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THE DEEP-ATMOSPHERE EULER EQUATIONS IN A NON-APPROXIMATED SHALLOW-ATMOSPHERE-ALIKE FORM

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ABSTRACT

The deep-atmosphere Euler equation set has been derived into a form similar to the shallow-atmosphere Euler equation set without any approximation, which makes it more convenient to evolve any existing shallow-atmosphere dynamics model to deep-atmosphere dynamics. The deep atmospheric dynamics system is a fully non-approximated Euler equation set, which includes all-dimensional Coriolis force, vertically expanded cells (r=a+z), and geocentric gravitational force with height. The deep-atmosphere Euler equation in spherical height coordinates was transferred into spherical generalized vertical coordinates with pseudo horizontal wind in NCEP Office Note 477 (Juang 2014). The deep-atmosphere continuity equation in generalized vertical coordinates can be easily converted into a shallow-atmosphere-alike form by introducing scaled horizontal winds and scaled height with a hydrostatic relationship in generalized coordinates to represent density. The relationship provides a coordinate pressure change with respect to scaled height for a given density. From this relationship we can use geopotential height to define scaled vertical motion, thus all the three-dimensional winds in deep-atmosphere dynamics can be replaced by scaled winds. Since all winds are using a scaled form, the momentum equations are converted into scaled momentum equations with scaled terms plus some add-on terms.

Since the deep-atmosphere Euler equation is applied to the whole atmosphere, the gas constituents should be flexible enough to consider gases in the lower atmosphere as well as the upper atmosphere; thus the generalized gas constants of R and Cp should be included in the thermodynamics as in Juang (2011). However, due to time splitting for the adiabatic dynamics solver and diabatic physics solver, enthalpy may not be necessary for thermodynamic variable, any form of temperature can be used as prognostic thermodynamic variable, as long as the thermodynamic equation is satisfied. Combined with the shallow-atmosphere-alike form of the deep-atmosphere continuity equation, thermodynamic equation, the three scaled momentum equations, and equation of state for the ideal gas law, the non-approximated shallow-atmosphere-alike form of deep-atmosphere Euler equation is given for easy implementation of deep-atmosphere dynamics into existing shallow-atmosphere dynamics models.

1. Introduction

We are in an era where nonhydrostatic atmospheric modeling is developing and used for operational numerical weather/climate prediction. Computing resources are increasing to the point of providing high-resolution computations, it is reasonable to consider nonhydrostatic modeling for the National Centers for Environmental Prediction (NCEP) operational model suite. However, certain aspects of nonhydrostatic modeling are implemented with limiting approximation such as deep-atmosphere requirements. The NCEP encompass SWPC (Space Weather Prediction Center) mission, which requires whole atmosphere modeling to provide vertical extended domain coupling with ionosphere models; thus, it is essential to model the atmosphere with deep-atmosphere dynamics.

The first to advocate for deep-atmosphere modeling appeared in the literature as Staniforth and Wood (2003) with a generalized vertical coordinate, Wood and Staniforth (2003) with a mass-based vertical coordinate, and furthermore they provided papers on the treatment of vector equations for a deep-atmosphere and semi-Lagrangian model in Staniforth and Wood (2010) for the momentum equation, and in Woods and Staniforth (2010) for the kinematic equation. Also deep-atmosphere dynamics has been used in an operational suite by UK Met Services (Davies et al. 2005) for global and regional modeling as a unified model. NCEP considered deep atmosphere dynamics in the Global Spectral Model with Office Note 477 (Juang 2014) and continue to develop atmosphere dynamics in Global Forecast System (GFS).

The difficulty in implementing deep-atmosphere dynamics is the necessity of allowing for a geo-centric radius, r, for all terms in the deep-atmosphere Euler equation set, which may require new approaches and/or new numerical techniques. The idea of using a mapping factor to have equal grid spacing to provide easy integration in the horizontal is similar to using scaled prognostic variables in the deep-atmosphere Euler equation to allow a shallow-atmosphere-alike form in the vertical. This idea makes deep-atmosphere dynamics more easily implemented in a shallow-atmosphere dynamics system, thus most or all of the dynamics routines in the shallow-atmosphere form can be re-used to solve deep-atmosphere dynamics. It is a convenient and effective way to implement deep-atmosphere dynamics into existing shallow-atmosphere models, such as the NCEP GSM (Global Spectral Model) and its enthalpy version, called the NCEP WAM (Whole Atmosphere Model).

