Variance–Covariance Matrix of Transformed GPS Positions: Case Study for the NAD 83 Geodetic Datum

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Abstract: This investigation introduces a rigorous mathematical transformation of variance–covariance matrices between a global geocentric frame and a plate-fixed geodetic frame. A practical example between the geocentric frame of International GNSS Service 2008 (IGS08) epoch 2005.00 and the geodetic frame North American Datum of 1983 (NAD 83) (2011) epoch 2010.00 was implemented. Although the theory is general, the transformation used here is controlled by the assumptions implicit in the definition of NAD 83. However, the same approach could be extended to future definitions of fixed-plate datums used by geodetic organizations for charting and mapping applications. Consequently, because the transformation between these two specific frames is assumed by definition to be a one-to-one errorless transformation, the uncertainties for the 14 Helmert transformation parameters between the two frames are assumed to be zero. Nevertheless, the formulation is complete and applicable to other specific datum-transformation situations. **DOI: 10.1061/(ASCE)SU.1943-5428.0000143.** © *2015 American Society of Civil Engineers.*

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Introduction

This paper expands well-known transformation equations between reference frames. Furthermore, rigorous transformations are extended to the variance–covariance (v–c) matrices of positions after transforming reference frames, which is an important step when one intends to use archived global positioning system (GPS)–determined data referred to a geodetic frame to be readjusted and constrained to a new more accurately defined geocentric reference frame. Not only should the old GPS coordinates or vector components be rigorously transformed between frames and epochs but, similarly, the transformed v–c matrices referred to the geodetic frame must be known to constrain selected fiducial points to a set of accuracies reflecting their position uncertainties on the geodetic frame. The main intent of this exercise was to describe in detail the mathematical theory necessary to achieve this rigorous transformation of v–c matrices between frames. In this particular case, the final frame is a plate-fixed geodetic frame implying that the coordinates referred to this frame are corrected for secular tectonic plate rotations to avoid, as much as possible, changes with time of the geodetic coordinates, a requisite for mapping applications. This investigation is restricted to the specific case of the North American Datum of 1983, epoch 2010.00, as defined in 2011, and hereafter denoted NAD 83. However, exactly the same logic could be applied to other cases where a rigorous relationship between a global GPS-derived frame and a plate-fixed geodetic datum defined for mapping applications is desired.

The particular illustration referred to NAD 83 was executed herein to qualify and validate the theory. This exercise starts with the original v–c matrix referred to a National Geodetic Survey (NGS) realization of the recently released geocentric frame of International GNSS Service 2008 (IGS08) epoch 2005.00 (Rebischung et al. 2012) assumed, for all practical purposes, to be equivalent to the International Terrestrial Reference Frame 2008 (ITRF2008) frame. This original v–c matrix resulted from a multiyear GPS solution of the continuously operating reference stations (CORS) data (Snay and Soler 2008) completed by NGS that contains observations from 1994 to 2011 (Griffiths et al. 2010). The transformed v–c matrix is desired in the NAD 83 geodetic frame. The initial IGS08 v–c matrix was extracted from the published SINEX file of the GPS-multiyear solution as derived by NGS (ftp://cors.ngs.noaa.gov/pub/Wsnx/IGS08-NGS-USED.SNX.Z). The most recent set of 14-parameter transformations between the two frames (IGS08 and NAD 83) was adapted from Pearson and Snay (2013).

The theory for this investigation was partially introduced in two previous papers by Soler and Marshall (2002, 2003); however, for the first time, this theory has been practically implemented for a geodetic datum, in particular the NAD 83 that, as will be seen later, implicitly contains a correction for plate rotation to make it plate-fixed. Over the years, there have been several realizations of the NAD 83 geodetic datum. For a history of the definition and development of NAD 83, consult the work by Snay (2012). This is still the geodetic datum currently adopted by Canada and the United States, and several authors have discussed the peculiarities of transformations involving a so-called plate-fixed geodetic datum such as NAD 83 (Craymer et al. 2000; Anon 2010; Pearson and Snay 2013; Chiu and Shih 2014).

Because the concept of plate-fixed datum is commonly a little fuzzy and somewhat difficult to comprehend, this article will elaborate further on the subject. In fact, as shown later in the article, the plate-fixed datum terminology is a misnomer. The plate

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encompassing the datum is not fixed but rotates according to their inherent geophysical constraints; however, the frame defining the geodetic datum is fixed and does not rotate with the plate.

After the mathematical development is introduced, the equations are particularized for NAD 83. At this stage, the procedures for correcting for the rotation effect of the North American plate are plainly described. This is followed by the derivation of the transformed v–c matrices from IGS08 to NAD 83 relating in detail the simplifications implicit in the definition of the NAD 83. Finally, using the same SINEX file, the general characteristics of the point-to-point, velocity-to-velocity, and point-to-velocity correlations are reviewed.

Mathematical Development

This article expands the basic theory previously given in Soler and Marshall (2003) in order to retain a formulation that avoids unnecessary simplifications and that is finally particularized to NAD 83 subject to the assumptions implicit in the definition of this particular datum. This will be discussed later in the section titled "Application to the NAD 83."

The final general equations to transform coordinates between any two Cartesian reference frames (e.g., IGS08 and NAD 83) can be written using a compact matrix notation as [Soler and Marshall 2003; Eq. (8)]

$$\{x(t_D)\}_{\mathrm{NAD83}} = \{T(t_k)\} + (t_D - t_k)\{\dot{T}\} + \Big[(1 + s(t_k))[\delta\mathfrak{R}] + (t_D - t_k) \times \Big[(1 + s(t_k))\big[\underline{\dot{\varepsilon}}\big]^T + \dot{s}\,[\delta\mathfrak{R}]\Big] + (t_D - t_k)^2 \dot{s}\big[\underline{\dot{\varepsilon}}\big]^T\Big] \times \big\{\{x(t)\}_{\mathrm{IGS08}} + (t_D - t)\{v\}_{\mathrm{IGS08}}\big\} \tag{1}$$

where $t =$ epoch of the coordinates in the initial frame IGS08 (e.g., 2005.00); $t_D =$ epoch of the transformed coordinates in the final frame NAD 83 (e.g., 2010.00); and $t_k =$ epoch at which the seven transformation parameters are given (e.g., 1997.00).

