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ESTIMATION OF THE INTENSITY OF A FILTERED POISSON PROCESS AND ITS APPLICATION TO ACOUSTIC ASSESSMENT OF MARINE ORGANISMS

by John E. Ehrenberg

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ABSTRACT

One of the primary goals of this research was to analyze the role of signal processing in acoustic systems for estimating the abundance of marine organisms. In such an analysis, the first step is to design a reasonable model for the scattering environment. The model used here assumes that the organisms are distributed according to a spatial Poisson distribution. With this model, it is shown that the scattered signal received is a filtered Poisson process with intensity $\lambda_0\beta(t)$, where $\beta(t)$ is a known deterministic function of time and λ_0 is the unknown spatial density of the scatterers. The problem of estimating the intensity factor of a filtered Poisson process may arise from a variety of physical models. For this reason, the general problem is considered first. Estimates obtained using independent samples from the filtered Poisson process are treated in detail. A lower bound on the variance of the independent sample estimate is derived and the structure of a recursive estimator for λ_0 is obtained.

The filtered Poisson process model for the acoustic scattered signal or reverberation is then developed in detail. An expression for the first-order density function for the reverberation process is obtained and its convergence to the Gaussian density is investigated.

The general theory developed is used to evaluate the performance of acoustic techniques for estimating the abundance of marine organisms. The performance of estimators using independent samples from the reverberation process is compared with the lower bound on the variance of the estimate. A Monte Carlo simulation shows that a recursive estimation technique satisfies the variance bound. Mean squared error expressions are derived for the two commonly used abundance estimation techniques, echo counting and echo integration.

A method for estimating the probability density function of the single scatterer target strength is discussed. The mean value of the target strength is required to scale the echo integrator output and obtain an absolute abundance estimate.

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CHAPTER 1

INTRODUCTION

Electrical engineers have for some time used the methods of statistical inference in the design and analysis of electronic systems. Most stochastic system theory developed to date has been concerned with processing of signals with Gaussian statistics. There are two reasons for this: (1) the Gaussian process model applies to a large class of physical phenomena; and (2) the Gaussian process has many mathematical properties that are analytically desirable. However, there are several physical phenomena that can not be suitably modeled by a Gaussian process. One type of stochastic process that arises often in nature is the filtered Poisson process.[1] This dissertation is concerned with the problem of estimating the spatial scattering density from acoustic volume reverberation. In this chapter, a brief account of previous research is given. The chapter is concluded with a preview of the material discussed in the following chapters.

1.1 Historical Account

A large-scale research effort to develop sonar systems began during World War II. Since that time, the U.S. Navy has sponsored a

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[1] Stochastic processes of this type are often referred to as shot noise processes because of their initial application as a model for noise in vacuum tubes [1]. The term filtered Poisson process is more descriptive of their main characteristic, that is, a process arising from a linear operation on a Poisson counting process.

great deal of research in the signal processing of underwater acoustic signals. One aspect of this research has been to characterize reverberation and develop signal processing techniques that minimize its adverse effects on sonar detection systems. Statistical models for reverberation were developed in the United States by Faure (1964) [2] and Middleton (1967 and 1972) [3]. Research on the statistical nature of reverberation was also carried on by Ol'shevskii [4] in the Soviet Union. A number of papers have dealt with the optimum processing of signals in a reverberation-limited environment. Treatment of some of the main results of this research can be found in the book by Van Trees [5] and the dissertation by Moose [6]. Two characteristics of the reverberation-related signal processing theory developed to date are that it assumes that the reverberation process has Gaussian statistics and that reverberation is an undesirable quantity whose effects are to be minimized.

During the period of rather extensive military-oriented acoustic research, biologists were beginning to use acoustic systems to study the distribution and relative abundance of fish populations. Until recently, the systems used by the biologist consisted of an echo sounder and chart recorder. Several articles were written that described techniques for determining the abundance of fish stocks from echograms. Two other abundance estimation techniques developed were echo integration [7] and echo counting [8].

The Marine Acoustics Group at the University of Washington was organized as part of the Sea Grant program in 1968. One of its initial goals was to develop, apply and evaluate an echo integration system. An analog echo integrator was built by Lahore as a master's thesis project in electrical engineering [7]. A digital echo integration system was later developed [9]. Statistical analysis of various acoustic abundance estimation techniques [10], [11] was conducted as a part of the research discussed in this dissertation. The model used in this statistical analysis was the reverberation model used earlier in military research. However, there are two main differences between the biological and military applications of the model: (1) the Gaussian process approximation is not in general valid for a signal scattered from fish; and (2) biologists are interested in extracting information from reverberation rather than minimizing its effect.

1.2 Preview

The problem of estimating the scattering density from volume reverberation is a special case of the general problem of estimating the intensity of a filtered Poisson process. For this reason, the general problem is considered first. Chapter 2 contains a summary of the mathematical theory used in the remainder of the dissertation. Some selected results from the theory of statistical parameter estimation are presented. The statistical properties of Poisson, filtered Poisson and Gaussian processes are stated, and in some cases derived.

Chapter 3 considers the general problem of estimating the intensity factor of a filtered Poisson process. The structures of signal processors that provide an estimate of the intensity factor are derived for a number of special cases. A problem that arises when the Gaussian process approximation is used for a filtered Poisson process with high intensity is discussed.

In Chapter 4, the filtered Poisson process model for volume reverberation is developed. The distribution of the random parameters that appear in the reverberation signal description is discussed. An expression for the first order density function for the reverberation process is obtained and its convergence to the Gaussian density is investigated.

In Chapter 5, the general theory developed in Chapter 3 is applied to the reverberation model. The performance of various estimators using independent samples from the reverberation process is compared with the lower bound on the variance of the intensity estimate. The two commonly used scattering density estimation techniques, echo integration and echo counting, are studied in detail. Expressions are obtained for the variance of the estimates for both processors for received signals in the absence and presence of additive noise. A method for estimating the probability density function of the single scatterer target strength is discussed.

Chapter 6 contains a summary of the major contributions of this research and suggests a number of areas for further investigation.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

2. Introduction

This chapter contains a brief summary of the mathematical theory required for an understanding of the results that appear later in the dissertation. The first part of the chapter deals with statistical estimation of parameters.[1] Gaussian and Poisson processes and processes related to the Poisson process are considered in the remainder of the chapter. The reader is assumed to have a knowledge of probability theory and some familiarity with mathematical statistics.

2.1 Statistical Parameter Estimation

Definition (2.1) (estimation model): The estimation model consists of three components:

i) parameter space: The true value of the parameter to be estimated, λ_o, is a point in the Euclidean space Ω.[2]

ii) observation space: The parameter λ_o is mapped into a set of observed random variables $R_1, R_2, \ldots, R_n$ in an observation space that is assumed to be a subset of an n-dimensional Euclidean space, R^n.

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[1] Detailed discussions of statistical estimation theory can be found in the books by Wilks [12], Cramér [13], and Hogg and Craig [14].

[2] The output of the parameter space can be a random variable or an unknown constant. In this dissertation, it will be assumed that λ_o is an unknown constant. The problem of estimating a random variable is discussed by Van Trees [15].

iii) estimation rule: On the basis of the observed random variables, the parameter λ_o is to be estimated. The mapping of the observation space into the estimate $\hat{\lambda}_o$ is called the estimation rule.

Definition (2.2) (unbiased estimate): An estimate $\hat{\lambda}_o$ is unbiased if $E[\hat{\lambda}_o] = \lambda_o$ where E[ ] denotes mathematical expectation.

Definition (2.3) (consistent estimate): An estimate $\hat{\lambda}_o$ is consistent if $\hat{\lambda}_o$ converges in probability to λ_o as n, the number of observations, goes to infinity.

Definition (2.4) (efficient estimate): Let $\hat{\lambda}_o$ be an unbiased estimate of λ_o having finite variance. If no other unbiased estimate has a smaller variance, then $\hat{\lambda}_o$ is called an efficient estimate for λ_o.

2.1.1 Sufficient Statistics

Definition (2.5) (sufficient statistic): Let $\underset{\sim}{R} = \{R_1, R_2, \ldots, R_n\}$ be a set of n random variables whose distribution function is given by $F(\underset{\sim}{r};\lambda_o)$.[3] A function $T(\underset{\sim}{R})$ is said to be a sufficient statistic for λ_o if the conditional distribution of $\underset{\sim}{R}$, given $T(\underset{\sim}{R}) = t$, is independent of λ_o.

Loosely speaking, a sufficient statistic contains all the information about λ_o that can be obtained from the sample, R.

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[3] Distribution and density functions are often written as $F_R(t)$ and $f_R(r)$ to make the reference to the random variable R explicit. In most cases, the notation F(r) and f(r) is self explanatory. This abbreviated notation will be used in the following except in cases where it leads to confusion.

Theorem (2.1) (factorization theorem): Let $F(\underset{\sim}{r};\lambda_o)$ be a family of distribution functions having probability density functions $f(\underset{\sim}{r};\lambda_o)$ with respect to a σ finite Borel measure, $M(d\underset{\sim}{r})$. Then $T(\underset{\sim}{R})$ is a sufficient statistic for λ_o if and only if there exist nonnegative measurable functions V and W such that

$$f(\underset{\sim}{r};\lambda_o) = V(T(\underset{\sim}{r});\lambda_o)W(\underset{\sim}{r}) \tag{2.1}$$

Proof: Cf. [12], Chapter 12 or [16], Chapter 3.

Definition (2.6) (exponential family of distributions): Let $F(\underset{\sim}{r};\lambda_o)$ be a family of distribution functions having probability density functions $f(\underset{\sim}{r};\lambda_o)$ with respect to a σ finite Borel measure, $M(d\underset{\sim}{r})$ and let λ_o be a point in a k dimensional parameter space. Then $F(r;\lambda_o)$ is said to be a k parameter exponential family if

$$f(\underset{\sim}{r};\lambda_o) = C(\lambda_o)h(\underset{\sim}{r})\exp\left[\sum_{i=1}^{k}\pi_i(\lambda_o)t_i(\underset{\sim}{r})\right] \tag{2.2}$$

Theorem (2.2): Let $\underset{\sim}{R}$ be a set of random variables from a k parameter exponential family. Then $T(\underset{\sim}{R}) = (t_1(\underset{\sim}{R}), \ldots, t_k(\underset{\sim}{R}))$ is a sufficient statistic for λ_o.

Proof: The proof follows directly from Theorem (2.1).

The probability density function of independent random variables taken from a Gaussian, a Poisson, a gamma distribution or many other common distributions can be written in the form of (2.2). Therefore, the above theorem provides an easy way of finding sufficient statistics for estimating parameters from these distributions.

2.1.2 Cramér-Rao Lower Bound

Definition (2.7) (regularity with respect to the first derivative): Let $F(\underset{\sim}{r};\lambda_o)$ be a family of distribution functions having probability density functions, $f(\underset{\sim}{r};\lambda_o)$ with respect to a fixed, σ finite, Borel measure, $M(d\underset{\sim}{r})$. $F(\underset{\sim}{r};\lambda_o)$ is regular with respect to its first λ_o derivative if for every estimate $\hat{\lambda}_o$ of λ_o with finite variance

$$\frac{d}{d\lambda_o} E[\hat{\lambda}_o(\underset{\sim}{r})] = E[\hat{\lambda}_o(\underset{\sim}{r})S(\underset{\sim}{r};\lambda_o)] \tag{2.3}$$

where

$$S(\underset{\sim}{r};\lambda_o) = \frac{\partial}{\partial\lambda_o} \ln f(\underset{\sim}{r};\lambda_o)$$

A sufficient condition for regularity is that $\frac{\partial}{\partial\lambda_o} \ln f(\underset{\sim}{r};\lambda_o)$ be dominated by some integrable function. (cf. [12], p. 347).

Theorem (2.3) (Cramér-Rao lower bound): Let $F(\underset{\sim}{r};\lambda_o)$ be regular with respect to its first λ_o derivative and let $\hat{\lambda}_o$ be an estimate of λ_o with bias $b(\hat{\lambda}_o) = E[\hat{\lambda}_o - \lambda_o]$, then

$$\text{Var}[\hat{\lambda}_o] \geq \frac{[1 + b(\hat{\lambda}_o)]^2}{E[S^2(\underset{\sim}{r};\lambda_o)]} \tag{2.4}$$

with equality if and only if

$$S(\underset{\sim}{r};\lambda_o) = K[\hat{\lambda}_o - \lambda_o - b(\hat{\lambda}_o)] \tag{2.5}$$

and where K is not a function of λ_o.

Proof: Cf. [12], Chapter 12.

2.1.3 Maximum Likelihood Estimates

Definition (2.8) (maximum likelihood estimates): Let $F(\underset{\sim}{r};\lambda_o)$, $\underset{\sim}{r} \in R^n$, $\lambda_o \in \Omega$ be a family of distribution functions having probability density functions $f(\underset{\sim}{r};\lambda_o)$ with respect to a σ finite Borel measure, $M(d\underset{\sim}{r})$. The parameter $\hat{\lambda}_o$, $\hat{\lambda}_o \in \Omega$ which maximizes $f(\underset{\sim}{r};\lambda_o)$ is called the maximum likelihood estimate of λ_o.

If the observed random variables $R_1, R_2, \cdots, R_n$ are independent and if $\hat{\lambda}_o$ is an interior point in the parameter space Ω, then the maximum likelihood estimate is a solution to

$$\sum_{i=1}^{n} \frac{d}{d\lambda_o} \ln f(R_i;\lambda_o) = 0 \qquad (2.6)$$

Maximum likelihood estimates are important because of the ease with which they are often obtained and because of their many desirable properties. Some of these properties are listed below (cf. [12], 362, 363):

i) asymptotically unbiased as sample size, $n \to \infty$

ii) asymptotically consistent as $n \to \infty$

iii) if an efficient estimate exists, it is given by the maximum likelihood estimate.

2.2 Stochastic Processes

Definition (2.9) (stochastic process): A stochastic process[4] $\{r(t), t \in T\}$ is a collection of random variables defined on a probability space. The set T is called the index set of the process. When T is a set of points in an Euclidean space, the process is said to be a discrete parameter process and when T is a region in an Euclidean space, the process is said to be a continuous parameter process.

2.2.1 Karhunen-Loève Expansion

The parameter estimation theory discussed in the first part of this chapter considered the problem of mapping a point in an n dimensional Euclidean observation space, R^n, into a parameter estimate $\hat{\lambda}_o$. The Karhunen-Loève expansion defined in this section allows stochastic processes that are mean squared continuous on a finite interval to be included in this estimation theory model. Readers not familiar with the Karhunen-Loève expansion can find a derivation and a discussion of its properties in the book by Van Trees [15].

Definition (2.10) (Karhunen-Loève expansion): Let $\{r(t), t \in [0,T]\}$ be a mean squared continuous process on the finite interval [0,T]. Then[5]

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[4] The stochastic processes considered in this chapter are assumed to be real processes. Much of the theory developed can be easily extended to apply to complex processes.

[5] l.i.m. denotes limit in the mean.

$$r(t) = \underset{N \to \infty}{\text{l.i.m.}} \sum_{i=1}^{n} R_i \phi_i(t) \qquad (2.7)$$

where the functions $\phi_i(t)$ are square integrable on [0,T] and the following relationships hold.[6]

i) $\int_0^T \phi_i(t) r_i(t) dt = R_i$

ii) $\int_0^T \phi_i(t) \phi_j(t) dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

iii) $\int_0^T \phi_i(u) K_r(t,u) du = \mu_i \phi_i(t)$

iv) $E[R_i R_j] = \begin{cases} \mu_i & i = j \\ 0 & i \neq j \end{cases}$

2.2.2 Gaussian Processes

Definition (2.11) (Gaussian process): A stochastic process $\{r(t), t \in T\}$ is said to be a Gaussian process if for any integer n and any subset $\{t_1, t_2, \ldots, t_n\}$ of T, the n random variables $r(t_1), r(t_2), \ldots, r(t_n)$ are jointly Gaussian.

The Karhunen-Loève expansion described in the previous section is particularly useful when dealing with mean squared continuous Gaussian processes. The random coefficients in (2.7) are obtained by a linear operation on the Gaussian process and are therefore Gaussian random variables. Using this fact and property (iv) it follows that $R_1, R_2, \ldots, R_n$

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[6] $K_r(t,u) = E[r(t)r(u)] - E[r(t)]\, E[r(u)]$

are independent random variables with joint density function given by[7]

$$f(r_1,r_2, \ldots ,r_n) = \frac{1}{\prod_{i=1}^{n} \sqrt{2\pi\mu_i}} \exp\left[- \sum_{i=1}^{n} \frac{(r_i - E[r_i])^2}{2\mu_i} \right] \qquad (2.8)$$

2.2.3 Counting Processes

Definition (2.12) (counting process): A counting process N(S) is a real integer-valued process which counts the number of occurrence points in the region S in an Euclidean space, R^n.

Definition (2.13) Poisson process)[8]: An integer-valued counting process N(S) is said to be a Poisson process with intensity function $\nu(s)$ if the following conditions are fulfilled[9]:

i) for any integer n and n nonoverlapping regions, $S_1,S_2, \ldots ,S_n$, the random variables $N(S_1),N(S_2), \ldots ,N(S_n)$ are independent.

ii) for any region S, the number of counts in the region S is governed by the following probability law:

---

[7] When the process is not Gaussian, the coefficients in (2.7) are pair-wise uncorrelated but not independent.

[8] Much of the material on Poisson processes has been taken from the book by Parzen [17]. Some of Parzen's results have been extended to apply to the problems. considered in this dissertation. Proofs of the theorems contained in Parzen are not repeated here.

[9] In many cases of interest, the region S is an interval in a one-dimensional Euclidean space, R^1. The Poisson process defined on R^1 will be designated as $\{N(t), t \geq 0\}$.

$$P[N(S) = k] = \frac{\left[\int_S \nu(s)ds\right]^k}{k!} \exp\left[-\int_S \nu(s)ds\right] \tag{2.9}$$

For k = 0,1,2, ... where $\nu(s)$ is a nonnegative function with bounded integral.

The process is called a homogeneous Poisson process if the intensity function satisfies the following relationship

$$\nu(s) = \nu \tag{2.10}$$

The process is called nonhomogeneous when (2.10) does not hold.

Theorem (2.4) (characteristic function of a Poisson process): Let $\{N(t), t \geq 0\}$ be a Poisson process defined on R^1. The characteristic function of $\{N(t), t \geq 0\}$ is

$$\phi_{N(t)}(u) = \exp\left[\int_0^t \nu(x)dx(e^{ju} - 1)\right] \tag{2.11}$$

Proof: See Chapter 4 of Parzen [17].

The moments of a Poisson process can be determined from the derivative of the logarithm of the characteristic function. In particular, the mean and variance of $\{N(t), t \geq 0\}$ are

$$E[N(t)] = j \frac{d}{du}\left[\ln \phi_{N(t)}(u)\right]\Bigg|_{u=0}$$

$$= \int_0^t \nu(x)dx \tag{2.12}$$

and

$$\mathrm{Var}[N(t)] = -\frac{d^2}{du^2}\left[\ln \phi_{N(t)}(u)\right]\Bigg|_{u=0}$$

$$= \int_0^t \nu(x)dx \tag{2.13}$$

The relationships between various order moments and the logarithm of the characteristic function are given in the book by Kendall and Stuart [18].

Theorem (2.5) (arrival time distribution): Let $\{N(t), t \geq 0\}$ be a Poisson process defined on the interval (0,T] and let $\tau_1, \ldots, \tau_n$ where $0 < \tau_1 < \ldots < \tau_n < t$ be the points at which events have occurred. The density function of $\tau_1, \ldots, \tau_n$ with respect to Lebesgue measure is

$$f(\tau_1, \ldots, \tau_n) = \nu(\tau_1) \ldots \nu(\tau_n) \exp\left[-\int_0^t \nu(x)dx\right] \tag{2.14}$$

Proof: From property (ii) in Definition (2.13), it follows that the distribution function of the arrival time of the first event is

$$F(\tau_1) = 1 - \text{Prob}\,[\text{no event in } (0, \tau_1]]$$

$$= 1 - \exp\left[-\int_0^{\tau_1} \nu(x)dx\right] \tag{2.15}$$

and the corresponding density function is

$$f(\tau_1) = \frac{d}{d\tau_1} F(\tau_1) = \nu(\tau_1) \exp\left[-\int_0^{\tau_1} \nu(x)dx\right] \tag{2.16}$$

The number of events in disjoint time intervals is independent by (i) of Definition (2.13). Therefore, the joint density of $\tau_1,\ldots,\tau_n$ is

$$f(\tau_1,\ldots,\tau_n) = \nu(\tau_1)\, e^{-\int_0^{\tau_1}\nu(x)dx} \cdots \nu(\tau_n) e^{-\int_{\tau_{n-1}}^{\tau_n} \nu(x)dx}\, e^{-\int_{t_n}^{t}\nu(x)dx} \quad (2.17)$$

$$= \nu(\tau_1)\ldots\nu(\tau_n)\ \exp\left[-\int_0^t \nu(x)dx\right] \quad (2.18)$$

In (2.17), all terms except the last are density functions for the intervals and the last term is the probability of no event between τ_n and t.

Theorem (2.6): Let $\{N(t),t\geq 0\}$ and $\tau_1,\ldots,\tau_n$ be defined as in Theorem (2.5). The density function of $\tau_1,\ldots,\tau_n$ given that exactly n events occurred in (0,t] is

$$f(\tau_1,\ldots,\tau_n \mid n \text{ events in } (0,t])$$

$$= \frac{\nu(\tau_1)\ldots\nu(\tau_n)n!}{\left[\int_0^t \nu(x)dx\right]^n} \quad (2.19)$$

Proof: The proof follows directly from Theorem (2.5) and Bayes rule for conditional density functions.

2.2.4 Filtered Poisson Processes

Definition (2.14) (filtered Poisson process): A stochastic process $\{r(t), t \geq 0\}$ is defined to be a filtered Poisson process if it can be represented by

$$r(t) = \sum_{m=1}^{N(t)} z(t, \tau_m, \underset{\sim}{\Theta}_m) \tag{2.20}$$

where $\{N(t), t \geq 0\}$ is a Poisson process, $\{\underset{\sim}{\Theta}_m\}$ is a sequence of independent random vectors that are independent of $\{N(t), t \geq 0\}$, τ_m is the occurrence time of the m^{th} event and $z(t, \tau, \underset{\sim}{\Theta})$ is a function of the variables t, τ and $\underset{\sim}{\Theta}$ where $z(t, \tau, \underset{\sim}{\Theta}) = 0$ for $t < \tau$.

Theorem (2.7): Let $\{r(t), t \geq 0\}$ be a filtered Poisson process. Then for any $t_n > t_{n-1} \ldots t_1 \geq 0$, $n \geq 1$ and real numbers $u_1, u_2, \ldots, u_n$, the joint characteristic function of $r(t_1), r(t_2), \ldots, r(t_n)$ is

$$\phi_{r(t_1),\ldots,r(t_n)}(u_1, u_2, \ldots, u_n)$$

$$\begin{aligned}
= \exp\Big\{ &\int_o^{t_1} \nu(x) E\left[e^{ju_1 z(t_1, x, \underset{\sim}{\Theta}) + \ldots + ju_n z(t_n, x, \underset{\sim}{\Theta})} - 1\right] dx \\
&+ \int_{t_1}^{t_2} \nu(x) E\left[e^{ju_2 z(t_2, x, \underset{\sim}{\Theta}) + \ldots ju_n z(t_n, x, \underset{\sim}{\Theta})} - 1\right] dx \\
&\;\;\vdots \\
&+ \int_{t_{n-1}}^{t_n} \nu(x) E\left[e^{ju_n z(t_n, x, \underset{\sim}{\Theta})} - 1\right] dx \Big\}
\end{aligned} \tag{2.21}$$

Proof: The joint characteristic function of the random variables $r(t_1), r(t_2), \ldots, r(t_n)$ is by definition

$$\phi_{r(t_1),\ldots,r(t_n)}(u_1,\ldots,u_n) = E \exp\left[\left(j \sum_{i=1}^{n} u_i r(t_i)\right)\right]$$

$$= E\left[\exp\left(j \sum_{m=1}^{N(t_n)} g_m(\tau_m)\right)\right] \qquad (2.22)$$

where

$$g_m(\tau_m) = u_1 z(t_1,\tau_m,\underset{\sim}{\Theta}_m)+\ldots+u_n z(t_n,\tau_m,\underset{\sim}{\Theta}_m)$$

Using conditional expectations, (2.22) can be rewritten as

$$\phi_{r(t_1),\ldots,r(t_n)}(u_1,\ldots,u_n) =$$

$$\sum_{k=0}^{\infty} E\left[\exp\left(j \sum_{m=1}^{N(t_n)} g_m(\tau_m)\right)\Big/ N(t_n) = k\right] P[N(t_n) = k] \qquad (2.23)$$

The conditional expectation in (2.23) can be written as

$$E\left[\exp\left(j \sum_{m=1}^{N(t_n)} g_m(\tau_m)\right)\Big/ N(t_n) = k\right]$$

$$= \int_0^{t_n}\int_{\tau_1}^{t_n}\cdots\int_{\tau_{k-1}}^{t_n} E\left[\exp\left(j \sum_{m=1}^{N(t_n)} g_m(\tau_m)\right)\Big/ N(t_n) = k,\ \tau_1,\ldots,\tau_k\right] \qquad (2.24)$$

$$f\left(\tau_1,\ldots,\tau_k / N(t_n) = k\right) d\tau_1,\ldots,d\tau_k$$

Using the arrival time density function obtained in Theorem (2.6), the conditional expectation becomes

$$E\left[\exp\left(j\sum_{m=1}^{N(t_n)} g_m(\tau_m)\right)/N(t_n) = k\right]$$

$$= \frac{k!}{\left[\int_o^{t_n}\nu(x)dx\right]^k}\int_o^{t_n}\cdots\int_{\tau_{k-1}}^{t_n}\prod_{m=1}^{k}\{E[\exp(jg_m(\tau_m))]\nu(\tau_m)\}\,d\tau_1\ldots d\tau_k \tag{2.25}$$

The multiple integral in (2.25) can be written as a single integral raised to the k^{th} power (cf. [17], p. 155).

$$E\left[\exp\left(j\sum_{m=1}^{N(t_n)} g_m(\tau_m)\right)/N(t_n) = k\right]$$

$$= \frac{1}{\left[\int_o^{t_n}\nu(x)dx\right]^k}\left[\int_o^{t_n}\nu(x)E[\exp(jg(x))]dx\right]^k \tag{2.26}$$

Substituting (2.26) into (2.23), it follows that

$$\phi_{r(t_1),\ldots,r(t_n)}(u_1,\ldots,u_n) =$$

$$\exp\left[-\int_o^{t_n}\nu(x)dx\right]\sum_{k=o}^{\infty}\frac{\left[\int_o^{t_n}\nu(x)E[\exp[jg(x)]\,dx\right]^k}{k!} \tag{2.27}$$

The form of the characteristic function in (2.21) is obtained by writing the summation in (2.27) as an exponential.