This manuscript is an extension of NCEP Office Note 477 on a deep-atmosphere nonhydrostatic dynamics system with a practical approach. The deep-atmosphere dynamics with a pseudo-spherical coordinate in the horizontal and generalized vertical coordinates in the vertical is illustrated in Section 2. The introduction of scaled prognostic variables, including height and three-dimensional winds is given in Section 3. The scaled momentum equations in shallow-atmosphere-alike form are given in Section 4. The inclusion of the thermodynamic equation with multi gas constituents due to deep-atmosphere concerns is given in Section 5. Section 6 wraps up the manuscript with conclusions and discussion.

2. The deep-atmosphere Euler equation in pseudo-spherical and generalized vertical coordinates

Based on deep-atmosphere Euler equation in spherical coordinates with a generalized vertical coordinate transfer as shown in NCEP Office Note 477 (Juang 2014), the deep-atmospheric Euler equation in pseudo-spherical coordinate can be written as

$$\frac{du^*}{dt}+\frac{u^*w}{r}-f_s v^*+f_c^* w+\frac{1}{\rho}\left(\frac{\partial p}{r\partial\lambda}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{r\partial\lambda}\right)=F_u \tag{2.1}$$

$$\frac{dv^*}{dt}+\frac{v^*w}{r}+f_s u^*+m^2\frac{s^{*2}}{r}\sin\phi+\frac{1}{\rho}\left(\frac{\partial p}{r\partial\varphi}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{r\partial\varphi}\right)=F_v \tag{2.2}$$

$$\frac{dw}{dt}-m^2\frac{s^{*2}}{r}-m^2 f_c^* u^*+\frac{1}{\rho}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}+g=F_w \tag{2.3}$$

$$\frac{dCpT}{dt}-\frac{1}{\rho}\frac{dp}{dt}=F_h \tag{2.4}$$

$$\frac{\partial\rho^*}{\partial t}+m^2\frac{\partial\rho^*\frac{u^*}{r}}{\partial\lambda}+m^2\frac{\partial\rho^*\frac{v^*}{r}}{\partial\varphi}+\frac{\partial\rho^*\dot{\zeta}}{\partial\zeta}=F_\rho^* \tag{2.5}$$

$$\frac{dq_i}{dt}=F_{q_i} \tag{2.6}$$

$$p=\rho RT \tag{2.7}$$

where

$$\frac{d()}{dt}=\frac{\partial()}{\partial t}+m^2u^*\frac{\partial()}{r\partial\lambda}+m^2v^*\frac{\partial()}{r\partial\varphi}+\dot{\zeta}\frac{\partial()}{\partial\zeta} \tag{2.8}$$

$$u^*=u\cos\phi\ ;\quad u=r\cos\phi\frac{d\lambda}{dt} \tag{2.9}$$

$$v^*=v\cos\phi\ ;\quad v=r\frac{d\phi}{dt} \tag{2.10}$$

$$m=\frac{1}{\cos\phi} \tag{2.11}$$

$$\frac{\partial}{\partial\varphi}=\cos\phi\frac{\partial}{\partial\phi}=\frac{1}{m}\frac{\partial}{\partial\phi} \tag{2.12}$$

$$\rho^*=\rho\frac{r^2}{a^2}\frac{\partial r}{\partial\zeta} \tag{2.13}$$

$$f_s=2\Omega\sin\phi \tag{2.14}$$

$$f_c^*=2\Omega\cos^2\phi \tag{2.15}$$

$$s^{*2} = u^{*2} + v^{*2} \tag{2.16}$$

$$g = g_0 \frac{a^2}{r^2} \tag{2.17}$$

These equations and how they are derived from height vertical coordinates can be found in detail in Staniforth and Wood (2003) and Juang (2014).