Assuming differential rotations, the matrix $[\delta\mathfrak{R}]$ is given explicitly by

$$[\delta\mathfrak{R}] = \begin{bmatrix} 1 & \varepsilon_z(t_k) & -\varepsilon_y(t_k) \\ -\varepsilon_z(t_k) & 1 & \varepsilon_x(t_k) \\ \varepsilon_y(t_k) & -\varepsilon_x(t_k) & 1 \end{bmatrix} = [I] + \big[\underline{\varepsilon(t_k)}\big]^T \tag{2}$$

The compact notation used herein to represent skew-symmetric (antisymmetric) matrices is

$$\big[\underline{\dot{\varepsilon}}\big]^T = \begin{bmatrix} 0 & \dot{\varepsilon}_z & -\dot{\varepsilon}_y \\ -\dot{\varepsilon}_z & 0 & \dot{\varepsilon}_x \\ \dot{\varepsilon}_y & -\dot{\varepsilon}_x & 0 \end{bmatrix} \tag{3}$$

where the superscript $T =$ transpose; $[I] =$ a 3×3 unit matrix; and $[\underline{\varepsilon}]^T =$ a skew-symmetric matrix containing the differential rotations around the three axes of the IGS08 frame. To complete the description of Eq. (1), it should be mentioned that all 3×3 matrices are represented between brackets, 3×1 column vectors between braces, e.g., $\{x\} = \{x_1\ x_2\ x_3\}^T \equiv \{x\ y\ z\}^T$, and scalars between parentheses. Eq. (1) is consistent with counterclockwise (or anticlockwise) positive rotations of the IGS08 frame around the x_1, x_2, and x_3 axes by angular amounts ε_x, ε_y, and ε_z (expressed in radians), respectively. The transformation parameters involved in Eq. (1) are the so-called Helmert transformation parameters (three shifts, T_x, T_y, and T_z, three differential rotations, ε_x, ε_y, and ε_z, and one differential scale factor s) and their derivatives with respect to time, all of them given in the sense IGS08 to NAD 83, written symbolically as IGS08 $\rightarrow$ NAD 83. The time derivatives of the parameters are marked, as usual, with a dot. This abridged symbolic matrix notation will become particularly advantageous later when taking the partial derivatives to determine the elements of the Jacobian matrix required in the error propagation process. Neglecting in Eq. (1) the terms with products $s\big[\underline{\dot{\varepsilon}}\big]^T$, $\dot{s}\big[\underline{\varepsilon}\big]^T$, and $\dot{s}\big[\underline{\dot{\varepsilon}}\big]^T$, the expression simplifies to

$$\{x(t_D)\}_{\mathrm{NAD83}} = \{T(t_k)\} + (t_D - t_k)\{\dot{T}\} + \Big[(1 + s(t_k))[\delta\mathfrak{R}] + (t_D - t_k)\Big[\big[\underline{\dot{\varepsilon}}\big]^T + \dot{s}[I]\Big]\Big] \times \big\{\{x(t)\}_{\mathrm{IGS08}} + (t_D - t)\{v\}_{\mathrm{IGS08}}\big\} \tag{4}$$

An abbreviated form of Eq. (4) after neglecting the products of the time derivatives by the velocities and $s[\underline{\varepsilon}]^T$ was given by Altamimi et al. [2002, Eq. (A9)]; however, the sign convention used for the rotations was not unequivocally stated (see Soler 1997). In order to avoid any possible confusion, the clarification of the nomenclature used when applying rotations is very important and, as is emphasized later, critical when frame rotations and geophysical tectonic plate rotations affecting plate-fixed datums are combined into a unique equation.

The components of the velocity vector appearing on the right-hand side of Eq. (4) can be computed by the following equation, assuming only the effect of the (Euler) rotation of an arbitrary tectonic plate p:

$$\{v\}_{\mathrm{IGS08}} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}_{\mathrm{IGS08}} \approx \big[\underline{\dot{\Omega}}\big]_p \{x\}_{\mathrm{IGS08}} = \begin{bmatrix} 0 & -\dot{\Omega}_z & \dot{\Omega}_y \\ \dot{\Omega}_z & 0 & -\dot{\Omega}_x \\ -\dot{\Omega}_y & \dot{\Omega}_x & 0 \end{bmatrix}_p \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{\mathrm{IGS08}} \tag{5}$$

Here, the column vector $\{v\}$ contains the three components of the velocity of a point (along the x-axis, y-axis, and z-axis of IGS08) whose position vector is $\{x\}$. This velocity is the resultant of an infinitesimal angular rotation of amount $|\vec{\dot{\Omega}}|$ about an axis going through the origin of the IGS08 frame intersecting the rotation (Euler) pole of the tectonic plate. The elements of $\big[\underline{\dot{\Omega}}\big]_p$ have units of rad/year and contain the angular velocity components $\{\dot{\Omega}\}_p$ for the particular plate p on which the point is located. These components are given, for example, by McCarthy (1996, p. 14) in units of rad/Myear for several macroplates for the geophysical model (no net rotation) NNR-NUVEL-1A. The spherical longitude (λ) and latitude (ϕ) of the axis along the vector $\{\dot{\Omega}\}$ defining the rotation pole is straightforward from the equations

$$\lambda = \arctan\frac{\dot{\Omega}_y}{\dot{\Omega}_x};\quad 0 \le \lambda \le 2\pi \tag{6}$$

$$\phi = \arctan\frac{\dot{\Omega}_z}{\sqrt{\dot{\Omega}_x^2 + \dot{\Omega}_y^2}};\quad -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \tag{7}$$

A note of caution, Eq. (5) is consistent with the nomenclature popularized in the geophysical literature that uses a counterclockwise rotation of geocentric vectors [frame remains fixed ⇄ points (the tip of the vector) move relative to the frame]. However, in the geodetic literature it is customary to use counterclockwise rotation of axes (points remain fixed ⇄ frames rotate). These two types of rotations are opposite in sign. In other words, a counterclockwise rotation of geocentric vectors is equivalent to a clockwise rotation of axes and vice versa (see Soler 1998; Soler and Marshall 2002). Furthermore, Eq. (6) gives a general analytical expression for the longitude value as a function of the angular velocity components. However, when it comes to numerical computations, the signs of the input components should be used to determine the proper quadrant of the resulting longitude value.