The characteristic function in (2.21) is used to obtain some moments of a filtered Poisson process in Appendix A. It is easily shown that the mean, $m(t)$, and variance, $\sigma^2(t)$, of $r(t)$ are

$$m(t) = \int_0^t \nu(x)E[z(t,x,\underset{\sim}{\Theta})]dx \tag{2.28}$$

$$\sigma^2(t) = \int_0^t \nu(x)E[z^2(t,x,\underset{\sim}{\Theta})]dx \tag{2.29}$$

Definition (2.15): A stochastic process $\{r(t),t\in T\}$ with finite second moments and whose distribution depends on some parameter is said to be asymptotically Gaussian as the parameter tends to a given limit if the following conditions hold for every set of indices $t_1,t_2, \ldots ,t_n$: the joint characteristic function of the standarized random variables, $r^*(t_1),r^*(t_2), \ldots ,r^*(t_n)$, where

$$r^*(t_i) = \frac{r(t_i) - E[r(t_i)]}{\sigma[r(t_i)]} \tag{2.30}$$

satisfies

$$\ln \phi_{r^*(t_1),\ldots,r^*(t_n)}(u_1,\ldots,u_n) \longrightarrow$$

$$-\frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \rho[r(t_i)r(t_j)] \tag{2.31}$$

as the parameter tends to the given limit, where

$$\rho[r(t_i)r(t_j)] = \frac{E[r(t_i)r(t_j)] - E[r(t_i)]E[r(t_j)]}{\sigma[r(t_i)]\sigma[r(t_j)]} \tag{2.32}$$

and where $\sigma[r(t_i)]$ is the standard deviation of $r(t_i)$.

Theorem (2.8): Let $\{r(t),t\geq 0\}$ be a filtered Poisson process with intensity function $\nu(x) = \lambda_o\beta(x)$ where $\beta(x)>0$ for $x>0$, and let

$$m(t) = \lambda_o\int_0^t \beta(x)E[z(t,x,\underset{\sim}{\Theta})]dx \tag{2.33}$$

$$\sigma^2(t) = \lambda_o\int_0^t \beta(x)E[z^2(t,x,\underset{\sim}{\Theta})]dx \tag{2.34}$$

$$\rho(t_i,t_j) = \frac{\int_0^t \beta(x)E[z(t_i,x,\underset{\sim}{\Theta})z(t_j,x,\underset{\sim}{\Theta})]dx}{\sqrt{\int_0^t \beta(x)E[z^2(t_i,x,\underset{\sim}{\Theta})]dx\int_0^t \beta(x)E[z^2(t_j,x,\underset{\sim}{\Theta})]dx}} \tag{2.35}$$

$$k(t) = \lambda_o\int_0^t \beta(x)E[z^3(t,x,\underset{\sim}{\Theta})]dx \tag{2.36}$$

If the mean, $m(t)$, the variance, $\sigma^2(t)$, and the third cumulant, $k(t)$ are finite for all t, then $\{r(t),t\geq 0\}$ is asymptotically Gaussian as $\lambda_o \to \infty$.

Proof: The characteristic function of the filtered Poisson process as determined in Theorem (2.7) can be written as

$$\phi_{r(t_1),\ldots,r(t_n)}(u_1,\ldots,u_n) =$$

$$\exp\Big[\int_0^t \nu(x)E\ e^{ju_1z(t_1,x,\underset{\sim}{\Theta})+\ldots+ju_nz(t_n,x,\underset{\sim}{\Theta})} - 1\Big]dx \tag{2.37}$$

where $z(t_i,x,\underset{\sim}{\Theta}) = 0$ for $t_i<x$ and $i = 1,2, \ldots ,n$. Expanding the logarithm in a power series, it follows that

$$\ln \phi_{r(t_1),\ldots,r(t_n)}(u_1,\ldots,u_n)$$

$$= j\sum_{i=1}^{n} u_i\ m(t_i) - \frac{1}{2}\sum_{\ell,k=1}^{n} u_\ell u_k\ \rho(t_\ell,t_k)\sigma(t_\ell)\sigma(t_k) \tag{2.38}$$

$$+ C_o\sum_{i=1}^{n} |u_i|^3 K(t_i)$$

where $|C_o|<1$. Define a new vector random process $\{r^*(t), t\geq 0\}$ where

$$r^*(t_i) = \frac{r(t_i) - m(t_i)}{\sigma(t_i)} \tag{2.39}$$

It follows from the definition of the characteristic function, (2.38) and (2.39) that

$$\ln \phi r^*(t_1),\ldots,r^*(t_n)(u_1,\ldots,u_n) = -\frac{1}{2}\sum_{\ell,k=1}^{n} u_\ell u_k \rho(t_\ell,t_k)$$

$$+ \frac{Co}{\sqrt{\lambda_o}}\sum_{\ell=1}^{n} |u_\ell|^3 \frac{\int_0^t \beta(x)E[z^3(t,x,\underset{\sim}{\Theta})]dx}{\left[\int_0^t \beta(x)E[z^2(t,x,\underset{\sim}{\Theta})]dx\right]^{3/2}} \tag{2.40}$$

and

$$\lim_{\lambda_o\to\infty} \phi_{r^*(t_1),\ldots,r^*(t_n)}(u_1,\ldots,u_n) = -\frac{1}{2}\sum_{\ell,k=1}^{n} u_\ell u_k \rho(t_\ell,t_k) \tag{2.41}$$

CHAPTER 3

POISSON INTENSITY ESTIMATION

3. Introduction

A filtered Poisson process is a reasonable model for many random phenomena occurring in nature. As was mentioned in Chapter 1, the filtered Poisson process was originally used as a model for shot noise. It will be shown in the next chapter that acoustic volume reverberation can also be modeled by a filtered Poisson process. Some other examples of filtered Poisson processes are given in reference [17]. This chapter deals with the problem of estimating the Poisson intensity factor from a filtered Poisson process. Such an estimate often provides information about the physical environment that produced the process.

A number of aspects of the Poisson intensity estimation problem are considered in this chapter. The problem of estimating the intensity of a Poisson counting process is discussed in the first section. It is then shown in the second section that the problem of estimating the intensity of a filtered Poisson process is in many cases equivalent (in theory) to the problem considered in the first section. A problem that arises when the Gaussian process approximation is used for a filtered Poisson process with high intensity is discussed. The maximum likelihood estimator for the intensity factor using independent samples from the filtered Poisson process is derived and a bound on the variance of the estimate is obtained.

3.1 Poisson Counting Process Intensity Estimation

The estimation problem considered in this section is the following. Let $\{N(t), t \geq 0\}$ be a nonhomogeneous Poisson counting process with intensity function $\nu(t)$ where

$$\nu(t) = \lambda_o \beta(t) \tag{3.1}$$

and $\beta(t)$ is a known deterministic real positive function defined in the interval $[0,t]$. We wish to find an estimate of λ_o from a realization of the process $\{N(t), t \geq 0\}$. The available information or statistics that can be used in obtaining the estimate of λ_o are the occurrence times of the Poisson events $\tau_1, \tau_2, \ldots, \tau_k$. The joint density function of $\tau_1, \tau_2, \ldots, \tau_k$ with respect to Lesbegue measure obtained in Theorem (2.5) is

$$\begin{aligned} f(\tau_1, \tau_2, \ldots, \tau_k) &= \lambda_o^k \beta(\tau_1) \ldots \beta(\tau_k) \exp\left[- \lambda_o \int_0^t \beta(x)dx \right] \\ &= \exp\left[k \ln\lambda_o + \ln[\beta(\tau_1) \ldots \beta(\tau_k)] - \lambda_o \int_0^t \beta(x)dx \right] \end{aligned} \tag{3.2}$$

The second form of the density function shows that the distribution of the arrival times is a one parameter exponential family. Using Theorem (2.2), it follows that k, the number of Poisson events in $[0,t]$, is a sufficient statistic for λ_o. In other words, the number of events, k, contains all the useful information about the parameter λ_o and the estimate of λ_o can be based solely on k.

The probability density function of k given in Definition (2.13) is

$$f(k;\lambda_o) = \frac{\lambda_o\left[\int_0^t \beta(x)dx\right]^k}{k!} \exp\left[- \lambda_o \int_0^t \beta(x)dx\right] \tag{3.3}$$

The maximum likelihood estimate for λ_o in terms of k is given by

$$\frac{d}{d\lambda_o} f(k;\lambda_o)\Big|_{\lambda_o = \hat{\lambda}_o} = 0$$

$$\frac{d}{d\lambda_o}\left\{\frac{\left[\lambda_o\int_0^t \beta(x)dx\right]^k}{k!}\exp\left[-\lambda_o\int_0^t \beta(x)dx\right]\right\}\Bigg|_{\lambda_o = \lambda_o} = 0 \qquad (3.4)$$

The solution to (3.4) is

$$\hat{\lambda}_o = \frac{k}{\int_0^t \beta(x)dx} \qquad (3.5)$$

The expected value of the estimate is

$$E[\hat{\lambda}_o] = \frac{E[k]}{\int_0^t \beta(x)dx} = \lambda_o \qquad (3.6)$$

and the variance is

$$Var[\hat{\lambda}_o] = \frac{E[k^2]}{\left[\int_0^t \beta(x)dx\right]^2} - \lambda_o^{\,2} = \frac{\lambda_o}{\int_0^t \beta(x)dx} \qquad (3.7)$$

Therefore, the estimate is unbiased and is consistent.[1] The Cramér-Rao lower bound on the variance of an estimate of λ_o stated in Theorem (2.3) is

$$Var[\hat{\lambda}_o] = \left(E\left[\frac{\partial}{\partial\lambda_o}\ln f(k;\lambda_o)^{\,2}\right]\right)^{-1}$$

---

[1] Since $\beta(x) > 0$, $x\epsilon[o,t]$, $\hat{\lambda}_o$ converges to λ_o in the mean squared sense as $t \to \infty$. The consistency of $\hat{\lambda}_o$ follows by recalling that convergence in the mean squared sense implies convergence in probability.

$$= \left(E\left[\left\{\frac{k\int_0^t \beta(x)dx}{\lambda_o\int_0^t \beta(x)dx} - \int_o^t \beta(x)dx\right\}^2\right]\right)^{-1} = \frac{\lambda_o}{\int_o^t \beta(x)dx} \tag{3.8}$$

The variance in (3.7) satisfies the Cramér-Rao bound with equality and therefore, λ_o is an efficient estimate of λ_o.

3.2 Filtered Poisson Process Intensity Estimation

Let $\{r(t), t \geq 0\}$ be a filtered Poisson process defined by

$$r(t) = \sum_{m=1}^{N(t)} z(t, \tau_m, \underset{\sim}{\Theta}_m) \tag{3.9}$$

where $\{N(t), t \geq 0\}$ is a Poisson counting process with intensity $\nu(t) = \lambda_o$ $\beta(t)$. When the individual Poisson events are discernible, the problem of estimating λ_o reduces to the problem considered in the previous section and the intensity factor estimate is given by (3.5). The following theorem shows that under some rather general conditions, the Poisson occurrence times, τ_i, can in theory be uniquely determined from the observed process $r(t)$.

<u>Theorem (3.1)</u>: Let $\{r_1(t), t \geq 0\}$ and $\{r_2(t), t \geq 0\}$ be filtered Poisson processes defined by

$$r_1(t) = \sum_{m=1}^{N_1(t)} z_1(t, \tau_{1m}, \underset{\sim}{\Theta}_{1m}) \tag{3.10}$$

$$r_2(t) = \sum_{\ell=1}^{N_2(t)} z_2(t, \tau_{2\ell}, \underset{\sim}{\Theta}_{2\ell}) \tag{3.11}$$

where the response functions $z_1(t,\tau_{1m},\underset{\sim}{\Theta}_{1m})$ and $z_2(t,\tau_{2m},\underset{\sim}{\Theta}_{2m})$ satisfy the following conditions:

i) $z_j(t,\tau_{jk},\underset{\sim}{\Theta}_{jk})$ is analytic[2] in the interval $(\tau_{jk},T_{jk}]$ and is zero outside the interval $(\tau_{jk},T_{jk}]$ with probability 1 for $j = 1,2$ and all k.

ii) $z_j(t,\tau_{jk},\underset{\sim}{\Theta}_{jk})>0$ with probability 1 for some $t\in(\tau_{jk},T_{jk}]$ for $j = 1,2$ and all k.

Then if $r_1(t) = r_2(t)$ for all $t\in[0,T]$ it follows that

$$z_1(t,\tau_{1i},\underset{\sim}{\Theta}_{1i}) = z_2(t,\tau_{2i},\underset{\sim}{\Theta}_{2i}) \qquad (3.12)$$

for all i.

<u>Proof</u>:

Since $\{N_1(t),t\geq 0\}$ and $\{N_2(t),t\geq 0\}$ are Poisson processes, it follows that

$$\tau_{\ell 1} > 0 \qquad \ell = 1,2 \qquad (3.13)$$

$$|\tau_{\ell i} - \tau_{\ell i-1}| > 0 \qquad \ell = 1,2 \text{ all } i > 1 \qquad (3.14)$$

with probability 1. Without loss of generality, it can be assumed that

$$\tau_{21} \geq \tau_{11} > 0. \qquad (3.15)$$

---

[2] A function is analytic at a point z_o if and only if it can be represented by a power series that is convergent throughout a neighborhood of z_o (cf.[19], p.192).

Over the interval $(\tau_{11}, \min(\tau_{22}, \tau_{12}))$ the only contribution to $r_1(t)$ is $z_1(t, \tau_{11}, \underset{\sim}{\Theta}_{11})$ and the only contribution to $r_2(t)$ is $z_2(t, \tau_{21}, \underset{\sim}{\Theta}_{21})$. Since $r_1(t) = r_2(t)$ for all $t \in [0,T]$, it follows that $z_1(t, \tau_{21}, \underset{\sim}{\Theta}_{21}) = z_2(t, \tau_{11}, \underset{\sim}{\Theta}_{11})$ over the interval $(\tau_{11}, \min(\tau_{22}, \tau_{12}))$ which has nonzero length with probability 1 by (3.14). By analytic continuation (cf. [19], p. 206), $z_1(t, \tau_{11}, \underset{\sim}{\Theta}_{11}) = z_2(t, \tau_{21}, \underset{\sim}{\Theta}_{21})$ for all $t \in (\tau_{11}, T_{11}]$ and by assumption (i) will be equal to zero outside $[\tau_{11}, T_{11}]$.

Since $z_1(t, \tau_{11}, \underset{\sim}{\Theta}_{11})$ (which equals $z_2(t, \tau_{21}, \underset{\sim}{\Theta}_{21})$) is known over its entire nonzero time interval it can be substracted from $r_1(t)$ and $r_2(t)$. The same procedure can now be used to show that $z_1(t, \tau_{12}, \underset{\sim}{\Theta}_{22}) = z_2(t, \tau_{22}, \underset{\sim}{\Theta}_{22})$, $z_1(t, \tau_{13}, \underset{\sim}{\Theta}_{13}) = z_2(t, \tau_{23}, \underset{\sim}{\Theta}_{23})$ etc.

The above theorem shows that if the response functions are analytic, the Poisson occurrence times can be obtained from the observed process $\{r(t), t \geq 0\}$. In this case, the estimate of λ_o is obtained using (3.5) and the variance of the estimate is given by (3.7).

The class of analytic functions is quite general. One subclass is the generalized exponential functions which include real and complex exponentials, sinesoids and sums and products of these functions.

Theorem (3.1) does not provide a practical solution to the problem of estimating the intensity factor of a filtered Poisson process since it is not in general possible to implement a signal processor that will resolve all individual Poisson events. However, the theorem shows that the variance of any estimate of λ_o is lower bounded by the variance in (3.7).

$$\mathrm{Cov}[r(t),r(u)] = \lambda_o \rho_r(t,u) \tag{3.18}$$

where $\rho_r(t,u)$ is a deterministic function of t and u defined by

$$\rho_r(t,u) = \int_o^{\min(t,u)} \beta(x)E[z(t,x,\underset{\sim}{\Theta})z(u,x,\underset{\sim}{\Theta})]dx \tag{3.19}$$

The problem of estimating the intensity factor, λ_o, is equivalent to the problem of estimating the scale factor of the covariance function of the Gaussian process approximation for $\{r(t),t\geq 0\}$.

The maximum likelihood estimate of λ_o can be obtained using the Karhunen-Loéve expansion described in Section 2.2.2 Define the random variables R_i by

$$R_i = \int_o^T r(t)\phi_i(t)dt \tag{3.20}$$

with the functions $\phi_i(t)$ chosen to satisfy

$$\gamma_i\phi_i(t) = \int_o^T \rho_r(t,u)\phi_i(u)du \tag{3.21}$$

Using the Gaussian approximation for r(t) and the properties of the Karhunen-Loéve expansion, it follows that R_i are independent Gaussian random variables with $E[R_i] = 0$ and $\mathrm{Var}[R_i] = \lambda_o\gamma_i$. The process

$$r^k(t) \triangleq \sum_{i=1}^{k} R_i\phi_i(t) \tag{3.22}$$

converges to $\{r(t),t\geq 0\}$ in the mean square sense as k approaches infinity. The joint density function for $R_1, \ldots ,R_k$ is

$$f(r_1,\ldots,r_k;\lambda_o) = \prod_{j=1}^{k} (2\pi\lambda_o\gamma_j)^{-\frac{1}{2}}\exp\{-r_j^{\,2}/2\lambda_o\gamma_j\} \tag{3.23}$$

The maximum likelihood estimate of λ_o in terms of $R_1, \ldots, R_k$ is given by

$$\frac{d}{d\lambda_o} f(R_1,\ldots,R_k;\lambda_o)\Big|_{\lambda_o = \hat{\lambda}_o} = 0 \tag{3.24}$$

Using the density in (3.23) it follows that

$$\hat{\lambda}_o = \frac{1}{k}\sum_{j=1}^{k} R_j^{\,2}/\gamma_j \tag{3.25}$$

where $R_j/\sqrt{\gamma_j}$ are independent Gaussian random variables with

$$E\left[R_j/\sqrt{\gamma_j}\right] = 0 \tag{3.26}$$

$$\mathrm{Var}\left[R_j/\sqrt{\gamma_j}\right] = \lambda_o$$

It can be shown that $k\hat{\lambda}_o/\lambda_o$ is chi-squared distributed with k-1 degrees of freedom (cf. [12], p. 208). The variance of the estimate, $\hat{\lambda}_o$, which is obtained from the moments of the chi-squared random variable is

$$\mathrm{Var}[\hat{\lambda}_o] = \frac{\lambda_o^{\,2}}{k^2} E\left[\left\{\frac{k\lambda_o}{\lambda_o}\right\}^2\right] - \frac{\lambda_o^{\,2}}{k^2}\left\{E\left[\frac{k\lambda_o}{\lambda_o}\right]\right\}^2$$

$$= \frac{\lambda_o^{\,2}}{k^2}(k^2-1) + \frac{\lambda_o^{\,2}}{k^2}\left(\frac{k^2-2k-1}{k^2}\right)$$

$$= \frac{2\lambda_o^{\,2}}{k^2}(k-1) \tag{3.27}$$

For a positive definite covariance function, the number of terms in the Karhunen-Loève expansion is infinite (cf. [15], p. 303) and

$$\lim_{k \to \infty} \text{Var } [\hat{\lambda}_o] = 0 \tag{3.28}$$

With the Gaussian process assumption, (3.28) shows that it is possible to obtain a zero variance estimate of λ_o from any finite observation interval $[0,T]$.[3] However, the Poisson counting process analysis has shown that the minimum variance attainable is

$$\text{Var } [\hat{\lambda}_o] = \frac{\lambda_o}{\int_o^t \beta(x)dx} \tag{3.29}$$

The source of the dilemma is the two contradicting assumptions made at the beginning of the analysis. The assumption that $r(t)$ has finite variance requires that λ_o is bounded. On the other hand, the convergence to a Gaussian process requires that λ_o goes to infinity. Bounded moments and Gaussian convergence can be obtained for the modified process $\{r(t)/\sqrt{\lambda_o}, t \geq 0\}$. The covariance of this new process is not a function of λ_o and therefore does not provide a means of estimating λ_o. The rather disturbing existence of a consistent estimate of λ_o from a finite observation

[3] The existence of a zero variance estimate for the covariance function scale factor has been discussed previously. Scharf and Lytle [20] have investigated the stability of the estimate and have found that the number of terms in the Karhunen-Loève expansion should not exceed the time-bandwidth product of the stationary process. The reader is referred to [20] for further discussion and a list of related references.

interval of a Gaussian process can be avoided in a number of ways. For example, the estimate can be based on a finite number of samples from the process. The problem of estimating λ_0 using independent samples from the process is considered in the next section. The presence of some white Gaussian noise in the observed data also results in an estimate with non-zero variance. Although practical methods for estimating λ_0 can be derived using the Gaussian model, the performance of the estimators should be evaluated using the statistical properties of the filtered Poisson process.

3.4 Intensity Estimates Using Independent Samples

In many cases of interest, the response functions, $z(t,\tau,\underset{\sim}{\Theta})$, of the filtered Poisson process $\{r(t), t \geq 0\}$ defined in (3.10) are of fixed duration. That is,

$$z(t,\tau,\underset{\sim}{\Theta}) = 0 \quad \text{for} \quad t > \tau_i + T \tag{3.30}$$

When (3.30) is satisfied, the random variables obtained by sampling $r(t)$ at a rate less than or equal to $1/T$ are independent. The independence of the samples follows from the independent increment property of the Poisson process and the assumed independence of the random vectors $\underset{\sim}{\Theta}$.

The problem of estimating λ_0 using independent samples from $\{r(t), t \geq 0\}$ is considered in the following. Two limiting cases of the estimation problem are investigated first.

3.4.1 Low and High Density Estimates

Let R_1, R_2, ..., R_n be the variables obtained by sampling $\{r(t), t \geq 0\}$ at times t_1, t_2, ..., t_n. The process will be designated as a low density process when

$$\int_{t_i-T}^{t_i} \lambda_o \beta(x) dx \ll 1 \qquad i = 1, \ldots, n \tag{3.31}$$

and as a high density process when

$$\int_{t_i-T}^{t_i} \lambda_o \beta(x) dx \gg 1 \qquad i = 1, \ldots, n \tag{3.32}$$

For the low density process, the probability of more than one Poisson event occurring in the interval (t_i-T, t_i) is very small. Therefore, if the individual response functions, $z(t,\tau,\underset{\sim}{\Theta})$ are nonzero with probability 1 on the interval (τ_i, τ_i+T), the number of Poisson events, k_i, occurring in the interval (t_i-T, t_i) can be approximated by

$$k_i = \begin{cases} 0 & R_i = 0 \\ 1 & R_i \neq 0 \end{cases} \tag{3.33}$$

That is, the individual events are resolved by sampling the low density process at a rate of 1/T. The optimum estimate of λ_o for this case is given by (3.5). The block diagram for the corresponding signal processor is shown in Figure 3.1.

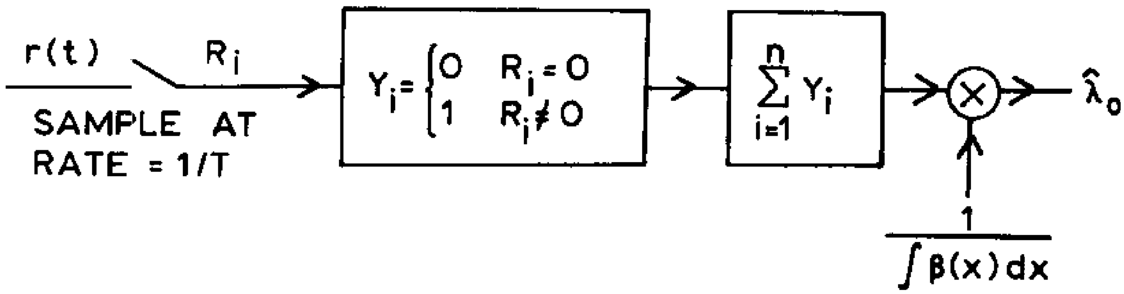

Figure 3.1. Low density estimator

The mean and variance of the low density estimate are

$$E[\hat{\lambda}_0] = \frac{1}{\int_{t_1}^{t_n} \beta(x)dx} \sum_{i=1}^{n} E[Y_i] \tag{3.34}$$

$$\mathrm{Var}[\hat{\lambda}_0] = \frac{1}{\left[\int_{t_1}^{t_n} \beta(x)dx\right]^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \{E[Y_i Y_j] - E[Y_i]\ E[Y_j]\}$$

$$= \sum_{i=1}^{n} \{E[Y_i^2] - (E[Y_i])^2\} \tag{3.35}$$

The last equality follows from the independence of the Y_i. The first and second moments of Y_i are

$$E[Y_i] = E[Y_i^2] = 1 \cdot \text{Prob [one or more events in } (t_i - T, t_i)]$$

$$= 1 - \exp\left[-\lambda_0 \int_{t_i - T}^{t_i} \beta(x)dx\right] \tag{3.36}$$

It follows from (3.34), (3.35) and (3.36) that

$$E[\hat{\lambda}_o] = \frac{n - \sum_{i=1}^{n} \exp\left[-\lambda_o \int_{t_i-T}^{t_i} \beta(x)dx\right]}{\int_{t_1}^{t_n} \beta(x)dx} \tag{3.37}$$

$$\mathrm{Var}\ [\hat{\lambda}_o] = \frac{\sum_{i=1}^{n} \left(\exp\left[-\lambda_o \int_{t_i-T}^{t_i} \beta(x)dx\right] - \exp\left[-2\lambda_o \int_{t_i-T}^{t_i} \beta(x)dx\right]\right)}{\left[\int_{t_1}^{t_n} \beta(x)dx\right]^2} \tag{3.38}$$

The samples R_1, R_2, ... R_n from a high density process have a Gaussian distribution as λ_o approaches infinity. The mean and variance of the independent samples from $\{r(t), t \geq 0\}$ are

$$E[R_i] = \lambda_o m_i$$

$$\mathrm{Var}\ [R_i] = \lambda_o \sigma_i^2 \tag{3.39}$$

where

$$m_i = \int_{t_i-T}^{t_i} \beta(x) E[z(t_i, x, \underset{\sim}{\Theta})] dx$$

and

$$\sigma_i^2 = \int_{t_i-T}^{t_i} \beta(x) E[z^2(T_i, x, \underset{\sim}{\Theta})] dx$$

In many cases of interest, such as the reverberation problem considered in the following chapters, $\{r(t), t \geq 0\}$ is zero mean. The joint density function for the zero mean samples using the Gaussian assumption is

$$f(r_1,\ldots,r_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\lambda_o\sigma_i^2}} \exp\left[\frac{-r_i^2}{2\lambda_o\sigma_i^2}\right]$$

$$= \frac{1}{(2\pi\lambda_o)^{n/2}\sigma_1\sigma_2\ldots\sigma_n} \exp\left[-\frac{1}{2\lambda_o}\sum_{i=1}^{n}\frac{r_i^2}{\sigma_i^2}\right] \tag{3.40}$$

It follows from Theorems (2.1) and (2.2) that $\sum_{i=1}^{n}(r_i^2/\sigma_i^2)$ is a sufficient statistic for estimating λ_o. The maximum likelihood estimate is the solution to

$$\frac{\partial}{\partial\lambda_o} \ln f(R_1,\ldots,R_n)\Big|_{\lambda_o = \hat{\lambda}_o} = 0 \tag{3.41}$$

and

$$\hat{\lambda}_o = \frac{1}{n}\sum_{i=1}^{n}\frac{R_i^2}{\sigma_i^2} \tag{3.42}$$

A block diagram of the high density processor is shown in Figure 3.2.