The original continuity equation in spherical coordinates explicitly shows the vertical expanded structure as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{r\cos\phi \partial\lambda} + \frac{\partial \rho v \cos\phi}{r\cos\phi\partial\phi} + \frac{\partial \rho r^2 w}{r^2 \partial r} = F_\rho \tag{2.18}$$

But the fourth term on the left hand side is numerically complicated to solve. Transferring it into a generalized vertical coordinate can convert the vertical expanding form into an implicit form without approximation as

$$\frac{\partial\left(\rho r^2 \cos\phi \frac{\partial r}{\partial \zeta}\right)}{\partial t} + \frac{\partial\left(\rho r^2 \cos\phi \frac{\partial r}{\partial \zeta}\right)\dot{\lambda}}{\partial \lambda} + \frac{\partial\left(\rho r^2 \cos\phi \frac{\partial r}{\partial \zeta}\right)\dot{\phi}}{\partial \phi} + \frac{\partial\left(\rho r^2 \cos\phi \frac{\partial r}{\partial \zeta}\right)\dot{\zeta}}{\partial \zeta} = F_\rho \tag{2.19}$$

and it can be transferred into Eq. (2.5) by dividing $a^2 \cos\phi$ and with Eq. (2.12) (see Appendix A in Juang 2014 for details). Since we are dealing with deep-atmosphere dynamics, we can set entire equation to be only for dynamics, so the entire right hand side of F can be set to be zero for the rest of the manuscript with a few exceptions which we will mention.

3. The introduction of scaled prognostic variables

From the continuity equation in pseudo-spherical coordinate, it is easy to assign a pressure with respect to the generalized vertical coordinate as

$$\frac{\partial \breve{p}}{\partial \zeta} = -\rho^* g_0 \tag{3.1}$$

where the pressure (pressure with smile on top) may be called "coordinate pressure" which we will use to define coordinates in relation to generalized coordinates. Putting Eq. (3.1) into Eq. (2.5), we have the continuity equation as

$$\frac{\partial \frac{\partial \breve{p}}{\partial \zeta}}{\partial t} + m^2 \frac{\partial \frac{\partial \breve{p}}{\partial \zeta}\frac{u^*}{r}}{\partial \lambda} + m^2 \frac{\partial \frac{\partial \breve{p}}{\partial \zeta}\frac{v^*}{r}}{\partial \varphi} + \frac{\partial \frac{\partial \breve{p}}{\partial \zeta}\dot{\zeta}}{\partial \zeta} = 0 \tag{3.2}$$

which is very close to the shallow-atmosphere form but is not quite complete. To make the deep-atmosphere continuity equation into the same form as the shallow-atmosphere equation, we define scaled horizontal wind as

$$\breve{u} = \frac{a}{r} u^* \tag{3.3}$$

$$\breve{v} = \frac{a}{r} v^* \tag{3.4}$$

and by putting them into Eq.(3.2), we obtain

$$\frac{\partial \frac{\partial \breve{p}}{\partial \zeta}}{\partial t} + m^2 \frac{\partial \frac{\partial \breve{p}}{\partial \zeta}\breve{u}}{a\partial \lambda} + m^2 \frac{\partial \frac{\partial \breve{p}}{\partial \zeta}\breve{v}}{a\partial \varphi} + \frac{\partial \frac{\partial \breve{p}}{\partial \zeta}\dot{\zeta}}{\partial \zeta} = 0 \tag{3.5}$$

which is completely the same form as the shallow-atmosphere Euler equation for the continuity equation, except we are using coordinate pressure and scaled horizontal winds. In this form, Eq. (3.5), we can apply all existing numerical techniques and computational methods to solve it, since all terms are the same as in shallow-atmosphere system. The scaled winds can be obtained if we have the vertical location of r, which can be obtained by our definition of coordinate pressure in Eq. (3.1).