Application to the NAD 83

Here, Eq. (5) is particularized for the specific case of the NAD 83. Fig. 1 will help to understand how tectonic plate motions are taken into consideration in the computation of coordinates referred to three-dimensional geodetic datums such as NAD 83. The 14 transformation parameters described before will help to transform coordinates (and velocities) from the IGS08 frame at epoch t to the geodetic datum frame (e.g., NAD 83) at epoch t_D. However, in the interval $(t_D - t)$, owing to tectonic plate rotations, the spatial location of an arbitrary point A(t) will move to point B(t_D) by an amount $(t_D - t)\left[\underline{\dot{\Omega}}\right]_p \{x(t)\}_{\mathrm{IGS08}}$. Historically, datum coordinates have been considered fixed in time to avoid the necessity of constantly updating maps and surveys to account for plate rotations. The achievement of this presumption has been recently made more difficult by the introduction of accurate geospatial technologies, such as, GPS and the high accuracy with which spatial point displacements caused by continental tectonic plate velocities can be currently determined. However, owing to mapping considerations and other inherent practical issues, geodetic datum coordinates are still assumed largely unaffected by secular tectonic plate rotations. Consequently, to assure that geodetic datum points retain as constant coordinates as possible, a correction opposite to the displacement caused by the plate rotation during the interval $(t_D - t)$ is implemented—in essence, the expression $(t_D - t)\left[\underline{\dot{\Omega}}\right]_p^T \{x(t)\}_{\mathrm{IGS08}}$ in Fig. 1. Recall that the geodetic frame $(x_{1D}, x_{2D}, x_{3D})_{\mathrm{NAD83}}$ always remains fixed, attached to the Earth, and rotating with it, whereas distinct rigid spherical tectonic plates, comprising the crust, rotate individually on its surface. To avoid, as much as possible, changes to the positional coordinates [referred to the $(x_{1D}, x_{2D}, x_{3D})_{\mathrm{NAD83}}$ frame] of the points located on the surface of a moving plate, a correction to bring back the location of point B(t_D) as close as possible to the original point A(t) [resulting in $A_D(t_D)$ in Fig. 1] is applied to compensate for the displacement caused by plate motion. To compute this correction, as mentioned before, a geophysical tectonic plate rotation model is adopted by convention. This model, which in this context is assumed to have error-free rotations, should be retained as long as the basic definition of the geodetic datum is not changed in order to keep a set of consistent plate-fixed coordinates. In reality, the plate has moved physically in space, i.e., point A has moved spatially to point B; however, the coordinates of point B are then corrected for the motion of the plate and brought back as close as possible to the original position of point A. That way, the locations of point A at time t and time t_D have practically the same coordinates when referred to the datum frame $(x_{1D}, x_{2D}, x_{3D})_{\mathrm{NAD83}}$. To further clarify the concept, it should be mentioned that, presently, NGS has defined a unique geodetic datum frame, NAD 83, which is based on the IGS08 frame. Depending on the location of the NAD 83 GPS-observed points on the surface of the Earth, NGS applies one of three tectonic plate rotation corrections (for North America, Pacific, and Mariana plates) that use the NNR-NUVEL-1A geophysical model velocities (see Pearson and Snay 2013). Since 1998, the Geodetic Survey Division of Canada and the NGS of the United States have adopted, by convention, the plate angular rotation components given in the geophysical model NNR-NUVEL-1A as errorless. At the time, no stochastic values for these parameters were available (Craymer et al. 2000). The plate rotation model cannot be truly errorless, possibly leading to a set of residual rotational velocities in NAD 83, although the magnitude of these residuals should not severely impact the main conclusions of this investigation.

To qualify the statements made in the preceding sections, the specific case of the regional NAD 83 datum covering the contiguous United States is considered here. Implicit in the definition of this geodetic datum and to take account of the North

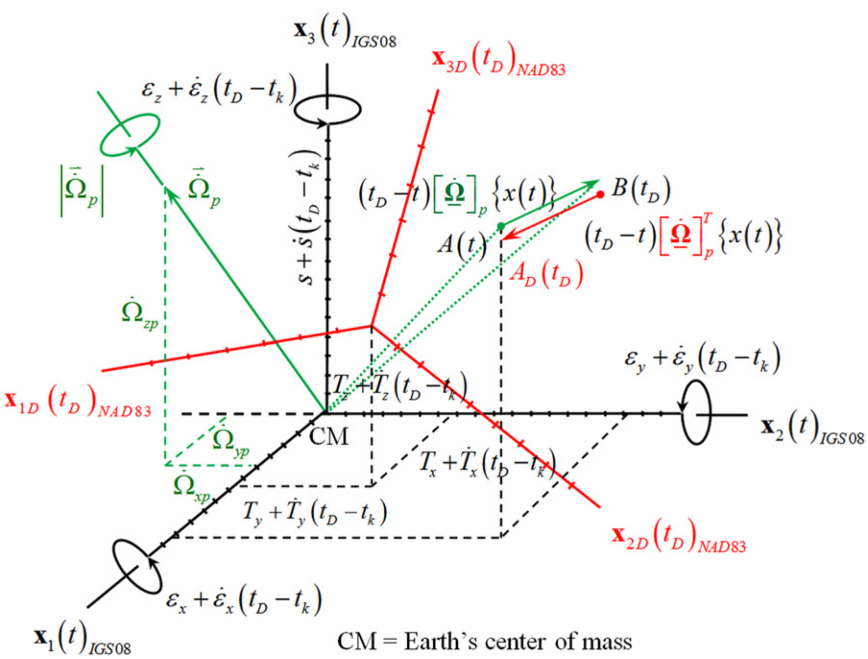

Fig. 1. Correction for tectonic plate rotations in geodetic datum coordinates

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American plate rotation, the transformation ITRF96→NAD 83 includes time variations of the rotations required to reach the NAD 83 frame that are based on the rotation rates of the NNR-NUVEL-1A geophysical plate rotation model. For example, following Pearson and Snay (2013, Table 7), it is possible to define the selected counterclockwise rotations of the transformation ITRF2008→NAD 83 written step-by-step as follows:

$$
\begin{aligned}
\text{ITRF 2008} \to \text{NAD 83} = {} & (\text{ITRF2008} \to \text{ITRF2005}) \\
& + (\text{ITRF2005} \to \text{ITRF2000}) \\
& + (\text{ITRF2000} \to \text{ITRF97}) \\
& + (\text{ITRF97} \to \text{ITRF96}) \\
& + (\text{ITRF96} \to \text{NAD83})
\end{aligned}
$$

Concentrating exclusively on the time variations of the three rotations along the ITRF 2008 axes given at some common epoch $t_k = 1997.00$ (in units of mas/year), one can write

$$
\underset{\substack{\text{ITRF 2008}\to\\ \text{NAD 83}}}{\begin{Bmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \dot{\varepsilon}_z \end{Bmatrix}} = \underset{\substack{\text{ITRF 2008}\to\\ \text{ITRF 2005}}}{\begin{Bmatrix} 0.00 \\ 0.00 \\ 0.00 \end{Bmatrix}} + \underset{\substack{\text{ITRF 2005}\to\\ \text{ITRF 2000}}}{\begin{Bmatrix} 0.00 \\ 0.00 \\ 0.00 \end{Bmatrix}} + \underset{\substack{\text{ITRF 2000}\to\\ \text{ITRF 97}}}{\begin{Bmatrix} 0.00 \\ 0.00 \\ -0.020 \end{Bmatrix}}
$$

$$
+ \underset{\substack{\text{IGS 97}\to\\ \text{IGS 96}}}{\begin{Bmatrix} +0.01347 \\ -0.01514 \\ +0.00027 \end{Bmatrix}} + \underset{\substack{\text{ITRF 96}\to\\ \text{NAD 83}}}{\begin{Bmatrix} +0.0532 \\ -0.7423 \\ -0.0316 \end{Bmatrix}} = \begin{Bmatrix} \dot{\varpi}_x \\ \dot{\varpi}_y \\ \dot{\varpi}_z \end{Bmatrix} + \begin{Bmatrix} \dot{\Omega}_x \\ \dot{\Omega}_y \\ \dot{\Omega}_z \end{Bmatrix}
$$

$$
= \begin{Bmatrix} +0.01347 \\ -0.01514 \\ -0.01973 \end{Bmatrix} + \begin{Bmatrix} +0.0532 \\ -0.7423 \\ -0.0316 \end{Bmatrix}_{\text{NUVEL}-1\text{A}} = \begin{Bmatrix} +0.06667 \\ -0.75744 \\ -0.05133 \end{Bmatrix} \tag{8}
$$