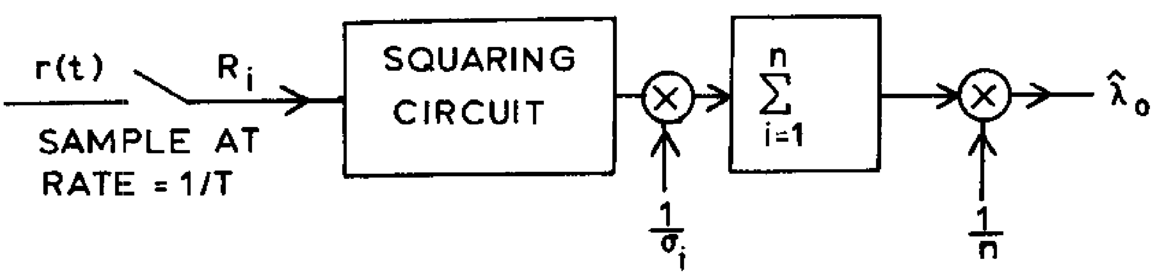

Figure 3.2. High density estimator

The expected value of the estimate in (3.42) is

$$E[\hat{\lambda}_o] = \frac{1}{n} \sum_{i=1}^{n} \frac{E[R_i^2]}{\sigma_i^2} = \lambda_o \tag{3.43}$$

The variance of $\hat{\lambda}_o$ is

$$\begin{aligned} \mathrm{Var}[\hat{\lambda}_o] &= E[(\hat{\lambda}_o - E[\hat{\lambda}_o])^2] \\ &= \frac{1}{n^2} E\left[\left(\sum_{i=1}^{n} \frac{R_i^2}{\sigma_i^2}\right)^2\right] - \lambda_o^2 \\ &= \frac{1}{n^2} \sum_{i=1}^{n} \frac{E[R_i^4]}{\sigma_i^4} - \frac{1}{n} \lambda_o^2 \end{aligned} \tag{3.44}$$

The fourth moment of R_i can be calculated using the assumed Gaussian distribution or the actual filtered Poisson distribution. For the Gaussian distribution,

$$E[R_i^4] = 3(E[R_i^2])^2 = 3\lambda_o^2 \sigma_i^4 \tag{3.45}$$

and

$$\mathrm{Var}[\hat{\lambda}_o] = \frac{2}{n} \lambda_o^2 \tag{3.46}$$

It can be shown that for the density function in (3.40) the variance of λ_o satisfies the Cramér-Rao bound with equality. Therefore, $\hat{\lambda}_o$ is the minimum variance, unbiased estimate of λ_o for the assumed Gaussian distribution of the samples $R_1, R_2, \ldots, R_n$.

The fourth moment for a sample from a zero mean filtered Poisson process given in (A.11) is

$$E[R_i^4] = \lambda_o m_{4i} + 3\lambda_o^2 \sigma_i^4 \tag{3.47}$$

where

$$m_{4i} = \int_{t_i - T}^{t_i} \beta(x) E[z^4(t_i, x, \underset{\sim}{\theta})] dx$$

Therefore, the variance of $\hat{\lambda}_o$ is

$$\mathrm{Var}[\hat{\lambda}_o] = \frac{\lambda_o}{n^2} \sum_{i=1}^{n} \frac{m_{4i}}{\sigma_i^4} + \frac{2}{n} \lambda_o^2 \tag{3.48}$$

The second term in (3.48) dominates in high densities and the variance expression reduces to (3.46). Unlike the continuous process estimate discussed in Section 3.3, the Gaussian approximation for independent samples from a high density process produces a result that asymptotically agrees with the Poisson process result.

3.4.2 General Estimator Using Independent Samples

In the general case, the distribution function of the random variable obtained by sampling $\{r(t), t \geq 0\}$ at time t' can be written as

$$F(r_{t'}; \lambda_o) = \sum_{k=o}^{\infty} F(r_{t'}/k) \mathrm{Prob}(k; \lambda_o, t') \tag{3.49}$$

where $F(r_{t'}/k)$ is the distribution of $R_{t'}$ conditioned on the occurrence of k Poisson events in the interval [t'-T,t'] and $\text{Prob}(k;\lambda_o,t')$ is the probability of k events occurring in [t'-T,t'], that is,

$$\text{Prob}(k;\lambda_o,t') = \frac{\left[\int_{t'-T}^{t'}\lambda_o\beta(x)dx\right]^k}{k!}\exp\left[-\int_{t'-T}^{t'}\lambda_o\beta(x)dx\right] \tag{3.50}$$

If it is assumed that $F(r_{t'}/k)$ has a density, $f(r_{t'}/k)$ with respect to a fixed measure, M(dr), then the density of $F(r_{t'})$ (with respect to M(dr) is

$$f(r_{t'};\lambda_o) = \sum_{k=o}^{\infty}\frac{\left[\int_{t'-T}^{t'}\lambda_o\beta(x)dx\right]^k}{k!}\exp\left[-\int_{t'-T}^{t'}\lambda_o\beta(x)dx\right]f(r_{t'}/k) \tag{3.51}$$

The maximum likelihood estimate of λ_o in terms of the independent samples $R_1,R_2, \ldots ,R_n$ is the solution to (2.6). Using the density function in (3.51),

$$-\sum_{i=1}^{n}\int_{t_i-T}^{t_i}\beta(x)dx + \sum_{i=1}^{n}\left.\frac{\frac{1}{\lambda_o}\sum_{k=1}^{\infty}\frac{\left[\int_{t_i-T}^{t_i}\lambda_o\beta(x)dx\right]^k}{(k-1)!}f(R_i/k)}{\sum_{k=o}^{\infty}\frac{\left[\int_{t_i-T}^{t_i}\lambda_o\beta(x)dx\right]^k}{k!}f(R_i/k)}\right|_{\lambda_o=\hat{\lambda}_o} = 0 \tag{3.52}$$

Assuming regularity with respect to the first derivative of $F(t_{t'};t)$, the Cramér-Rao bound on the variance of the estimate is

$$\text{Var}[\hat{\lambda}_o] \geq \frac{\{1+b[\hat{\lambda}_o]\}^2}{E\left[\left(\sum_{i=1}^{n}\frac{d}{d\lambda_o}\ln f(R_i;\lambda_o)\right)^2\right]} \tag{3.53}$$

where $b[\hat{\lambda}_o]$ is the bias in the estimate. Since the bias is in general unknown, the following lower bound on the variance of the estimate can be used.

$$\mathrm{Var}[\hat{\lambda}_o] \geq \frac{1}{E\left[\left(\sum_{i=1}^{n} \frac{d}{d\lambda_o} \ln f(R_i;\lambda_o)\right)^2\right]} \tag{3.54}$$

3.4.3 Recursive Estimation of λ_o

Maximum likelihood estimation of λ_o is not practical for real time data processing systems. The maximum likelihood estimate must be determined by numerically solving (3.52) for $\hat{\lambda}_o$ for a set of observed sample values $R_1, R_2, \ldots, R_n$. If additional samples are obtained, the previous estimate can not simply be updated. A new solution to (3.52) must be obtained using both the old and new sample values. Numerical solution of (3.52) is complicated by the infinite series representation for the sample value density function.

The problems connected with the maximum likelihood estimate make a recursive estimation algorithm for $\hat{\lambda}_o$ desirable. The technique of stochastic approximation introduced by Robbins and Monro [21] can be used in many cases to obtain such an algorithm.

The stochastic approximation algorithm to be used is the following. Let $R_1, R_2, \ldots, R_n$ be a sequence of independent, identically distributed random observations and let $Z(R,\lambda)$ be a random variable whose distribution depends on R and the parameter λ. If $E[Z(R,\lambda_o)] = 0$, then the algorithm

$$\lambda_{n+1} = \lambda_n - a_n Z(R_n, \lambda_n) \tag{3.55}$$

where

$$\sum_{n=1}^{\infty} a_n = \infty \tag{3.56}$$

and

$$\sum_{n=1}^{\infty} a_n^2 < \infty \tag{3.57}$$

converges to λ_o with probability one provided that $E[Z(R,\lambda)]$ satisfy certain weak conditions [21].

If $R_1, R_2, \ldots, R_n$ are samples from a homogeneous Poisson process and $Z(R,\lambda)$ is defined as

$$Z(R,\lambda) = -\frac{d}{d\lambda} \ln f\ (R,\lambda)$$

then

$$E[Z(R,\lambda)] = -\int_{-\infty}^{\infty} f(r,\lambda)\ \frac{d}{d\lambda} \ln\ f(r,\lambda) M(dr) \tag{3.58}$$

If it is assumed that $F(r,\lambda)$ is regular with respect to its first derivative, then

$$\begin{aligned} \frac{d}{d\lambda_o}\int_{\infty}^{\infty} f(r;\lambda_o) M(dr) &= \int_{-\infty}^{\infty} \ln[f(r;\lambda_o)] f(r;\lambda_o) M(dr) \\ &= E[Z(R;\lambda_o)] \end{aligned} \tag{3.59}$$

However,

$$\frac{d}{d\lambda_o}\int_{-\infty}^{\infty} f(r;\lambda_o)M(dr) = \frac{d}{d\lambda_o}(1) = 0 \qquad (3.60)$$

and

$$E[Z(R;\lambda_o)] = 0 \qquad (3.61)$$

Therefore, the recursive estimation algorithm for λ_o is

$$\lambda_{n+1} = \lambda_n + a_n\left[\frac{d}{d\lambda_n} f(R_n;\lambda_n)\right] \qquad (3.62)$$

Any choice of coefficients, a_n, may be used, provided that the two conditions in (3.56) and (3.57) are satisfied. One choice for a_n is

$$a_n = \frac{1}{n} G_n(\lambda_n) \qquad (3.63)$$

where

$$G_n(\lambda_n) = \frac{1}{E\left[\left(\frac{d}{d\lambda_n} \ln f(R_n;\lambda_n)\right)^2\right]}$$

That is, $G_n(\lambda_n)$ is the Cramér-Rao bound for an unbiased estimate assuming that λ_n is the true parameter value. Sakrison [22] has shown that under certain weak conditions, the stochastic approximation algorithm in (3.62) with a_n defined in (3.63) is asymptotically efficient. A block diagram of the recursive estimator defined by (3.62) and (3.63) is shown in Figure 3.3.

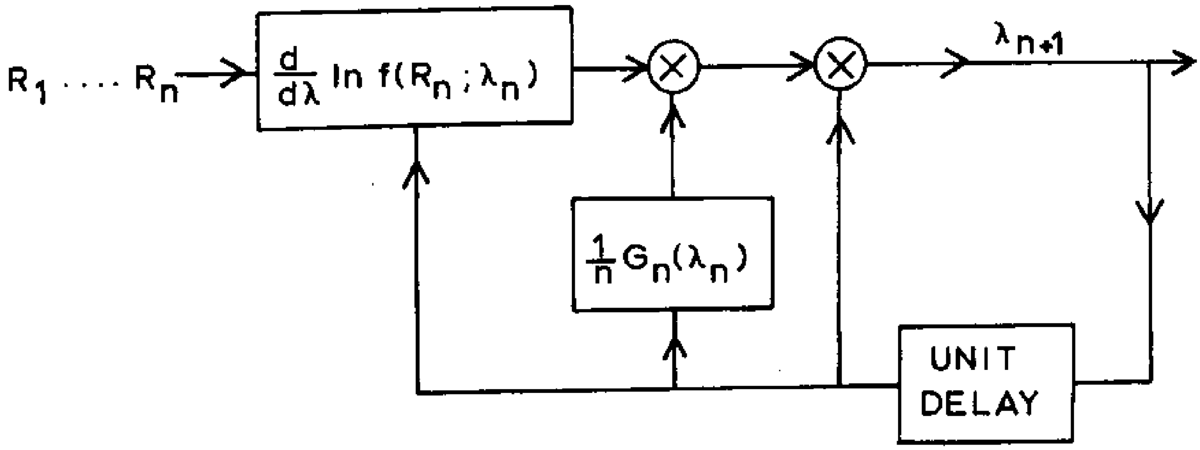

Figure 3.3. Recursive estimator

It is interesting to study the structure of the recursive estimator for low and high density processes. For the low density process,

$$f(r_n,\lambda_n) \cong \begin{cases} e^{-\lambda_n T} & r_n = 0 \\ \lambda_n T e^{-\lambda_n T} & r_n \neq 0 \end{cases} \tag{3.64}$$

and

$$\frac{d}{d\lambda_n}\left[\ln f(r_n;\lambda_n)\right] \cong \begin{cases} -T & r_n = 0 \\ -T+\frac{1}{\lambda_n} & r_n \neq 0 \end{cases} \tag{3.65}$$

The n^{th} term in the gain sequence, a_n, is approximated by

$$a_n = \frac{1}{n} G_n(\lambda_n) \cong \frac{\lambda_n}{nT} \tag{3.66}$$

Therefore, the total algorithm is

$$\lambda_{n+1} = \lambda_n + \begin{cases} -\dfrac{\lambda_n}{n} & R_n = 0 \\ -\dfrac{\lambda_n}{n} + \dfrac{1}{nT} & R_n \neq 0 \end{cases} \tag{3.67}$$

It is easily shown that the algorithm in (3.67) is equivalent to the maximum likelihood estimator for a low density process (see Figure 3.1). The equation for the maximum likelihood estimate in Figure 3.1 is

$$\lambda_{n+1} = \frac{1}{nT} \sum_{i=1}^{n} Y_i \tag{3.68}$$

where

$$Y_i = \begin{cases} 0 & R_i = 0 \\ 1 & R_i \neq 0 \end{cases}$$

Equation (3.68) can be rewritten as

$$\lambda_{n+1} = \left(\frac{n-1}{n}\right) \frac{1}{(n-1)T} \sum_{i=1}^{n-1} Y_i + \frac{1}{nT} Y_i$$

$$= \frac{n-1}{n} \lambda_n + \frac{1}{nT} Y_n$$

$$= \lambda_n + \begin{cases} -\dfrac{\lambda_n}{n} & R_n = 0 \\ -\dfrac{\lambda_n}{n} + \dfrac{1}{nT} & R_n \neq 0 \end{cases}$$

Therefore, (3.67) and (3.68) are equivalent.

For the high density process, the density function for the n^{th} sample is approximately

$$f(r_n;\lambda_n) = \frac{1}{\sqrt{2\pi\sigma^2 \lambda_n}} \exp\left[-\frac{1}{2}\frac{r_n^2}{\lambda_n\sigma^2}\right]$$

where

$$\sigma^2 = \int_0^T E[z^2(t,x,\underset{\sim}{\Theta})]dx$$

and the function $Z(r_n,\lambda_n)$ is

$$Z(r_n,\lambda_n) = -\frac{d}{d\lambda}[\ln f(r_n,\lambda_n)] = -\frac{1}{2\lambda_n}\left[\frac{r_n^2}{\lambda_n\sigma^2} - 1\right]$$

The n^{th} term in the gain sequence is

$$a_n = \frac{1}{n} G_n(\lambda_n) = \frac{1}{n\, E\left[\left(\frac{d}{d\lambda_n} \ln f(r_n;\lambda_n)\right)^2\right]}$$

$$= 2\lambda_n^2$$

Therefore, the high density recursive estimation algorithm is

$$\lambda_{n+1} = \lambda_n + \frac{1}{n}\left[\frac{R_n^2}{\sigma^2} - \lambda_n\right] \tag{3.69}$$

It can be easily shown that (3.69) is equivalent to the high density maximum likelihood estimator shown in Figure 3.2.

The recursive algorithm in (3.62) requires that all the parameters of the sample value density function $f(r_n;\lambda_o)$, except λ_o, be known *a priori*. When this assumption is not satisfied, the algorithm can be modified to estimate all the unknown parameters simultaneously [22]. The algorithm also requires that the sample values be from a stationary process. Recent work on stochastic approximation algorithms [23] has shown that this condition can be relaxed if the form of the nonstationariness is known. When the samples are nonstationary, $Z(R_n;\lambda_n)$ is a function of R_n,λ_n and t_n and the gain sequence, a_n, is a function of λ_n and t_n.

CHAPTER 4

REVERBERATION MODEL

4. Introduction

The initial purpose of the research described in this dissertation was to analyze and develop active sonar systems for estimating the spatial abundance of marine organisms. By making certain assumptions about the spatial distribution of the organisms, the problem reduced to a special case of the general problem described in the last chapter. The remainder of the dissertation deals with the specific problem of using an active sonar to estimate the abundance of scatterers with a spatial Poisson distribution. In this chapter, the relationship between the physical model and the statistical characteristics of the scattered signal is investigated.

4.1 Scattering Model

There are two general approaches to the problem of characterizing the signal reflected from a scattering field: the classical approach and the phenomenological approach. The classical approach to the problem starts with the wave equation for propagation in an inhomogeneous medium and attempts to determine a solution or approximate solution satisfying the necessary boundary conditions. It is extremely difficult to obtain the detailed statistical characteristics of the reverberation with this approach. The phenomenological approach to the problem starts with some random distribution of point scatterers and assumes that the rest of the medium is homogeneous and isotropic[1]. This approach is valid if the sizes of the scatterers are of the order of a wavelength or less and if the effects of multiple scattering are negligible. This type of scattering is often called first order point scattering. It is assumed that the conditions for first order point scattering are satisfied in the following development. The following additional assumptions about the scattering model are also made:

1) co-location of the acoustic transducer and the receiver
2) constant velocity of propagation in the medium
3) the transmitted signal is narrowband[2]
4) the reflective properties of the individual scatterers are not a function of time or frequency.

Assumptions 1 to 3 are reasonable for typical sonar systems with a vertical beam pattern. Analysis of sonar systems with nonvertical beams should include the effects of sound velocity variations with depth. Assumption 4 implies that the signal reflected from a scatterer is a scaled replica

---

[1] A more detailed comparison of the two approaches to the scattering problem and a list of related references is contained in the papers by Middleton [3].

[2] A signal, $r(t)$, is narrowband if it can be written as

$$r(t) = \mathrm{Re}[f(t)e^{j\omega_c t}]$$

where $f(t)$, the complex envelope, is slowly varying with respect to $\exp(j\omega_c t)$. Several properties of narrowband signals are discussed in Appendix A of Volume 3 of Van Trees [5].

of the incident signal. Signal distortion occurs when the scatterer has a resonant structure in the frequency bandwidth of the incident signal or when the reflective boundary in the scatterer is not well defined. Middleton has suggested that this type of distortion can be accounted for by modeling the scatterer as a stochastic linear filter [3]. A reverberation model that accounts for nonzero velocity gradients and pulse distortion would apply to a larger class of problem than the model presented in this chapter but would not provide any additional insight into the problem of statistically estimating the scattering density.

4.1.1 Single Scatterer

The following model is used for the sonar system. A narrowband signal, $s_t(t)$, is applied to an acoustic transmitter at t = 0. The signal can be written in complex envelope notation as

$$S_t(t) = \begin{cases} \sqrt{2E_t}\ \mathrm{Re}\left[f(t)e^{j\omega_c t}\right] & t \geq 0 \\ 0 & t<0 \end{cases} \tag{4.1}$$

where E_t is the energy in the signal, ω_c is the carrier frequency and f(t) is the complex envelope normalized to have unity energy. The acoustic transmitter is assumed to have ideal linear response characteristics, that is, the pressure wave out of the transmitter is proportional to the input voltage. Assume that a single scatterer is located at polar coordinates (r,Θ,ϕ) relative to the transmitting source and has an inward radial velocity of v_r as shown in Figure 4.1.[3] Further, assume that the

[3] The source and transmitter may have any general velocity vectors. The only part of the velocity that must be specified is v_r.

horizontal distance traveled by the transmitter between the time of transmission and the time of reception of the signal is negligible.

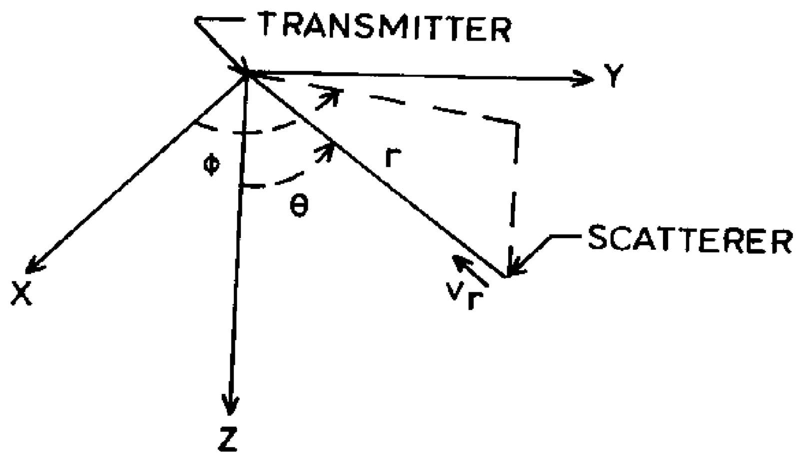

Figure 4.1. Geometry for scattering model

The received signal resulting from the scatterer at (r,θ,ϕ) is

$$r(t) = \frac{Ag^2(\theta,\phi)e^{-\alpha_o ct}}{(ct/2)^2} \mathrm{Re}\left[f(t - 2r/c)e^{j\{(\omega_c + \omega_D)t + \psi\}}\right] \tag{4.2}$$

where c is the velocity of propagation, $\exp(-\alpha_o ct)/(ct/2)^2$ is the propagation loss due to spherical spreading and absorption, $g(\theta,\phi)$ is the value of the beam pattern directivity function at angular coordinates (θ,ϕ), ω is the phase of the returned signal, ω_D is the Doppler shift introduced by the scatterer and A is an amplitude factor that accounts for the size of the scatterer, its angle of illumination, etc.

The Doppler shift can be approximated by

$$\omega_D = \frac{2V_r}{c} \tag{4.3}$$

provided that $v_r/c \ll 1$. The radial component of the scatterer velocity, v_r, also causes a compression or stretching of the time scale of the pulse envelope, f(t). This effect can be ignored provided that the time-bandwidth product[4] WT, satisfies the following relationship (cf. [5], p. 241)

$$WT \ll \frac{c}{2v_r} \qquad (4.4)$$

4.1.2 Several Scatterers

An expression for the received signal due to several scatterers follows directly from the first order point scattering assumption and (4.2).

$$r(t) = \sum_{i=1}^{N(t)} z(t, \tau_i, \underset{\sim}{\Theta}_i)$$

where

$$z(t, \tau_i, \underset{\sim}{\Theta}_i) = \frac{A_i g^2(\theta_i, \phi_i) e^{-\alpha_o ct}}{(ct/2)^2} \, \mathrm{Re}\left[f(t-\tau_i) e^{j\{(\omega_c + \omega_{Di})t + \psi_i\}}\right] \qquad (4.5)$$

---

[4] In the case of a time limited signal, the bandwidth, W, is defined such that

$$\int_{-W}^{W} |\mathfrak{F}\{f(t)\}|^2 dt = .9 \int_{-\infty}^{\infty} |\mathfrak{F}\{f(t)\}|^2 dt.$$

A similar definition can be made for the time duration of a bandlimited signal.

and N(t) is the number of scattered signals (echoes) arriving at the receiver in the time interval [0,t]. The statistical properties of the parameter vector, $\underset{\sim}{\Theta}_i = \{\theta_i, \phi_i, A_i, \omega_{o_i}, \psi_i\}$, and N(t) must be specified to complete the model.

The number of echoes received in the time interval, [0,t], is a function of the spatial distribution of the scatterers. It will be assumed that the scatterers have a nonhomogeneous Poisson distribution in volume with a spatial intensity function, $\nu_s(V)$, that is,

$$P[N(S)=k] = \frac{\left[\int \nu_s(V)\,dV\right]^k}{k!} \exp\left[-\int \nu_s(V)\,dV\right] \tag{4.6}$$

The underlying assumptions for the Poisson distribution were given in Definition (2.13). The Poisson intensity as a function of time t is

$$\nu(t) = \frac{d}{dt}\left[\int_0^t \nu_s(V(t'))\ \frac{\partial V(t')}{\partial t'}\,dt'\right] \tag{4.7}$$

where V(t') is the volume insonified in the time interval [0,t'].

In many applications, the sonar system has a vertical beam and a spatial density that is only a function of the depth, d. In this case, (4.7) becomes

$$\nu(t) = \frac{d}{dt}\left[\int_0^t \nu_s\left(\frac{ct\ \cos\theta}{2}\right) \frac{\partial V(t')}{\partial t'}\,dt'\right] \tag{4.8}$$

where θ is shown in Figure 4.1. Some particular cases are given below.

Case 1):

constant density, $\nu_s(V) = \lambda_o$

$$\nu(t) = \frac{d}{dt}\left[\int_o^t \lambda_o \frac{\partial V(t')}{\partial t'} dt'\right]$$

$$= \frac{d}{dt}\left[\lambda_o \int_o^t dt' \int_o^{\pi/2} \frac{\pi c^3}{4} t'^2 \sin\theta d\theta\right]$$

$$= \frac{\lambda_o \pi c^3 t^2}{4} \tag{4.9}$$

Case 2):

scattering layer,

$$\nu_s(V) = \nu_s(d) = \begin{cases} 0 & \text{if } d < d_o \\ \lambda_o & \text{if } d \geq d_o \end{cases}$$

$$\nu(t) = \begin{cases} \frac{d}{dt}\left[2\pi\lambda_o \int_{\frac{2d_o}{c}}^{t} dt' \int_o^{\cos^{-1}(2d_o/ct')} t'^2 (c/2)^3 \sin\theta d\theta\right] & \text{if } \frac{ct}{2} \geq d_o \\ 0 & \text{if } \frac{ct}{2} < d_o \end{cases}$$

$$
= \begin{cases} 2\pi\lambda_o \left[\left(\frac{c}{2}\right)^3 t^2 - \left(\frac{c}{2}\right)^2 d_o t \right] & \text{if } \frac{ct}{2} \geq d_o \\ 0 & \text{if } \frac{ct}{2} < d_o \end{cases} \tag{4.10}
$$

The distribution of the sequence of random vectors, $\underset{\sim}{\Theta}_i$, must be specified to complete the statistical description of r(t). If it is assumed that $\underset{\sim}{\Theta}_i$ is a sequence of independent, identically distributed, random vectors and that $\underset{\sim}{\Theta}_i$ is independent of $\{N(t), t \geq 0\}$, then r(t) is a filtered Poisson process. The independence of the parameter vectors, $\underset{\sim}{\Theta}_i$, follows directly from the spatial Poisson assumption. The independence of $\{N(t), t \geq 0\}$ and $\underset{\sim}{\Theta}_i$ also seems reasonable for most cases of interest.[5] In the remainder of this dissertation, it is assumed that a filtered Poisson process is a satisfactory statistical model for reverberation.

4.1.3 Characterization of the Random Parameter Vector

The random parameter vector, $\underset{\sim}{\Theta}_i$, has components $(\theta_i, \phi_i, A_i, \omega_{Di}, \psi_i)$. The random phase, ψ_i, will be assumed to be uniformly distributed from 0 to 2π. A physical interpretation of this assumption is that the ranges of the scatterers, modulo one wavelength, are uniform random variables.

---

[5] In very high scattering densities, the size of the individual scatterers is inversely proportional to N(t) and A_i, the echo amplitude, and is not independent of $\{N(t), t \geq 0\}$. The Poisson assumption is questionable in this case.

From the uniform phase assumption, it follows that all the odd order moments of the received process $\{r(t),t\geq 0\}$ are equal to zero.