Though Eq. (3.1) is used to replace density in Eq. (2.5) to get a generalized form, it is also defined in the relationship between coordinate pressure and height based on given density as

$$\frac{\partial \breve{p}}{\partial \zeta} = -\rho^* g_0 = -\rho \frac{r^2}{a^2}\frac{\partial r}{\partial \zeta} g_0 = -\rho g_0 \frac{\partial \frac{r^3}{3a^2}}{\partial \zeta} = -\rho \frac{\partial \breve{\Phi}}{\partial \zeta} \tag{3.6}$$

and we can give a solution for the scaled geopotential as

$$\breve{\Phi} = \frac{g_0\left(r^3 - a^3\right)}{3a^2} = \frac{g_0\left(r-a\right)\left(r^2 + ra + a^2\right)}{3a^2} \tag{3.7}$$

thus, when r is close to a, $\breve{\Phi} \cong g_0\left(r-a\right) = g_0 z$ is the same as the shallow-atmosphere definition. And the vertical motion in the scaled geopotential term can be written as

$$\frac{d\breve{\Phi}}{dt} = \frac{g_0 r^2}{a^2}\frac{dr}{dt} = g_0 \frac{r^2}{a^2} w \tag{3.8}$$

by analogy, we can define the scaled vertical motion from scaled geopotential as

$$\frac{d\breve{\Phi}}{dt} = g_0 \breve{w} \tag{3.9}$$

thus, we define scaled vertical motion as

$$\breve{w} = \frac{r^2}{a^2} w \tag{3.10}$$

Thus, coordinate pressure or mass pressure, and three-dimensional winds are all in scaled form, as the deep-atmosphere continuity equation becomes a shallow-atmosphere-alike form.

4. The deep-atmosphere Euler equation in shallow form

From Eqs. (3.5) and (3.9), we realize that the variables for coordinate height and three dimensional winds are in scaled form. And continuity equation is already in shallow-atmosphere-alike form in Eq. (3.5), thus we only have to provide scaled prognostic momentum equations to close the system. The total derivative of the three-dimensional winds can be derived as the following

$$\frac{d\breve{u}}{dt} = \frac{a}{r}\frac{du^*}{dt} - \frac{au^*}{r^2}\frac{dr}{dt} = \frac{1}{\varepsilon}\frac{du^*}{dt} - \frac{\breve{u}\breve{w}}{\varepsilon^3 a} \quad (4.1)$$

$$\frac{d\breve{v}}{dt} = \frac{a}{r}\frac{dv^*}{dt} - \frac{av^*}{r^2}\frac{dr}{dt} = \frac{1}{\varepsilon}\frac{dv^*}{dt} - \frac{\breve{v}\breve{w}}{\varepsilon^3 a} \quad (4.2)$$

$$\frac{d\breve{w}}{dt} = \frac{r^2}{a^2}\frac{dw}{dt} + \frac{2r}{a^2}w^2 = \varepsilon^2\frac{dw}{dt} + \frac{2\breve{w}^2}{a\varepsilon^3} \quad (4.3)$$

where

$$\varepsilon = \frac{r}{a} \quad (4.4)$$

Next, we put Eqs. (2.1), (2.2) and (2.3) into Eqs. (4.1), (4.2), and (4.3) respectively, and we obtain

$$\frac{d\breve{u}}{dt} = -2\frac{\breve{u}\breve{w}}{\varepsilon^3 a}\delta - f_c^*\frac{\breve{w}}{\varepsilon^3}\delta + f_s\breve{v} - \frac{1}{\varepsilon^2}\left(\frac{1}{\rho}\frac{\partial p}{a\partial\lambda} + \frac{\partial p}{\partial\breve{p}}\frac{\partial\breve{\Phi}}{a\partial\lambda}\right) \quad (4.5)$$

