.2 m above the sea surface adjacent to the reference point 'ref' marked by the **X** drilled into the deck, and the decks rise by about 0.64 m to the flux tower at the bow. The NOAA/K radar and Univ. Miami radars are close enough to point 'ref' to use the 1.2 m height above sea level of the main deck. The lidar is close enough to the bow so as to estimate its height above sea level as an additional 0.6 m. According to Appendix A, the X-band antenna is 0.11 m higher than the W-band antenna. Although $\widehat{\mathbf{p}}^E$, and therefore also its third component $\widehat{p}^E_{down}$, are the same for the X-band and W-band radars, the X- and W-band radars record their own range, r_{ange}; hence they have different Δz from (22), as well as different Δlat and Δlon from (20) and (21). Using the surveyed heights of the radars and lidar given in Appendix A, we have

$$H_{lidar} = 1.8 + 4.84 - (\Delta z)_{lidar} \tag{26}$$

$$H_{NOAA/K} = 1.2 + 5.30 - (\Delta z)_{NOAA/K} \tag{27}$$

$$H_{W-band} = 1.2 + 2.88 - (\Delta z)_{W-band} \tag{28}$$

$$H_{X-band} = 0.11 + 1.2 + 2.88 - (\Delta z)_{X-band} \quad . \tag{29}$$

These heights are given in meters and are positive for positions above the sea surface; Δz is subtracted above because Δz is the negative of the increment of altitude.

5. COMPARISON OF MOTION CORRECTIONS USING THE SHIP'S DATA VERSUS THE LIDAR'S DATA

There are several reasons to compare the correction of the various instruments' data using the ship's motion-detection data with the same correction using the lidar's motion-detection data. The manufacturer of the lidar's system referred to the lidar's master GPS as the point where the velocity is reported. The accuracy of that claim can be checked by comparison with the ship's motion-detection data. There were dropouts of the lidar's motion-detection when the GPS lost signal. The ship's POS MV data are more continuous. It is important to assess the accuracy of the lidar's motion-detection data as compared to the ship's motion-detection data. In this section, methods for quantifying such comparisons are given. First, more specific notation is needed that includes both the coordinate system and the measuring system for each type of data. That specific notation is described in the next section. Subsequent sections give regression algorithms that determine quantities of interest and examples of uses for the results.

5.1 Notation for Quantities Measured by the Motion Detection Systems

The superscript L will denote a vector's components in the lidar's motion-detection coordinate system. The superscript S will denote a vector's components in the ship's motion-detection coordinate system. The superscript E will denote a vector's components in the Earth's coordinate system; that is, north, east, down. The phrase "in the coordinate system" means that the vector's components are obtained by projection of the vector along the coordinate axes; that is, by inner product of the vector with the unit vectors that are aligned with the positive direction of each axis. The lidar's and ship's coordinate systems have their origins spatially displaced from one another by a fixed separation vector, and they are in fixed orientation relative to one another. The ship's coordinate system is forward (x), starboard (y) and down (z). Both coordinate systems are assumed to be right handed in the order (x, y, z) , which are the names of the axes; (x, y, z) corresponds to and can be replaced by numerical indices $(1, 2, 3)$. For the ship's coordinate system, rotation about the x axis in the right-handed sense is called 'roll' ϕ; rotation about the y axis in the right-handed sense is called 'pitch' θ; rotation about the z axis in the right-handed sense is called 'heading'.

Both lidar and ship coordinate systems are translating relative to the Earth's coordinate system which is fixed relative to the Earth's lithosphere. A given velocity $\mathbf{v}$ is denoted by $\mathbf{v}^S$ when its components are in the ship's coordinate system and by $\mathbf{v}^L$ when its components are in the lidar's coordinate system and by $\mathbf{v}^E$ when its components are in the Earth's coordinate system. Both the lidar's and ship's coordinate systems are rotating relative to a coordinate system that is fixed relative to the Earth's lithosphere by angular rate denoted by $\mathbf{\Omega}^S$ when its components are in the ship's coordinate system and by $\mathbf{\Omega}^L$ when its components are in the lidar's coordinate system. $\mathbf{\Omega}^L$ and $\mathbf{\Omega}^S$ are the same vector because the ship is a rigid body.

The subscript L denotes a quantity measured by the lidar's motion-detection system. The subscript S denotes a quantity measured by the ship's motion-detection system. No subscript appears on quantities calculated from the measured quantities. Although $\mathbf{\Omega}^L$ and $\mathbf{\Omega}^S$ are the same vector, $\mathbf{\Omega}^S_S$ and $\mathbf{\Omega}^L_L$ differ because of measurement errors; that difference is a function of time because of random measurement errors. The analogous statement is not true of $\mathbf{v}^E_S$ and $\mathbf{v}^E_L$ because the velocity at the lidar differs from the velocity at the origin of the coordinate system of the ship's motion-detection system.

5.2 Measured Quantities and Units

If the data are in units other than those stated here, then the data should be changed to the units stated here. In particular, angles and angular rates are in degrees and degrees per second, respectively.

Ω^L_L is the vector of angular rates in radians per second measured by the lidar's motion-detection system;

Ω^S_S is the vector of angular rates in radians per second measured by the ship's motion-detection system;

v^E_L is the velocity vector in meters per second measured by the lidar's motion-detection system with its components in the Earth's coordinate system (north, east, down);

v^E_S is the velocity vector in meters per second measured by the ship's motion-detection system with its components in the Earth's coordinate system (north, east, down);

$(\phi_L, \theta_L, \psi_L)$ are the Euler angles from the lidar's motion-detection system determined from integration of the angular rates Ω^L_L;

$(\phi_S, \theta_S, \psi_S)$ are the Euler angles from the ship's motion-detection system determined from integration of the angular rates Ω^S_S;

t_L is the sequence of times in seconds at which Ω^L_L and $\mathbf{v}^E_L$ are measured;

t_S is the sequence of times in seconds at which Ω^S_S and $\mathbf{v}^E_S$ are measured.

5.3 Quantities to be Determined by Regression: $\mathbf{r}^S(SL)$ and R

$\mathbf{r}^S(SL)$ is the position vector in meters that points from the location on the ship where the velocity is $\mathbf{v}_S^E$ to the location at the lidar where the velocity is $\mathbf{v}_L^E$. Since it has superscript S, the components of $\mathbf{r}^S(SL)$ are in the ship's motion-detection coordinate system. **R** is the 3 by 3 coordinate transformation matrix that transforms a vector from the lidar's coordinate system to the ship's coordinate system. **R** is expressed in terms of the Euler angles of that coordinate transformation by:
first column:

$$\begin{array}{c} \cos\theta^{SL}\cos\psi^{SL} \\ -\cos\phi^{SL}\sin\psi^{SL}+\sin\theta^{SL}\cos\psi^{SL}\sin\phi^{SL} \\ \sin\phi^{SL}\sin\psi^{SL}+\cos\phi^{SL}\sin\theta^{SL}\cos\psi^{SL} \end{array}$$

second column:

$$\begin{array}{c} \cos\theta^{SL}\sin\psi^{SL} \\ \cos\phi^{SL}\cos\psi^{SL}+\sin\theta^{SL}\sin\phi^{SL}\sin\psi^{SL} \\ -\cos\psi^{SL}\sin\phi^{SL}+\cos\phi^{SL}\sin\theta^{SL}\sin\psi^{SL} \end{array}$$

third column:

$$\begin{array}{c} -\sin\theta^{SL} \\ \cos\theta^{SL}\sin\phi^{SL} \\ \cos\theta^{SL}\cos\phi^{SL} \end{array} \tag{30}$$

The quantities to be determined by regression are the 3 components of $\mathbf{r}^S(SL)$ and the 3 Euler angles $\left(\phi^{SL},\theta^{SL},\psi^{SL}\right)$. Of course, **R** is also the matrix of 9 direction cosines formed by the inner products of the unit vectors aligned along the positive axes of the ship's coordinate system with the unit vectors aligned along the positive axes of the lidar's coordinate system.

The time sequences must coincide, but the sample times are not necessarily the same, that is, $t_L \neq t_S$. Assume that the data rates are unequal. If the lidar has the faster time series, then linearly interpolate the values of Ω_L^L and $\mathbf{v}_L^E$ to the slower time sequence t_S. If the ship has the faster time series, then linearly interpolate the values of Ω_S^S and $\mathbf{v}_S^E$ to the times t_L. If the data are synchronized, then no interpolation is needed.

5.4 Equations Used for the Regression

First, consider the equations for the rotation matrix **R**. At every position on the ship, the angular rates are the same because the ship is a rigid body. Therefore, by definition of **R**, and neglecting the measurement errors,

$$\mathbf{\Omega}_S^S = \mathbf{R}\mathbf{\Omega}_L^L \ . \tag{31}$$

This constitutes 3 nonlinear transcendental equations for the 3 unknown Euler angles. It can be solved numerically at each time. The solution will vary with time because of random errors in Ω_S^S and Ω_L^L. It is easier to use all of the time series in a regression routine to obtain the Euler angles and their random errors. It may be yet easier to ignore the Euler angle formulation and obtain all 9 components of **R** by regression.

At this point, we know $\mathbf{R}$. Further, $\mathbf{R}$ should be an orthogonal transformation such that its transpose is its inverse. Therefore, test that

$$\mathbf{R}^T = \mathbf{R}^{-1} \quad . \tag{32}$$

Let $\mathbf{I}$ be the identity matrix. The test of (32) is that the 9 elements of the matrix

$$\mathbf{R}\mathbf{R}^T - \mathbf{I} \tag{33}$$

should be small compared to unity.