The distribution of the angular location variables, θ and ϕ, is a function of the spatial distribution of the scatterers. The probability that an incremental volume, ΔV, at spherical coordinates (r,θ,ϕ) contains a point scatterer is

$$\begin{aligned} P\,[\text{scatter in } \Delta V] &= \nu_s(r,\theta,\phi)\Delta V \\ &= \nu_s(r,\theta,\phi)r^2 \sin\theta d\theta d\phi \end{aligned} \tag{4.11}$$

Therefore, the joint density function of the angular location (θ,ϕ) conditioned on the range, r, is

$$f(\theta,\phi;r) = \frac{\nu_s(r,\theta,\phi)\sin\theta}{\int_0^{2\pi} d\phi \int_0^{\pi/2} \nu_s(r,\theta,\phi)\sin\theta d\theta} \tag{4.12}$$

The denominator in (4.12) is a normalizing constant that insures that the density function integrates to 1. In the special case of a vertical pointing sonar and a spatial density that is a function of depth, the density function is

$$f(\theta,\phi;d) = \frac{\nu_s(d)\sin\theta}{\int_0^{2\pi} d\phi \int_0^{\pi/2} \nu_s(d)\sin\theta d\theta}$$

$$f(\theta,\phi;t) = \frac{\nu_s\left(\frac{ct}{2}\cos\theta\right)\sin\theta}{2\pi \int_o^{\pi/2} \nu_s\left(\frac{ct}{2}\cos\theta\right)\sin\theta d\theta} \tag{4.13}$$

Expressions for the density, $f(\theta,\phi,t)$, for a constant scattering density and a scattering layer are given below.

Case 1):

constant density, $\nu_s(d) = \lambda_o$

$$f(\theta,\phi;t) = \frac{\sin\theta}{2\pi} \tag{4.14}$$

Case 2):

scattering layer,

$$\nu_s(d) = \begin{cases} 0 & \text{if } d < d_o \\ \lambda_o & \text{if } d \geq d_o \end{cases}$$

$$f(\theta,\phi;t) = \begin{cases} 0 & \text{if } \theta < \cos^{-1}\left(\frac{2d_o}{ct}\right) \\ \dfrac{\sin\theta}{2\pi \int_o^{\cos^{-1}\left(\frac{2d_o}{ct}\right)} \sin\theta d\theta} & \text{if } \theta \geq \cos^{-1}\left(\frac{2d_o}{ct}\right) \end{cases}$$

$$= \begin{cases} 0 & \text{if } \theta < \cos^{-1}\left(\frac{2d_o}{ct}\right) \\ \dfrac{\sin\theta}{2\pi \quad 1-\left[\frac{2d_o}{ct}\right]} & \text{if } \theta \geq \cos^{-1}\left(\frac{2d_o}{ct}\right) \end{cases} \tag{4.15}$$

The density function for θ and ϕ can be used to obtain the moments of the beam pattern weighting term, $g^2(\theta,\phi)$, which appears in the expression for r(t). The second and fourth moments for $g^2(\theta,\phi)$ for a circular piston transducer and a constant scattering density are evaluated in Appendix B.

The random amplitude factor, A, is a function of the target strength of the scatterer. The target strength depends on the size and structure of the scatterer and the angle of acoustic illumination. Because of the many physical factors that can affect A, it is difficult to obtain a general analytical expression for $f_A(a)$. The target strength of individual fish for various species and various target angles have been measured by Love [24]. While these measurements point out some of the sources of variability, they do not provide a means of obtaining an expression for $f_A(a)$. A technique for obtaining $f_A(a)$ from the single fish echoes is described in Chapter 5.

The Doppler shift density function is a function of the relative motion of the scatterers and the acoustic source. The second order properties of the reverberation signal have been studied for several different types of Doppler shift by Moose [6] and Swarts [25].

4.1.4 First Order Density Function

The probability density function of a sample from the scattered signal, $\{r(t), t \geq 0\}$ can be written as an infinite series using (3.46). If the transmitted signal is of duration T, the density function of a sample of $\{r(t), t \geq 0\}$ at time t' is

$$f(r_{t'};\lambda_o) = \sum_{k=o}^{\infty} \frac{\left[\lambda_o \int_{t'-T}^{t'} \beta(x)dx\right]^k}{k!} \exp\left[-\lambda_o \int_{t'-T}^{t'} \beta(x)dx\right] f(r_{t'}/k) \qquad (4.16)$$

where $f(r_{t'}/k)$ is the density of $r_{t'}$ conditioned on the occurrence of k scatterers contributing to r(t) at time t' and $\beta(t) = \nu(t)/\lambda_o$. The received signal due to k scatterers can be written as

$$r(t') = \begin{cases} 0 & \text{if } k = 0 \\ \sum_{i=1}^{k} B_i \operatorname{Re}\left[f(t'-\tau_i)e^{-j(\omega_i t' + \psi_i)}\right] & \text{if } k > 0 \end{cases} \qquad (4.17)$$

where B_i is a random amplitude term that includes the effects of target strength, beam pattern and losses.[6] When the magnitude of the complex envelope, f(t), is a square pulse, (4.17) can be written as

$$r(t') = \begin{cases} 0 & \text{if } k = 0 \\ \sum_{i=1}^{k} B_i \cos(\omega_i t' + \psi_i) & \text{if } k > 0 \end{cases} \qquad (4.18)$$

The density function for k = 0 with respect to $M(dr_{t'})$ is

$$f(r_{t'}/k = 0) = 1 \qquad (4.19)$$

---

[6] Unless propagation losses are compensated for by a time-varying-gain amplifier, the statistics of B_i will be a function of time. Since we are concerned with the density at a particular time, t', the time dependence has not been indicated.

where $M(dr_{t'})$ is counting measure at $r_{t'} = 0$, that is[7]

$$\int_{-\infty}^{\infty} f(r_{t'}/k = 0)\; M(dr_{t'}) = f(r_{t'}/k = 0)\Big|_{r_{t'} = 0}$$

If it is assumed that the B_i are independent, identically distributed random variables, the characteristic function of $r_{t'}$ given k scatterers can be written as

$$\phi_{r_{t'}}(u/k) = [\phi_1(u)]^k \tag{4.20}$$

where $\phi_1(u)$ is the characteristic function of an individual term in the sum in (4.18). That is,

$$\phi_1(u) = E\left[e^{ju\,B\cos(\omega t + \psi)}\right] \tag{4.21}$$

Using conditional expectations, (4.21) becomes

$$\phi_1(u) = E_B\left[E_{\psi/B}\left(e^{ju\,B\cos(\omega t + \psi)}\right)\right]$$

$$= E_B\left[\frac{1}{2\pi}\int_0^{2\pi} e^{ju\,B\cos(\omega t + \psi)}\,d\psi\right] \tag{4.22}$$

---

[7] The concept of probability measure is often avoided in engineering literature. Counting measure can be avoided by using delta functions. In this case, we would write $f(r_{t'}/k = 0) = \delta(r_{t'})$ and use Lebesgue measure. For convenience, the delta function notation will be used in the remainder of this dissertation.

The integral can be expressed as a Bessel function (cf. [26], Eq. 9.1.21) and

$$\phi_1(u) = E_B[J_0(uB)] \tag{4.23}$$

The distribution of the amplitude variable, B, must be specified to evaluate the expectation in (4.23). There are two approaches that can be used to obtain the distribution of B. One is to determine the density of B from its contributing factors: the beam pattern term, $g^2(\theta,\phi)$, the target strength term, A, and the losses. Additional data must be collected and analyzed to obtain a valid model for f(a). The other approach is to directly measure the distribution for B. Since f(b) is a function of the spatial distribution and type of scatterer, the density, f(b), should be experimentally determined each time a particular scattering environment is investigated. Jobst [27] has made measurements of the echo strength distribution and found that the distribution of his data resembled an exponential distribution. The density for exponentially distributed B is

$$f(b) = \sigma e^{-\sigma b} \qquad b \geq 0 \tag{4.24}$$

Since the exponential distribution leads to an analytically pleasing expression for the first order density, it will be used for f(b) in the remainder of this section. From (4.20), (4.23) and (4.24), it follows that

$$\phi_1(u) = \int_0^\infty \sigma e^{-\sigma b} J_0(ub)\,db = \frac{\sigma}{\sqrt{\sigma^2 + u^2}} \tag{4.25}$$

and that

$$\phi_{r_{t'}}(u/k) = \frac{\sigma^k}{(\sigma^2 + u^2)^{k/2}} \tag{4.26}$$

The corresponding density function is obtained by Fourier transforming (4.26),

$$f(r_{t'}/k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{r_{t'}}(u/k) e^{-jur_{t'}} du$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{\sigma^k}{(\sigma^2 + u^2)^{k/2}} \cos(ur_{t'}) du \tag{4.27}$$

Evaluating the integral (cf. [26], Eq. 9.6.25), (4.27) can be written as

$$f(r_{t'}/k) = \frac{\sigma^{\frac{k+1}{2}} r_{t'}^{\frac{k-1}{2}}}{\sqrt{\pi}\, 2^{\frac{k-1}{2}}\, \Gamma(k/2)} K_{\frac{k-1}{2}}(\sigma r_{t'}) \tag{4.28}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind of order ν. The complete density function for $r_{t'}$ for exponentially distributed amplitudes is

$$f(r_{t'};\lambda_o) = \exp\left[\lambda_o \int_{t'-T}^{t'} \beta(x) dx \right] \delta(r_{t'})$$

$$+ \sum_{k=1}^{\infty} \frac{\left[-\lambda_o \int_{t'-T}^{t'} \beta(x) dx\right]^k}{k!} \frac{\sigma K_{(k-1)/2}(\sigma r_{t'})}{\sqrt{\pi}\Gamma(k/2)} \left(\frac{\sigma r_{t'}}{2}\right)^{\frac{k-1}{2}} \exp\left[-\lambda_o \int_{t'-T}^{t'} \beta(x) dx\right] \tag{4.29}$$

The density function, $f(r_{t'};\lambda_o)$ is plotted in Figure 4.2 for several different Poisson intensities. The Gaussian density function is shown for comparison. For high Poisson intensities the Gaussian density and $f(r_{t'};\lambda_o)$ appear almost identical. The difference between the Gaussian density and $f(r_{t'};\lambda_o)$ is better illustrated by the semi-logarithmic plot in Figure 4.3. This figure shows that $f(r_{t'};\lambda_o)$ has heavier tails than the Gaussian density. This heavy-tailed behavior of the first order density function is present in all reverberation signals and is not dependent on the assumed exponential amplitude distribution. This can be shown by expanding the density in an Edgeworth series expansion (cf. [6], Chapter 3).

In many acoustic signal processing systems, the signal is envelope detected before it is sampled. The density function of the sampled envelope can also be represented by the infinite series in (4.16). The density function for the sampled envelope conditioned on the presence of k terms in the sum is a special case of the random flight problem [28] originally considered by Lord Rayleigh. If $\tilde{r}_{t'}$ is the sampled envelope random variable, then

$$f(\tilde{r}_{t'}/k) = \tilde{r}_{t'} \int_0^\infty J_1(\tilde{r}_{t'}x) \prod_{i=1}^{k} J_0(B_i x)dx \tag{4.30}$$

where B_i is the amplitude of the return from the i^{th} scatterer. If it is assumed that the B_i are independent, identically distributed, exponential random variables, the density in (4.30) becomes

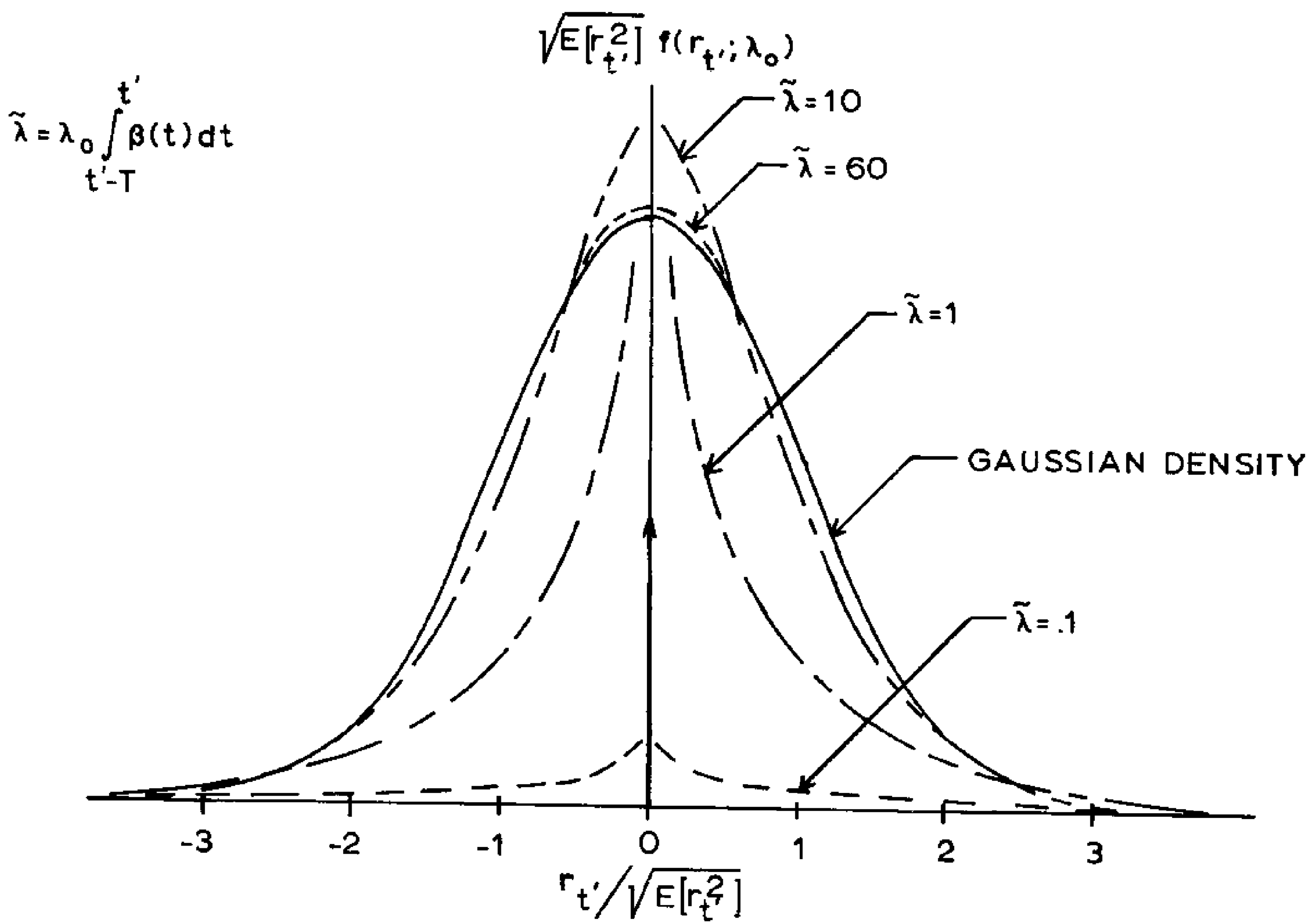

Figure 4.2. Reverberation first order density function

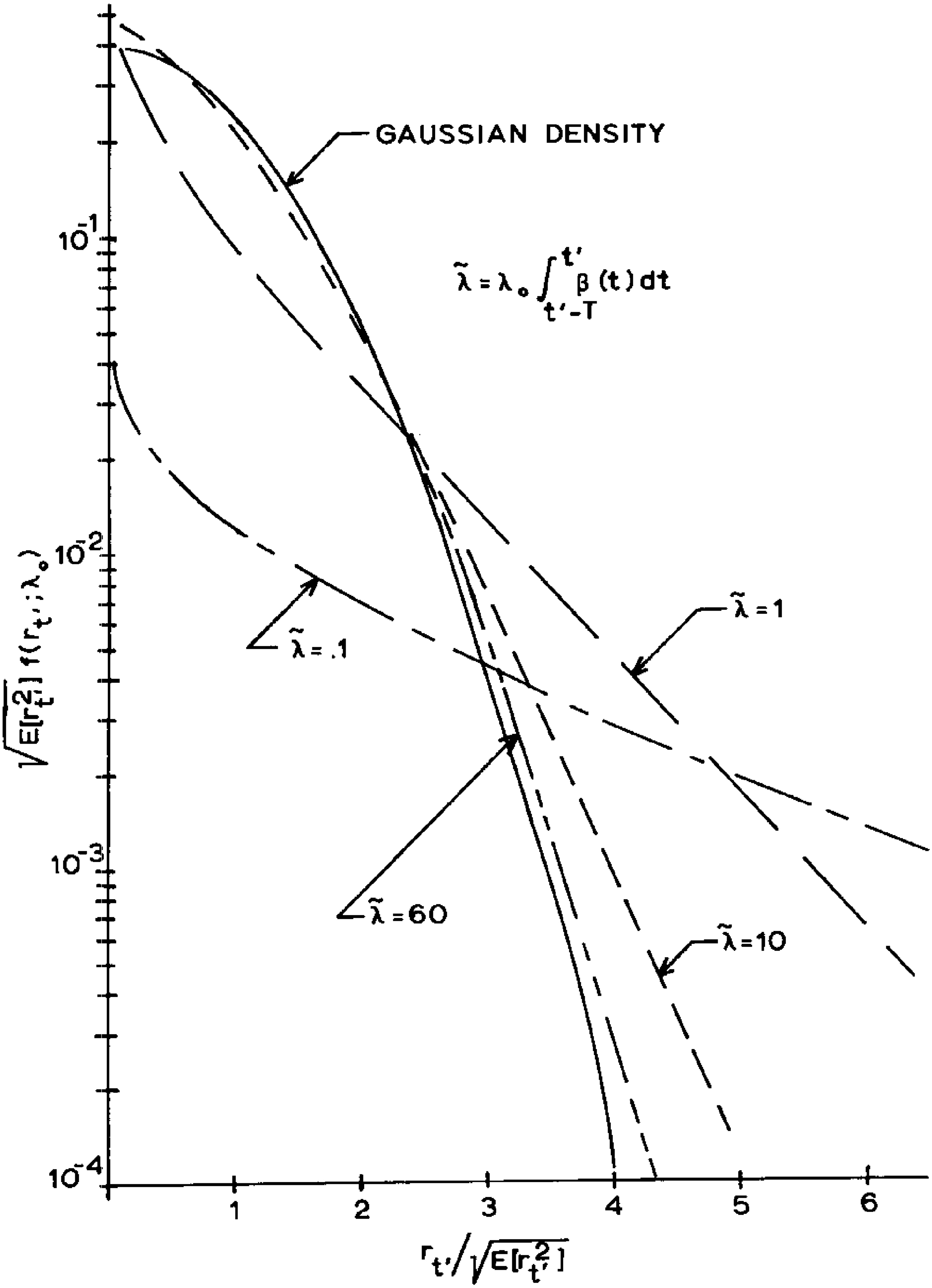

Figure 4.3. Semi-logarithmic plot of reverberation first order density function

$$f(\tilde{r}_{t'}/k) = \tilde{r}_{t'} \int_0^\infty J_1(\tilde{r}_{t'}x)\left[\int_0^\infty \sigma e^{-\sigma b} J_0(bx)\,db\right]^k dx \qquad (4.31)$$

Evaluating the inner integral (cf. [29], Eq. 6.611.1), (4.31) becomes

$$f(\tilde{r}_{t'}/k) = \tilde{r}_{t'}\sigma^k \int_0^\infty \frac{J_1(\tilde{r}_{t'}x)}{(\sigma^2 + x^2)^{k/2}}\,dx \qquad (4.32)$$

The complete density function, $f(r_{t'};\lambda_0)$, can be obtained using the above expression in (4.16).

CHAPTER 5

SPATIAL DENSITY ESTIMATION

5. Introduction

The theory of Poisson intensity estimation and a mathematical model for volume scattering were developed in the previous two chapters. These results are now used to determine the structure and performance of active sonar systems that are capable of estimating the spatial density of volume scatterers.

Density estimates using independent samples from a reverberation process are considered in the first part of the chapter. The relative performance of various estimation techniques is determined. The remainder of the chapter is devoted to a study of some signal processing techniques that are presently being used to estimate the scattering density of marine organisms. Expressions are obtained for the expected value and the mean squared error of the estimate obtained with echo integration and echo counting. A method for extracting the target strength distribution from single scatterer echo amplitudes is discussed. The mean value of the target strength is required to scale the integrator output and obtain an absolute abundance estimate. The target strength distribution can also be used by marine biologists to determine the size distribution of organisms in the scattering volume.

5.1 Sampled Signal Estimates

The problem of estimating the intensity of a filtered Poisson process using independent samples from the process was discussed in Section 3.4. These results will now be applied to the reverberation model developed in Chapter 4. The performance of the various sampled signal estimates will be compared for the following model:

i) returned echoes are rectangular pulses with exponentially distributed amplitudes,

$$r(t) = \sum_{i=1}^{N(t)} B_i[u(t-\tau_i)-u(t-\tau_i-T_p)]\cos(\omega t+\psi_i)$$

where $p_B(b) = \sigma e^{-\sigma b} \qquad b \geq 0$

ii) constant Poisson intensity, $\nu(t) = \lambda_o$

iii) no noise in the received signal

This is the basic model used to study the first order density function of the reverberation signal in Section 4.1.4. The assumption of a constant Poisson intensity is not valid for most cases of interest. However, this assumption greatly simplifies the analysis and the results obtained illustrate the basic properties of the various estimation techniques.

The performance criterion used in this section is the normalized mean squared error in the estimate of the density factor, $\hat{\lambda}_o$

$$\frac{e^2}{\lambda_o^2} = \frac{E[(\hat{\lambda}_o-\lambda_o)^2]}{\lambda_o^{\,2}}$$

When the estimate is unbiased, $e^2 = \mathrm{Var}[\hat{\lambda}_o]$.

5.1.1 Sampled Echo Integrator

The block diagram of the sampled echo integrator is shown in Figure 5.1. The envelope detector removes the carrier from the signal.

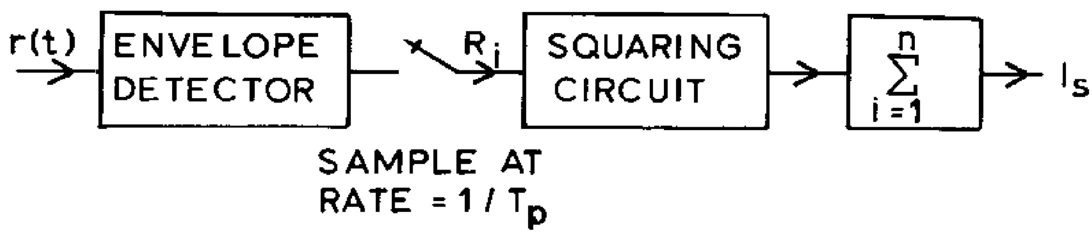

Figure 5.1. Sampled echo integrator

The mean and variance of I_s are

$$E[I_s] = \sum_{i=1}^{n} E[R_i^2] \tag{5.1}$$

$$Var[I_s] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[R_i^2 R_j^2] - \left\{ \sum_{i=1}^{n} E[R_i^2] \right\}^2 \tag{5.2}$$

From the moment expressions in (3.39) and (3.47) it follows that

$$E[I_s] = \sum_{i=1}^{n} \lambda_o E[B^2] T_p/2$$

$$= \lambda_o n T_p/\sigma^2 \tag{5.3}$$

and

$$Var[I_s] = \sum_{i=1}^{n} Var\ [R_i^2]$$

$$= 6n\lambda_o T_p/\sigma^4 + 2n\ \lambda_o^2 T_p^2/\sigma^4 \tag{5.4}$$

If the parameter σ is known, the sampled integrator output, I_s, can be scaled to provide an unbiased estimate of λ_o.

$$\lambda_o = \frac{I_s}{nT_p/\sigma^2} \tag{5.5}$$

Since the estimate is unbiased, the variance and mean squared error are equivalent and

$$\frac{e^2}{\lambda_o^2} = \frac{Var[I_s]}{(E[I_s])^2} = \frac{6}{n\lambda_o T_p} + \frac{2}{n} \tag{5.6}$$

5.1.2 Sampled Echo Counter

A block diagram of the sampled echo counter is shown in Figure 5.2.

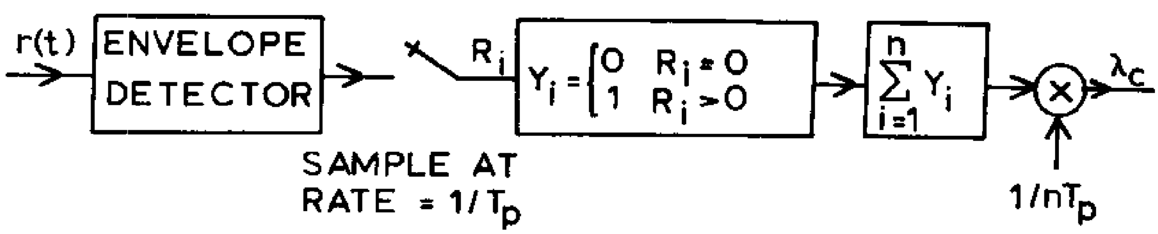

Figure 5.2. Sampled echo counter

The structure shown in Figure 5.2 is basically the same as the low density estimator described in Section 3.4.1. The principal shortcoming of the sampled echo counter is its inability to provide an accurate estimate in high scattering densities. When two or more echoes occur in the same sampling interval, T_p, only a single echo is counted. Expressions for the mean and variance of the estimated density, λ_o, are given by (3.37) and (3.38). For the model in Section 5.1,

$$E[\lambda_c] = \frac{1-e^{-\lambda_o T_p}}{T_p} \tag{5.7}$$

$$Var[\lambda_c] = \frac{e^{-\lambda_o T_p} - e^{-2\lambda_o T_p}}{nT_p^2} \tag{5.8}$$

Unlike the sampled echo integrator, the sampled echo counter has a bias, b, given by,

$$b = \lambda_o - E[\lambda_c]$$

$$= \lambda_o - \frac{1-e^{-\lambda_o T_p}}{T_p} \tag{5.9}$$

The normalized mean squared error of the estimated density is

$$\frac{e_c^2}{\lambda_o^2} = \frac{b^2 + Var[\lambda_c]}{\lambda_o^2}$$

$$= 1 - \left[\frac{1}{T_p} + \frac{e^{-\lambda_o T_p}}{\lambda_o T_p}\right]^2 + \frac{e^{-\lambda_o T_p}}{n(\lambda_o T_p)^2}\left(1-e^{-\lambda_o T_p}\right) \tag{5.10}$$

5.1.3 Bound on Sampled Estimates

An equation for the maximum likelihood estimate of λ_o in terms of independent samples $R_1, R_2, \ldots, R_n$ is given in (3.53). The corresponding lower bound on the variance of the estimate is given in (3.55). Assuming the samples, $r(t_1), r(t_2), \ldots, r(t_3)$, are taken from the process $\{r(t), t \geq 0\}$ defined in Section 5.1, the maximum likelihood estimate is given by

$$-nT_p + \sum_{i=1}^{n} \left\{ \frac{\sum_{k=1}^{\infty} \frac{(\lambda_o T_p)^k}{(k-1)!\lambda_o} f(r(t_i)/k)}{\sum_{k=0}^{\infty} \frac{(\lambda_o T_p)^k}{k!} f(r(t_i)/k)} \right\} \Bigg|_{\lambda=\hat{\lambda}_o} = 0 \tag{5.11}$$

where

$$f(r/k=0) = \delta(r)$$

$$f(r/k) = \frac{\sigma^{(k+1)/2} r^{(k-1)/2}}{\sqrt{\pi}\, 2^{(k-1)/2}\, (k/2)} K_{(k-1)/2}(\sigma r) \qquad k \neq 0 \tag{5.12}$$

and the bound on the normalized mean squared error is

$$\frac{e^2}{\lambda_o^2} \geq \frac{\mathrm{Var}[\hat{\lambda}_o]}{\lambda_o^2} \geq \left\{ n\lambda_o^2 E[\frac{d}{d\lambda_o}[\ln f(r(t_i);\lambda_o)]^2 \right\}^{-1}$$

$$= \left\{ n\lambda_o^2 \int_{-\infty}^{\infty} \left[-T_p + \frac{\sum_{k=1}^{\infty} \frac{(\lambda_o T_p)^k}{(k-1)!\lambda_o} f(r/k)}{\sum_{k=0}^{\infty} \frac{(\lambda_o T_p)^k}{k!} f(r/k)} \right]^2 e^{-\lambda_o T_p} \sum_{k=0}^{\infty} \frac{(\lambda_o T_p)^k}{k!} f(r/k)\, dr \right\}^{-1} \tag{5.13}$$

One measure of the performance of an estimate of λ_o is to compare its mean squared error with the mean squared error of the estimate obtained when all the echoes are resolvable. If k echoes are received in the time interval (o, nT_p), the optimum estimate of λ_o from (3.5) is

$$\hat{\lambda}_o = \frac{k}{nT_p} \tag{5.14}$$

and the normalized mean squared error is

$$\frac{e^2}{\lambda_o^2} \geq \frac{\mathrm{Var}[\hat{\lambda}_o]}{\lambda_o^2} = \frac{1}{\lambda_o n T_p} \tag{5.15}$$

The normalized mean squared error expressions in (5.6), (5.10), (5.13) and (5.15) are plotted in Figure 5.3 as a function of the expected number of echoes per sample, $\lambda_o T_p$. The sample echo counter is the optimum processor in low scattering densities ($\lambda_o T_p$<.2 for the model used in obtaining Figure 5.3). On the other hand, in high densities ($\lambda_o T_p$>10), the sampled echo integrator is the best of all the possible processors that use independent samples from the received signal. The choice of the signal processor is not so simple in the medium density range (.2>$\lambda_o T_p$>10). In many cases, the choice of the processor for the medium density range will depend upon the required degree of accuracy. The normalized mean squared error of the sampled echo integrator is inversely proportional to the number of samples, n. Therefore, if an unlimited number of samples are available, the sample echo integrator could be used to estimate λ_o to any accuracy. The sample echo counter does not have this desirable property. For large n, the mean squared error of the counter is determined by the bias term (5.9) which does not decrease with n. If only a limited number of samples are available and if neither the sampled echo integrator nor counter provides the required degree of accuracy in the estimate of λ_o, then a more complex signal processor must be used. One such processor is the recursive estimator that will be discussed in the next section.