The components of the vector $\{\dot{\Omega}\}$ shown above are exactly the conversion from rad/Myear given in McCarthy (1996, Table 3.2) to mas/year for the tectonic plate model NNR-NUVEL-1A. Accordingly, the value of the skew-symmetric matrix $[\underline{\dot{\varepsilon}}]^T$ to be used in Eq. (4), in rad/year, is

$$
[\underline{\dot{\varepsilon}}]^T = \begin{bmatrix} 0 & \dot{\varpi}_z & -\dot{\varpi}_y \\ -\dot{\varpi}_z & 0 & \dot{\varpi}_x \\ \dot{\varpi}_y & -\dot{\varpi}_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & \dot{\Omega}_z & -\dot{\Omega}_y \\ -\dot{\Omega}_z & 0 & \dot{\Omega}_x \\ \dot{\Omega}_y & -\dot{\Omega}_x & 0 \end{bmatrix}
$$

$$
= [\underline{\dot{\varpi}}]^T + [\underline{\dot{\Omega}}]_p^T \tag{9}
$$

These are the values implicit in the formulation given in Soler and Snay (2004) and in the Appendix of Pearson and Snay (2013) although using an alternative form of Eq. (4). These last two papers, after adopting the notation introduced herein, use the short form

$$
\{x(t_D)\}_{\text{NAD83}} = \{T(t_D)\} + (1 + s(t_D))\{x(t_D)\}_{\text{IGS08}} + [\underline{\varepsilon(t_D)}]^T\{x(t_D)\}_{\text{IGS08}} \tag{10}
$$

However, as previously discussed, the transformation parameters are always given at some arbitrary epoch, e.g., $t_k =$ 1997.00, which is different from t_D. Thus, to implement Eq. (10) it is necessary to compute the transformation parameters at epoch t_D as follows:

$$
\{T(t_D)\} = \{T(t_k)\} + (t_D - t_k)\{\dot{T}\} \tag{11a}
$$

$$
[\underline{\varepsilon(t_D)}]^T = [\underline{\varepsilon(t_k)}]^T + (t_D - t_k)[\underline{\dot{\varepsilon}}]^T \tag{11b}
$$

$$
\{s(t_D)\} = s(t_k) + (t_D - t_k)\dot{s} \tag{11c}
$$

Note that the sense of the rotations implicit throughout this paper, as adopted by NGS, contrasts with the clockwise positive rotation selected, for example, by Craymer et al. (2000), where the assumptions $\dot{s} = 0$, $[\underline{\dot{\varepsilon}}]^T = [0]$, and $\dot{s} = 0$ were introduced. There is one more consideration that should be mentioned in conjunction with Eq. (10), namely that the three coordinates of the vector $\{x(t_D)\}_{\text{IGS08}}$ should be known as indicated at epoch t_D. However, originally they are given at the initial epoch t. These coordinates could be updated as follows:

$$
\{x(t_D)\}_{\text{IGS08}} = \{x(t)\}_{\text{IGS08}} + (t_D - t)\{v\}_{\text{IGS08}} \tag{12}
$$

Therefore, Eq. (10) after substituting Eqs. (11) and (12) is equivalent to Eq. (4). The main advantage of Eq. (4) is, perhaps, that a single equation explicitly provides a full view of the transformation of coordinates involved and that the terms of the order $s[\underline{\varepsilon}]^T$ were not neglected. Eq. (10), implemented as described above, comprises the conceptual original version of the software horizontal time-dependent positioning (*HTDP, version 3.1, NGS*). When this program was initially released in 1992, the value of the velocities of the points required in Eq. (12) were computed using Eq. (5) and the NNR-NUVEL-1A rigid plate rotation model. However, over the years many improvements and refinements have been incorporated into this user-friendly utility until reaching its current version 3.2 (see Snay et al. 2013).

The velocities referred to the NAD 83 frame could be obtained by taking the derivatives of Eq. (4) with respect to t_D, resulting in

$$
\{v\}_{\text{NAD83}} = \{\dot{T}\} + \left[[\underline{\dot{\varepsilon}}]^T + \dot{s}[I]\right]\{x(t)\}_{\text{IGS08}} + \left[[(1 + s(t_k))[\delta\mathfrak{R}]] + (2t_D - t_k - t)\left[[\underline{\dot{\varepsilon}}]^T + \dot{s}[I]\right]\right]\{v\}_{\text{IGS08}} \tag{13}
$$

Eq. (13) given here in explicit matrix form is equivalent to the equation in Altamimi et al. (2011), except that these authors neglect smaller-than-second-order products, assuming $s(t_k)\{v\}_{\text{IGS08}} = [\underline{\dot{\varepsilon}}]^T\{v\}_{\text{IGS08}} = \dot{s}[I]\{v\}_{\text{IGS08}} = \{0\}$. Notice that with these substitutions the above equation also simplifies to the formulation given in Soler and Snay (2004), namely

$$
\{v\}_{\text{NAD83}} = \{\dot{T}\} + \left[[\underline{\dot{\varepsilon}}]^T + \dot{s}[I]\right]\{x(t)\}_{\text{IGS08}} + \{v\}_{\text{IGS08}} \tag{14}
$$

If it is assumed that the velocities of the surveyed stations are only affected by the rotation of a particular plate p where the mark is located, then after substituting Eq. (5) into Eq. (4), one can write

$$
\{x(t_D)\}_{\text{NAD83}} = \{T\} + (t_D - t_k)\{\dot{T}\}\left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[[\underline{\dot{\varepsilon}}]^T + \dot{s}[I]\right]\right]\left\{\{x(t)\}_{\text{IGS08}} + (t_D - t)[\underline{\dot{\Omega}}]_p\{x(t)\}_{\text{IGS08}}\right\} \tag{15}
$$

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Notice that because according to Eq. (9) $\left[\underline{\dot{\varepsilon}}\right]^T = \left[\underline{\dot{\varpi}}\right]^T + \left[\underline{\dot{\Omega}}\right]_p^T$, the transformation implicit in the software *HTDP* to transform between IGS08 and NAD 83 applies a clockwise rotation of the vector $\{x(t)\}_{\mathrm{IGS08}}$ by the amount $(t_D - t_k)\left[\underline{\dot{\Omega}}\right]_p^T \{x(t)\}_{\mathrm{IGS08}}$, which is compensated by the counterclockwise rotation of the vector $\{x(t)\}_{\mathrm{IGS08}}$, $(t_D - t_k)\left[\underline{\dot{\Omega}}\right]_p \{x(t)\}_{\mathrm{IGS08}}$, where it is assumed that the displacement of the point is only due to the velocity generated by the rotation of the plate. In essence, this implies that the physical rotation of the plate is approximately canceled by a rotation in the opposite sense implicit in the $\left[\underline{\dot{\varepsilon}}\right]^T$ matrix [see Eqs. (8) and (9)]. Thus, in reality, the location of the point is moved back to its original position in the NAD 83 frame after the corrections caused by the 14 transformation parameters separating the two frames are applied. One more note of caution: in this reasoning it has been assumed that the displacement of the point is only affected by the angular velocity of rotation of the North American plate using the NNR-NUVEL-1A. However, *HTDP* has two options: (1) to use as velocities the default value as modeled by the software; (2) to explicitly include the velocities as selected by the user. In either case *HTDP* continues to use for the rotation rates in Eq. (11) the assumptions explicitly stated in Eq. (8), which clearly shows the dependence on the NNR-NUVEL-1A model. One more clarification about *HTDP* should be mentioned. In active tectonic areas the current version of *HTDP* introduces models to correct for displacements caused by earthquakes using the parameters of geophysically determined dislocation models as well as other improvements (Snay et al. 2013). These predicted velocities at each point in tectonically active regions could also be determined independently using *HTDP*. To compute the transformation of v–c matrices in this investigation, it was assumed that the only displacements of the points caused by the velocities with respect to the IGS08 frame were those generated by the rotation of the North American plate using the NNR-NUVEL-1A geophysical model. As can be seen in Eq. (8), these are the major secular contributors to the velocities, primarily on the y-component; accordingly, other refinements to obtain the real value of the velocities were neglected.