Another way to determine $\mathbf{R}$ is to use the Euler angles $(\phi_L, \theta_L, \psi_L)$ and $(\phi_S, \theta_S, \psi_S)$ which are functions of time. By definition, any vector $\mathbf{U}$ has its Earth, lidar, and ship's components related by

$$\begin{aligned} \mathbf{R}(\phi_L, \theta_L, \psi_L)\, \mathbf{U}^E &= \mathbf{U}^L \\ \mathbf{R}(\phi_S, \theta_S, \psi_S)\, \mathbf{U}^E &= \mathbf{U}^S \quad . \end{aligned}$$

where $\mathbf{R}(\phi_L, \theta_L, \psi_L)$ is (30) with (ϕ, θ, ψ) replaced by $(\phi_L, \theta_L, \psi_L)$ and $\mathbf{R}(\phi_S, \theta_S, \psi_S)$ is (30) with (ϕ, θ, ψ) replaced by $(\phi_S, \theta_S, \psi_S)$. Thus,

$$\begin{aligned} \mathbf{U}^E &= \mathbf{R}^T(\phi_L, \theta_L, \psi_L)\, \mathbf{U}^L \\ \mathbf{U}^E &= \mathbf{R}^T(\phi_S, \theta_S, \psi_S)\, \mathbf{U}^S \quad . \end{aligned}$$

Eliminating $\mathbf{U}^E$ gives $\mathbf{R}^T(\phi_L, \theta_L, \psi_L)\, \mathbf{U}^L = \mathbf{R}^T(\phi_S, \theta_S, \psi_S)\, \mathbf{U}^S$, thus,

$$\mathbf{U}^S = \mathbf{R}(\phi_S, \theta_S, \psi_S)\, \mathbf{R}^T(\phi_L, \theta_L, \psi_L)\, \mathbf{U}^L$$

from which the definition of $\mathbf{R}$ in (31) gives

$$\mathbf{R} = \mathbf{R}(\phi_S, \theta_S, \psi_S)\, \mathbf{R}^T(\phi_L, \theta_L, \psi_L) \quad . \tag{34}$$

$\mathbf{R}$ should be independent of time, whereas $\mathbf{R}(\phi_L, \theta_L, \psi_L)$ and $\mathbf{R}(\phi_S, \theta_S, \psi_S)$ vary with time. $\mathbf{R}$ will vary with time because of random errors in $(\phi_L, \theta_L, \psi_L)$ and $(\phi_S, \theta_S, \psi_S)$.

Now, consider the velocity at a given point on the ship. Consider a position vector $\mathbf{r}^S$ that points from the ship's coordinate origin where the velocity is $\mathbf{v}_S^E$, as measured by the ship's motion-detection system, to any other location. Let $\mathbf{V}^E(\mathbf{r})$ be the velocity of that location on the ship. Since $\mathbf{V}^E$ has superscript E, $\mathbf{V}^E$ is expressed in the Earth's coordinate system. The velocity $\mathbf{V}^E(\mathbf{r}^S)$ as determined by the ship's motion-detection system is

$$\mathbf{V}^E(\mathbf{r}^S) = \mathbf{v}_S^E + \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S\right)^E \quad . \tag{35}$$

How to calculate $\left(\mathbf{\Omega}^S \times \mathbf{r}^S\right)^E$ is given in Section 2.2. In particular, the velocity at the lidar's motion-detection system, as determined by the ship's motion-detection system, is

$$\mathbf{V}^E\left(\mathbf{r}^S(SL)\right) = \mathbf{v}_S^E + \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(SL)\right)^E \quad , \tag{36}$$

where the argument (SL) of $\mathbf{r}^S(SL)$ denotes ship origin to lidar origin. Recall that the velocity of that point as measured by the lidar's motion-detection system is $\mathbf{v}_L^E$. Equating the ship's and lidar's measured velocities, i.e., $\mathbf{V}^E\left(\mathbf{r}^S(SL)\right)=\mathbf{v}_L^E$, gives an equation for $\mathbf{r}^S(SL)$, namely,

$$\mathbf{v}_L^E=\mathbf{v}_S^E+\left(\boldsymbol{\Omega}_S^S\times\mathbf{r}^S(SL)\right)^E \quad . \tag{37}$$

Note that we have neglected the measurement errors in $\mathbf{v}_L^E$ and $\mathbf{v}_S^E$; those errors produce errors in $\mathbf{r}^S(SL)$, as do errors in $\mathbf{R}$ and Ω_S. Equation (37) constitutes 3 linear algebraic equations for the 3 unknown components of $\mathbf{r}^S(SL)$. However, the equations are linearly dependent because there is no component of the vector $\boldsymbol{\Omega}_S^S\times\mathbf{r}^S(SL)$ in the directions of vectors $\boldsymbol{\Omega}_S^S$ or $\mathbf{r}^S(SL)$. In other words, $\boldsymbol{\Omega}_S^S\times\mathbf{r}^S(SL)$ can be written as an antisymmetric 3-by-3 matrix multiplying $\mathbf{r}^S(SL)$, but the determinant of any 3-by-3 antisymmetric matrix is zero.

The algorithm to determine the components of $\mathbf{r}^S(SL)=r_x\widehat{\mathbf{x}}^S+r_y\widehat{\mathbf{y}}^S+r_z\widehat{\mathbf{z}}^S$ from (37) is given in Appendix B. The quantities $a(t)$, $b(t)$, $c(t)$, $f(t)$, $v_x(t)$, $v_y(t)$, $w_x(t)$, $w_y(t)$, $w_z(t)$, $u_x(t)$, $u_y(t)$, $u_z(t)$ that appear below are defined in terms of data at time t in Appendix B. The algorithm is:

$$r_x=\frac{-b(t)\pm\sqrt{[b(t)]^2-4a(t)\,c(t)}}{2a(t)} \quad . \tag{38}$$

Then determine r_y from r_x as follows:

$$r_y=\frac{f(t)}{v_y(t)}-r_x\frac{v_x(t)}{v_y(t)} \quad .$$

Obviously $v_y(t)$ must not be zero. Take the solution for r_x and r_y to determine r_z from either

$$r_z=\frac{r_xw_x(t)+r_yw_y(t)}{-w_z(t)}$$

(obviously $w_z(t)$ must not be zero), or

$$r_z=\frac{f(t)-[r_xu_x(t)+r_yu_y(t)]}{u_z(t)}$$

(obviously $u_z(t)$ must not be zero).

We have two solutions for $\mathbf{r}^S(SL)$ corresponding to the + or − sign in (38); the correct solution must be chosen. Now we have equations that determine the components of $\mathbf{r}^S(SL)$ from data at any time t for which $w_z(t)\neq 0$ and $v_z(t)\neq 0$. We can remove outliers from the set of solutions at many times t and average the remaining values over time. Although we defined subscripts $(x,\ y,\ z)$ to denote (forward, starboard, down), cyclic permutation of the subscripts $(x,\ y,\ z)$ could give any one of the components of $\mathbf{r}^S(SL)$ from the quadratic equation; then the other two components are subsequently determined. One advantage of this is that if $w_z(t)=0$ or $v_z(t)=0$ then cyclic permutation of the subscripts allows solution. As the ship moves we get information from different times on the port-starboard and vertical components of $\mathbf{r}^S(SL)$. However, we get information on the forward component of $\mathbf{r}^S(SL)$ only when the ship changes its heading.

6. CORRECTING DATA FOR SHIP MOTION

6.1 Correct the Lidar's Data Using the Lidar's Motion Detection System

The lidar's motion-correction system consists of 4 GPS antennas positioned a few feet beyond the corners of the roof of the lidar's sea container and an accelerometer box attached to the inside ceiling of the sea container. That system reports the velocity in the Earth's coordinate system for some point. It was not clear where that point was at the beginning of the cruise of the R/V *Seward Johnson*. Later, the manufacturer informed us that the velocity was reported for the position of the master GPS antenna; the accuracy of the manufacturer's statement is unknown, but it can be tested with the present formulation. It is the motion of the scanning mirror that contaminates the Doppler measurement (Hill, 2005). Unfortunately the lidar's master GPS antenna is diagonally opposite the scanning mirror. Therefore, consider the determination of the velocity of the scanning mirror using the lidar's data.

Let $\mathbf{r}^L$ be the position vector in the lidar's coordinate system that points from the lidar's master GPS antenna to the scanning mirror. Because of the rotation of the scanning mirror's periscope, $\mathbf{r}^L$ is a function of time. The variant of (37) that gives the velocity of the scanning mirror, i.e., $\mathbf{v}^E(\mathbf{r}_{\text{scanning}})$, from the velocity reported at the master GPS antenna, i.e., $\mathbf{v}_L^E$, is

$$\mathbf{v}^E(\mathbf{r}_{\text{scanning}}) = \mathbf{v}_L^E + \left(\mathbf{\Omega}_L^L \times \mathbf{r}^L\right)^E \quad . \tag{39}$$

Here the angular rate of rotation, if nonzero, of the lidar's periscope is neglected; that effect is given in Hill (2005).

6.2 Correct the Lidar's Data Using the Ship's Motion-Detection System

Multiply (31) by $\mathbf{R}^{-1}$ to obtain

$$\mathbf{\Omega}^L = \mathbf{R}^{-1}\mathbf{\Omega}_S^S \quad . \tag{40}$$

Since we now know $\mathbf{r}^S(SL)$, we use it in (37) to give $\mathbf{v}^E\left(\mathbf{r}^S(SL)\right)$ as a substitute for $\mathbf{v}_L^E$ from

$$\mathbf{v}^E\left(\mathbf{r}^S(SL)\right) = \mathbf{v}_S^E + \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(SL)\right)^E \quad . \tag{41}$$

We thereby obtain the quantities needed by the lidar from the ship's motion-detection, namely $\mathbf{v}^E$ at the lidar from (41) and Ω^L from (40).

6.3 Correct NOAA/K and Univ. Miami's X- and W-Band Doppler Velocities Using the Lidar's Motion-Detection System

It is clear that the velocity of the center of the three radar antennas can be calculated using the lidar's motion-detection system data. The method is similar to that in Section 6.2. The equations will not be given in detail because the POS MV data are recorded for those radar antennas. The use of the lidar's motion-detection data is therefore unlikely.

6.4 Correct the Sonic Anemometer Data Using the Ship's Motion-Detection System

The sonic anemometer's coordinate axes might be tilted relative to the ship's coordinate system. Unlike the ship's coordinate system, that anemometer uses a (forward, left of forward, up) coordinate system. The coordinate transformation matrix between the two coordinate systems, $\mathbf{R}$, can be determined by the methods discussed in Section 5.4. Those possibilities and two others are given below. The equations given here are used by Bariteau (2005) to compare the POS MV dynamics with those of the ETL sonic anemometer's accelerometer system for the RICO data. The ETL sonic system is used to correct the momentum and heat flux for ship motion.

POS MV data are recorded for the location of both the NOAA/K antenna and the lidar's accelerometer box. From the Appendix, the position vector from NOAA/K to the sonic anemometer's volume in meters, is $\mathbf{r}^S(Ks) = \begin{bmatrix} 26.56 & -3.34 & -7.13 \end{bmatrix}$, and the position vector from the lidar's accelerometer box to the sonic anemometer's volume in meters, is $\mathbf{r}^S(Ls) = \begin{bmatrix} 5.35 & -3.32 & -7.59 \end{bmatrix}$. The lower-case symbol s denotes the sonic anemometer.