$$\frac{d\breve{v}}{dt} = -2\frac{\breve{v}\breve{w}}{\varepsilon^3 a}\delta - f_s\breve{u} - m^2\frac{\breve{s}^2}{a}\sin\phi - \frac{1}{\varepsilon^2}\left(\frac{1}{\rho}\frac{\partial p}{a\partial\varphi} + \frac{\partial p}{\partial\breve{p}}\frac{\partial\breve{\Phi}}{a\partial\varphi}\right) \quad (4.6)$$

$$\frac{d\breve{w}}{dt} = 2\frac{\breve{w}^2}{\varepsilon^3 a}\delta + m^2\varepsilon^3\frac{\breve{s}^2}{a}\delta + m^2\varepsilon^3 f_c^*\breve{u}\delta + g_0\left(\frac{\partial p}{\partial\breve{p}}\varepsilon^4 - 1\right) \quad (4.7)$$

where $\delta = 1$ is the default for deep-atmosphere dynamics. When $\varepsilon = 1$ everywhere in a shallow-atmosphere Euler equation set, $\delta = 0$ is required for the deep-atmosphere Euler equation set to reduce it to a traditional shallow-atmosphere Euler equation.

Eqs. (3.5), (4.5), (4.6), and (4.7) are a deep-atmosphere dynamics Euler equation set in shallow-atmosphere-alike form for the continuity equation and three momentum equations in three dimensions, to form the deep-atmosphere dynamics in shallow-atmosphere-alike form. To close the system, we need a thermodynamics equation and idea gas law.

5. Generalized gas tracer in thermodynamic equation for deep-atmosphere

For deep-atmosphere dynamics equations, the so-called dry air standard atmosphere may not be precisely right for the vertically extended whole atmosphere. The flexibility of different gas constituents present in the low atmosphere as well as deep atmosphere up to several hundred kilometers should be considered. To illustrate this, we should start from the ideal gas law and internal energy equation (thermodynamics equation), for example, as in Juang (2011).

The ideal gas rule and composited multi constituent gases can be written as

$$p = \rho RT \quad (5.1)$$

$$R = \left(1 - \sum_{i=1}^{n} q_i\right)R_0 + \sum_{i=1}^{n} q_i R_i \quad (5.2)$$

and thermodynamic equation should be

$$\frac{dC_PT}{dt} - \frac{RT}{p}\frac{dp}{dt} = F_T \tag{5.3}$$

$$C_P = \left(1 - \sum_{i=1}^{n} q_i\right) C_{P_0} + \sum_{i=1}^{n} q_i C_{P_i} \tag{5.4}$$

where R_0 and C_{P_0} are for the base gas constituent in modeling and R_i and C_{P_i} are for the ith gas constituent with total n gas constituents as model tracers. For example in Table 1, the first columns with values are the base air gases in the current operational GFS GSM (upper panel) and in the experimental WAM GSM (bottom panel). And cloud water is not a gas constituent but a model tracer, so there is no Ri and Cpi values in the table, others are gas cinstituents with values of Ri and Cpi.

To solve Eq. (5.3) including the source term in the right hand side, combining C_PT as the enthalpy may be the only way to simplify the solution as in Juang (2011). However, in practice, most numerical weather and climate models separate adiabatic dynamic solver and diabatic model physics solver, as a time splitting scheme, thus C_P can be separated out from the total derivative. We can illustrate it under adiabatic dynamics first as follows. While for adiabatic conditions

$$\frac{dq_i}{dt} = 0 \tag{5.5}$$

so

$$\frac{dCp}{dt} = \sum_{i=0}^{n}\left(Cp_i \frac{dq_i}{dt}\right) = 0 \tag{5.6}$$

$$\frac{dR}{dt} = \sum_{i=0}^{n}\left(R_i \frac{dq_i}{dt}\right) = 0 \tag{5.7}$$

then

$$\frac{dC_PT}{dt} - \frac{RT}{p}\frac{dp}{dt} = C_P\frac{dT}{dt} + T\frac{dC_P}{dt} - \frac{RT}{p}\frac{dp}{dt} = C_P\frac{dT}{dt} - \frac{RT}{p}\frac{dp}{dt} = 0 \tag{5.8}$$