5.1.4 Recursive Estimation of λ_o

A method for recursively estimating the intensity of a filtered Poisson process was discussed in Section 3.4.3. Let $r(t_n)$ be the n^{th} sample from the process $\{r(t), t \geq 0\}$ defined in Section 5.1 and let λ_n

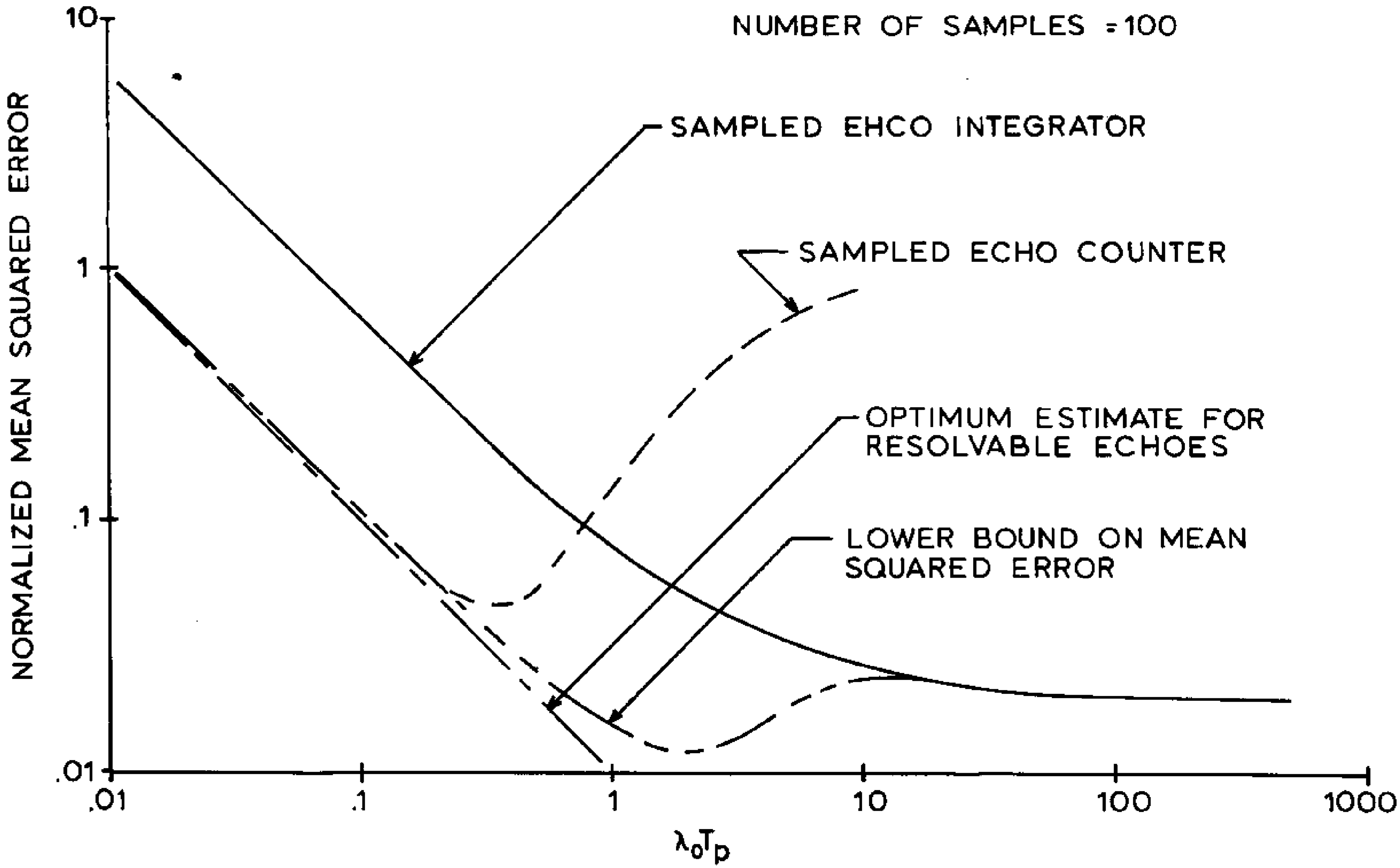

Figure 5.3. Normalized mean squared errors of sampled signal estimators

be the estimate of λ_o at time t_n. Then, the form of the recursive algorithm is

$$\lambda_{n+1} = \lambda_n + a_n \left[\frac{d}{d\lambda} \ln f(r(t_n),\lambda)\Big|_{\lambda=\lambda_n}\right] \tag{5.16}$$

where

$$f(r;\lambda) = \sum_{k=0}^{\infty} \frac{(\lambda_o T_p)^k}{k!} f(r/k)$$

$$a_n = \frac{1}{nE\left[\left(\frac{d}{d\lambda} \ln f(r;\lambda)\Big|_{\lambda=\lambda_n}\right)^2\right]}$$

and where $f(r/k)$ is defined in (5.12).

The sampled echo integrator can also be written in recursive form using (3.69).

$$\lambda_{n+1} = \lambda_n + \frac{1}{n}\left(\frac{r^2(t_n)}{\sigma_r^{\,2}} - \lambda_n\right) \tag{5.17}$$

where

$$\sigma_r^{\,2} = \mathrm{Var}[r^2(t_i)] \;.$$

It would be very difficult, if not impossible, to analytically determine the finite sample performance of the algorithm in (5.16). The performance can best be evaluated by means of a Monte Carlo simulation. The two algorithms in (5.16) and (5.17) were simulated for a sequence of 100 samples and a Poisson intensity of 2. The mean squared error in the estimates was obtained by averaging the errors for a hundred different sequences. The initial estimate, λ_1, was determined from the first sample as follows.

$$\lambda_1 = \frac{r^2(t_1)}{\sigma_r^{\ 2} T_p} \tag{5.18}$$

The results of the simulation are shown in Figure 5.4. The 95% confidence interval for the estimated mean squared error for the two algorithms is shown at various points along the curves. The estimated error of the recursive algorithm is less than the Cramér Rao bound on the variance. This is due to the inaccuracy in the estimate obtained from the Monte Carlo simulation which has used a finite number of sequences.

It was mentioned in Section 3.4.3 that stochastic approximation algorithms can be extended to independent samples from a nonstationary process. The applicability of the recursive estimation algorithm to nonstationary processes is essential in the problem of acoustic estimation of scattering abundance. Recall from the model developed in Chapter 4 that the acoustic reverberation process is nonstationary because of the increase in the cross sectional area of the acoustic beam with time.

5.2 Echo Integration

The echo integrator[1] is a continuous time version of the sampled echo integrator described in Section 5.1.1. A block diagram of the system to be considered is shown in Figure 5.5.

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[1] The performance of the echo integrator has been discussed in two papers by the present author [10], [11]. The approach in this section is more general than the previous analyses. However, the referenced papers have a more detailed discussion of the effects of the various system parameters on the performance of the integrator.

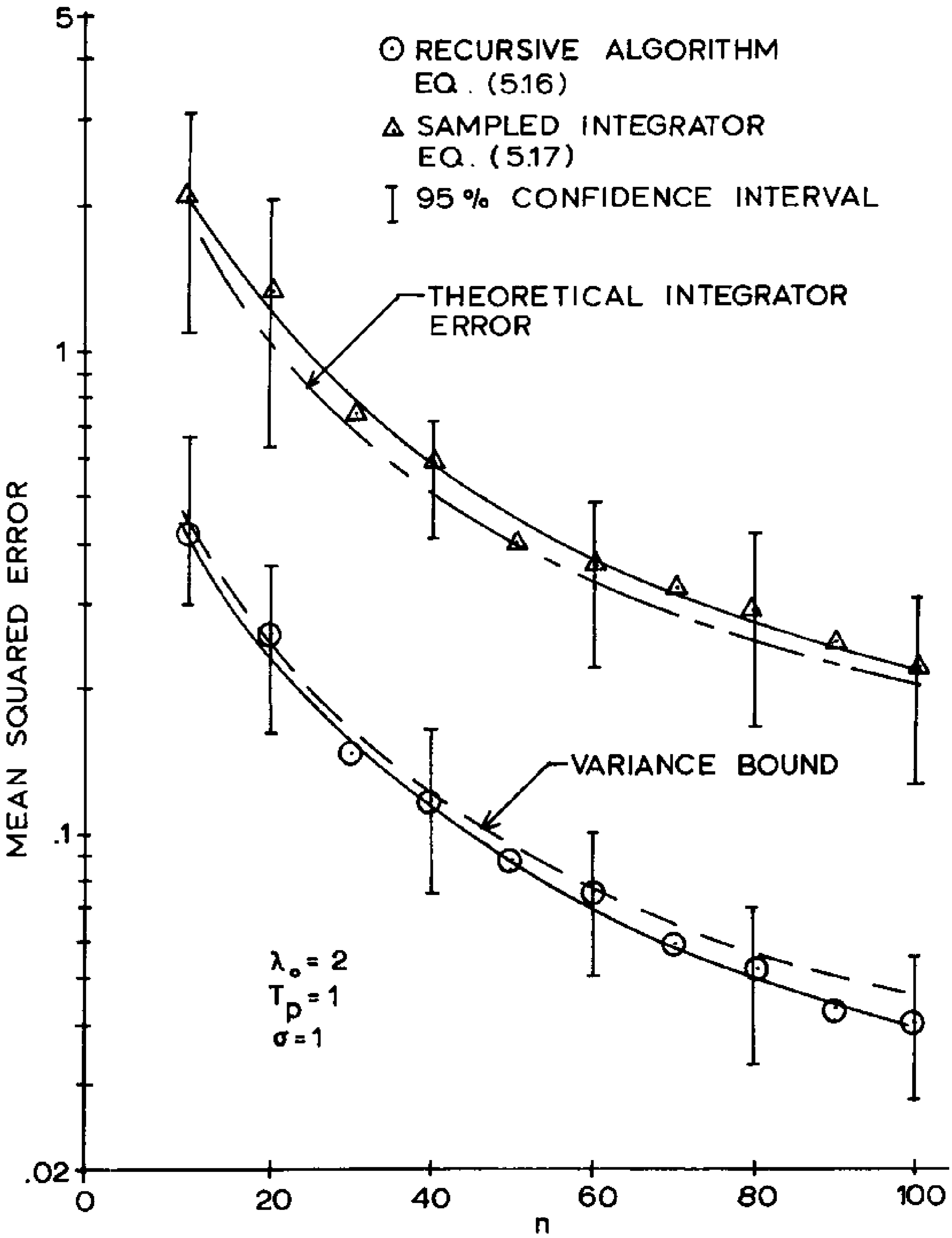

Figure 5.4. Monte Carlo simulation of estimation algorithms in Equations (5.16) and (5.17)

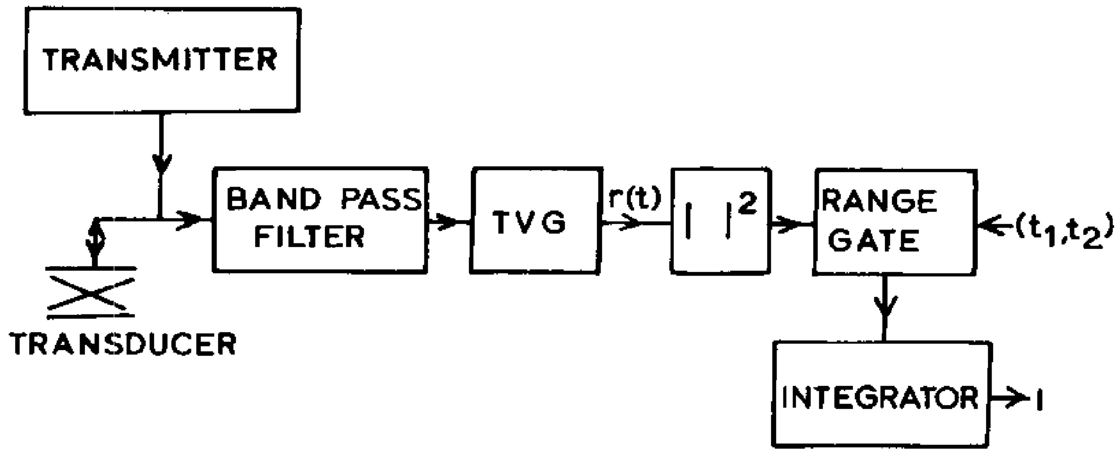

Figure 5.5. Echo integrator

The T.V.G. (time-varying-gain) amplifier equalizes the signals received from scatterers at different depths. The range gate controls the depth interval over which the scattering density is estimated.

5.2.1 Single Pulse Analysis

The analysis begins with a special problem: the integration of returns from a single transmitted acoustic pulse. From (4.5), the received signal at time t is

$$r(t) = \sum_{i=1}^{N(t)} A_i g^2(\theta_i,\phi_i) \frac{e^{-\alpha_o ct}}{(ct/2)^2} G(t)\mathrm{Re}[f(t-\tau_i)e^{j\{(\omega_c+\omega_{Di})t+\psi_i\}}] \quad (5.19)$$

where G(t) is the time varying gain function of the T.V.G. amplifier. The other parameters have been defined in Chapter 4. The output of the integrator for a single transmitted pulse can be represented by

$$I = \int_{t_1}^{t_2} |r(t)|^2 dt \tag{5.20}$$

The performance of the echo integrator can be determined from the expressions for E_I and Var_I[2] which follow from

$$E_I = E[I] = \int_{t_1}^{t_2} E[|r(\alpha)|^2] d\alpha \tag{5.21}$$

$$Var_I = Var[I] = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \{E[|r(\alpha)|^2 |r(\beta)|^2] - E[|r(\alpha)|^2]E[|r(\beta)|^2]\} d\alpha d\beta \tag{5.22}$$

by interchanging the order of integration and expectation. The moments of $|r(t)|^2$ that appear in (5.21) and (5.22) which are evaluated in Appendix C are

$$E[|r(\alpha)|^2] = \int_{t_1}^{\alpha} \nu(\tau) E\{|\tilde{z}(\alpha,\tau,\underset{\sim}{\Theta})|^2\} d\tau \tag{5.23}$$

and

---

[2] The integration time is usually long compared to the pulse length. Under these conditions, the total integrator output can be approximated by a sum of statistically independent components and is approximately Gaussian distributed because of the central limit theorem. A Gaussian random variable is completely characterized by its mean and variance. Therefore, E_I and Var_I contain all the information necessary to statistically characterize the estimate obtained using the echo integrator.

$$\mathrm{Cov}[|r(\alpha)|^2,|r(\beta)|^2] = E[|r(\alpha)|^2|r(\beta)|^2]-E[|r(\alpha)|^2]E[|r(\beta)|^2]$$

$$= \int_{t_1}^{\min(\alpha,\beta)} \nu(\tau)E\{|\tilde{z}(\alpha,\tau,\underset{\sim}{\Theta})|^2|\tilde{z}(\beta,\tau,\underset{\sim}{\Theta})|^2\}$$

$$+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(\tau)E[\tilde{z}_c(\alpha,\tau,\underset{\sim}{\Theta})\tilde{z}_c(\beta,\tau,\underset{\sim}{\Theta})]d\tau\right\}^2$$

$$+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(\tau)E[\tilde{z}_c(\alpha,\tau,\underset{\sim}{\Theta})\tilde{z}_s(\beta,\tau,\underset{\sim}{\Theta})]d\tau\right\}^2$$

$$+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(\tau)E[\tilde{z}_s(\alpha,\tau,\underset{\sim}{\Theta})\tilde{z}_c(\beta,\tau,\underset{\sim}{\Theta})]d\tau\right\}^2$$

$$+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(\tau)E[\tilde{z}_s(\alpha,\tau,\underset{\sim}{\Theta})\tilde{z}_s(\beta,\tau,\underset{\sim}{\Theta})]d\tau\right\}^2 \qquad (5.24)$$

where

$$\tilde{z}(t,\tau,\underset{\sim}{\Theta}) = \tilde{z}_c(t,\tau,\underset{\sim}{\Theta}) - j\tilde{z}_s(t,\tau,\underset{\sim}{\Theta})$$

and

$$\tilde{z}_c(t,\tau,\underset{\sim}{\Theta}) = \mathrm{Re}\left[\frac{Ag^2(\theta,\phi)e^{-\alpha_0 ct}}{(ct/2)^2}G(t)f(t-\tau)e^{j(\omega_D t+\psi)}\right]$$

$$\tilde{z}_s(t,\tau,\underset{\sim}{\Theta}) = \mathrm{Im}\left[\frac{Ag^2(\theta,\phi)e^{-\alpha_0 ct}}{(ct/2)^2}G(t)f(t-\tau)e^{j(\omega_D t+\psi)}\right] \qquad (5.25)$$

When the envelope of the transmitted signal, $f(t)$, is real and the density function of the Doppler shift variable ω_D is symmetric about zero, the complex envelope, $\tilde{z}(t,\tau,\underset{\sim}{\Theta})$ is real and (5.23) and (5.24) simplify to

$$E[|r(\alpha)|^2] = \int_t^{\alpha} \nu(\tau) E[\tilde{z}_c^2(\alpha,\tau,\underset{\sim}{\Theta})]d\tau \tag{5.26}$$

$$\begin{aligned} \mathrm{Cov}[|r(\alpha)|^2|r(\beta)|^2] &= \int_{t_1}^{\min(\alpha,\beta)} \nu(\tau) E[\tilde{z}_c^2(\alpha,\tau,\underset{\sim}{\Theta})\tilde{z}_c^2(\beta,\tau,\underset{\sim}{\Theta})]d\tau \\ &+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(\tau) E[\tilde{z}_c(\alpha,\tau,\underset{\sim}{\Theta})\tilde{z}_c(\beta,\tau,\underset{\sim}{\Theta})]d\tau\right\}^2 \end{aligned} \tag{5.27}$$

For simplicity, (5.26) and (5.27) are used to obtain expressions for E_I and Var_I. The derivation for a complex $f(t)$ and asymmetric Doppler denstiy function is rather involved because of the number of terms that must be retained. From (5.25), (5.26) and (5.27) and the assumed uniform distribution of the random phase variable ψ, it follows that

$$E[|r(\alpha)|^2] = \frac{E[A^2 g^4(\theta,\phi)]e^{-2\alpha_0 ct}G(\alpha)}{2(c\alpha/2)^4} \int_{t_1}^{\alpha} \nu(\tau) f^2(\alpha-\tau)d\tau \tag{5.28}$$

$$\begin{aligned} \mathrm{Cov}[|r(\alpha)|^2|r(\beta)|^2] &= \frac{E[A^4 g^8(\theta,\phi)]G^2(\alpha)G^2(\beta)e^{-2\alpha_0 c(\alpha+\beta)}}{4(c\alpha\beta/4)^8} \\ &\quad E[1+\cos\{2\omega_D(\alpha-\beta)\}] \int_{t_1}^{\min(\alpha,\beta)} \nu(\tau) f^2(\alpha-\tau) f^2(\beta-\tau)d\tau \\ &+ \frac{1}{2}\left\{ E[A^2 g^4(\theta,\phi)\cos\{\omega_D(\alpha-\beta)\}] \frac{G(\alpha)G(\beta)c^{-\alpha_0 c(\alpha+\beta)}}{(c\alpha\beta/4)^4} \right. \\ &\quad \left. \int_{t_1}^{\min(\alpha,\beta)} \nu(\tau) f(\alpha-\tau) f(\beta-\tau)d\tau\right\}^2 \end{aligned} \tag{5.29}$$

The mean and variance of the integrator output can be calculated using (5.28) and (5.29) in (5.21) and (5.22). In the general case, the expressions for E_I and Var_I must be numerically evaluated. Closed form expressions can be obtained by making the following assumptions:

i) rectangular pulse, $f(t) = u(t) - u(t-T_p)$ where $u(t)$ is the unit step function and T_p is the pulse length.

ii) no Doppler shift, $\omega_D = 0$

iii) $t_2-t_1>>T_p$ and $t_1>>T_p$

iv) uniform spatial density, $\nu(t) = \dfrac{\lambda_o \pi c^3 t^2}{4}$

v) $T_p>>2\pi/\omega$

vi) attenuation losses are negligible ($\alpha_o=0$) and $G(t)=t^{g_o}$.

The expressions for Var_I and E_I with these assumptions are[3]

$$E_I = \frac{E[A^2 g^4(\theta,\phi)]kT_p[t_2^{\,2g_o-1}-t_1^{\,2g_o-1}]}{2(2g_o-1)} \qquad (5.30)$$

$$Var_I = \frac{E[A^4 g^8(\theta,\phi)]T_p^{\,2}k[t_2^{\,4g_o-5}-t_1^{\,4g_o-5}]}{4(4g_o-5)}$$

$$+ \frac{(E[A^2 g^4(\theta,\phi)])^2 k^2 T_p^{\,3}[t_2^{\,4g_o-3}-t_1^{\,4g_o-3}]}{6(4g_o-3)} \qquad (5.31)$$

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[3] The steps leading to these results are contained in [10]. The previous analysis assumes an ideal elliptical transducer beam pattern.

where

$$k = \frac{\lambda_o \pi c^3}{4}$$

It is reasonable to assume that the target strength random variable, A^2, and the beam pattern random variable $g(\theta,\phi)$ are uncorrelated. The evaluation of the moments of $g(\theta,\phi)$ is considered in Appendix B. The density function and moments of A^2 can be estimated by the method discussed in Section 5.4. The integrator output can be scaled to produce an estimate of λ_o after $E[A^2]$ and $E[g^4(\theta,\phi)]$ have been determined.

$$\hat{\lambda}_o = \frac{1}{E[A^2]E[g^4(\theta,\phi)]T_p \ (c/2)^3 \ \dfrac{(t_2^{2g_o-1} - t_1^{2g_o-1})}{2g_o-1}} \tag{5.32}$$

The relative size of the mean, E_I, and the variance, Var_I, can be determined by normalizing the variance with E_I^2. From (5.30), (5.31) and the assumption that A and $g(\theta,\phi)$ are uncorrelated, it follows that

$$\frac{Var_I}{E_I^2} = \frac{E[A^4]E[g^8(\theta,\phi)](2g_o-1)^2[1-(t_1/t_2)^{4g_o-5}]}{(E[A^2]E[g^4(\theta,\phi)])^2(4g_o-5)kt_2^3[1-(t_1/t_2)^{2g_o-1}]^2}$$

$$+ \frac{2(2g_o-1)^2T_p[1-(t_1/t_2)^{4g_o-3}]}{3(4g_o-3)t_2[1-(t_1/t_2)^{2g_o-1}]^2} \tag{5.33}$$

5.2.2 Multiple Pulse Analysis

The results of the previous section can be generalized to include integration of returns from multiple soundings. Define I_T as the integrator output after the integration of the returns from N_p pulses. Then

$$I_T = I_1 + I_2 + \dots + I_{N_p} \tag{5.34}$$

where I_j is the integrator output for the j^{th} pulse. If it is assumed that the scattering density is constant for all N_p pulses and that each pulse is range gated to the same interval, then

$$E[I_T] = N_p E_I \tag{5.35}$$

and

$$\text{Var}(I_T] = N_p \text{Var}_I + \sum_{\substack{i=1 \\ i \neq j}}^{Np} \sum_{j=1}^{Np} \text{Cov}[I_i, I_j] \tag{5.36}$$

The covariance term in (5.36) can be written in terms of $|r_i(\alpha)|$ and $|r_j(\beta)|$, the envelopes of the returned signal for the i^{th} and j^{th} pulses.

$$\text{Cov}[I_i, I_j] = \int_{t_1}^{t_2} \int_{t_1}^{t_2} \Big\{ E[|r_i(\alpha)|^2 |r_j(\beta)|^2] - E[|r_i(\alpha)|^2] E[|r_j(\beta)|^2] \Big\} d\alpha d\beta \tag{5.37}$$

If a new volume of water is insonified with each acoustic pulse, the returned signals, $r_i(\alpha)$ and $r_j(\beta)$ are independent and

$$\mathrm{Cov}[I_i, I_j] = 0$$

and

$$\mathrm{Var}[I_T] = N_p \mathrm{Var}_I \tag{5.38}$$

To evaluate (5.36) for overlapping volumes of water, the returned signals can be split into two parts as follows,

$$r_i(\alpha) = r_{oi}(\alpha) + r_{ni}(\alpha)$$

$$r_j(\beta) = r_{oj}(\beta) + r_{nj}(\beta) \tag{5.39}$$

where $r_{oi}(\alpha)$ and $r_{oj}(\beta)$ are produced by scatterers in overlapping volumes of water. Since $r_{ni}(\alpha)$ and $r_{nj}(\beta)$ are produced by returns from nonoverlapping volumes, they are independent and do not contribute to $\mathrm{Cov}[I_i, I_j]$. Using the assumed spatial Poisson distribution for the scatterers, it follows that $r_{oi}(\alpha)$ and $r_{oj}(\beta)$ are filtered Poisson processes with intensity function

$$\nu_o(t) = \frac{\lambda_o c A_{ij}(t)}{2}$$

where $A_{ij}(t)$ is the cross sectional area of the overlapping volume for the i^{th} and j^{th} pulse at time t. An expression for $A_{ij}(t)$ in terms of the physical parameters of the system for an ideal elliptical beam is derived in Appendix D.