Let the velocities in the Earth's reference frame at NOAA/K as reported by the POS MV be denoted by $\mathbf{v}_S^E(K)$, and at the lidar by $\mathbf{v}_S^E(L)$. Recall that the subscript S means that those velocities are measured by the ship's POS MV system. Also, to determine $\mathbf{v}_S^E(L)$ from the POS MV's Group 102 data, "along", "across" and "down" velocity components require the calculation and correction described in Section 3 for the Group 102 data. The two determinations of the velocity at the sonic anemometer are denoted by the same symbol $\mathbf{v}_S^E(s)$ and $\mathbf{v}_S^E(s)$. Now, $\mathbf{v}_S^E(s)$ and $\mathbf{v}_S^E(s)$ are given by

$$\mathbf{v}_S^E(s) = \mathbf{v}_S^E(K) + \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ks)\right)^E \tag{42}$$

$$\mathbf{v}_S^E(s) = \mathbf{v}_S^E(L) + \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)^E \quad . \tag{43}$$

Recall from Section 2 that the POS MV outputs $\mathbf{v}_S^E$, $\mathbf{a}_S^S$, and $\mathbf{\Omega}_S^S$. Henceforth, use only the latter equation (43) because the former equation (42) is identical in form to (43). In the ship's coordinate system, $\mathbf{r}^S(Ls)$ is independent of time. Of course, $\mathbf{v}_S^S$, $\mathbf{v}_S^E$, $\mathbf{a}_S^S$, and $\mathbf{\Omega}_S^S$ are time dependent. Acceleration is defined in the Earth's coordinate system as

$$\mathbf{a}_S^E \equiv \frac{d\mathbf{v}_S^E}{dt} \quad .$$

Note that in the notation of Section 7, this derivative is denoted by $\mathbf{a}^E = \frac{d_E \mathbf{v}^E}{dt}$. This time derivative is taken in the coordinate system rotating with the Earth. Differentiating (43)

$$\frac{d\mathbf{v}_S^E(s)}{dt} = \frac{d\mathbf{v}_S^E(L)}{dt} + \frac{d}{dt}\left[\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)^E\right]$$

$$\mathbf{a}_S^E(s) = \mathbf{a}_S^E(L) + \frac{d}{dt}\left[\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)^E\right] \quad . \tag{44}$$

Multiply (44) by $\mathbf{Q}$ to obtain the equation in the ship's coordinate system

$$\mathbf{a}_S^S(s) = \mathbf{a}_S^S(L) + \mathbf{Q}\frac{d}{dt}\left[\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)^E\right] \quad .$$

By definition of $\mathbf{Q}$, for any vector $\mathbf{U}^E = \mathbf{Q}^{-1}\mathbf{U}^S$; thus,

$$\mathbf{a}_S^S(s) = \mathbf{a}_S^S(L) + \mathbf{Q}\frac{d}{dt}\left[\mathbf{Q}^{-1}\left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)\right] \quad . \tag{45}$$

Finally, (45) is the equation to be evaluated by computer. The POS MV system gives data for the right-hand side of (45), namely for $\mathbf{a}_S^S(L)$ and $\boldsymbol{\Omega}_S^S$ and the Euler angles that determine $\mathbf{Q}$ and $\mathbf{Q}^{-1}$ (see Section 2.1); also, $\mathbf{r}^S(Ls)$ is given above. The algorithm to calculate $\mathbf{Q}\frac{d}{dt}\left[\mathbf{Q}^{-1}\left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)\right]$ is to calculate $\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(Ls)$, multiply by the matrix $\mathbf{Q}^{-1}$, perform a numerical time derivative, then multiply by matrix $\mathbf{Q}$.

The second term in (45) can be evaluated by another means because

$$\mathbf{Q}\frac{d}{dt}\left[\mathbf{Q}^{-1}\left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)\right] = \mathbf{Q}\frac{d\mathbf{Q}^{-1}}{dt}\left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(Ls)\right) + \frac{d\boldsymbol{\Omega}_S^S}{dt} \times \mathbf{r}^S(Ls) \tag{46}$$

wherein $\mathbf{Q}\mathbf{Q}^{-1} = \mathbf{I}$ was used. On the right-hand side of (46) the only numerical derivative required is $\frac{d\boldsymbol{\Omega}_S^S}{dt}$. The expression for $\mathbf{Q}^{-1}$ in Section 2.1 can be differentiated with respect to time to provide the following analytic expression:

The derivative of the first column of $\mathbf{Q}^{-1}$ is

$$\frac{d}{dt}\begin{pmatrix} \cos\theta\cos\psi \\ \cos\theta\sin\psi \\ -\sin\theta \end{pmatrix} =$$

$$\begin{pmatrix} -(\sin\theta\cos\psi)\frac{d\theta}{dt} - (\cos\theta\sin\psi)\frac{d\psi}{dt} \\ (\cos\theta\cos\psi)\frac{d\psi}{dt} - (\sin\theta\sin\psi)\frac{d\theta}{dt} \\ -(\cos\theta)\frac{d\theta}{dt} \end{pmatrix} \quad .$$

The derivative of the second column of $\mathbf{Q}^{-1}$ is

$$\frac{d}{dt}\begin{pmatrix} -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi \\ \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi \\ \cos\theta\sin\phi \end{pmatrix} =$$

$$\begin{pmatrix} \begin{bmatrix} -(\cos\phi\cos\psi)\frac{d\psi}{dt} + (\sin\phi\sin\psi)\frac{d\phi}{dt} + (\cos\theta\cos\psi\sin\phi)\frac{d\theta}{dt} \\ +(\cos\phi\sin\theta\cos\psi)\frac{d\phi}{dt} - (\sin\theta\sin\phi\sin\psi)\frac{d\psi}{dt} \end{bmatrix} \\ \begin{bmatrix} -(\cos\psi\sin\phi)\frac{d\phi}{dt} - (\cos\phi\sin\psi)\frac{d\psi}{dt} + (\cos\theta\sin\phi\sin\psi)\frac{d\theta}{dt} \\ +(\cos\phi\sin\theta\sin\psi)\frac{d\phi}{dt} + (\sin\theta\cos\psi\sin\phi)\frac{d\psi}{dt} \end{bmatrix} \\ \left[(\cos\theta\cos\phi)\frac{d\phi}{dt} - (\sin\theta\sin\phi)\frac{d\theta}{dt}\right] \end{pmatrix} \quad .$$

The derivative of the third column of $\mathbf{Q}^{-1}$ is

$$\frac{d}{dt}\begin{pmatrix} \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ \cos\theta\cos\phi \end{pmatrix} =$$

$$\begin{pmatrix} \begin{bmatrix} (\cos\phi\sin\psi)\frac{d\phi}{dt} + (\cos\psi\sin\phi)\frac{d\psi}{dt} + (\cos\theta\cos\phi\cos\psi)\frac{d\theta}{dt} \\ -(\sin\theta\cos\psi\sin\phi)\frac{d\phi}{dt} - (\cos\phi\sin\theta\sin\psi)\frac{d\psi}{dt} \end{bmatrix} \\ \begin{bmatrix} -(\cos\phi\cos\psi)\frac{d\phi}{dt} + (\sin\phi\sin\psi)\frac{d\psi}{dt} + (\cos\theta\cos\phi\sin\psi)\frac{d\theta}{dt} \\ +(\cos\phi\sin\theta\cos\psi)\frac{d\psi}{dt} - (\sin\theta\sin\phi\sin\psi)\frac{d\phi}{dt} \end{bmatrix} \\ \left[-(\cos\phi\sin\theta)\frac{d\theta}{dt} - (\cos\theta\sin\phi)\frac{d\phi}{dt}\right] \end{pmatrix} \quad .$$

From (2)

$$\boldsymbol{\Omega}_S^S = \begin{pmatrix} \frac{d\phi}{dt} \\ \frac{d\theta}{dt} \\ \frac{d\psi}{dt} \end{pmatrix} \quad .$$

Thus, the angular rates that appear in the above derivative $\frac{d\mathbf{Q}^{-1}}{dt}$, namely, $\frac{d\phi}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, are the components of $\mathbf{\Omega}_S^S$.

The second term in (45) can be evaluated by a third, somewhat simpler, means. Analytically perform the matrix product $\mathbf{Q}\frac{d\mathbf{Q}^{-1}}{dt}$ and define a new matrix $\mathbf{M}$ as follows:

$$\mathbf{Q}\frac{d\mathbf{Q}^{-1}}{dt} \equiv \mathbf{M} = \begin{pmatrix} 0 & M_{12} & M_{13} \\ -M_{12} & 0 & M_{23} \\ -M_{13} & -M_{23} & 0 \end{pmatrix} \quad ,$$

where
$M_{12} \equiv \frac{d\theta}{dt}\sin\phi - \frac{1}{2}\frac{d\psi}{dt}\left[\cos(\theta+\phi) + \cos(\theta-\phi)\right]$,
$M_{13} \equiv \frac{d\theta}{dt}\cos\phi + \frac{1}{2}\frac{d\psi}{dt}\left[\sin(\theta+\phi) - \sin(\theta-\phi)\right]$, and
$M_{23} \equiv \frac{d\psi}{dt}\sin\theta - \frac{d\phi}{dt}$.

Then, the second term in (45), which is (46), becomes

$$\mathbf{Q}\frac{d}{dt}\left[\mathbf{Q}^{-1}\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)\right] = \mathbf{M}\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right) + \frac{d\mathbf{\Omega}_S^S}{dt} \times \mathbf{r}^S(Ls) \quad . \tag{47}$$

Let us express (45) in yet a fourth way as follows:

$$\begin{aligned}
\mathbf{a}_S^S(s) &= \mathbf{a}_S^S(L) + \mathbf{Q}\frac{d}{dt}\left[\mathbf{Q}^{-1}\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)\right] \\
&= \mathbf{a}_S^S(L) + \mathbf{Q}\frac{d}{dt}\left[\mathbf{\Omega}_S^E \times \mathbf{r}^E(Ls)\right] \\
&= \mathbf{a}_S^S(L) + \mathbf{Q}\left[\frac{d\mathbf{\Omega}_S^E}{dt} \times \mathbf{r}^E(Ls) + \mathbf{\Omega}_S^E \times \frac{d\mathbf{r}^E(Ls)}{dt}\right] \quad .
\end{aligned} \tag{48}$$

Let $\mathbf{x}^E(L)$ and $\mathbf{x}^E(s)$ denote the spatial positions of the lidar and sonic anemometer, respectively, in the Earth's coordinate system. Because $\mathbf{r}^E(Ls) = \mathbf{x}^E(s) - \mathbf{x}^E(L)$, we have

$$\frac{d\mathbf{r}^E(Ls)}{dt} = \frac{d\mathbf{x}^E(s)}{dt} - \frac{d\mathbf{x}^E(L)}{dt} \tag{49}$$

$$\begin{aligned}
&= \mathbf{v}_S^E(s) - \mathbf{v}_S^E(L) \\
&= \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)^E \quad ,
\end{aligned} \tag{50}$$

where (43) was used to obtain the last expression. Substitution of (50) in (48) gives

$$\begin{aligned}
\mathbf{a}_S^S(s) &= \mathbf{a}_S^S(L) + \mathbf{Q}\left[\frac{d\mathbf{\Omega}_S^E}{dt} \times \mathbf{r}^E(Ls) + \mathbf{\Omega}_S^E \times \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)^E\right] \\
&= \mathbf{a}_S^S(L) + \frac{d\mathbf{\Omega}_S^S}{dt} \times \mathbf{r}^S(Ls) + \mathbf{\Omega}_S^S \times \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right) \quad .
\end{aligned} \tag{51}$$