Eq. (15) could also be written as

$$\{x(t_D)\}_{\mathrm{NAD83}} = \{T\} + (t_D - t_k)\{\dot{T}\} + \left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}[I]\right]\right]\left[[I] + (t_D - t)\left[\underline{\dot{\Omega}}\right]_p\right]\{x(t)\}_{\mathrm{IGS08}} \quad (16)$$

or alternatively

$$\{x(t_D)\}_{\mathrm{NAD83}} = \{T\} + (t_D - t_k)\{\dot{T}\} + \left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}[I]\right]\right]\{x(t)\}_{\mathrm{IGS08}} + \left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}[I]\right]\right](t_D - t)\left[\underline{\dot{\Omega}}\right]_p\{x(t)\}_{\mathrm{IGS08}} \quad (17)$$

The velocities could be obtained after taking the derivatives with respect to t_D of the above equation. To facilitate taking the derivatives, Eq. (17) can be rewritten as follows:

$$\{x(t_D)\}_{\mathrm{NAD83}} = \{T\} + (t_D - t_k)\{\dot{T}\} + \left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}\,[I]\right]\right]\{x(t)\}_{\mathrm{IGS08}} + \left[(t_D - t)(1+s)[\delta\mathfrak{R}] + (t_D - t)(t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}\,[I]\right]\right]\left[\underline{\dot{\Omega}}\right]_p\{x(t)\}_{\mathrm{IGS08}} \quad (18)$$

or finally

$$\{x(t_D)\}_{\mathrm{NAD83}} = \{T\} + (t_D - t_k)\{\dot{T}\} + \left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}\,[I]\right]\right]\{x(t)\}_{\mathrm{IGS08}} + \left[(t_D - t)(1+s)[\delta\mathfrak{R}] + (t_D^2 - t_D t_k - t_D t + t t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}\,[I]\right]\right]\left[\underline{\dot{\Omega}}\right]_p\{x(t)\}_{\mathrm{IGS08}} \quad (19)$$

After taking the derivatives of the above equation with respect to t_D, the following general expression for the transformation of velocities is arrived at:

$$\{v\}_{\mathrm{NAD83}} = \{\dot{T}\} + \left[\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}[I]\right] + \left[(1+s)[\delta\mathfrak{R}] + (2t_D - t_k - t)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}[I]\right]\right]\left[\underline{\dot{\Omega}}\right]_p\right]\{x(t)\}_{\mathrm{IGS08}} \quad (20)$$

Notice that Eq. (20) reduces to Eq. (14) if one neglects the products $\left[\underline{\varepsilon}\right]^T\left[\underline{\dot{\Omega}}\right]$, $\left[\underline{\dot{\varepsilon}}\right]^T\left[\underline{\dot{\Omega}}\right]$, $s\left[\underline{\varepsilon}\right]^T$, $s\left[\underline{\dot{\Omega}}\right]_p$, and $\dot{s}\left[\underline{\dot{\Omega}}\right]_p$. Eqs. (16) and (20) are considered the mathematical models of the transformations pertaining to position to be used in the development of the v–c matrices.

Variance-Covariance Matrix Error Propagation

The main intent of this section is to advance a rigorous theoretical formalism for transforming v–c matrices of coordinates between geocentric and datum frames using Eq. (16). In the case at hand, the IGS08, epoch t, multiyear solution v–c matrix (SINEX file) is being transformed to the NAD 83 (2011) frame, epoch t_D, using the 14 parameters at epoch t_k connecting the two frames.

Mathematically, and according to the well-known error propagation law (see Mikhail and Ackermann 1976), one can write

$$\Sigma_{\mathrm{NAD83}} = [J]\,\Sigma_{\mathrm{IGS08}}\,[J]^T \quad (21)$$

where the a priori known symmetric v–c matrix Σ_{IGS08} is of the form

$$\Sigma_{\mathrm{IGS08}} = \begin{bmatrix} \Sigma_C & \Sigma_{CV} & \Sigma_{CP} & \Sigma_{C\dot{P}} \\ \Sigma_{VC} & \Sigma_V & \Sigma_{VP} & \Sigma_{V\dot{P}} \\ \Sigma_{PC} & \Sigma_{PV} & \Sigma_P & \Sigma_{P\dot{P}} \\ \Sigma_{\dot{P}C} & \Sigma_{\dot{P}V} & \Sigma_{\dot{P}P} & \Sigma_{\dot{P}} \end{bmatrix}_{(6n+14)\times(6n+14)} \quad \text{(symmetric)} \quad (22)$$

where the subindices $C =$ coordinates; $V =$ velocities; $P =$ transformation parameters; and $\dot{P} =$ rate of change of P. For clarity, the explicit form of the symmetric matrix Σ_C is given below:

$$\Sigma_C = \begin{bmatrix} \Sigma_{x_1} & \Sigma_{x_1x_2} & \cdots & \Sigma_{x_1x_n} \\ \Sigma_{x_2x_1} & \Sigma_{x_2} & \cdots & \Sigma_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{x_nx_1} & \Sigma_{x_nx_2} & \cdots & \Sigma_{x_n} \end{bmatrix}_{(3n\times3n)} \quad \text{(symmetric)} \quad (23)$$

where

$$\Sigma_{x_i} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 \end{bmatrix}_i; \quad \Sigma_{x_ix_j} = \begin{bmatrix} \sigma_{x_ix_j} & \sigma_{x_iy_j} & \sigma_{x_iz_j} \\ \sigma_{y_ix_j} & \sigma_{y_iy_j} & \sigma_{y_iz_j} \\ \sigma_{z_ix_j} & \sigma_{z_iy_j} & \sigma_{z_iz_j} \end{bmatrix} \quad (24)$$

Notice that for $i \neq j = 1, \ldots, n$ ($n =$ total number of points) $\Sigma_{x_jx_i} = \Sigma_{x_ix_j}^T$. Although not given here explicitly, the same logic applies to the matrix Σ_V, which is also $(3n \times 3n)$ and can be written from Eq. (23) after replacing the subindex x with v. The above arguments could be extended to the cross-covariance matrix Σ_{CV}, which also fulfills the property $\Sigma_{VC} = \Sigma_{CV}^T$.