Assume that the same scatterers are present in any given volume over a period of several transmission pulses. The positions of the scatterers relative to the beam pattern can be expected to change and these changes are assumed to cause the phases of the signals for different pulses to be uncorrelated. The expression for $Cov[I_i,I_j]$ is evaluated by means similar to those used in the single pulse analysis. The resulting expression for the six assumptions used in the single pulse analysis is

$$Cov[I_i,I_j] = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \frac{E[A_i^2 A_j^2 g_i^4(\theta,\phi) g_j^4(\theta,\phi)]}{4}$$

$$\alpha^2 g_o^{-4} \beta^2 g_o^{-4} \Gamma_{ij}(\alpha,\beta,T_p)\, d\alpha d\beta \qquad (5.40)$$

where

$$\Gamma_{ij}(\alpha,\beta,\ T_p) = \int_{t_1}^{\min(\alpha,\beta)} \frac{\lambda_o C}{2} A_{ij}(\tau)$$

$$\{u(\alpha-\tau)-u(\alpha-\tau-T_p)\}\{u(\beta-\tau)-u(\beta-\tau-T_p)\}d\tau \qquad (5.41)$$

Further assumptions must be made to obtain a closed form expression for $Var[I_T]$. These assumptions and the resulting expression are contained in a paper by Moose and Ehrenberg [10]. This paper also considers the effect of the various system parameters on the performance of the integrator.

5.2.3 Effect of Noise on the Integrator

The analysis in the previous two sections has assumed that only the signal reflected by the scatterers is present at the output of the transducer. In practice, the transducer output will also contain noise.

Define n(t) as the noise process at the output of the T.V.G. amplifier. If n(t) and r(t) are assumed to be narrowband processes with real envelopes, the integrator output for a single acoustic pulse can be written as

$$I = \int_{t_1}^{t_2} [r(t)+n(t)]^2 dt$$

$$= \int_{t_1}^{t_2} [r^2(t)+n^2(t)+2n(t)r(t)]dt \qquad (5.42)$$

The quantities of interest are again the mean and variance of I. Assuming that r(t) and n(t) are zero mean and statistically independent, it follows that

$$E_I = \int_{t_1}^{t_2} \{E[r^2(t)]+E[n^2(t)]\}dt \qquad (5.43)$$

and

$$\mathrm{Var}_I = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \{\mathrm{Cov}[n^2(\alpha),n^2(\beta)] + \mathrm{Cov}[r^2(\alpha),r^2(\beta)] \tag{5.44}$$

$$+ 4E[r(\alpha)r(\beta)]E[n(\alpha)n(\beta)\}d\alpha d\beta$$

The mean and covariance of the squared reverberation signal, $r^2(t)$ are defined in (5.30) and (5.31). The joint second moment of $r(t)$ can be obtained from (A.12) and (5.25),

$$E[r(\alpha)r(\beta)] = \frac{E[A^2g^4(\theta,\phi)]G(\alpha)G(\beta)\cos\omega_D(\alpha-\beta)}{2}$$

$$\frac{\epsilon^{-\alpha_0 c(\alpha+\beta)}}{(c\alpha\beta/4)^2} \int_{t_1}^{\min(\alpha,\beta)} \nu(\tau)f(\alpha-\tau)f(\beta-\tau)d\tau \tag{5.45}$$

The statistical properties of the noise process must be specified to evaluate the moments of $n(t)$ that appear in (5.43) and (5.44). The ambient noise in a sonar system usually consists of radiated signals from several independent noise sources [30]. It follows from central limit theorem arguments that the amplitude of the ambient noise is Gaussian distributed. The noise at the output of the bandpass filter, $n_o(t)$, will be assumed to be stationary with a flat spectral density within the bandpass. The assumed spectral density of $n_o(t)$ is shown in Figure 5.6. The autocorrelation of $n_o(t)$ is obtained by inverse Fourier transforming $S_{n_o}(f)$,

$$R_{n_o}(\tau) = \int_{-\infty}^{\infty} S_{n_o}(f)e^{j2\pi f\tau}df = 2N_oW\cos 2\pi f_o\tau \, \frac{\sin 2\pi\tau W}{2\pi\tau W} \tag{5.46}$$

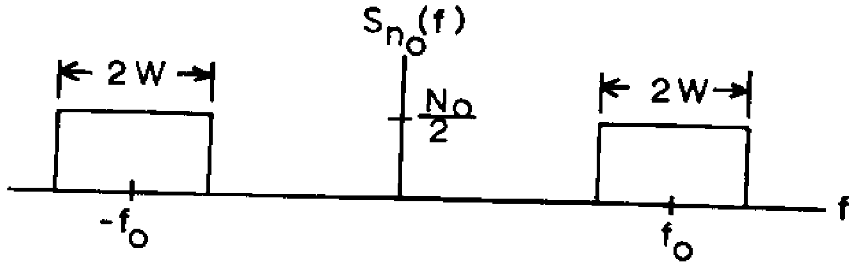

Figure 5.6. Filtered noise spectral density

For a pulse of length T_p, the filter bandwidth and the pulse length are approximately related by

$$W \approx \frac{1}{T_p} \tag{5.47}$$

Therefore,

$$R_{n_o}(\tau) \approx \frac{2n_o}{T_p} \cos 2\pi f_o \tau \, \frac{\sin 2\pi\tau/T_p}{2\pi\tau/T_p} \tag{5.48}$$

The noise process at the input of the integrator, $n^2(t)$, is related to $n_o(t)$ by

$$n^2(t) = G^2(t) n_o^{\,2}(t) \tag{5.49}$$

Therefore, the moments of $n^2(t)$ in (5.43) and (5.44) are

$$E[n^2(t)] = G^2(t) E[n_o^{\,2}(t)]$$

$$\mathrm{Cov}[n^2(\alpha) n^2(\beta)] = G^2(\alpha) G^2(\beta) \mathrm{Cov}[n_o^{\,2}(\alpha), n_o^{\,2}(\beta)]$$

$$E[n(\alpha) n(\beta)] = G(\alpha) G(\beta) R_{n_o}(\alpha,\beta) \tag{5.50}$$

The first and second moments of a square law detected Gaussian process are (cf. [1], pp. 253-255)

$$E[n_o^2(t)] = R_{n_o}(0) \tag{5.51}$$

$$\mathrm{Cov}[n_o^2(\alpha), n_o^2(\beta)] = 2R_{n_o}^2(\alpha,\beta) \tag{5.52}$$

From (5.48), (5.50) through (5.52), it follows that

$$E[n^2(t)] = G^2(t)2N_o/T_p \tag{5.53}$$

$$\mathrm{Cov}[n^2(\alpha), n^2(\beta)] = G^2(\alpha)G^2(\beta)8\left(\frac{N_o}{T_p}\right)^2 \cos^2 2\pi f_o\tau \left[\frac{\sin 2\pi\tau/T_p}{2\pi\tau/T_p}\right]^2 \tag{5.54}$$

The following expressions for the mean and variance of the integrator output for a reverberation signal in noise are obtained from (5.30), (5.31), (5.43), (5.44), (5.53), (5.54), the six assumptions in Section 5.2.1 and the assumption that A and $g(\theta,\phi)$ are uncorrelated.

$$E_I = \frac{E[A^2]E[g^4(\theta,\phi)]kT_p[t_2^{2g_o-1} - t_1^{2g_o-1}]}{2(2g_o-1)}$$

$$\frac{2N_o(t_2^{2g_o+1} - t^{2g_o+1})}{T_p(2g_o+1)} \tag{5.55}$$

$$\mathrm{Var}_I = \frac{E[A^4]E[g^8(\theta,\phi)]T_p^{\,2}k[t_2^{\,4g_o-5}-t_1^{\,4g_o-5}]}{4(4g_o-5)}$$

$$\frac{(E[A^2]E[g^4(\theta,\phi)])^2k^2T_p^{\,3}[t_2^{\,4g_o-3}-t_1^{\,4g_o-3}]}{6(4g_o-3)}$$

$$+8\pi\left\{\frac{N_o}{T_p}\right\}^2\frac{(t_2^{\,4g_o+1}-t_1^{\,4g_o+1})}{4g_o+1} \qquad (5.56)$$

It was shown in Section 5.2.1 that in the absence of noise, the integrator output could be scaled to produce an unbiased estimate of λ_o. This is no longer the case when the received signal contains ambient noise. An expression for the bias is obtained using (5.32) and (5.55)

$$b_I = E[\hat{\lambda}_o - \lambda_o]$$

$$= \frac{2N_o(t_2^{\,2g_o+1}-t_1^{\,2g_o+1})(2g_o-1)}{\pi T_p^{\,2}(t_2^{\,2g_o-1}-t_1^{\,2g_o-1})(2g_o+1)E[A^2g^4(\theta,\phi)](c/2)^3} \qquad (5.57)$$

The normalized mean squared error in the estimated density, e_I^2, is

$$e_I^{\,2} = \frac{\mathrm{Var}[\hat{\lambda}_o] + (E[\hat{\lambda}_o - \lambda_o])^2}{\lambda_o^{\,2}}$$

$$= \frac{E[A^4]E[g^8(\theta,\phi)](2g_o-1)^2[1-(t_1/t_2)^{4g_o-5}]}{(E[A^2]E[g^4(\theta,\phi)])^2(4g_o-5)kt_2^{\,3}[1-(t_1/t_2)^{2g_o-1}]^2}$$

$$+ \frac{2(2g_o-1)^2 T_p[1-(t_1/t_2)^{4g_o-3}]}{3(4g_o-3)t_2[1-(t_1/t_2)^{2g_o-1}]^2} + \frac{b_I^{\ 2}}{\lambda_o^{\ 2}}$$

$$+ \frac{32\pi N_o^{\ 2}(2g_o-1)^2 t_2^{\ 3}[1-(t_1/t_2)^{4g_o+1}]}{T_p^{\ 4}k^2(4g_o+1)(E[A^2]E[g^4(\theta,\phi)])^2[1-(t_1/t_2)^{2g_o-1}]^2} \qquad (5.58)$$

where b_I is defined in (5.57). The normalized variance of the integrator output in the absence of noise could be reduced by shortening the length, T_p (cf. Equation (5.33)). However, shortening the pulse length results in a wider bandwidth and increases the contribution of the ambient noise to the mean squared error (cf. Equation (5.58)).

5.3 Echo Counting

The echo counter is the continuous time version of the sampled echo counter described in Section 5.1.2. A block diagram of the echo counter is shown in Figure 5.7. The threshold device eliminates low level noise and amplitude variations from the signal.

5.3.1 Single Pulse Analysis

If the transmitted signal is a gated sine wave of duration T_p. the signal out of the T.V.G. amplifier can be written as

$$e(t) = \sum_{i=1}^{N(t)} B_i[u(t-\tau_i)-u(t-\tau_i-T_p)]\cos(\omega t+\psi_i) \qquad (5.59)$$

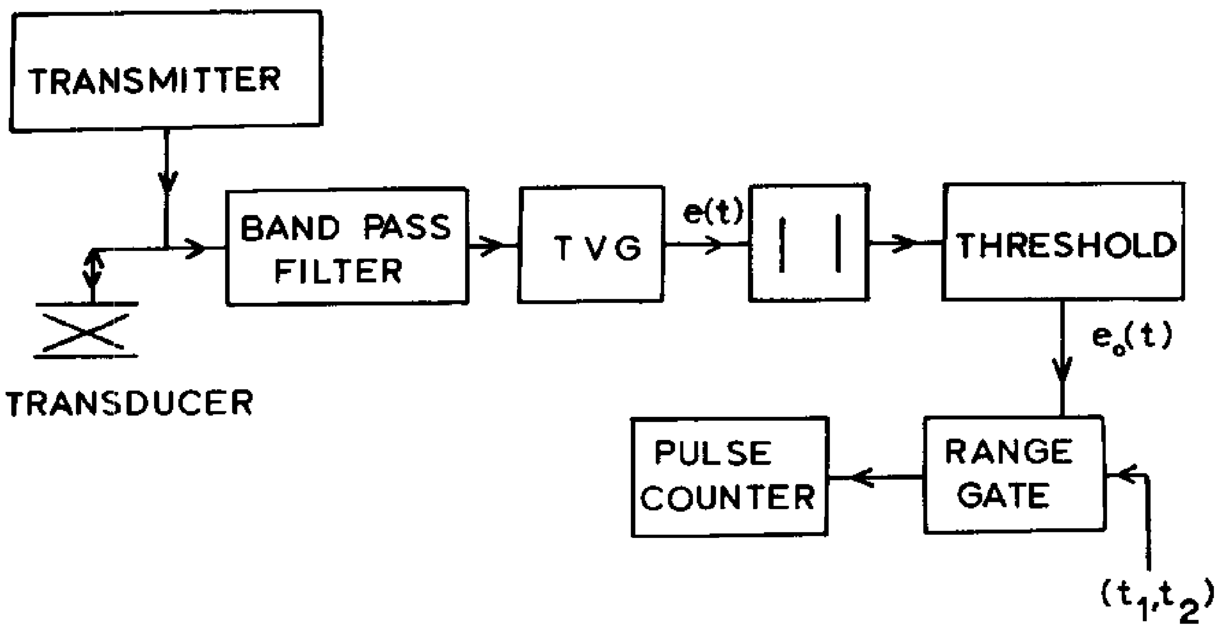

Figure 5.7. Echo counter

where B_i is the amplitude of the i^{th} echo, N(t) is a nonhomogeneous counting process with intensity $\nu(t)$ and τ_i is the time at which the i^{th} echo appears at the counter. The output of the threshold device, $e_o(t)$, is

$$e_o(t) = \begin{cases} 1 & |e(t)| \geq T \\ 0 & |e(t)| < T \end{cases} \tag{5.60}$$

The process, e(t), can be split into two parts

$$e(t) = e_A(t)+e_B(t) \tag{5.61}$$

where

$$e_A(t) = \sum_{j=1}^{N_A(t)} B_{Aj}[u(t-\tau_{Aj})-u(t-\tau_{Aj}-T_p)]\cos(\omega t+\psi_{Aj})$$

$$e_B(t) = \sum_{\ell=1}^{N_B(t)} B_{B\ell}[u(t-\tau_{B\ell})-u(t-\tau_{B\ell}-T_p)]\cos(\omega t+\psi_{B\ell})$$

such that $e_A(t)$ contains only echoes whose amplitudes are above the threshold, T, and $e_B(t)$ contains echoes whose amplitudes are below T. $N_A(t)$ and $N_B(t)$ are nonhomogeneous Poisson counting processes with intensities $\nu_A(t)$ and $\nu_B(t)$ defined by

$$\nu_A(t) = \nu(t)P[B \geq T]$$

$$\nu_B(t) = \nu(t)P[B < T] \qquad (5.62)$$

It will be assumed that the signal at the output of the threshold device is due solely to $e_A(t)$.[4]

The number of echoes registered by the counter for a single acoustic pulse is

$$C(t_2) = \sum_{i=1}^{N_A(t)} x_i(\tau_{Ai},t) \qquad (5.63)$$

[4] This assumption is an oversimplification of the effect of the envelope detector and the threshold device. In a previous analysis [11], the problem of the threshold was avoided by assuming an ideal transducer beam pattern with constant intensity within the beam. All the scatterers insonified by the ideal beam were assumed to have echo levels greater than the threshold.

where

$$x_i(\tau_{Ai},t) = \begin{cases} 1 & \text{if counter unlocked at } \tau_{Ai} \\ 0 & \text{if counter locked at } \tau_{Ai} \end{cases}$$

The counter will be locked at τ_{Ai} if

$$\tau_{Ai} - \tau_{Ai-1} < T_p \tag{5.64}$$

The output of the counter, $C(t_2)$, is a filtered Poisson process and its mean and variance can be determined from the characteristic function derived in Theorem (2.7). The resulting expressions for $E[C(t_2)]$ and $Var[C(t_2)]$ are (cf. [17], p. 156)

$$E[C(t_2)] = \int_{t_1}^{t_2} \nu_A(\tau) E[x(\tau,t)] d\tau$$

$$Var[C(t_2)] = \int_{t_1}^{t_2} \nu_A(\tau) E[x^2(\tau,t)] d\tau \tag{5.65}$$

The first and second moments of $x(\tau,t)$ are easily evaluated

$$E[x(\tau,t)] = E[x^2(\tau,t)]$$

$$= 1 \; P \; [\text{counter unlocked at time } \tau]$$

$$= \exp \left[-\int_{\tau-T_p}^{\tau} \nu_A(x) dx \right]$$

$$\approx \exp \left[-\nu_A(\tau) T_p \right] \tag{5.66}$$

The last approximation assumes $\nu_A(\tau)$ is nearly constant over a pulse length, T_p. It follows that

$$E[C(t_2)] = \mathrm{Var}[C(t_2)] = \int_{t_1}^{t_2} \nu_A(\tau) e^{-\nu_A(\tau) T_p} d\tau \tag{5.67}$$

When the scatterers are uniformly distributed in the insonified volume,

$$\nu_A(\tau) = \lambda_o K_C \tau^2$$

where

$$K_C = \frac{c^3 \pi}{4} \int_T^{\infty} f(b)\,db \tag{5.68}$$

and

$$E[C(t_2)] = \mathrm{Var}[C(t_2)] = \lambda_o K_C \int_{t_1}^{t_2} \tau^2 e^{-\tau^2 \lambda_o K_C T_p} d\tau \tag{5.69}$$

In low spatial densities,

$$e^{-\tau^2 \lambda_o K_C T_p} \approx 1$$

and the counter output is proportional to the density factor, λ_o. Therefore, using the low density assumption, the estimate of λ_o is

$$\hat{\lambda}_o = \frac{C(t_2)}{K_C(t_2^3 - t_1^3)/3} \tag{5.70}$$

The bias in the estimate is

$$b_C = E[\hat{\lambda}_o] - \lambda_o = \lambda_o \left[\frac{3\int_{t_1}^{t_2} \tau^2 e^{-\tau^2 \lambda_o K_C T_p} d\tau}{t_2^3 - t_1^3} - 1 \right] \qquad (5.71)$$

The normalized mean squared error of the estimate $\hat{\lambda}_o$ is

$$\frac{e_C^2}{\lambda_o^2} = \frac{b_C^2 + \mathrm{Var}[\hat{\lambda}_o]}{\lambda_o^2} = \left[\frac{3\int_{t_1}^{t_2} \tau^2 e^{-\tau^2 \lambda_o K_C T_p} d\tau}{t_2^3 - t_1^3} - 1 \right]^2 + \frac{9\int_{t_1}^{t_2} \tau^2 e^{-\tau^2 \lambda_o K_C T_p} d\tau}{\lambda_o K_C (t_2^3 - t_1^3)^2} \qquad (5.72)$$

5.3.2 Multiple Pulse Analysis

The total count registered by the counter, C_T, after N_p acoustic pulses is

$$C_T = C_1 + C_2 + \ldots + C_{N_p} \qquad (5.73)$$

where C_j is the number of echoes counted on the j^{th} acoustic pulse. The estimate of λ_o assuming a constant spatial density for all N_p pulses is

$$\hat{\lambda}_o = \frac{C_T}{N_p K_C (t_2^3 - t_1^3)/3} \qquad (5.74)$$

Processing the returns from several acoustic pulses does not reduce the bias obtained with the echo counter. The normalized mean squared error after N_p pulses is

$$\frac{e_C^2}{\lambda_o^2} = \frac{b^2_C + \mathrm{Var}[\hat{\lambda}_o]}{\lambda_o^2} = \frac{b^2_C}{\lambda_o^2} + \left[\frac{3}{\lambda_o N_p K_C (t_2^3 - t_1^3)}\right]^2 \sum_{i=1}^{N_p} \sum_{j=1}^{N_p} E[C_i C_j] - E[C_i]E[C_j]$$

$$= \frac{b_C^2}{\lambda_o^2} + \left[\frac{3}{\lambda_o N_p K_C (t_2^3 - t_1^3)}\right]^2 \left\{ N_p \ \mathrm{Var}[C(t_2)] \right.$$

$$\left. + \sum_{\substack{i=1 \\ i \neq j}}^{N_p} \sum_{j=1}^{N_p} \mathrm{Cov}[C_i, C_j] \right\} \tag{5.75}$$

where b_C and $\mathrm{Var}[C(t_2)]$ are defined in (5.71) and (5.69).

If a different volume of water is sampled with each pulse, $\mathrm{Cov}[C_i, C_j]=0$ and

$$\frac{e_C^2}{\lambda_o^2} = \frac{b_C^2}{\lambda_o^2} + \frac{1}{N_p}\left[\frac{3}{K_C (t_2^3 - t_1^3)}\right]^2 \mathrm{Var}[C(t_2)] \tag{5.76}$$

Evaluation of the covariance term in (5.75) for overlapping sampled volumes is very difficult unless some further simplifying assumptions are made. An ideal beam pattern with constant intensity within the beam was assumed in a previous analysis [11]. The interested reader is referred to Reference 11 for the overlapping volume analysis with this assumption.

The relative performance of the echo integrator and echo counter is illustrated by the following example.

Example (5.1):

Physical model:

i) constant spatial intensity, λ_o

ii) target strength variable, A, is Rayleigh distributed

$$f_A(a) = \frac{a}{\alpha^2} e^{-a^2/2\alpha^2} \qquad a \geq 0$$

iii) negligible absorption, $\alpha_o = 0$

iv) no ambient noise

v) sound velocity in water, c = 1500 m/sec

System parameters:

i) surveyed depth interval = 10-40 meters

ii) T.V.G. gain factor, g_o = 2

iii) pulse length, T_p = 1 msec

iv) number of pulses, N_p = 400 (no overlap)

v) circular piston transducer with $d/\lambda=6$ (3 dB beamwidth of 11.5°

vi) counter set to detect all scatterers in the region where $g^2(\theta,\phi) > -15$ dB.

The beam pattern moment ratio which is obtained from the curves in Figure B1 is

$$\frac{E[g^8(\theta,\phi)]}{(E[g^4(\theta,\phi)])^2} = 220$$

and the target strength moment ratio is

$$\frac{E[A^4]}{(E[A^2])^2} = 2$$

The probability of counting a single target is obtained by integrating the beam pattern density function derived in Appendix B,

$$\int_T^\infty f(b)\,db = \int_{.001}^{1} f_G(g)\,dg = .018$$

The normalized mean squared errors for the echo integrator and echo counter for the above set of conditions is plotted as a function of λ_o in Figure 5.8. The normalized mean squared errors in Figure 5.8 have the same characteristics as the corresponding sampled process error curves shown in Figure 5.3. At low densities, the mean squared error of the integrator and counter decreases as $1/\lambda_o$. For these densities, the individual echoes do not in general overlap and the principal source of variance is the variability of the scattering density about its mean value. In this region, the echo counter has a lower error since it is not affected by variations in the echo level.

5.3.3 Effect of Noise on the Counter

The echo counter like the echo integrator is adversely affected by ambient noise. An extraneous count will be recorded when the envelope of the noise process exceeds the threshold level, T. In this section, expressions will be derived for the mean and variance of the counter output when the input to the transducer is Gaussian distributed ambient noise.

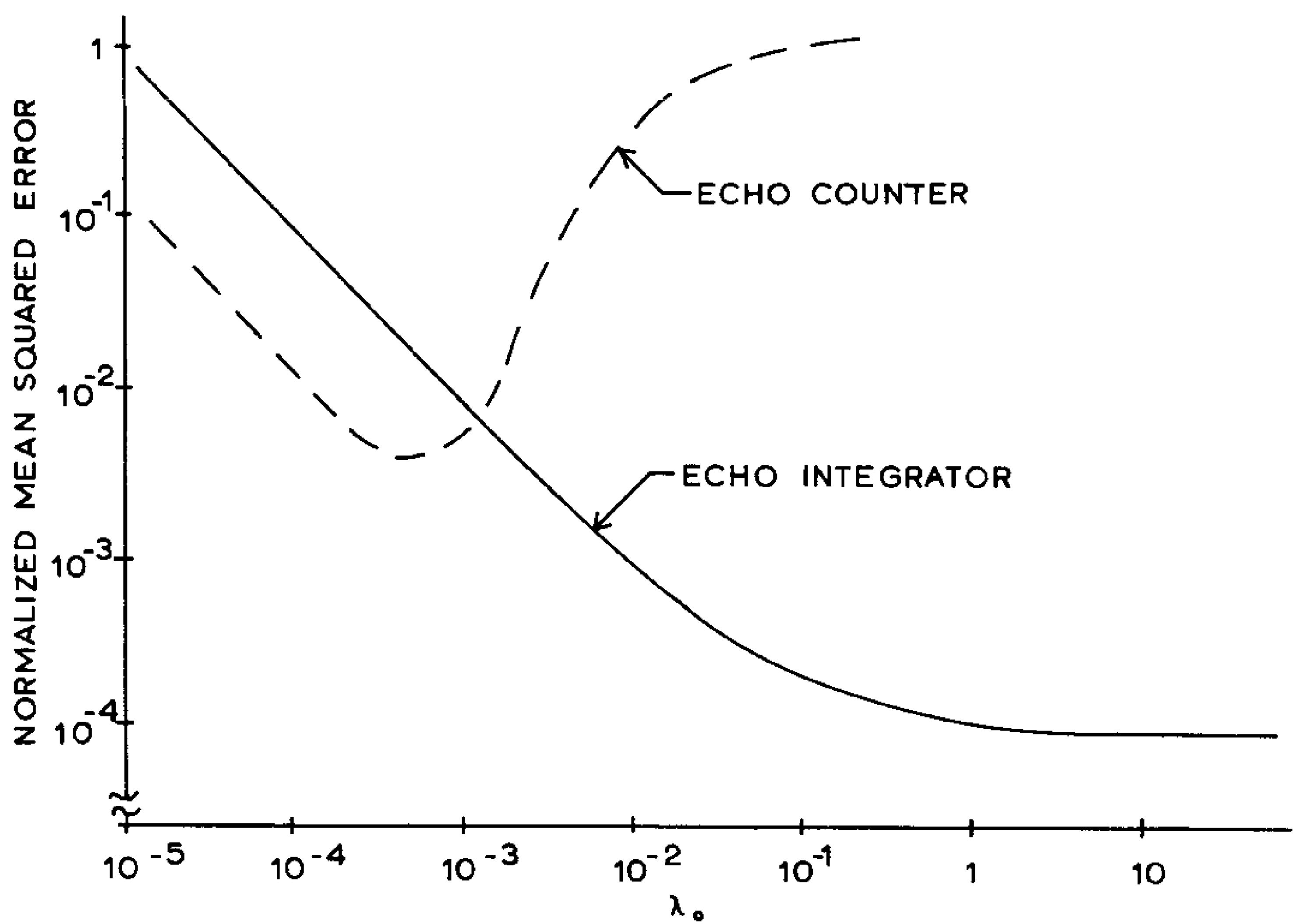

Figure 5.8. Normalized mean squared errors of integrator and counter

Assume that the threshold is set such that the mean time between counts due to the ambient noise is much greater than the correlation time of the noise process. In this case, the number of times the noise process exceeds the threshold can be approximated by a Poisson process with intensity $\nu_n(t)$ given by[5]

$$\nu_n(t) = \frac{T}{R_n(t,t)} \sqrt{\frac{|R_n''(t,t)|}{2\pi}}\, e^{-T^2/2R_n(t,t)} \tag{5.77}$$

where

$$R_n''(t,t) = \frac{d^2}{d\tau^2} R_n(t,t+\tau)\Big|_{\tau = 0}$$

and $R_n(t,t+\tau)$ is the autocorrelation function of the envelope of the noise process at the input to the linear detector. In order to evaluate the noise envelope correlation function, assume that the noise at the output of the bandpass filter has the spectrum shown in Figure 5.6. The time varying gain amplifier (T.V.G.) changes the height of the noise spectrum as a function of time. If the T.V.G. gain function, $G(t)$, is slowly varying relative to the correlation time of the noise, the height of the spectral density at time t is $G^2(t)N_o$ and the autocorrelation function is