To summarize, equations (45), (46), (47), (51) give us four means by which to calculate $\mathbf{a}_S^S(s)$; they are:

$$\begin{aligned}
\mathbf{a}_S^S(s) &= \mathbf{a}_S^S(L) + \mathbf{Q}\frac{d}{dt}\left[\mathbf{Q}^{-1}\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right)\right] \\
&= \mathbf{a}_S^S(L) + \frac{d\mathbf{\Omega}_S^S}{dt} \times \mathbf{r}^S(Ls) + \mathbf{Q}\frac{d\mathbf{Q}^{-1}}{dt}\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right) \\
&= \mathbf{a}_S^S(L) + \frac{d\mathbf{\Omega}_S^S}{dt} \times \mathbf{r}^S(Ls) + \mathbf{M}\left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right) \\
&= \mathbf{a}_S^S(L) + \frac{d\mathbf{\Omega}_S^S}{dt} \times \mathbf{r}^S(Ls) + \mathbf{\Omega}_S^S \times \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(Ls)\right) \quad .
\end{aligned}$$

The coordinate transformation matrix between the sonic anemometer's coordinate system and the ship's coordinate system, $\mathbf{R}$, can be determined by the methods in Section 5.4. For the method of equation (34), (34) is replaced by

$$\mathbf{R} = \mathbf{R}(\phi_S, \theta_S, \psi_S)\, \mathbf{R}^T(\phi_s, \theta_s, \psi_s) \quad , \tag{52}$$

where $(\phi_s, \theta_s, \psi_s)$ are the Euler angles measured by the anemometer's system. To evaluate the right-hand side of (52), the Euler angles are substituted into (30). Relationship (52) holds even though the Euler angles are defined differently for the ship's orientation as compared to the anemometer's orientation. For the method of equation (31), (31) is replaced by

$$\mathbf{\Omega}_S^S = \mathbf{R}\mathbf{\Omega}_s^s \quad ,$$

where $\mathbf{\Omega}_s^s$ is the angular rate measured by the anemometer's system, as denoted by superscript s, with its components in the sonic anemometer's coordinate system, as denoted by subscript s. One could also use acceleration or velocity because, to within measurement errors,

$$\mathbf{a}_S^S(s) = \mathbf{R}\mathbf{a}_s^s \quad ,$$

and

$$\mathbf{v}_S^S(s) = \mathbf{R}\mathbf{v}_s^s \quad ,$$

where $\mathbf{a}_s^s$ and $\mathbf{v}_s^s$ are acceleration and velocity measured by the anemometer's system with its components in the sonic anemometer's coordinate system.

7. METEOROLOGICAL RELATIONSHIPS

Section 5 deals with velocities and accelerations determined by the POS MV and the lidar's motion-detection system. As such, the Coriolis force is absent. The question arises as to the relationship of the above analysis to the standard meteorological equations for velocity and acceleration in a rotating reference frame. Those standard equations can be found in Chapter IV of Fleagle and Businger (1980).

We have previously defined, in (4), the Earth's coordinate system as being eastward, northward, down. As such, the Earth's coordinate system rotates once per day. Now define an almost inertial coordinate system called the celestial reference frame. Let the celestial reference frame be the reference frame with its origin at the Earth's center and rotating once per year with its rotation axis perpendicular to the plane of the Earth's orbit around the sun. The centrifugal force of that yearly rotation about the sun is approximately equal and opposite to the sun's gravitational force on the Earth. Hence, the celestial reference frame is an almost inertial reference frame. The centrifugal force caused by the Earth's yearly rotation about the sun is about 1/5 of the centrifugal force at the equator caused by the Earth's daily rotation. Let the rotating reference frame be any reference frame rotating relative to the celestial reference frame. The rotating reference frame need not be rotating with the Earth's lithosphere. In fact, the relevant rotating reference frame for present purposes is the ship's coordinate system (1); the ship's coordinate system rotates with the Earth once per day and rotates relative to the Earth's coordinate system because of ocean motion and ship maneuvering. Henceforth, the rotating reference frame will be referred to as the ship's reference frame. The lidar's coordinate system is at fixed angles relative to the ship's coordinate system, so the present discussion also applies to the lidar's reference frame. Let d_C/dt and d_S/dt denote time derivatives in the celestial and ship's reference frames, respectively. The standard relationship between the time derivatives is

$$\frac{d_C}{dt} = \frac{d_S}{dt} + \mathbf{\Omega}\times \quad , \tag{53}$$

where $\mathbf{\Omega}$ is the angular rate of the ship's reference frame relative to the celestial reference frame; $\mathbf{\Omega}$ is the sum of the Earth's rotation-rate vector relative to the celestial reference frame $\mathbf{\Omega}^E$ and that of the ship relative to the Earth's coordinate system, $\mathbf{\Omega}^S$, i.e.,

$$\mathbf{\Omega} = \mathbf{\Omega}^E + \mathbf{\Omega}^S \quad . \tag{54}$$

It is important to note that in this section the superscript does not denote the coordinate system in which the components of a vector are expressed. The Earth's rotation-rate vector causes the ship to rotate once per day. On the basis of the definition of the Euler angles for the POS MV and lidar motion-detection systems, those systems' output of the angular rate vector does not include the Earth's rotation-rate vector; those systems output $\mathbf{\Omega}^S$.

Let $\mathbf{x}$ be a position vector from the center of the Earth. Applying (53) twice gives

$$\frac{d_C\mathbf{x}}{dt} = \frac{d_S\mathbf{x}}{dt} + \mathbf{\Omega}\times\mathbf{x} \tag{55}$$

$$\begin{aligned}\frac{d_C^2\mathbf{x}}{dt^2} &= \left(\frac{d_S}{dt} + \mathbf{\Omega}\times\right)\frac{d_S\mathbf{x}}{dt} + \left(\frac{d_S}{dt} + \mathbf{\Omega}\times\right)(\mathbf{\Omega}\times\mathbf{x}) \\ &= \frac{d_S^2\mathbf{x}}{dt^2} + 2\mathbf{\Omega}\times\frac{d_S\mathbf{x}}{dt} + \frac{d_S\mathbf{\Omega}}{dt}\times\mathbf{x} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x}) \quad .\end{aligned} \tag{56}$$

Note that (55) relates velocities, $d_C\mathbf{x}/dt$ and $d_S\mathbf{x}/dt$, in the celestial and ship's reference frames, respectively; (56) relates accelerations, $d_C^2\mathbf{x}/dt^2$ and $d_S^2\mathbf{x}/dt^2$, in the celestial and ship's reference frames, respectively. Note the nomenclature:

$2\mathbf{\Omega}\times\frac{d_S\mathbf{x}}{dt}$ Coriolis acceleration

$\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x})$ centrifugal acceleration

Now consider the special case in which $\mathbf{x}$ is a position vector from the center of the Earth to any point fixed on the ship. Two examples of such a point are the sonic anemometer and the lidar's accelerometer box. In the meteorological case, the air flows relative to the rotating coordinate system. Unlike the meteorological case, a fixed point on the ship is rotating with the ship's reference frame such that

$$\frac{d_S \mathbf{x}}{dt} = 0 \text{ and } \frac{d_S^2 \mathbf{x}}{dt^2} = 0 \quad .$$

Hence, Coriolis acceleration vanishes; (55) and (56) become

$$\frac{d_C \mathbf{x}}{dt} = \mathbf{\Omega} \times \mathbf{x} \tag{57}$$

$$\frac{d_C^2 \mathbf{x}}{dt^2} = \frac{d_S \mathbf{\Omega}}{dt} \times \mathbf{x} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) \quad . \tag{58}$$

Now apply (57) and (58) to two fixed points on the ship denoted by $\mathbf{x}_1$ and $\mathbf{x}_2$, and define their relative position to be $\mathbf{r} \equiv \mathbf{x}_2 - \mathbf{x}_1$. The velocities and accelerations in the celestial reference frame are denoted by

$$\mathbf{v}^C(\mathbf{x}_1) \equiv \frac{d_C \mathbf{x}_1}{dt} \text{ and } \mathbf{v}^C(\mathbf{x}_2) \equiv \frac{d_C \mathbf{x}_2}{dt}$$

$$\mathbf{a}^C(\mathbf{x}_1) \equiv \frac{d_C^2 \mathbf{x}_1}{dt^2} \text{ and } \mathbf{a}^C(\mathbf{x}_2) \equiv \frac{d_C^2 \mathbf{x}_2}{dt^2} \quad .$$

Apply (57) to both points $\mathbf{x}_1$ and $\mathbf{x}_2$, and subtract the two equations; then

$$\mathbf{v}^C(\mathbf{x}_2) - \mathbf{v}^C(\mathbf{x}_1) = \mathbf{\Omega} \times \mathbf{r} \quad . \tag{59}$$

Likewise, (58) gives

$$\mathbf{a}^C(\mathbf{x}_2) - \mathbf{a}^C(\mathbf{x}_1) = \frac{d_S \mathbf{\Omega}}{dt} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \quad . \tag{60}$$

We now return to (55) and (56) and consider the same point $\mathbf{x}$, which is a position vector from the center of the Earth to any point fixed on the ship, but use the Earth's coordinate system on the right-hand side of (55) and (56). Hence, the angular rate vector is that of the Earth's rotation, denoted by $\mathbf{\Omega}^E$, which has magnitude equal to once per day. Thus, (55) and (56) give

$$\frac{d_C \mathbf{x}}{dt} = \frac{d_E \mathbf{x}}{dt} + \mathbf{\Omega}^E \times \mathbf{x} \tag{61}$$

$$\frac{d_C^2 \mathbf{x}}{dt^2} = \frac{d_E^2 \mathbf{x}}{dt^2} + 2\mathbf{\Omega}^E \times \frac{d_E \mathbf{x}}{dt} + \frac{d_E \mathbf{\Omega}^E}{dt} \times \mathbf{x} + \mathbf{\Omega}^E \times \left(\mathbf{\Omega}^E \times \mathbf{x}\right) \quad .$$

The angular rate $\mathbf{\Omega}^E$ in the Earth's reference frame is constant, i.e. $\frac{d_E \mathbf{\Omega}^E}{dt} = 0$; hence, the above is

$$\frac{d_C^2 \mathbf{x}}{dt^2} = \frac{d_E^2 \mathbf{x}}{dt^2} + 2\mathbf{\Omega}^E \times \frac{d_E \mathbf{x}}{dt} + \mathbf{\Omega}^E \times \left(\mathbf{\Omega}^E \times \mathbf{x}\right) \quad . \tag{62}$$