and the above equation also can be written as

$$\frac{d\ln T}{dt} - \frac{R}{C_P}\frac{d\ln p}{dt} = 0 \tag{5.9}$$

From Eq. (5.9), we can define potential temperature as the following

$$\theta = \frac{T}{\pi} \tag{5.10}$$

where

$$\pi = \left(\frac{p}{p_0}\right)^{\frac{R}{Cp}} \tag{5.11}$$

and R and Cp have to be total values including all gas constituents, so that

$$\frac{d\ln\theta}{dt}=\frac{d\ln T}{dt}-\frac{d\ln\pi}{dt}=\frac{d\ln T}{dt}-\frac{R}{Cp}\frac{d\left(\ln p-\ln p_0\right)}{dt}=\frac{d\ln T}{dt}-\frac{R}{Cp}\frac{d\ln p}{dt} \tag{5.12}$$

which is the same as Eq. (5.9), so

$$\frac{d\ln\theta}{dt}=0 \tag{5.13}$$

is a potential temperature conservation. And it is easy to expand the definition of virtual potential temperature by

$$p=\rho RT=\rho R_0 T_v \tag{5.14}$$

then putting the above definition of T and Tv into Eq. (5.9), we have

$$\frac{d\ln\frac{R_0T_v}{R}}{dt}-\frac{R}{C_P}\frac{d\ln p}{dt}=\frac{d\ln\frac{R_0}{R}}{dt}+\frac{d\ln T_v}{dt}-\frac{R}{C_P}\frac{d\ln p}{dt}=\frac{d\ln T_v}{dt}-\frac{R}{C_P}\frac{d\ln p}{dt}=0 \tag{5.15}$$

And we find the total R and Cp are still used to define pi for virtual potential temperature definition based on adiabatic conditions for the thermodynamics equation.

6. Conclusions and discussion

This manuscript provides further information and techniques to implement deep-atmosphere dynamics into any existing shallow-atmosphere model. Even though the dynamics are derived under spherical Gaussian horizontal coordinates and generalized vertical coordinates, the idea of scaled winds to allow a shallow-atmosphere-alike form of the deep-atmosphere Euler equation can be applied to any coordinates and any global and regional models. The shallow-atmosphere-alike form of the deep-atmosphere Euler equation with parameters can be reduced to shallow-atmosphere dynamics as long as delta=0 and epsilon=1 in the equation set.

In addition to the scaled momentum equations in shallow-atmosphere-alike form, the generalized multi-gas constituents thermodynamics equation has to accompany the deep-atmosphere dynamics, because deep-atmosphere dynamics should be used to resolve the whole atmosphere in the situation where the atmospheric gas constituents need to be flexible to include both the low atmosphere and upper atmosphere. And one thing to be noted is that the definition of potential temperature should follow the thermodynamic equation for adiabatic conditions, with the result that pi, the Exner function, should be defined by using total gas constants and the total gas specific heat for constant pressure of all gases. Using only the dry-air gas constant and specific heat to define potential temperature is an approximation, and not that accurate, because this definition of potential temperature doesn't satisfy the thermodynamic equation for adiabatic conditions, which requires using values of R and Cp for total gas constituents in modeling. Finally, this non-approximated shallow-atmosphere-alike form of the deep-atmosphere Euler equation set provides an easy way to implement deep-atmosphere dynamics into the shallow-atmosphere system and can be applied into existing NCEP shallow-atmosphere models.

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	N2+O2+…	vapor	O3	clwr
Ri	286.05	461.50	173.22	0.0
Cpi	1004.6	1846.0	820.24	0.0

	N2+…	vapor	O3	clwr	O	O2
Ri	296.80	461.50	173.22	0	519.67	259.84
Cpi	1039.6	1846.0	820.24	0	1299.2	918.10

Table 1: list of R and Cp for all model tracers starting in the third column and continuing to the right. The second column contains base air constituents for the GFS GSM in the upper panel and WAM GSM in the bottom panel. clwr is a tracer of cloud water, which is not gas constituent, so there are no R and Cp values assigned to it.