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[5] The expression for $\nu_n(t)$ is derived by Helstrom (cf. [31] pp. 253-257). Helstrom's analysis is used to obtain an expression for the probability of error in a radar system, a problem closely related to the problem treated in this section.

$$R_n(t,t+\tau) \approx G^2(t)N_o \int_{-W}^{W} e^{j2\pi f\tau}df \tag{5.78}$$

It follows that

$$R_n(t,t) \approx 2G^2(t)N_oW \tag{5.79}$$

$$R_n''(t,t) \approx \frac{-8G^2(t)N_oW^3\pi^2}{3} \tag{5.80}$$

The following expression for $\nu_n(t)$ is obtained from (5.77), (5.79), (5.80) and the approximation that $W \approx 1/T_p$.

$$\nu_n(t) = \sqrt{\frac{T^2}{3N_oG^2(t)T_p}}\, e^{-\frac{T^2T_p}{4G^2(t)N_o}} \tag{5.81}$$

The intensity, $\nu_n(t)$, is plotted in Figure 5.9 as a function of the threshold to noise ration, R, defined as

$$R = \frac{T^2}{\text{Var}[n(t)]} = \frac{T^2}{2N_oWG^2(t)} = \frac{T^2T_p}{2N_oG^2(t)} \tag{5.82}$$

Recall that the Poisson assumption was based on the condition that the mean time between counts is much greater than the correlation time of the process. In terms of $\nu_n(t)$ and T_p, this condition is equivalent to

$$\frac{1}{\nu_n(t)} >> T_p \tag{5.83}$$

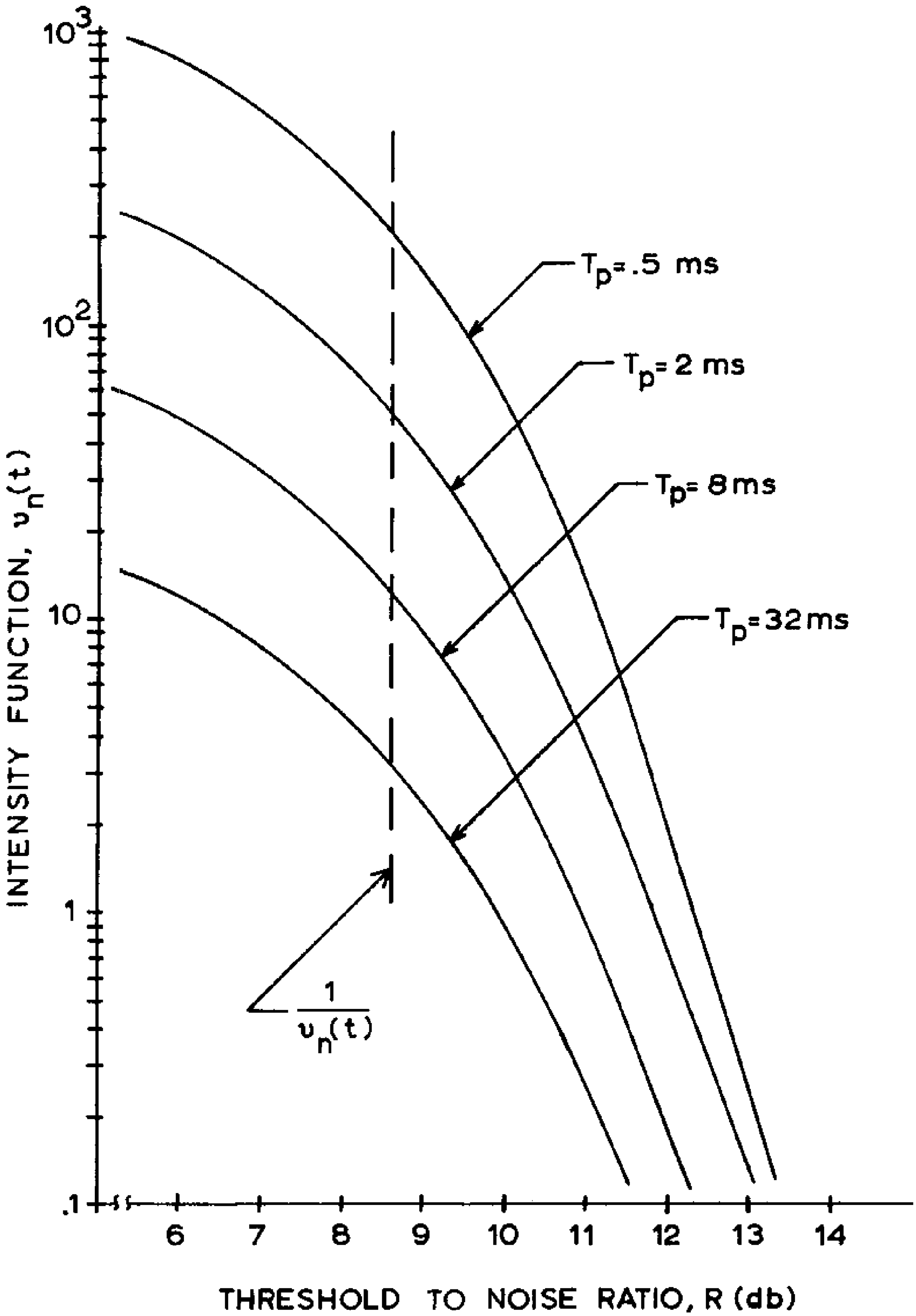

Figure 5.9. Intensity of noise induced counts

For $R > 9$ dB and $(1/\nu_n(t)) > 10\ T_p$, the Poisson model is a good approximation to the distribution of the noise produced counts.

Expressions for the mean and variance of the number of counts, $C_n(t_2)$ produced by the noise in a time interval (t_1,t_2) follow directly from the properties of a nonhomogeneous Poisson process (cf. (2.12) and (2.13)),

$$E[C_n(t_2)] = \mathrm{Var}[C_n(t_2)] = \int_{t_1}^{t_2} \nu_n(t)dt \tag{5.84}$$

where $\nu_n(t)$ is defined in (5.81). The density estimate for the counter is obtained by scaling the number of recorded counts. If the scale factor for uniformly distributed scatterers is used, the noise induced bias in the estimated density is

$$b_n = \frac{C_n(t_2)}{K_C(t_2{}^3 - t_1{}^3)/3} \tag{5.85}$$

where K_C is defined in (5.68). The ratio of noise induced biases for the integrator and the counter is a measure of the relative effect of the ambient noise on the two estimation techniques. The ratio, b_n/b_I, which is obtained from (5.57), (5.81) and (5.85) is

$$\frac{b_n}{b_I} = \frac{5\sqrt{\pi}TT_p{}^{3/2}E[A^2]E[g^4(\theta,\phi)] \int_{t_1}^{t_2} t^2 e^{-\frac{T^2T_p}{4N_0t^4}} dt}{4\sqrt{3}N_o{}^{3/2}(t_2{}^5 - t_1{}^5) \int_T^{\infty} f(b)db} \tag{5.86}$$

Equation (5.86) has been evaluated for the assumed set of operating conditions in Example (5.2).

Example (5.2):

Parameters:

i) surveyed depth interval = 10-40 meters

ii) circular transducer with $d/\lambda = 6$

iii) target strength term, A, is a constant

iv) sound velocity, c = 1500 m/sec

The definition of the threshold to noise ratio, R, will be modified as follows to account for the changing noise spectral density with time.

$$R = \frac{T^2}{\mathrm{Var}[n(t)]} = \frac{T^2 T_p}{2N_o E[G^2(t)]} \tag{5.87}$$

where $E[G^2(t)]$ is the average gain of the T.V.G. for the surveyed depth interval. The fourth moment of the beam pattern term, $E[g^4(\theta,\phi)]$ is obtained from Figure B1. Since the target strength variable, A, is assumed to be constant, the integral of the echo level density function, f(b), can be expressed in terms of the integral of the beam pattern density function, $f_G(g)$

$$\int_T^{\infty} f(b)db = \int_{T^2/A^2}^{1} f_G(g)dg$$

where $f_G(g)$ has been derived in Appendix B. A plot of the bias ratio, b_n/b_I, as a function of threshold to noise ratio and threshold setting relative to A is shown in Figure 5.10.

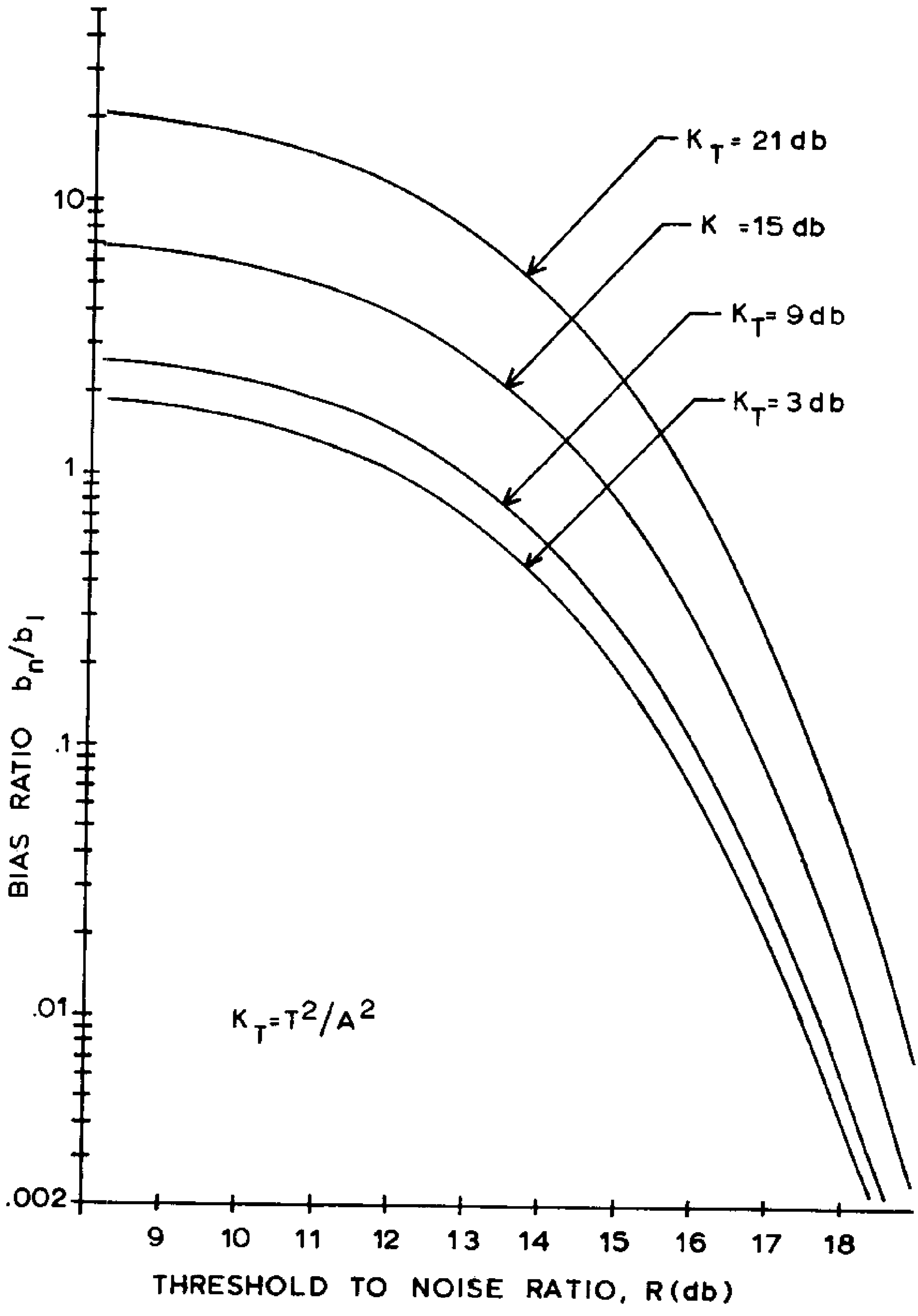

Figure 5.10. Comparison of bias errors in integrator and counter

5.4 Target Strength Density Function Estimation

A technique for making an in situ estimate of the density function of the target strength variable, A^2, is discussed in this section. The density function can be used to evaluate the moments of A^2 which appear in the expression for the mean and variance of the integrator output. The target strength density function can also be used to estimate the scatterer size distribution within the surveyed population. Another method for obtaining the target strength distribution has been proposed by Craig and Forbes [8]. Their technique is physically motivated and implicitly assumes that the scatterers are uniformly distributed in space. The method described in this section is statistically motivated and can be applied to any spatial distribution.

The estimate of the target strength density function is obtained in two steps. The first step estimates a portion of the density function of the integrated echo level, $A^2g^4(\theta,\phi)$. The partial integrated echo level density function is then used to evaluate the density of A^2. A block diagram of the system used to obtain the single scatterer integrated echo level, I_i, is shown in Figure 5.11. The time varying gain function of the T.V.G. amplifier is adjusted to cancel all propagation losses. The threshold device represents the inability of the single target recognition circuit to distinguish signals whose integrated echo level, $A^2g^4(\theta,\phi)$, is smaller than a certain level, T. If all the single echoes could be distinguished, it would be possible to estimate the moments of $A^2g^4(\theta,\phi)$ directly. However, a large percentage of the total

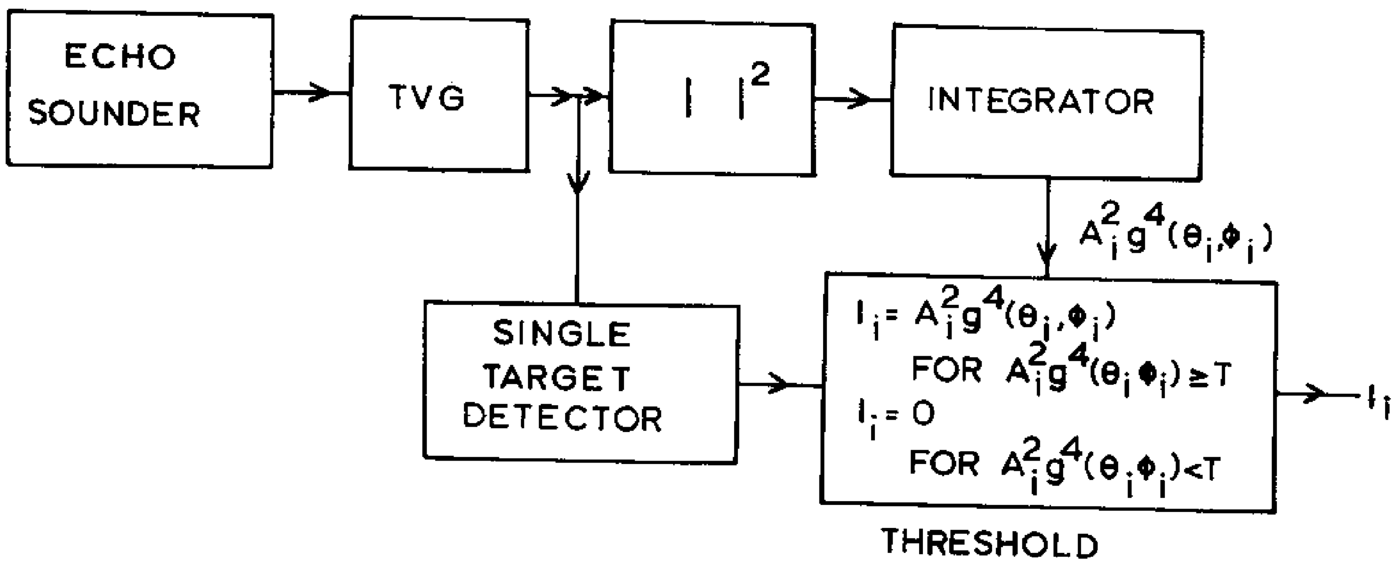

Figure 5.11. Single echo recovery system

number of single echoes are from scatterers located in the low gain portion of the beam pattern and fall below the threshold.

The single target recognition circuit is the most difficult portion of the system to implement. Two possible circuits are discussed in Appendix A of reference [32].

In order to simplify notation, define the random variables G and A_o as follows

$$G = g^4(\theta,\phi)$$
$$A_o = A^2 \tag{5.88}$$

with density functions $f_G(g)$ and $f_{A_o}(a)$. An expression for the density of G in terms of the directivity function of the transducer and the spatial distribution of the scatterers is obtained in Appendix B.

The relationship between the density functions for G and A_o and the density function of I, the echo level random variable, can be found using elementary probability theory (cf. [33], p. 205).

$$f(i) = \begin{cases} C_o \int_o^\infty \frac{1}{a} f_{A_o}(a) f_G(i/a)da & i \geq T \\ 0 & i < T \end{cases} \tag{5.89}$$

where C_o, a constant that depends on the threshold level T, insures that f(i) integrates to 1. The directivity function $g(\theta,\phi)$ and consequently the random variable G are contained in the interval [0,1]. The random variable A_o is finite and can be assumed to be contained in some interval $[0,A_{max}]$. Using these intervals for A_o and G, (5.89) can be rewritten as

$$f(i) = \begin{cases} C_o \int_i^{A_{max}} \frac{1}{a} f_{A_o}(a) f_G(i/a)da & i \geq T \\ 0 & i < T \end{cases} \quad (5.90)$$

since $f_{A_o}(a) = 0$ for $a>A_{max}$ and $f_G(g) = 0$ for $g>1$. Equation (5.90) is a Volterra integral equation of the first kind for the unknown function $f_{A_o}(a)$. For the general class of real functions, this type of integral equation does not have a unique solution. However, the equation has a unique solution within the class of positive real functions. This fact is easily shown in a proof by contradiction.

The target strength density function estimate is obtained by numerically solving the integral equation (5.90). The following procedure is one of the many methods that can be used to solve an integral equation. Some other techniques are described in reference [34].

The unknown density function is first approximated by an n^{th} degree polynomial,

$$f_{A_o} = \frac{1}{C_o} \sum_{j=1}^{n} \alpha_j a^j \quad (5.91)$$

With this representation for the unknown density, the integral equation becomes

$$\hat{f}(i) = \sum_{j=0}^{n} \alpha_j \beta_j (i,A_{max}) \quad (5.92)$$

where

$$\beta_j(i,A_{max}) = \int_i^{A_{max}} a^{j-1} f_G(i/a)da \tag{5.93}$$

and where $\hat{f}(i)$ is an estimate of the density function for I. The maximum value of the target strength density, A_{max} , is not known a priori. It can be estimated by the greatest single scatterer integrated echo level, $\max(I_i)$. A Monte Carlo simulation has shown that the choice of A_{max} does not greatly affect the estimate of $f_{A_o}(a)$. The unknown coefficients, α_j, in (5.91) are evaluated by a least squares fit of the functions $\beta_j(i,A_{max})$ to be estimated density, $\hat{f}(i)$. The normalizing coefficient, C_o, in (5.91) is chosen such that $f_{A_o}(a)$ integrates to 1.

The procedure described above has been investigated using a Monte Carlo simulation. Random variables, I, representing the integrated squared echo values were generated by taking the product of a beam pattern variable, G, and a target strength variable, A_o. The distribution for G was derived assuming a piston transducer and a uniform spatial distribution of the scatterers producing single echoes. A number of ways of estimating the echo level density, f(i), were investigated. The technique which worked the best was to estimate f(i) by the derivative of a least squares polynomial approximation to the empirical distribution function of I. The results of the Monte Carlo simulation of the density estimation technique are shown in Figure 5.12. The target strength variable, A_o, was lognormally distributed for the simulation. That is,

$$A_o = e^y \tag{5.94}$$

where y was a Gaussian random variable. The mean and variance of y were

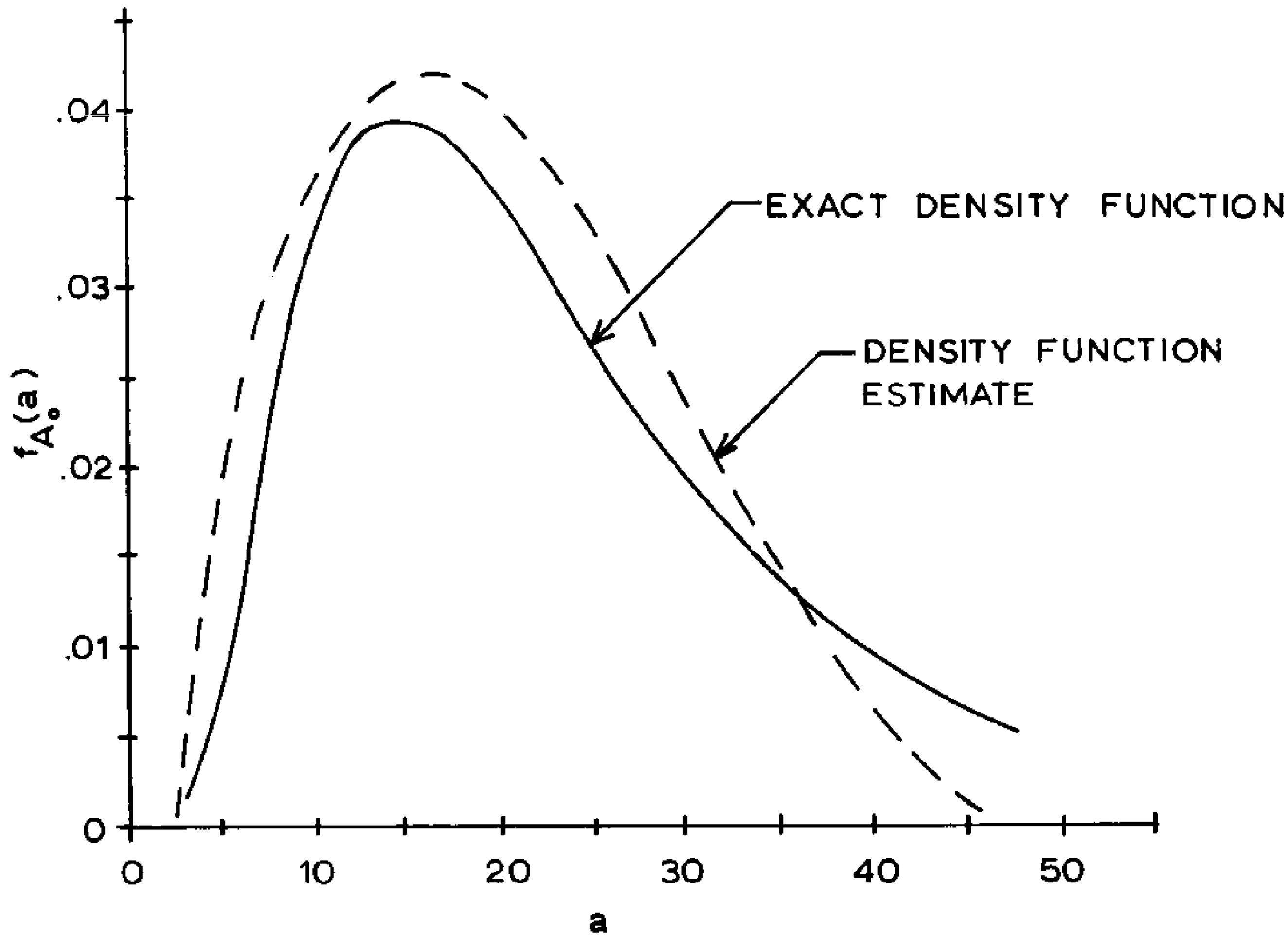

Figure 5.12. Simulated density function estimate

3 and .36 respectively. The order of the polynomial fit to f(i) was 7 and the degree of the polynomial approximation to $f_{A_o}(a)$ was 4. Three hundred samples were used to obtain the density estimate.

CHAPTER 6

CONCLUSIONS

The purpose of this dissertation has been to develop methods for statistically estimating the intensity factor of a filtered Poisson process and to use the methods developed to determine the structure and analyze the performance of signal processors that estimate the spatial density of marine organisms. In this chapter, the principal contributions of this research are reviewed and some problems requiring further investigation are suggested.

6.1 Principal Contributions

The application of statistical estimation theory to Poisson counting processes is not new (cf. [35], chapters 2 and 3). However, the problem of estimating the intensity factor of a filtered Poisson process has not been treated previously. Therefore, the theoretical development in Chapter 3 is one of the main contributions of this dissertation. Because of the wide usage of digital techniques, most signal processors operate on samples from the signal of interest. For this reason, the sampled process intensity factor estimators derived in Section 3.4 are not only of theoretical interest but are of practical importance.

Previous authors [3], [4], [6] have used a Poisson distribution of point scatterers as a model for reverberation. These authors, however, did not take advantage of the many mathematical properties that can be derived when the received scattered signal is described in terms of a filtered Poisson process. For example, Moose [6] has only shown that the first order density function of reverberation is asymptotically Gaussian as the scattering density goes to infinity. Using the theory developed in Chapter 2, it is easily shown that all order densities are asymptotically Gaussian.

The expression for the reverberation process first order density function derived in Section 4.1.4 is a new result. The common approach used previously has been to use the Edgeworth series expansion for the first order density function. While this former approach allows one to easily show that the density is asymptotically Gaussian, it does not provide any insight into the form of the density function for low scattering densities.

Chapter 5 contains the most practical contributions of this dissertation. Nearly all the material in Chapter 5 is new and cannot be found elsewhere except in other publications by the present author. The results of the Monte Carlo simulation of the sampled process recursive estimator in Section 5.1 show that the mean squared error of the intensity factor estimate is, in some cases, significantly lower than the mean square errors produced by the two commonly used techniques of echo counting and echo integration.

The echo integrator and echo counter error expressions derived in Chapter 5 can be used by the marine biologist in planning an acoustic assessment survey. If there is some prior knowledge of the approximate density of the species to be surveyed, the two error expressions show which technique is superior and provide the biologist with a means of determining the amount of acoustic sampling required to obtain the desired degree of accuracy.

The target strength density function estimation technique described in Section 5.4 is more general and more accurate than the commonly used technique which was originally proposed by Craig and Forbes [8].

6.2 Topics for Further Study

There are a number of unsolved problems related to the research reported in this dissertation. Some of these problems are cited in this section.

The recursive estimation algorithm which uses independent samples from the reverberation process has a significantly smaller variance than either the echo counter or the echo integrator for medium scattering densities (see Section 5.1.4). However, a number of assumptions were made in evaluating the performance of the recursive algorithm. The sensitivity of the performance of the algorithm to these assumptions should be investigated. Some of the factors that should be considered in the study are: (1) the effect of differences in the form of the assumed and actual echo level density function; (2) algorithm performance for a process generated by a non-Poisson spatial distribution; and (3) convergence of the algorithm when parameters besides the intensity factor must be estimated.