Again considering two points on the ship, $\mathbf{x}_1$ and $\mathbf{x}_2$ and $\mathbf{r} \equiv \mathbf{x}_2 - \mathbf{x}_1$, we have

$$\mathbf{v}^C(\mathbf{x}_2) - \mathbf{v}^C(\mathbf{x}_1) = \frac{d_E \mathbf{r}}{dt} + \mathbf{\Omega}^E \times \mathbf{r} \tag{63}$$

$$\mathbf{a}^C(\mathbf{x}_2) - \mathbf{a}^C(\mathbf{x}_1) = \frac{d_E^2 \mathbf{r}}{dt^2} + 2\mathbf{\Omega}^E \times \frac{d_E \mathbf{r}}{dt} + \mathbf{\Omega}^E \times \left(\mathbf{\Omega}^E \times \mathbf{r}\right) \quad . \tag{64}$$

The velocities and accelerations in the Earth's reference frame are denoted by

$$\mathbf{v}^E(\mathbf{x}_1) \equiv \frac{d_E \mathbf{x}_1}{dt} \text{ and } \mathbf{v}^E(\mathbf{x}_2) \equiv \frac{d_E \mathbf{x}_2}{dt} \text{ such that } \mathbf{v}^E(\mathbf{x}_2) - \mathbf{v}^E(\mathbf{x}_1) = \frac{d_E \mathbf{r}}{dt} \tag{65}$$

$$\mathbf{a}^E(\mathbf{x}_1) \equiv \frac{d_E^2 \mathbf{x}_1}{dt^2} = \frac{d_E \mathbf{v}^E(\mathbf{x}_1)}{dt} \text{ and } \mathbf{a}^E(\mathbf{x}_2) \equiv \frac{d_E^2 \mathbf{x}_2}{dt^2} \text{ such that } \mathbf{a}^E(\mathbf{x}_2) - \mathbf{a}^E(\mathbf{x}_1) = \frac{d_E^2 \mathbf{r}}{dt^2} \quad . \tag{66}$$

Equate (59) and (63) and equate (60) to (64) to obtain

$$\frac{d_E \mathbf{r}}{dt} + \boldsymbol{\Omega}^E \times \mathbf{r} = \boldsymbol{\Omega} \times \mathbf{r} \tag{67}$$

$$\frac{d_E^2 \mathbf{r}}{dt^2} + 2\boldsymbol{\Omega}^E \times \frac{d_E \mathbf{r}}{dt} + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) = \frac{d_S \boldsymbol{\Omega}}{dt} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \quad . \tag{68}$$

Hence, from (67) and (2)

$$\mathbf{v}^E(\mathbf{x}_2) - \mathbf{v}^E(\mathbf{x}_1) + \boldsymbol{\Omega}^E \times \mathbf{r} = \left[\boldsymbol{\Omega}^E + \boldsymbol{\Omega}^S\right] \times \mathbf{r} \quad ,$$

which is

$$\mathbf{v}^E(\mathbf{x}_2) - \mathbf{v}^E(\mathbf{x}_1) = \boldsymbol{\Omega}^S \times \mathbf{r} \quad , \tag{69}$$

which has the same meaning as (42) and (43). Also, from (68) and (2)

$$\begin{aligned} &\mathbf{a}^E(\mathbf{x}_2) - \mathbf{a}^E(\mathbf{x}_1) + 2\boldsymbol{\Omega}^E \times \left[\mathbf{v}^E(\mathbf{x}_2) - \mathbf{v}^E(\mathbf{x}_1)\right] + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) \\ &= \frac{d_S\left[\boldsymbol{\Omega}^E + \boldsymbol{\Omega}^S\right]}{dt} \times \mathbf{r} + \left[\boldsymbol{\Omega}^E + \boldsymbol{\Omega}^S\right] \times \left(\left[\boldsymbol{\Omega}^E + \boldsymbol{\Omega}^S\right] \times \mathbf{r}\right) \quad . \end{aligned} \tag{70}$$

Apply (53) to both ship and Earth reference frames and equate the two expressions for d_C/dt; we then have

$$\begin{aligned} \frac{d_C}{dt} &= \frac{d_S}{dt} + \boldsymbol{\Omega} \times \\ \frac{d_C}{dt} &= \frac{d_E}{dt} + \boldsymbol{\Omega}^E \times \quad , \end{aligned}$$

so

$$\frac{d_E}{dt} + \boldsymbol{\Omega}^E \times = \frac{d_S}{dt} + \boldsymbol{\Omega} \times \quad ,$$

so

$$\begin{aligned} \frac{d_S}{dt} &= \frac{d_E}{dt} + \left[\boldsymbol{\Omega}^E - \boldsymbol{\Omega}\right] \times \\ \frac{d_S}{dt} &= \frac{d_E}{dt} - \boldsymbol{\Omega}^S \times \quad . \end{aligned} \tag{71}$$

Substitute (69) and (71) into (70), and recall that $\frac{d_E \boldsymbol{\Omega}^E}{dt} = 0$; then

$$\begin{aligned} &\mathbf{a}^E(\mathbf{x}_2) - \mathbf{a}^E(\mathbf{x}_1) + 2\boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) \\ &= -\left(\boldsymbol{\Omega}^S \times \boldsymbol{\Omega}^E\right) \times \mathbf{r} + \frac{d_E \boldsymbol{\Omega}^S}{dt} \times \mathbf{r} - \left(\boldsymbol{\Omega}^S \times \boldsymbol{\Omega}^S\right) \times \mathbf{r} \\ &+ \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) \quad . \end{aligned}$$

Simplify, noting that $\boldsymbol{\Omega}^S \times \boldsymbol{\Omega}^S = 0$, then

$$\begin{aligned} &\mathbf{a}^E(\mathbf{x}_2) - \mathbf{a}^E(\mathbf{x}_1) + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) \\ &= -\left(\boldsymbol{\Omega}^S \times \boldsymbol{\Omega}^E\right) \times \mathbf{r} + \frac{d_E \boldsymbol{\Omega}^S}{dt} \times \mathbf{r} + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) \quad . \end{aligned}$$

Using $-\left(\boldsymbol{\Omega}^S \times \boldsymbol{\Omega}^E\right) \times \mathbf{r} = \mathbf{r} \times \left(\boldsymbol{\Omega}^S \times \boldsymbol{\Omega}^E\right)$ and the identity for the triple cross product, we have

$$\begin{aligned} &-\boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) - \left(\boldsymbol{\Omega}^S \times \boldsymbol{\Omega}^E\right) \times \mathbf{r} + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) \\ &= -\left[\boldsymbol{\Omega}^S\left(\boldsymbol{\Omega}^E \cdot \mathbf{r}\right) - \mathbf{r}\left(\boldsymbol{\Omega}^E \cdot \boldsymbol{\Omega}^S\right)\right] + \left[\boldsymbol{\Omega}^S\left(\mathbf{r} \cdot \boldsymbol{\Omega}^E\right) - \boldsymbol{\Omega}^E\left(\mathbf{r} \cdot \boldsymbol{\Omega}^S\right)\right] + \left[\boldsymbol{\Omega}^E\left(\boldsymbol{\Omega}^S \cdot \mathbf{r}\right) - \mathbf{r}\left(\boldsymbol{\Omega}^S \cdot \boldsymbol{\Omega}^E\right)\right] \\ &= \mathbf{0} \quad . \end{aligned} \tag{72}$$

Thus,

$$\mathbf{a}^E(\mathbf{x}_2) - \mathbf{a}^E(\mathbf{x}_1) = \frac{d_E \boldsymbol{\Omega}^S}{dt} \times \mathbf{r} + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) \quad ,$$

which has the same meaning as (44). We can also substitute (71) and (72) into (60) to obtain

$$\begin{aligned} &\mathbf{a}^C(\mathbf{x}_2) - \mathbf{a}^C(\mathbf{x}_1) \\ &= \frac{d_E \boldsymbol{\Omega}}{dt} \times \mathbf{r} - \left(\boldsymbol{\Omega}^S \times \boldsymbol{\Omega}^E\right) \times \mathbf{r} + \left\{\boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right)\right\} \\ &= \frac{d_E \boldsymbol{\Omega}}{dt} \times \mathbf{r} - \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) \\ &\quad + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) \\ &= \frac{d_E \boldsymbol{\Omega}}{dt} \times \mathbf{r} + 2\boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) \quad . \end{aligned}$$

Since

$$\begin{aligned} \frac{d_E \boldsymbol{\Omega}}{dt} \times \mathbf{r} &= \frac{d_E \boldsymbol{\Omega}^S}{dt} \times \mathbf{r} + \frac{d_E \boldsymbol{\Omega}^E}{dt} \times \mathbf{r} \\ &= \frac{d_E \boldsymbol{\Omega}^S}{dt} \times \mathbf{r} \end{aligned}$$

and substituting (69), we have

$$\begin{aligned} \mathbf{a}^C(\mathbf{x}_2) - \mathbf{a}^C(\mathbf{x}_1) &= \frac{d_E \boldsymbol{\Omega}^S}{dt} \times \mathbf{r} + 2\boldsymbol{\Omega}^E \times \left[\mathbf{v}^E(\mathbf{x}_2) - \mathbf{v}^E(\mathbf{x}_1)\right] + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) + \boldsymbol{\Omega}^S \times \left(\boldsymbol{\Omega}^S \times \mathbf{r}\right) \\ &= \mathbf{a}^E(\mathbf{x}_2) - \mathbf{a}^E(\mathbf{x}_1) + 2\boldsymbol{\Omega}^E \times \left[\mathbf{v}^E(\mathbf{x}_2) - \mathbf{v}^E(\mathbf{x}_1)\right] + \boldsymbol{\Omega}^E \times \left(\boldsymbol{\Omega}^E \times \mathbf{r}\right) \quad . \end{aligned}$$

This last equation contains the Coriolis acceleration and the Earth's centrifugal force.