The v–c matrix of the transformation parameters can be explicitly written as

$$\Sigma_P = \begin{bmatrix} \sigma^2_{T_x} & \sigma_{T_xT_y} & \sigma_{T_xT_z} & \sigma_{T_x\varepsilon_x} & \sigma_{T_x\varepsilon_y} & \sigma_{T_x\varepsilon_z} & \sigma_{T_xs} \\ & \sigma^2_{T_y} & \sigma_{T_yT_z} & \sigma_{T_y\varepsilon_x} & \sigma_{T_y\varepsilon_y} & \sigma_{T_y\varepsilon_z} & \sigma_{T_ys} \\ & & \sigma^2_{T_z} & \sigma_{T_z\varepsilon_x} & \sigma_{T_z\varepsilon_y} & \sigma_{T_z\varepsilon_z} & \sigma_{T_zs} \\ & & & \sigma^2_{\varepsilon_x} & \sigma_{\varepsilon_x\varepsilon_y} & \sigma_{\varepsilon_x\varepsilon_z} & \sigma_{\varepsilon_xs} \\ & & & & \sigma^2_{\varepsilon_y} & \sigma_{\varepsilon_y\varepsilon_z} & \sigma_{\varepsilon_ys} \\ & & & & & \sigma^2_{\varepsilon_z} & \sigma_{\varepsilon_zs} \\ \text{sym} & & & & & & \sigma^2_s \end{bmatrix}_{(7\times 7)} \tag{25}$$

An equation similar to Eq. (25) applies to $\Sigma_{\dot{P}}$.

In the majority of practical cases, some of the cross-covariances in Eq. (22) are not known and are assumed to be zero; thus, independent of matrix dimensions

$$\Sigma_{CP} = \Sigma_{C\dot{P}} = \Sigma_{VP} = \Sigma_{V\dot{P}} = \Sigma_{P\dot{P}} = [0] \tag{26}$$

Furthermore, the block matrices Σ_P and $\Sigma_{\dot{P}}$ should be full matrices with corresponding cross-covariance matrix $\Sigma_{P\dot{P}}$. However, the international scientific agencies disseminating the values of the 14 transformation parameters do not publish even the diagonal elements of Σ_P and $\Sigma_{\dot{P}}$. To benefit v–c analyses, all elements of matrices Σ_P, $\Sigma_{\dot{P}}$, and $\Sigma_{P\dot{P}}$ should be made available; this will be the only way to rigorously know the full impact of assuming zero the v-c matrices of the 14 transformation parameters and their time derivatives on the results. Furthermore, the current SINEX format does not include specifications for these estimated parameters. In other words, ITRF2008 should include the full v–c matrix of the 14 transformation parameters with respect to the previous frames of the series ITRFxxxx.

The Jacobian matrix $[J]$ of Eq. (21) involves two mathematical models, one related to the positions and the other to the velocities —basically, Eqs. (16) and (20). Eq. (16) may be expressed as the compact functional relationship:

$$\mathfrak{I} = \mathfrak{I}\left(C, V, P, \dot{P}\right) \tag{27}$$

Similarly, Eq. (20) takes the functional form

$$\mathfrak{A} = \mathfrak{A}\left(C, V, P, \dot{P}\right) \tag{28}$$

Consequently, because only the computation of the v–c matrix of the positions is the aim of the present investigation, the Jacobian $[J]$ required in Eq. (21) is composed of the following submatrices of partial derivatives:

$$\begin{aligned}[J] &= \left[[\partial\mathfrak{I}/\partial C]_{(3n\times 3n)} \quad [\partial\mathfrak{I}/\partial V]_{(3n\times 3n)} \quad [\partial\mathfrak{I}/\partial P]_{(3n\times 7)} \quad [\partial\mathfrak{I}/\partial \dot{P}]_{(3n\times 7)} \right]_{(3n)\times(3n+14)} \\ &= \left[[\mathfrak{I}_C]_{(3n\times 3n)} \quad [\mathfrak{I}_V]_{(3n\times 3n)} \quad [\mathfrak{I}_P]_{(3n\times 7)} \quad [\mathfrak{I}_{\dot{P}}]_{(3n\times 7)} \right]_{(3n)\times(3n+14)} \\ &= \left[[\mathfrak{I}_{CV}] \quad [\mathfrak{I}_{P\dot{P}}] \right]_{(3n)\times(3n+14)} \end{aligned} \tag{29}$$

Contribution of Eq. (16) to the Jacobian

Eq. (16) can be represented explicitly by the following functional relationship:

$$\begin{aligned} X = \mathfrak{I}(Y) = \mathfrak{I}\,(&x_i, y_i, z_i, v_{xi}, v_{yi}, v_{zi}, T_x, T_y, T_z, \varepsilon_x, \varepsilon_y, \varepsilon_z, s, \\ &\dot{T}_x, \dot{T}_y, \dot{T}_x, \dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_z, \dot{s}) \end{aligned} \tag{30}$$

where $i = 1, \ldots, n$ ($n =$ total number of points in the transformation). The partial derivatives of Eq. (16) with respect to all the parameters given in the right-hand side of Eq. (30) must be determined to know the corresponding contribution to the Jacobian matrix $[J]$ of Eq. (29). To facilitate this computation, the following symbolic vector and matrix partial differentiation definitions are introduced:

$$\partial(\{x\})/\partial\{x\} = [I] \tag{31}$$

For any arbitrary 3×3 scalar matrix $[A]$

$$\partial([A]\{x\})/\partial\{x\} = [A] \tag{32}$$

and finally

$$\partial\left([\underline{\varepsilon}]^T\{x\}\right)/\partial\{\varepsilon\} = \partial\left(-[\underline{x}]^T\{\varepsilon\}\right)/\partial\{\varepsilon\} = \partial\left([\underline{x}]\{\varepsilon\}\right)/\partial\{\varepsilon\} = [\underline{x}] \tag{33}$$

$$\partial\left([\underline{\varepsilon}]^T\left[\underline{\dot{\Omega}}\right]_p\{x\}\right)/\partial\{\varepsilon\} = \partial\left([\underline{\varepsilon}]^T\{a\}\right)/\partial\{\varepsilon\} = [\underline{a}] \tag{34}$$

$$\partial\left([\underline{\dot{\varepsilon}}]^T\left[\underline{\dot{\Omega}}\right]_p\{x\}\right)/\partial\{\dot{\varepsilon}\} = \partial\left([\underline{\dot{\varepsilon}}]^T\{a\}\right)/\partial\{\dot{\varepsilon}\} = [\underline{a}] \tag{35}$$

where the components of vector $\vec{a} = (a_1, a_2, a_3)$ required in the skew-symmetric matrices of Eqs. (34) and (35) are explicitly given by

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \left[\underline{\dot{\Omega}}\right]_p\{x\} = \begin{Bmatrix} -\dot{\Omega}_z y + \dot{\Omega}_y z \\ \dot{\Omega}_z x - \dot{\Omega}_x z \\ -\dot{\Omega}_y x + \dot{\Omega}_x y \end{Bmatrix} \tag{36}$$