This dissertation has dealt with the problem of estimating the multiplicative factor of a known intensity function of a filtered Poisson process. A natural extension of this research would be to allow the intensity function itself to be a stochastic process. Poisson processes with stochastic intensity functions are called doubly stochastic Poisson processes. The theory for this type of process is still in its early stages of development and to date no results have been published on doubly stochastic filtered Poisson processes.

It is shown in Chapters 3 and 4 that the echo level density function must be known to obtain a complete statistical characterization of the reverberation process. Since little is known about the statistics of the echo level or target strength of marine organisms, some assumptions were made before the mathematical model for reverberation was completed. Target strength and echo level data for difference sizes and species of marine organisms need to be collected so that accurate statistical models can be constructed.

In the practical application of acoustic techniques to fisheries resource assessment, the marine acoustics group at the University of Washington has found that the effect of background noise is almost negligible. For this reason, this dissertation has not considered the problem of optimally estimating the intensity of a filtered Poisson process received in the presence of noise. However, there are other applications of the filtered Poisson process model where background noise is significant. The problem of optimal intensity estimation in the presence of noise is important in these cases and should therefore be further considered.

The validity of the point scattering assumption is questionable in high scattering densities. Both analytical and experimental studies should be conducted to determine the effect of multiple scattering as a function of the spatial density of various types of marine organisms.

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APPENDIX A

MOMENTS OF A FILTERED POISSON PROCESS

The following relationships between the bivariate characteristic function, the moments and the cumulants are used in this appendix.[1] The joint characteristic function of two random variables, X_1 and X_2, is defined as

$$\phi_{X_1,X_2}(u_1,u_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp[jx_1u_1 - jx_2u_2]\,dF(x_1,x_2)$$

$$= \sum_{r,s=0}^{\infty} \mu'_{rs} \frac{(ju_1)^r}{r!}\frac{(ju_2)^s}{s!} \qquad \text{(A.1)}$$

$$= \exp\left[\sum_{r,s=1}^{\infty} k_{rs} \frac{(ju_1)^r}{r!}\frac{(ju_2)^s}{s!}\right] \qquad \text{(A.2)}$$

where μ'_{rs} is the rs^{th} bivariate moment and k_{rs} is the rs^{th} bivariate cumulant. It follows from (A.2) that

$$k_{rs} = j^{-(r+s)} \frac{\partial^{r+s}}{\partial u_1{}^r \partial u_2{}^s} \ln \phi_{X_1,X_2}(u_1,u_2)\Big|_{u_1 = u_2 = 0} \qquad \text{(A.3)}$$

Formulae that express the moments in terms of the cumulants and vise versa can be obtained from the characteristic function. The following relationships apply when X_1 and X_2 are zero mean

$$E[X_1{}^2] = \mu'_{20} = k_{20} \qquad \text{(A.4)}$$

---

[1] The material in the first part of this appendix has been taken from Volume 1 of Kendall and Stuart (cf. [12] Chapter 3).

$$E[X_1^4] = \mu'_{40} = k_{40} + 3k_{20}^2 \tag{A.5}$$

$$E[X_1X_2] = \mu'_{11} = k_{11} \tag{A.6}$$

$$E[X_1^2X_2^2] = \mu'_{22} = k_{22} + k_{20}k_{02} + k_{11}^2 \tag{A.7}$$

The bivariate characteristic function of two samples, $r(t_1)$ and $r(t_2)$, taken from a filtered Poisson process is (cf. Theorem (2.7))

$$\phi_{r(t_1),r(t_2)}(u_1,u_2) = \exp\left\{\int_0^{t_1} \nu(x)E[e^{ju_1z(t_1,x,\underset{\sim}{\Theta})} e^{ju_2z(t_2,x,\underset{\sim}{\Theta})} - 1]dx + \int_{t_1}^{t_2} \nu(x)\, e^{ju_2z(t_2,x,\underset{\sim}{\Theta})}\, dx\right\} \tag{A.8}$$

where $t_2 \geq t_1$. The following expression for $k_{rs}(r(t_1),r(t_2))$, the rs^{th} cumulant of $r(t_1)$ and $r(t_2)$, is obtained from (A.3) and (A.8).

$$k_{rs}(r(t_1),r(t_2)) = j^{-(r+s)}\left\{\int_0^{t_1} \nu(x)\left[\frac{\partial^{r+s}}{\partial u_1^r \partial u_2^s}\phi_{z_1,z_2}(u_1,u_2)\right]\Bigg|_{u_1=u_2=0} dx + \int_{t_1}^{t^2} \nu(x)\left[\frac{\partial^s}{\partial u_2^s}\phi_{z_1,z_2}(0,u_2)\right]\Bigg|_{u_2=0} dx\right\} \tag{A.9}$$

where $\phi_{z_1,z_2}(u_1,u_2)$ is the joint characteristic function of $z(t_1,x,\underset{\sim}{\Theta})$ and $z(t_2,x,\underset{\sim}{\Theta})$. When $r(t)$ is a zero mean process, it follows from (A.4) through (A.7) and (A.9) that

$$E[r^2(t)] = \int_0^t \nu(x)E[z^2(t,x,\underset{\sim}{\Theta})]dx \tag{A.10}$$

$$E[r^4(t)] = \int_0^t \nu(x)E[z^4(t,x,\underset{\sim}{\Theta})]dx + 3\left(\int_0^t \nu(x)E[z^2(t,x,\underset{\sim}{\Theta})]dx\right)^2 \tag{A.11}$$

$$E[r(t_1)r(t_2)] = \int_0^{\min(t_1,t_2)} \nu(x)E[z(t_1,x,\underset{\sim}{\Theta})z(t_2,x,\underset{\sim}{\Theta})]dx \tag{A.12}$$

$$E[r^2(t_1)r^2(t_2)] - E[r^2(t_1)]E[r^2(t_2)]$$

$$= \int_0^{\min(t_1,t_2)} \nu(x)E[z^2(t_1,x,\underset{\sim}{\Theta})z^2(t_2,x,\underset{\sim}{\Theta})]dx$$

$$+ 2\left(\int_0^{\min(t_1,t_2)} \nu(x)E[z(t_1,x,\underset{\sim}{\Theta})z(t_2,x,\underset{\sim}{\Theta})]dx\right)^2 \tag{A.13}$$

APPENDIX B

SOME STATISTICAL PROPERTIES OF $g(\theta,\phi)$

The magnitude of the signal reflected by a scatterer at angular location θ and ϕ is dependent on the transducer directivity function $g(\theta,\phi)$. Since the position of the scatterers is random, the directivity function term, $g(\theta,\phi)$, is also random. In this appendix, some of the statistical properties of $g(\theta,\phi)$ are investigated.

The moments of $g(\theta,\phi)$ can be expressed in terms of the density of the angular location variables, $f(\theta,\phi)$,

$$E[g^n(\theta,\phi)] = \int_0^{2\pi} \int_0^{\pi/2} g^n(\theta,\phi) f(\theta,\phi) d\theta d\phi \tag{B.1}$$

where the spherical coordinate system defining the angles is shown in Figure 4.1. Some typical density functions for θ and ϕ are given in Section 4.1.3. In particular, when the scattering density is constant, the moment expression is

$$E[g^n(\theta,\phi)] = \frac{1}{2\pi} \int_0^{\pi/2} \sin\theta \int_0^{2\pi} g^n(\theta,\phi) d\phi d\theta \tag{B.2}$$

One common type of transducer is the circular piston which has the following directivity function

$$g(\theta,\phi) = \frac{2J_1(\frac{\pi d}{\lambda} \sin\theta)}{\frac{\pi d}{\lambda} \sin\theta} \tag{B.3}$$

where

λ = acoustic wavelength

d = diameter of the piston

$J_1(x)$ = first order Bessel function

A plot of $E[g^4(\theta,\phi)]$ and $E[g^8(\theta,\phi)]$ as a function of d/λ for a piston transducer and a uniform spatial scattering density is shown in Figure B1.

The density function of the beam pattern random variable $G = g^4(\theta, \phi)$ is used in Examples (5.1) and (5.2) and the target strength density function estimation technique described in Section 5.4. The density function is obtained from the joint density for θ and ϕ and the general expression for a function of two random variables (cf. [33] pp. 199-205).

$$f_G(g) = \int_0^{2\pi} \left\{ \frac{f_{\theta,\phi}(\alpha_1,\beta_1)}{\left|\frac{\partial}{\partial\theta} g^4(\theta,\phi)\right|} + \cdots + \frac{f_{\theta,\phi}(\alpha_j,\beta_j)}{\left|\frac{\partial}{\partial\theta} g^4(\theta,\phi)\right|} \right\} d\phi \qquad \text{(B.4)}$$

where

$(\alpha_1,\beta_1), \ldots, (\alpha_j,\beta_j)$ = solutions to equations $g = g^4(\theta,\phi)$

The beam pattern density function, $f_G(g)$, for a piston transducer and a uniform spatial scattering density is shown in Figure B2.

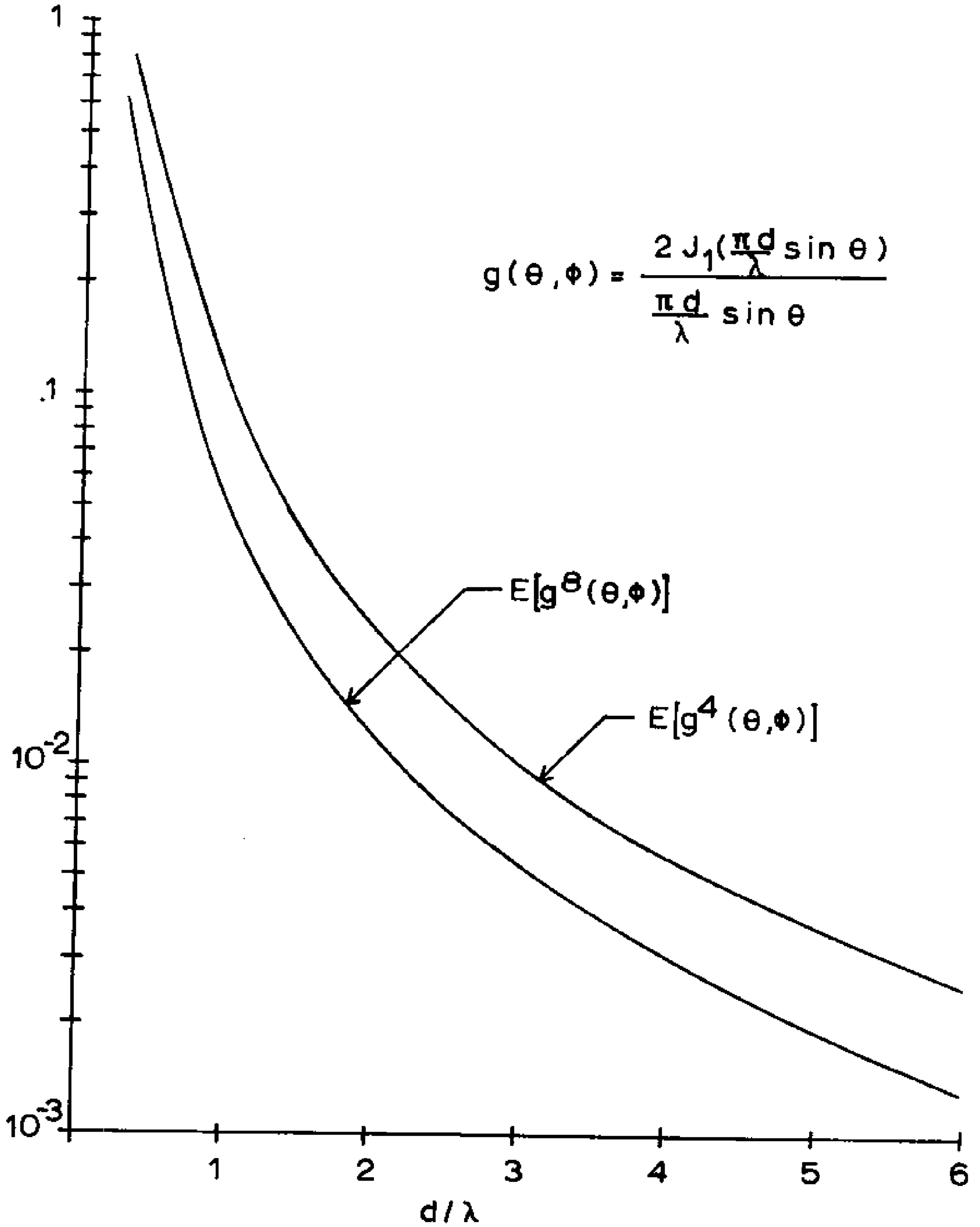

Figure B1. Moments of $g(\theta,\phi)$ for piston transducer and uniform spatial scattering distribution

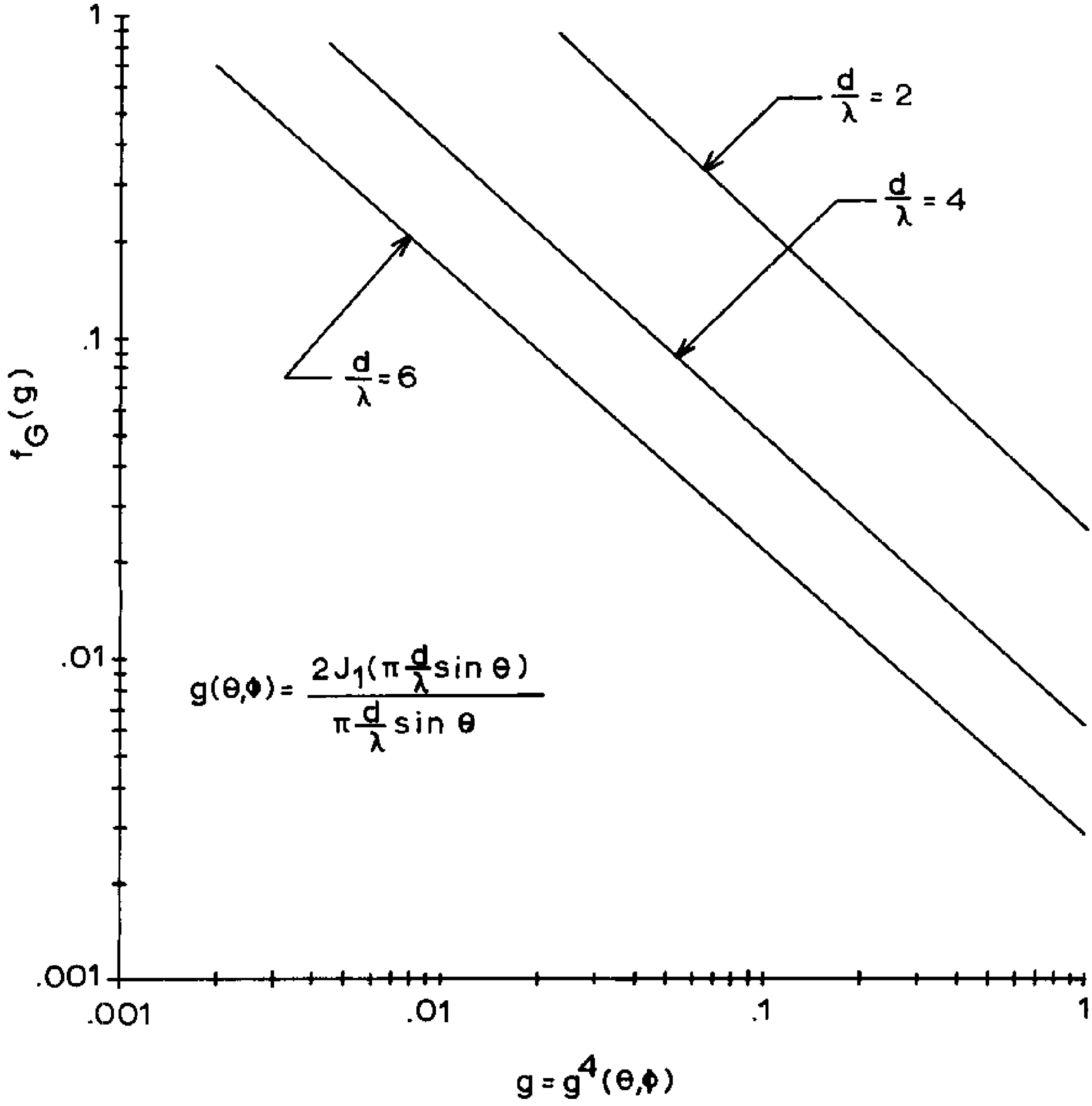

Figure B2. Beam pattern density function for piston transducer and uniform spatial scattering distribution

APPENDIX C

MOMENTS OF $|r(\alpha)|^2$

Let $\{r(\alpha), \alpha \geq t_1\}$ be a zero mean filtered Poisson process defined by

$$r(\alpha) = \sum_{i=1}^{N(\alpha)} z(\alpha, \tau_i, \underset{\sim}{\Theta}_i) \tag{C.1}$$

where $z(\alpha, \tau_i, \underset{\sim}{\Theta}_i)$ is a narrow-band signal for all choices of the parameter vector, $\underset{\sim}{\Theta}$. The signal $z(\alpha, \tau, \underset{\sim}{\Theta})$ can be written in complex envelope notation as

$$z(\alpha, \tau, \underset{\sim}{\Theta}) = \text{Re}[\tilde{z}(\alpha, \tau, \underset{\sim}{\Theta}) e^{j\omega_c \alpha}] \tag{C.2}$$

where

$$\tilde{z}(\alpha, \tau, \underset{\sim}{\Theta}) = \tilde{z}_c(\alpha, \tau, \underset{\sim}{\Theta}) - j\tilde{z}_s(\alpha, \tau, \underset{\sim}{\Theta})$$

The squared envelope of $r(t)$ can be expressed in terms of its two low-pass quadrature components, $r_c(\alpha)$ and $r_s(\alpha)$,

$$|r(\alpha)|^2 = r_c^2(\alpha) + r_s^2(\alpha) \tag{C.3}$$

It can easily be shown using the structure of a quadrature demodulator that

$$r_c(\alpha) = \sum_{i=1}^{N(\alpha)} \tilde{z}_c(\alpha, \tau_i, \underset{\sim}{\Theta}_i)$$

$$r_s(\alpha) = \sum_{i=1}^{N(\alpha)} \tilde{z}_s(\alpha, \tau_i, \underset{\sim}{\Theta}_i) \tag{C.4}$$

The mean and covariance of $|r(\alpha)|^2$ are

$$E[|r(\alpha)|^2] = E[r_c^2(\alpha)] + E[r_s^2(\alpha)] \tag{C.5}$$

$$\begin{aligned} \mathrm{Cov}[|r(\alpha)|^2,|r(\beta)|^2] &= \mathrm{Cov}[r_c^2(\alpha),r_c^2(\beta)] \\ &+ \mathrm{Cov}[r_c^2(\alpha),r_s^2(\beta)] + \mathrm{Cov}[r_s^2(\alpha),r_c^2(\beta)] \\ &+ \mathrm{Cov}[r_s^2(\alpha),r_s^2(\beta)] \end{aligned} \tag{C.6}$$

It can be shown that the joint characteristic function of $r_c(\alpha), r_c(\beta)$, $r_s(\alpha)$ and $r_s(\beta)$ is

$$\begin{aligned} &\varphi_{r_c(\alpha),r_s(\alpha),r_c(\beta),r_s(\beta)}(u_1,u_2,u_3,u_4) = \\ &\exp\left\{\int_{t_1}^{\alpha} \nu(x)E[e^{ju_1\tilde{z}_c(\alpha,x,\underset{\sim}{\theta})+\ldots+ju_4\tilde{z}_s(\beta,x,\underset{\sim}{\theta})}-1]dx\right. \\ &\left.+\int_{\alpha}^{\beta} \nu(x)E[e^{ju_3\tilde{z}_c(\beta,x,\underset{\sim}{\theta})+ju_4z_s(\beta,x,\underset{\sim}{\theta})}-1]dx\right\} \end{aligned} \tag{C.7}$$

where it has been assumed that $\alpha<\beta$ and that the process originated at time t_1. The derivation of (C.7) is almost identical to the proof of Theorem (2.7). The moments of $r_c(\alpha)$ and $r_s(\alpha)$ are obtained using (C.7) and the moment relationships in Appendix A.

$$E[r_c^2(\alpha)] = \int_{t_1}^{\alpha} \nu(x)\tilde{z}_c^2(\alpha,x,\underset{\sim}{\theta})dx \tag{C.8}$$

$$\mathrm{Cov}[r_c^2(\alpha), r_c^2(\beta)] = \int_{t_1}^{\min(\alpha,\beta)} \nu(x)\tilde{z}_c^2(\alpha,x,\underset{\sim}{\Theta})$$

$$\tilde{z}_c^2(\beta,x,\underset{\sim}{\Theta})\,dx + 2\left[\int_{t_1}^{\min(\alpha,\beta)} \nu(x)\tilde{z}_c(\alpha,x,\underset{\sim}{\Theta})\, z_c(\beta,x,\underset{\sim}{\Theta})\,dx\right]^2 \tag{C.9}$$

The expressions for $E[r_s^2(\alpha)]$ and $\mathrm{Cov}[r_c^2(\alpha), r_s^2(\beta)]$, $\mathrm{Cov}[r_s^2(\alpha), r_c^2(\beta)]$ and $\mathrm{Cov}[r_s^2(\alpha), r_s^2(\beta)]$ are found in an analogous way. Using the moment expressions for the quadrature components, it follows that

$$E[|r(\alpha)|^2] = \int_{t_1}^{\alpha} \nu(x)|\tilde{z}(\alpha,x,\underset{\sim}{\Theta})|^2 dx \tag{C.10}$$

and

$$E[|r(\alpha)|^2|r(\beta)|^2] - E[|r(\alpha)|^2]E[|r(\beta)|^2]$$

$$= \int_{t_1}^{\min(\alpha,\beta)} \nu(x)E[|\tilde{z}(\alpha,x,\underset{\sim}{\Theta}|^2|\tilde{z}(\beta,x,\underset{\sim}{\Theta}|^2]dx$$

$$+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(x)E[\tilde{z}_c(\alpha,x,\underset{\sim}{\Theta})\tilde{z}_c(\beta,x,\underset{\sim}{\Theta})]dx\right\}^2$$

$$+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(x)E[\tilde{z}_c(\alpha,x,\underset{\sim}{\Theta})\tilde{z}_s(\beta,x,\underset{\sim}{\Theta})]dx\right\}^2$$

$$+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(x)E[\tilde{z}_s(\alpha,x,\underset{\sim}{\Theta})\tilde{z}_c(\beta,x,\underset{\sim}{\Theta})]dx\right\}^2$$

$$+ 2\left\{\int_{t_1}^{\min(\alpha,\beta)} \nu(x)E[\tilde{z}_s(\alpha,x,\underset{\sim}{\Theta})\tilde{z}_s(\beta,x,\underset{\sim}{\Theta})]dx\right\}^2 \tag{C.11}$$

where

$$\left|\tilde{z}(\alpha,x,\underset{\sim}{\Theta})\right|^2 = \tilde{z}_c^{\,2}(\alpha,x,\underset{\sim}{\Theta}) + \tilde{z}_s^{\,2}(\alpha,x,\underset{\sim}{\Theta})$$

APPENDIX D

EXPRESSION FOR $A_{ij}(t)$

Assume that the transducer has a beam pattern with an elliptical cross section and a uniform intensity across the beam. The cross sections for the i^{th} and j^{th} acoustic pulses at a distance ct/2 from the transducer are shown in Figure D1.

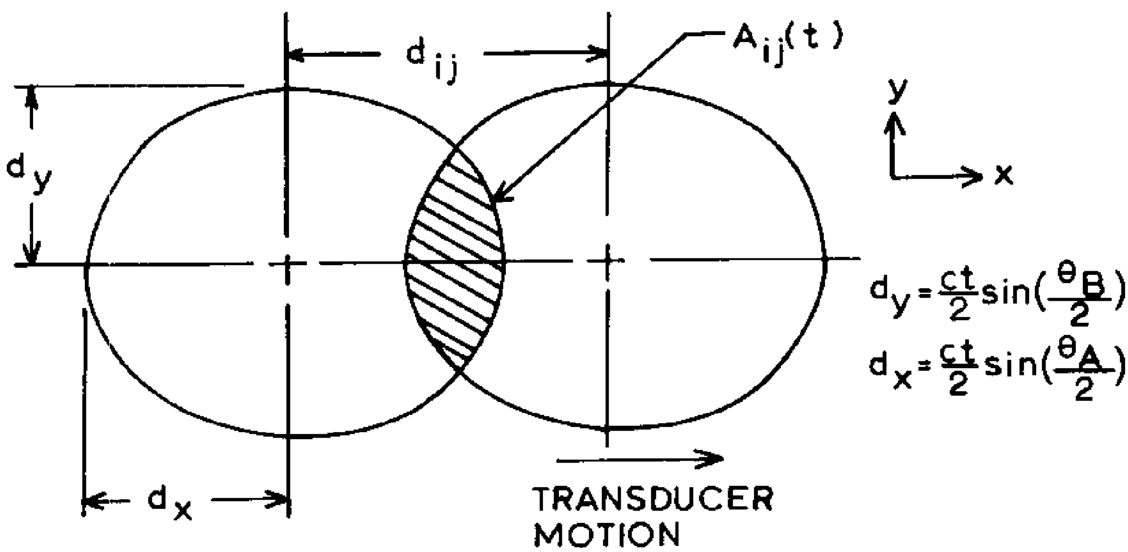

Figure D1. Beam pattern cross sections

If the speed of the transducer is v_T and the time between pulses is T_s, then the distance between the centers of the beam patterns for the i^{th} and j^{th} pulses, d_{ij}, is

$$d_{ij} = v_T T_s |i-j| \tag{D.1}$$

The equation for the ellipse shown in Figure D1 is

$$\frac{x^2}{\left[\frac{ct}{2}\sin\left(\frac{\theta_A}{2}\right)\right]^2} + \frac{y^2}{\left[\frac{ct}{2}\sin\left(\frac{\theta_B}{2}\right)\right]^2} = 1$$

or

$$y = \left[1 - \frac{x^2}{\frac{ct}{2}\sin\left(\frac{\theta_A}{2}\right)}\right]^{\frac{1}{2}} \frac{ct}{2}\sin\frac{\theta_B}{2} \tag{D.2}$$

An integral expresson for $A_{ij}(t)$ is

$$A_{ij}(t) = \frac{4\sin(\theta_B/2)}{\sin(\theta_A/2)} \int_{\frac{d_{ij}}{2}}^{\frac{ct}{2}\sin\left(\frac{\theta_A}{2}\right)} \sqrt{\left(\frac{ct}{2}\right)^2 \sin^2\left(\frac{\theta_A}{2}\right) - x^2}\, dx \tag{D.3}$$

Evaluating the integral, it follows that

$$A_{ij}(t) = \left(\frac{ct}{2}\right)^2 \sin\left(\frac{\theta_A}{2}\right) \sin\left(\frac{\theta_B}{2}\right) \sin^{-1}\left[1 - \left(\frac{|i-j|v_T T_s}{ct\sin(\theta_A/2)}\right)^2\right]^{\frac{1}{2}}$$

$$- \frac{ct}{2}\sin\left(\frac{\theta_B}{2}\right) |i-j| v_T T_s \left[1 - \left(\frac{|i-j|v_T T_s}{ct\sin(\theta_A/2)}\right)^2\right]^{\frac{1}{2}}$$

for $|i-j|v_T T_s \leq ct\sin(\theta_A/2)$

$$= 0 \quad \text{for } |i-j|v_T T_s > ct\sin(\theta_A/2) \tag{D.4}$$