7.1 Summary for Meteorological Relationships

In summary, beginning with an almost inertial celestial reference frame, we have the following:

For a position vector $\mathbf{x}$ from the center of the Earth to any point fixed on the ship, (57) and (58) give

$$\mathbf{v}^{C}(\mathbf{x}) = \boldsymbol{\Omega} \times \mathbf{x}$$

$$\mathbf{a}^{C}(\mathbf{x}) = \frac{d_S\boldsymbol{\Omega}}{dt}\times\mathbf{x} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) \quad ,$$

and, from (61) and (62) we have

$$\mathbf{v}^{C}(\mathbf{x}) = \mathbf{v}^{E}(\mathbf{x}) + \boldsymbol{\Omega}^{E}\times\mathbf{x}$$

$$\mathbf{a}^{C}(\mathbf{x}) = \mathbf{a}^{E}(\mathbf{x}) + 2\boldsymbol{\Omega}^{E}\times\mathbf{v}^{E}(\mathbf{x}) + \boldsymbol{\Omega}^{E}\times\left(\boldsymbol{\Omega}^{E}\times\mathbf{x}\right) \quad .$$

For velocity and acceleration differences between two points on the ship we have:

$$\mathbf{v}^{C}(\mathbf{x}_2) - \mathbf{v}^{C}(\mathbf{x}_1) = \boldsymbol{\Omega}\times\mathbf{r}.$$

Likewise, (58) gives

$$\begin{aligned}\mathbf{a}^{C}(\mathbf{x}_2) - \mathbf{a}^{C}(\mathbf{x}_1) &= \frac{d_S\boldsymbol{\Omega}}{dt}\times\mathbf{r} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\\ &= \mathbf{a}^{E}(\mathbf{x}_2) - \mathbf{a}^{E}(\mathbf{x}_1) + 2\boldsymbol{\Omega}^{E}\times\left[\mathbf{v}^{E}(\mathbf{x}_2) - \mathbf{v}^{E}(\mathbf{x}_1)\right] + \boldsymbol{\Omega}^{E}\times\left(\boldsymbol{\Omega}^{E}\times\mathbf{r}\right)\end{aligned}$$

$$\mathbf{v}^{E}(\mathbf{x}_2) - \mathbf{v}^{E}(\mathbf{x}_1) = \boldsymbol{\Omega}^{S}\times\mathbf{r}$$

$$\mathbf{a}^{E}(\mathbf{x}_2) - \mathbf{a}^{E}(\mathbf{x}_1) = \frac{d_E\boldsymbol{\Omega}^{S}}{dt}\times\mathbf{r} + \boldsymbol{\Omega}^{S}\times\left(\boldsymbol{\Omega}^{S}\times\mathbf{r}\right) \quad .$$

The Coriolis acceleration appears above when time derivatives are performed in the Earth's reference frame, but not when time derivatives are performed in the ship's reference frame. The reason is that a point fixed on the ship is not moving in the ship's reference frame. Specifically, the coordinates of a point on the ship are at a fixed number of meters forward of any chosen origin on the ship, a fixed number of meters starboard of the origin, and a fixed number of meters down from the origin. The rates of change with time of those fixed coordinate values are zeros.

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9. REFERENCES

Bariteau, L., 2005. Comparison of the ETL motion system data with the ship's POS MV system (unpublished).

Bosart, B. L., W. C. Lee, and R. M. Wakimoto, 2002. Procedures to improve the accuracy of airborne Doppler radar data. *J. Atmos. Ocean. Technol.* 19 (3):322–339.

Corcoran, R., and R. Pronk, 2003. POS MV Model 320 V30 Ethernet & SCSI ICD, Document No. PUBS-ICD-000033, Rev. 1.02 (available from APPLANIX Corp.)

Edson, J. B., A. A. Hinton, K. E. Prada, J. E. Hare, and C. W. Fairall, 1998. Direct covariance flux estimates from mobile platforms at sea. *J. Atmos. Ocean. Technol.* 15:547–562.

Fleagle, R. G., and J. A. Businger, 1980. *An Introduction to Atmospheric Physics*, Academic Press, New York 432pp.

Heymsfield, G. M., S. W. Bidwell, I. J. Caylor, S. Ameen, S. Nicholson, W. Boncyk, L. Miller, D. Vandemark, P. E. Racette, and L. R. Dod, 1996. The EDOP radar system on the high-altitude NASA ER-2 aircraft. *J. Atmos. Ocean. Technol.* 13 (4):795–809.

Hill, R. J., 2005. Correction of the Doppler velocity of the NOAA Mini-MOPA lidar for ship motions. NOAA Technical Memorandum OAR PSD-308.

Law, D. C., S. A. McLaughlin, M. J. Post, B. L. Weber, D. C. Welsh, D. E. Wolfe, and D. A Merritt, 2002. An electronically stabilized phased array system for shipborne atmospheric wind profiling. *J. Atmos. Ocean. Technol.* 19 (6):924–933.

Lee, W. C., P. Dodge, F. D. Marks, and P. H. Hildebrand, 1994. Mapping of airborne Doppler radar data. *J. Atmos. Ocean. Technol.* 11 (4):572–578.

Lee, W. C., F. D. Marks, and C. Walther, 2003. Airborne Doppler radar data analysis workshop. *Bull. Amer. Meteorol. Soc.* 84 (8):1063.

Schulz, E. W., B. G. Sanderson, and E. F. Bradley, 2005. Motion correction for shipborne turbulence sensors. *J. Atmos. Ocean. Technol.* 22:55–69.

Testud, J., P. H. Hildebrand, and W. C. Lee, 1995. A procedure to correct airborne Doppler radars for navigation errors using the echo returned from the Earth's surface. *J. Atmos. Ocean. Technol.* 12:800–820.

Tucker, S., 2005. Evaluation of data from Applanix POS MV system on-board the Research Vessel *Seward Johnson* during the RICO 2005 Experiment. (unpublished).

Appendix A: Surveyed Coordinates to Use in the Ship's Inertial Measurement Unit

The POS MV consists of an Inertial Measurement Unit (IMU), a Primary GPS unit, and the ship's GPS2 unit. The Primary GPS unit has its antenna above the aft 01 deck. The ship's GPS2 unit, also designated POS MV GPS2, has its antenna above the aft 01 deck and is connected to a GPS receiver box in the Dry Lab. The IMU is within the accelerometer's box on the floor of the computer lab. The POS MV allows a point on the ship to be designated as the reference point and two other points designated as Sensor 1 and Sensor 2. For each of those three points, the POS MV unit outputs the velocities of those three points, the angular acceleration, and the orientation (Euler angles), as well as accuracy metrics for those quantities. Details of the POS MV data are given in Corcoran and Pronk (2003).

Reference point **X** drilled into the deck outside the compressor room is referred to as point 'ref'. Other points on the ship are surveyed from that reference point. That reference point is about 4 feet = 48/39 m= 1.23 m above sea level during measurements. Recall that the ship's coordinate system is forward (x), starboard (y) and down (z) and that coordinates of a point are given as $\left[\begin{array}{ccc} x & y & z \end{array}\right]$. The following are coordinates of position vectors from ref. to other points as measured in meters:

ref. to Primary GPS antenna $G = \left[\begin{array}{ccc} 0.0 & -3.784 & -6.144 \end{array}\right]$
ref. to POS MV GPS2 $A = \left[\begin{array}{ccc} 9.440 & -6.379 & -3.974 \end{array}\right]$
ref. to IMU is $I = \left[\begin{array}{ccc} 19.567 & -4.128 & -0.15 \end{array}\right]$
ref. to lidar is $L = \left[\begin{array}{ccc} 32.61 & 0.82 & -4.84 \end{array}\right]$
ref. to NOAA/K radar is $K = \left[\begin{array}{ccc} 11.40 & 0.84 & -5.30 \end{array}\right]$
ref. to roof W-band radar is $W = \left[\begin{array}{ccc} -9.19 & -2.88 & -2.88 \end{array}\right]$
ref. to compensated W-band radar is $C = \left[\begin{array}{ccc} -6.22 & -1.18 & -1.42 \end{array}\right]$
ref. to roof X-band radar is $R = \left[\begin{array}{ccc} -10.26 & -2.03 & -2.99 \end{array}\right]$
ref. to center of rotation of the ship (estimated) is $Z = \left[\begin{array}{ccc} 0.0 & -0.94 & 0.0 \end{array}\right]$

The center of rotation of the ship is estimated to be at the wall that is 0.94 m toward port side of the drilled **X** reference point. The position of the lidar is the red accelerometer box on the ceiling of the lidar sea container. The position of any of the radars is an estimate of the phase center of the radar's antenna.

Usually, the POS MV system would report the ship's data for point ref. For our purposes it is more convenient to redefine the reference point as the phase center of the NOAA/K radar antenna. Let NOAA/K radar be the new reference point entered into the IMU. The 'lever' arms to be entered into the POS MV computer data base for its GPS antennas, its new reference point 'ref', and Sensor 1 and Sensor 2 positions are:

NOAA/K to Primary GPS antenna: $G - K = \left[\begin{array}{ccc} -11.4 & -4.624 & -0.844 \end{array}\right]$
NOAA/K to POS MV GPS2: $A - K = \left[\begin{array}{ccc} -1.96 & -7.219 & 1.326 \end{array}\right]$
NOAA/K to IMU: $I - K = \left[\begin{array}{ccc} 8.167 & -4.968 & 5.15 \end{array}\right]$
NOAA/K to lidar: $L - K = \left[\begin{array}{ccc} 21.21 & -0.02 & 0.46 \end{array}\right]$
NOAA/K to roof W-band radar: $W - K = \left[\begin{array}{ccc} -20.59 & -3.72 & 2.42 \end{array}\right]$
NOAA/K to compensated W-band radar $C - K = \left[\begin{array}{ccc} -17.62 & -2.02 & 3.88 \end{array}\right]$
NOAA/K to roof X -band radar: $R - K = \left[\begin{array}{ccc} -21.66 & -2.87 & 2.31 \end{array}\right]$
NOAA/K to center of rotation of the ship $Z - K = \left[\begin{array}{ccc} -11.4 & -1.78 & 5.3 \end{array}\right]$

Sensor 1 has been assigned to the lidar accelerometer box. At 13:40 UTC on January 13, Sensor 2 was changed to the roof X-band. Prior to that time, Sensor 2 was the roof W-band radar (which ceased to function after January 13). Note that velocity data recorded for Sensor 1 and Sensor 2 positions are in the coordinate system (along track, across track, down) and that the "down" component has the wrong sign. Therefore, the extra analysis steps indicated in (6) and (7) as well as multiplication of the "down" velocity component by -1 are required to determine the velocity vector in the Earth's coordinate system for Sensor 1 and Sensor 2 positions. The position of the lidar accelerometer box center is recorded above for comparison with lidar motion data. The lidar's scanning mirror changes its position relative to the lidar accelerometer as its periscope rotates. The periscope center is displaced about 21 inches forward, 15 inches starboard, and the mirror center is $37 + 5$ inches above the IMU box.