In the following derivations and to simplify the notation, the identities $\{x(t)\} \equiv \{x\}$ and $\{v(t)\} \equiv \{v\}$ are introduced. Taking partial derivatives of Eq. (16) with respect to the 14 parameters, after using the above definitions and recalling that $\left[\underline{\dot{\Omega}}\right]_p\{x(t)\} = \{v\}$, the following expressions are obtained:

$$\begin{aligned} \partial\mathfrak{I}/\partial\{x\} &= \left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[[\underline{\dot{\varepsilon}}]^T + \dot{s}[I]\right]\right]\left[[I] + (t_D - t)\left[\underline{\dot{\Omega}}\right]_p\right] \\ &= [\partial x] \end{aligned} \tag{37}$$

$$\partial\mathfrak{I}/\partial\{v\} = (t_D - t)\left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}[I]\right]\right] = [\partial v] \tag{38}$$

$$\partial\mathfrak{I}/\partial\{T\} = [I] \tag{39}$$

$$\partial\mathfrak{I}/\partial\{\varepsilon\} = (1+s)\left[[\underline{x}] + (t_D - t)[\underline{a}]\right] = [\partial\varepsilon] \tag{40}$$

$$\partial\mathfrak{I}/\partial s = [\delta\mathfrak{R}]\left[[I] + (t_D - t)\left[\underline{\dot{\Omega}}\right]_p\right]\{x\} = \{\partial s\} \tag{41}$$

$$\partial\mathfrak{I}/\partial\{\dot{T}\} = (t_D - t_k)\,[I] \tag{42}$$

$$\partial\mathfrak{I}/\partial\{\dot{\varepsilon}\} = (t_D - t_k)\left[[\underline{x}] + (t_D - t)[\underline{a}]\right] = [\partial\dot{\varepsilon}] \tag{43}$$

$$\partial\mathfrak{I}/\partial\dot{s} = (t_D - t_k)\left[[I] + (t_D - t)\left[\underline{\dot{\Omega}}\right]_p\right]\{x\} = \{\partial\dot{s}\} \tag{44}$$

Notice that some of the final matrices presented above are dependent on the coordinates of each point; thus, one will have to compute them at each point i: $[\partial\varepsilon]_i$, $\{\partial s\}_i$, $[\partial\dot{\varepsilon}]_i$, $\{\partial\dot{s}\}_i$, after replacing the corresponding vectors $\{x\}_i$ when appropriate. Thus, the final elements (submatrices) of the contribution to the Jacobian matrix $[J]$ can be written in compact form as

$$[\mathfrak{I}_{CV}] = \begin{bmatrix} [\partial x] & [0] & \cdots & [0] & [\partial v] & [0] & \cdots & [0] \\ [0] & [\partial x] & \cdots & [0] & [0] & [\partial v] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & [\partial x] & [0] & [0] & \cdots & [\partial v] \end{bmatrix}_{(3n\times 6n)} = \Big[\underset{3n\times 3n}{[\mathfrak{I}_C]} \vdots \underset{3n\times 3n}{[\mathfrak{I}_V]}\Big] \tag{45}$$

and

$$[\mathfrak{I}_{P\dot{P}}] = \begin{bmatrix} [I] & [\partial\varepsilon]_1 & \{\partial s\}_1 & (t_D - t_k)\,[I] & [\partial\dot{\varepsilon}]_1 & \{\partial\dot{s}\}_1 \\ [I] & [\partial\varepsilon]_2 & \{\partial s\}_2 & (t_D - t_k)\,[I] & [\partial\dot{\varepsilon}]_2 & \{\partial\dot{s}\}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [I] & [\partial\varepsilon]_n & \{\partial s\}_n & (t_D - t_k)\,[I] & [\partial\dot{\varepsilon}]_n & \{\partial\dot{s}\}_n \end{bmatrix}_{(3n\times 14)} = \Big[\underset{3n\times 7}{[\mathfrak{I}_P]} \vdots \underset{3n\times 7}{[\mathfrak{I}_{\dot{P}}]}\Big] \tag{46}$$

Final Transformed v–c Matrix

The v–c matrix of the transformed coordinates and velocities on the frame NAD 83 at time t is computed using Eqs. (21) and (29). If the cross-covariances of Eq. (22) except Σ_{CV} are assumed to be zero, as is currently the case, then the methodology is simplified and the calculations can be performed point by point. The matrix expression in Eq. (21) now, for an arbitrary station i, takes the form

$$\underset{3\times 3}{\Sigma_{\text{NAD83}}} = \underset{3\times 20}{[J]} \begin{bmatrix} \underset{3\times 3}{\Sigma_{x_i}} & \Sigma_{x_i v_i} & [0] & [0] \\ & \underset{3\times 3}{\Sigma_{v_i}} & [0] & [0] \\ & & \underset{7\times 7}{\Sigma_P} & [0] \\ \text{sym} & & & \underset{7\times 7}{\Sigma_{\dot{P}}} \end{bmatrix}_{\text{IGS08}} \underset{20\times 3}{[J]^T} \tag{47}$$

where the Jacobian matrix is given by

$$\underset{3\times 20}{[J]} = \Big[\underset{3\times 3}{[\partial x]} \quad \underset{3\times 3}{[\partial v]} \quad \underset{3\times 7}{[I] \vdots [\partial\varepsilon] \vdots \{\partial s\}} \quad \underset{3\times 7}{(t_D - t_k)[I] \vdots [\partial\dot{\varepsilon}] \vdots \{\partial\dot{s}\}}\Big] \tag{48}$$

The matrices required in the blocks of Eq. (48) are given explicitly in Eqs. (37)–(44).

Final Simplifications Relative to the NAD 83 Frame

The coordinates of NAD 83 are defined in terms of a 14-parameter transformation from IGS08 coordinates [see Eq. (16)]. That is, if one knows the IGS08 coordinates of a point at a given epoch, one can exactly compute the corresponding NAD 83 coordinates for this point at another epoch, and vice versa. By definition, this transformation is assumed to be a one-to-one transformation without errors. Therefore, the variances of the 14 parameters in the transformations between IGS08 and NAD 83 are all zero in value. This definition simplifies even further the formulation for transforming v–c matrices from IGS08 to NAD 83. Consequently, the final Jacobian expression in Eq. (48) will be reduced to a 3×6 matrix that will not contain any derivatives with respect to the 14 transformation parameters. Thus, the Jacobian to be used in Eq. (47) reduces explicitly to

$$\underset{3\times 6}{[J]} = \Big[\underset{3\times 3}{[\partial x]} \quad \underset{3\times 3}{[\partial v]}\Big]$$
$$= \Bigg[\left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}[I]\right]\right]\left[[I] + (t_D - t)\left[\underline{\dot{\Omega}}\right]_p\right]$$
$$(t_D - t)\quad\left[(1+s)[\delta\mathfrak{R}] + (t_D - t_k)\left[\left[\underline{\dot{\varepsilon}}\right]^T + \dot{s}[I]\right]\right]\Bigg] \tag{49}$$

Note that owing to the simplifications inherent to the definition of NAD 83, the elements of the Jacobian matrix in Eq. (49) are independent of the coordinates and velocities of the points in the IGS08 frame; thus, all block elements are constant. Therefore, the same Jacobian matrix $[J]$ is used in Eq. (47) for transforming the v–c matrices of every point between IGS08 and NAD 83. Recall also that in this particular case $\Sigma_P = \Sigma_{\dot{P}} = [0]$. Table 1 (Pearson and Snay 2013) gives the 14 parameters necessary to transform v–c matrices between IGS08 and the geodetic frame currently adopted by NGS (NAD 83).