The instrumented tower was surveyed on January 28, 2005, which was after the above coordinates were recorded. The base of the tower is on the 01 deck 4.20 m above the water line. The sonic anemometer measurement volume is 10.1 m above the tower base. The center of the tower base was 6.35 m forward from the rear edge of the lidar sea container. The lidar's accelerometer box is 1.00 m forward of the rear edge of the lidar sea container. Thus, the center of the tower base is $6.35 - 1.00$ m forward of the lidar's accelerometer box. The center of the tower base was 1.55 m to port of the ship's center, and the lidar's accelerometer box is 1.77 m starboard of the ship's center to within 2 cm. Thus, the center of the tower base is $-1.55 - 1.77$ m starboard of the accelerometer box. From the top of the lidar sea container, that accelerometer box is 0.13 m down. The top of the lidar sea container is 2.64 m above the 01 deck. Thus, the center of the tower base is $2.64 - 0.13$ m downward of the accelerometer box. The lidar's accelerometer box is its reference point in this appendix, although the lidar's motion-detection system reports its velocity data at the position of the lidar system's master GPS antenna, according to the manufacturer. Relative to the lidar reference point, the coordinates of the base of the tower are:

lidar to tower base $T - L = \begin{bmatrix} 6.35 - 1.00 & -1.55 - 1.77 & 2.64 - 0.13 \end{bmatrix}$
$= \begin{bmatrix} 5.35 & -3.32 & 2.51 \end{bmatrix}$.

Hence the coordinates of NOAA/K to the tower base are

$T - K = [T - L] + [L - K] = \begin{bmatrix} 5.35 & -3.32 & 2.51 \end{bmatrix} + \begin{bmatrix} 21.21 & -0.02 & 0.46 \end{bmatrix}$.
$= \begin{bmatrix} 26.56 & -3.34 & 2.97 \end{bmatrix}$.

Thus, NOAA/K to tower base is $T - K = \begin{bmatrix} 26.56 & -3.34 & 2.97 \end{bmatrix}$.

An alternative survey method is to use the measurement that the rear of the lidar sea container is $15.55 - 0.59 + 5.25 = 20.21$ m forward of NOAA/K, and that NOAA/K is 1.80 m starboard of the ship's center, and NOAA/K is 2.97 m above the 01 deck. Then the tower is $1.55 + 1.80 = 3.35$ m to port of NOAA/K, and $6.35 + 20.21 = 26.56$ m forward of NOAA/K. The resulting coordinates of NOAA/K to tower base are $\begin{bmatrix} 26.56 & -3.35 & 2.97 \end{bmatrix}$. The difference with the above coordinates is only 1 cm in the starboard direction. Let $S - T = \begin{bmatrix} 0 & 0 & -10.1 \end{bmatrix}$ denote the position vector from the tower base to the sonic anemometer measurement volume. The coordinates of NOAA/K to the sonic anemometer's measurement volume are

$[T - K] + [S - T] = \begin{bmatrix} 26.56 & -3.34 & 2.97 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -10.1 \end{bmatrix} = \begin{bmatrix} 26.56 & -3.34 & -7.13 \end{bmatrix}$.

Thus, NOAA/K to sonic anemometer is $S - K = \begin{bmatrix} 26.56 & -3.34 & -7.13 \end{bmatrix}$.

The coordinates of lidar to the sonic anemometer's measurement volume are

$T - L = \begin{bmatrix} 5.35 & -3.32 & 2.51 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -10.1 \end{bmatrix} = \begin{bmatrix} 5.35 & -3.32 & -7.59 \end{bmatrix}$.

The reference point ref. to the tower base is

$$[T-L]+L=\left[\,5.35\ \ -3.32\ \ 2.51\,\right]+\left[\,32.61\ \ 0.82\ \ -4.84\,\right]=\left[\,37.96\ \ -2.5\ \ -2.33\,\right]\quad .$$

The above vertical coordinate of -2.33 m agrees exactly with the surveyed height of the 01 deck above the reference point. The reason for the agreement is that the POS MV's coordinate system has its axes toward bow and port parallel to the main deck. On the other hand, the trim of the ship determines the tilt of the ship relative to sea level. Recall from above that the reference point 'ref.' is 1.23 m above sea level. From the above, one expects the tower base to be $(2.33+1.23)$ m $=3.56$ m above sea level if the ship was trimmed level to sea level. The measurement of the base of the tower on the 01 deck to the water line was 4.20 m, as noted above. We surmise that the decks rise toward the bow by $4.20-3.56$ m $=0.64$ m relative to the reference point. The corresponding angle is approximately $0.64/37.96=1.6860\times10^{-2}$ radians, which is one degree.

Appendix B: Algorithm for Determining $\mathbf{r}^{S}(SL)$

As will be shown below, there are two solutions for $\mathbf{r}^{S}(SL)$. Only one of the two solutions is correct. For brevity, define the velocity difference expressed in the ship's coordinate system as

$$\boldsymbol{\Delta}^{S}\equiv\mathbf{Q}\left(\mathbf{v}_{L}^{E}-\mathbf{v}_{S}^{E}\right)\quad .$$

Now (37) can be written as

$$\boldsymbol{\Delta}^{S}=\boldsymbol{\Omega}_{S}^{S}\times\mathbf{r}^{S}(SL)$$

$$\begin{aligned}\text{so }\boldsymbol{\Delta}^{S}\times\boldsymbol{\Omega}_{S}^{S}&=\left[\boldsymbol{\Omega}_{S}^{S}\times\mathbf{r}^{S}(SL)\right]\times\boldsymbol{\Omega}_{S}^{S}\\&=-\boldsymbol{\Omega}_{S}^{S}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]+\mathbf{r}^{S}(SL)\left[\boldsymbol{\Omega}_{S}^{S}\cdot\boldsymbol{\Omega}_{S}^{S}\right]\quad .\end{aligned}\tag{73}$$

The following shows that $\left|\boldsymbol{\Delta}^{S}\times\boldsymbol{\Omega}_{S}^{S}\right|$ is not always zero:

$$\begin{aligned}\left|\boldsymbol{\Delta}^{S}\times\boldsymbol{\Omega}_{S}^{S}\right|^{2}&=\left\{-\boldsymbol{\Omega}_{S}^{S}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]+\mathbf{r}^{S}(SL)\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\right\}\cdot\left\{-\boldsymbol{\Omega}_{S}^{S}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]+\mathbf{r}^{S}(SL)\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\right\}\\&=\begin{array}{c}\boldsymbol{\Omega}_{S}^{S}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]\cdot\boldsymbol{\Omega}_{S}^{S}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]-\boldsymbol{\Omega}_{S}^{S}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]\cdot\mathbf{r}^{S}(SL)\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\\-\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\mathbf{r}^{S}(SL)\cdot\boldsymbol{\Omega}_{S}^{S}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]+\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\mathbf{r}^{S}(SL)\cdot\mathbf{r}^{S}(SL)\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\end{array}\\&=\begin{array}{c}\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]^{2}-\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]^{2}\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\\-\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]^{2}+\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\left|\mathbf{r}^{S}(SL)\right|^{2}\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\end{array}\\&=-\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]^{2}+\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\left|\mathbf{r}^{S}(SL)\right|^{2}\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\quad .\end{aligned}\tag{74}$$

Hence,

$$\begin{aligned}\left|\boldsymbol{\Delta}^{S}\times\boldsymbol{\Omega}_{S}^{S}\right|&=\left|\boldsymbol{\Omega}_{S}^{S}\right|\sqrt{\left|\mathbf{r}^{S}(SL)\right|^{2}\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}-\left[\boldsymbol{\Omega}_{S}^{S}\cdot\mathbf{r}^{S}(SL)\right]^{2}}\\&=\left|\mathbf{r}^{S}(SL)\right|\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\sqrt{1-\left[\cos\vartheta\right]^{2}}\\&=\left|\mathbf{r}^{S}(SL)\right|\left|\boldsymbol{\Omega}_{S}^{S}\right|^{2}\left|\sin\vartheta\right|\quad ,\end{aligned}$$

where ϑ is the angle between $\boldsymbol{\Omega}_{S}^{S}$ and $\mathbf{r}^{S}(SL)$.

Proof that $\boldsymbol{\Delta}^{S}$ is perpendicular to $\mathbf{r}^{S}(SL)$ is

$$\mathbf{r}^{S}(SL)\cdot\boldsymbol{\Delta}^{S}=\mathbf{r}^{S}(SL)\cdot\left[\boldsymbol{\Omega}_{S}^{S}\times\mathbf{r}^{S}(SL)\right]=\boldsymbol{\Omega}_{S}^{S}\cdot\left[\mathbf{r}^{S}(SL)\times\mathbf{r}^{S}(SL)\right]=0\ \text{ QED.}\tag{75}$$

Proof that $\mathbf{\Delta}^S$ is perpendicular to $\mathbf{\Omega}_S^S$ is

$$\mathbf{\Omega}_S^S \cdot \mathbf{\Delta}^S = \mathbf{\Omega}_S^S \cdot \left[\mathbf{\Omega}_S^S \times \mathbf{r}^S (SL)\right] = \mathbf{r}^S (SL) \cdot \left[\mathbf{\Omega}_S^S \times \mathbf{\Omega}_S^S\right] = 0 \ \text{ QED.}$$

Likewise, both $\mathbf{\Delta}^S$ and $\mathbf{\Omega}_S^S$ are perpendicular to $\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S$. Therefore, $\mathbf{\Delta}^S$ and $\mathbf{\Omega}_S^S$ and $\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S$ make an orthogonal set of vectors. Now,

$$\left|\mathbf{\Delta}^S\right|^2 = \mathbf{\Delta}^S \cdot \mathbf{\Delta}^S = \mathbf{\Delta}^S \cdot \left[\mathbf{\Omega}_S^S \times \mathbf{r}^S (SL)\right] = \mathbf{r}^S (SL) \cdot \mathbf{\Delta}^S \times \mathbf{\Omega}_S^S \quad .$$

Hence,

$$\frac{\mathbf{r}^S (SL) \cdot \mathbf{\Delta}^S \times \mathbf{\Omega}_S^S}{\left|\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S\right|} = \frac{\left|\mathbf{\Delta}^S\right|^2}{\left|\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S\right|} \quad . \tag{76}$$

The equations that we use to calculate $\mathbf{r}^S (SL)$ are (76), (75), and (74). From (1), $\left(\widehat{\mathbf{x}}^S, \widehat{\mathbf{y}}^S, \widehat{\mathbf{z}}^S\right)$ are the unit vectors in the (forward, starboard, down) coordinate system. Alternatively, they can denote unit vectors in the lidar's coordinate system for present purposes. Let us simplify notation by denoting the $\left(\widehat{\mathbf{x}}^S, \widehat{\mathbf{y}}^S, \widehat{\mathbf{z}}^S\right)$ components of $\mathbf{r}^S (SL)$ (or the components in the lidar's coordinate system) by r_x, r_y, and r_z, that is

$$\mathbf{r}^S (SL) = r_x \widehat{\mathbf{x}}^S + r_y \widehat{\mathbf{y}}^S + r_z \widehat{\mathbf{z}}^S \quad .$$

Note that the r_x, r_y, and r_z are not functions of time. At each time we calculate the unit vector $\frac{\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S}{\left|\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S\right|}$ and express it as its components in the $\left(\widehat{\mathbf{x}}^S, \widehat{\mathbf{y}}^S, \widehat{\mathbf{z}}^S\right)$ coordinate system (or the lidar's coordinate system). Let us simplify by denoting those components at each time t by $u_x (t)$, $u_y (t)$, and $u_z (t)$, that is

$$\frac{\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S}{\left|\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S\right|} = u_x (t) \, \widehat{\mathbf{x}}^S + u_y (t) \, \widehat{\mathbf{y}}^S + u_z (t) \, \widehat{\mathbf{z}}^S \quad .$$

At each time t we calculate the scalar $\frac{\mathbf{\Delta}^S \cdot \mathbf{\Delta}^S}{\left|\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S\right|}$ and denote it by $f (t)$, that is

$$f (t) \equiv \frac{\mathbf{\Delta}^S \cdot \mathbf{\Delta}^S}{\left|\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S\right|} \quad .$$

Our equation (76) becomes

$$r_x u_x (t) + r_y u_y (t) + r_z u_z (t) = f (t) \quad . \tag{77}$$

The values of $u_x (t)$, $u_y (t)$, $u_z (t)$, and $f (t)$ are known; the r_x, r_y, and r_z are the unknowns. If we simplify notation by defining $w_x (t)$, $w_y (t)$, and $w_z (t)$ by

$$\frac{\mathbf{\Delta}^S}{\left|\mathbf{\Delta}^S\right|} = w_x (t) \, \widehat{\mathbf{x}}^S + w_y (t) \, \widehat{\mathbf{y}}^S + w_z (t) \, \widehat{\mathbf{z}}^S \quad ,$$

then (75) becomes

$$r_x w_x (t) + r_y w_y (t) + r_z w_z (t) = 0 \quad . \tag{78}$$

The values of $w_x (t)$, $w_y (t)$, $w_z (t)$ are known; the r_x, r_y, and r_z are the unknowns. Simplify notation by defining $q_x (t)$, $q_y (t)$, and $q_z (t)$ by

$$\frac{\mathbf{\Omega}^S}{\left|\mathbf{\Omega}^S\right|} \equiv q_x (t) \, \widehat{\mathbf{x}}^S + q_y (t) \, \widehat{\mathbf{y}}^S + q_z (t) \, \widehat{\mathbf{z}}^S \quad .$$

At each time t we calculate the scalar $\frac{|\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S|^2}{|\mathbf{\Omega}_S^S|^4}$ and denote it by $g(t)$, that is

$$g(t) \equiv \frac{|\mathbf{\Delta}^S \times \mathbf{\Omega}_S^S|^2}{|\mathbf{\Omega}_S^S|^4} \quad .$$

Then, (74) can be written as

$$r_x^2 + r_y^2 + r_z^2 - [q_x(t) r_x + q_y(t) r_y + q_z(t) r_z]^2 = g(t) \quad . \tag{79}$$

The values of $q_x(t)$, $q_y(t)$, $q_z(t)$ are known; the r_x, r_y, and r_z are the unknowns. When $w_z(t) \neq 0$, we can substitute (78) into (77) to obtain

$$\begin{aligned} r_x u_x(t) + r_y u_y(t) + \left[\frac{r_x w_x(t) + r_y w_y(t)}{-w_z(t)}\right] u_z(t) &= f(t) \quad ; \\ r_x \left[u_x(t) - \frac{w_x(t)}{w_z(t)} u_z(t)\right] + r_y \left[u_y(t) - \frac{w_y(t)}{w_z(t)} u_z(t)\right] &= f(t) \quad . \end{aligned} \tag{80}$$

Hence, to be concise we define

$$v_x(t) \equiv \left[u_x(t) - \frac{w_x(t)}{w_z(t)} u_z(t)\right], \text{ and } v_y(t) \equiv \left[u_y(t) - \frac{w_y(t)}{w_z(t)} u_z(t)\right]$$

such that simplified notation for (80) is

$$r_x v_x(t) + r_y v_y(t) = f(t) \quad . \tag{81}$$

Similarly, when $w_z(t) \neq 0$, we can substitute (78) into (79) to obtain

$$\left[\begin{array}{c} r_x^2 + r_y^2 + r_x^2 \left(\frac{w_x(t)}{w_z(t)}\right)^2 + r_y^2 \left(\frac{w_y(t)}{w_z(t)}\right)^2 + 2 r_x r_y \frac{w_x(t)}{w_z(t)} \frac{w_y(t)}{w_z(t)} \\ -r_x^2 p_x^2(t) - r_y^2 p_y^2(t) - 2 r_x r_y p_x(t) p_y(t) \end{array}\right] = g(t) \tag{82}$$

where

$$p_x(t) = \left[q_x(t) - \frac{w_x(t)}{w_z(t)} q_z(t)\right], \text{ and } p_y(t) = \left[q_y(t) - \frac{w_y(t)}{w_z(t)} q_z(t)\right] \quad .$$

In (82), the coefficient of r_x^2 is

$$X \equiv 1 + \left(\frac{w_x(t)}{w_z(t)}\right)^2 - p_x^2(t) \quad ,$$

the coefficient of r_y^2 is

$$Y \equiv 1 + \left(\frac{w_y(t)}{w_z(t)}\right)^2 - p_y^2(t) \quad ,$$

and the coefficient of $r_x r_y$ is

$$Z \equiv 2 \frac{w_x(t)}{w_z(t)} \frac{w_y(t)}{w_z(t)} - 2 p_x(t) p_y(t) \quad ;$$

so (82) is

$$Xr_x^2 + Yr_y^2 + Zr_xr_y = g(t) \quad . \tag{83}$$

Eliminating r_y using (81) in (83) gives a quadratic equation to solve for r_x; that equation, namely, equation (84) in the summary below.

Let us summarize the above algorithm for calculating $\mathbf{r}^S(SL)$. To simplify notation, start with 5 definitions defining 11 quantities from data. They are:

$$f(t) \equiv \frac{\boldsymbol{\Delta}^S \cdot \boldsymbol{\Delta}^S}{\left|\boldsymbol{\Delta}^S \times \boldsymbol{\Omega}_S^S\right|} \quad .$$

$$g(t) \equiv \frac{\left|\boldsymbol{\Delta}^S \times \boldsymbol{\Omega}_S^S\right|^2}{\left|\boldsymbol{\Omega}_S^S\right|^4} \quad .$$

$$\frac{\boldsymbol{\Delta}^S}{\left|\boldsymbol{\Delta}^S\right|} \equiv w_x(t)\,\widehat{\mathbf{x}}^S + w_y(t)\,\widehat{\mathbf{y}}^S + w_z(t)\,\widehat{\mathbf{z}}^S \quad .$$

$$\frac{\boldsymbol{\Omega}^S}{\left|\boldsymbol{\Omega}^S\right|} \equiv q_x(t)\,\widehat{\mathbf{x}}^S + q_y(t)\,\widehat{\mathbf{y}}^S + q_z(t)\,\widehat{\mathbf{z}}^S \quad .$$

$$\frac{\boldsymbol{\Delta}^S \times \boldsymbol{\Omega}_S^S}{\left|\boldsymbol{\Delta}^S \times \boldsymbol{\Omega}_S^S\right|} \equiv u_x(t)\,\widehat{\mathbf{x}}^S + u_y(t)\,\widehat{\mathbf{y}}^S + u_z(t)\,\widehat{\mathbf{z}}^S \quad .$$

Then, for conciseness, define

$$v_x(t) = \left[u_x(t) - \frac{w_x(t)}{w_z(t)}u_z(t)\right], \text{ and } v_y(t) = \left[u_y(t) - \frac{w_y(t)}{w_z(t)}u_z(t)\right] \quad .$$

$$p_x(t) = \left[q_x(t) - \frac{w_x(t)}{w_z(t)}q_z(t)\right], \text{ and } p_y(t) = \left[q_y(t) - \frac{w_y(t)}{w_z(t)}q_z(t)\right] \quad ,$$

and

$$X(t) \equiv 1 + \left(\frac{w_x(t)}{w_z(t)}\right)^2 - p_x^2(t) \quad ,$$

$$Y(t) \equiv 1 + \left(\frac{w_y(t)}{w_z(t)}\right)^2 - p_y^2(t) \quad ,$$

$$Z(t) \equiv 2\frac{w_x(t)}{w_z(t)}\frac{w_y(t)}{w_z(t)} - 2p_x(t)\,p_y(t) \quad .$$

Obviously, $w_z(t)$ must not be zero.

Substitution of equations (81) in (83) gives the quadratic equation

$$a(t)\,r_x^2 + b(t)\,r_x + c(t) = 0 \quad , \tag{84}$$

where

$$a(t) \equiv X(t) + Y(t)\left(\frac{v_x(t)}{v_y(t)}\right)^2 - Z(t)\,\frac{v_x(t)}{v_y(t)} \quad ,$$

$$b(t) \equiv -2\left(\frac{f(t)}{v_y(t)}\frac{v_x(t)}{v_y(t)}\right) + Z(t)\,\frac{f(t)}{v_y(t)} \quad .$$

$$c(t) \equiv Y(t)\left(\frac{f(t)}{v_y(t)}\right)^2 - g(t) \quad .$$

The solution of (84) is:

$$r_x = \frac{-b(t) \pm \sqrt{[b(t)]^2 - 4a(t)\,c(t)}}{2a(t)} \quad . \tag{85}$$

Then, determine r_y from r_x using (81) as follows:

$$r_y = \frac{f(t)}{v_y(t)} - r_x\frac{v_x(t)}{v_y(t)} \quad .$$

Obviously, $v_y(t)$ must not be zero. Take the solution for r_x and r_y to determine r_z from either (75)

$$r_z = \frac{r_x w_x(t) + r_y w_y(t)}{-w_z(t)}$$

($w_z(t)$ must not be zero), or from (77)

$$r_z = \frac{f(t) - [r_x u_x(t) + r_y u_y(t)]}{u_z(t)}$$

($u_z(t)$ must not be zero).

We have two solutions for $\mathbf{r}^S(SL)$ corresponding to the + or − sign in (85); the correct solution must be chosen. Now we have equations that determine the components of $\mathbf{r}^S(SL)$ from data at any time t for which $w_z(t) \neq 0$ and $v_z(t) \neq 0$. We can remove outliers from the set of solutions at many times t and average the remaining values over time. We defined subscripts (x, y, z) to denote (forward, starboard, down) or the equivalent components in the lidar coordinate system However, cyclic permutation of the subscripts (x, y, z) could give any one of the components of $\mathbf{r}$ from the quadratic equation; then the other two components are subsequently determined. One advantage of this is that if $w_z(t) = 0$ or $v_z(t) = 0$ then cyclic permutation of the subscripts.