The components of the angular rotation vector of the North American tectonic plate were given in rad/Myear by McCarthy (1996), referred to the geophysical model NNR-NUVEL-1A. The corresponding values converted to mas/year are tabulated in Table 2. Using Eq. (8), the variations with respect to time of the three rotations around the IGS08 axes were calculated. They are also included in Table 2.

Table 1. Fourteen Transformation Parameters between Frames IGS08 and NAD 83: IGS08 ≡ ITRF2008 → NAD83 (2011) (Epoch $t_k = 1997.00$) (Data from Pearson and Snay 2013)

Parameter	Value
T_x (m)	0.99343
T_y (m)	− 1.90331
T_z (m)	− 0.52655
ε_x (mas)	25.91467
ε_y (mas)	9.42645
ε_z (mas)	11.59935
s (ppb)	1.71504
$\dot{T}_x$ (m/year)	0.00079
$\dot{T}_y$ (m/year)	− 0.00060
$\dot{T}_z$ (m/year)	− 0.00134
$\dot{\varepsilon}_x$ (mas/year)	0.06667
$\dot{\varepsilon}_y$ (mas/year)	− 0.75744
$\dot{\varepsilon}_z$ (mas/year)	− 0.05133
$\dot{s}$ (ppb/year)	− 0.10201

Note: mas ≡ milli arc second; ppb ≡ parts per billion = 10^{-3} ppm. Rotations consistent with the notation adopted in this article of anticlockwise rotations of ε_x, ε_y, ε_z, $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, and $\dot{\varepsilon}_z$ assumed positive.

The epochs to be used in Eq. (49) are $t_D = 2010.00$, $t = 2005.00$, and $t_k = 1997.00$. The transformation of the full v–c matrix of the IGS08 frame into NAD 83 can be implemented through the matrix operation

$$\underset{3n\times 3n}{\Sigma_{\text{NAD83}}} = \begin{bmatrix} \underset{3\times 6}{[J]} & [0] & \cdots & [0] \\ & \underset{3\times 6}{[J]} & \cdots & [0] \\ & & \ddots & \vdots \\ \text{sym} & & & \underset{3\times 6}{[J]} \end{bmatrix} \times \begin{bmatrix} \underset{6\times 6}{\begin{bmatrix} \Sigma_{C_1} & \Sigma_{C_1V_1} \\ \text{sym} & \Sigma_{V_1} \end{bmatrix}} & \begin{bmatrix} \Sigma_{C_1C_2} & \Sigma_{C_1V_2} \\ \Sigma_{V_1C_2} & \Sigma_{V_1V_2} \end{bmatrix} & \cdots & \begin{bmatrix} \Sigma_{C_1C_n} & \Sigma_{C_1V_n} \\ \Sigma_{V_1C_n} & \Sigma_{V_1V_n} \end{bmatrix} \\ & \underset{6\times 6}{\begin{bmatrix} \Sigma_{C_2} & \Sigma_{C_2V_2} \\ \text{sym} & \Sigma_{V_2} \end{bmatrix}} & \cdots & \begin{bmatrix} \Sigma_{C_2C_n} & \Sigma_{C_2V_n} \\ \Sigma_{V_2C_n} & \Sigma_{V_2V_n} \end{bmatrix} \\ & & \ddots & \vdots \\ \text{sym} & & & \underset{6\times 6}{\begin{bmatrix} \Sigma_{C_n} & \Sigma_{C_nV_n} \\ \text{sym} & \Sigma_{V_n} \end{bmatrix}} \end{bmatrix}_{\text{IGS08}} \begin{bmatrix} \underset{6\times 3}{[J]^T} & [0] & \cdots & [0] \\ & \underset{6\times 3}{[J]^T} & \cdots & [0] \\ & & \ddots & \vdots \\ \text{sym} & & & \underset{6\times 3}{[J]^T} \end{bmatrix} \quad (50)$$

Furthermore, owing to the small values of the variation with respect to time of the seven transformation parameters relating the IGS08 and NAD 83 frames given in Tables 1 and 2, Eq. (49) could be further simplified to

$$\underset{3\times 6}{[J]} = \begin{bmatrix} \underset{3\times 3}{[\partial x]} & \underset{3\times 3}{[\partial v]} \end{bmatrix} = \begin{bmatrix} (1+s)[\delta\mathfrak{R}] & (t_D - t)(1+s)[\delta\mathfrak{R}] \end{bmatrix} \quad (51)$$

which implies that for a single station the transformed v–c matrix of the coordinates take the explicit short form

$$\Sigma_{C_{i\,\text{NAD83}}} = (1+s)^2[\delta\mathfrak{R}]\Big[\Sigma_{C_{i\,\text{IGS08}}} + 2(t_D - t)\Sigma_{C_iV_{i\,\text{IGS08}}} + (t_D - t)^2\Sigma_{V_{i\,\text{IGS08}}}\Big][\delta\mathfrak{R}]^T \quad (52)$$

and

$$\Sigma_{C_iC_{j\,\text{NAD83}}} = (1+s)^2[\delta\mathfrak{R}]\Big[\Sigma_{C_iC_{j\,\text{IGS08}}} + (t_D - t)\Big[\Sigma_{C_iV_{j\,\text{IGS08}}} + \Sigma_{V_iC_{j\,\text{IGS08}}}\Big] + \Sigma_{V_iV_{j\,\text{IGS08}}}\Big][\delta\mathfrak{R}]^T \quad (53)$$

Notice that the above two expressions make use of the full v–c matrix included in the SINEX file, where the v–c matrix of the velocities and the cross-covariances between position and velocities are given. Sometimes (e.g., NGS's OPUS-RS solutions) the v–c matrix of the velocities is not known and is assumed to be equal to zero; consequently, Eqs. (52) and (53) further simplify to

$$\Sigma_{C_{i\,\text{NAD83}}} = (1+s)^2[\delta\mathfrak{R}]\,\Sigma_{C_{i\,\text{IGS08}}}\,[\delta\mathfrak{R}]^T = \Sigma_{C_{i\,\text{IGS08}}} \quad (54)$$

Table 2. Values of the Variations with Respect to Time of the Angles $\dot{\varpi}$ and the Components of the Angular Velocity $\dot{\Omega}$ of the North American Plate [see Eq. (8)]

Parameter	Value
$\dot{\Omega}_x$ (mas/year)	+0.0532
$\dot{\Omega}_y$ (mas/year)	−0.7423
$\dot{\Omega}_z$ (mas/year)	−0.0316
$\dot{\varpi}_x$ (mas/year)	+0.01347
$\dot{\varpi}_y$ (mas/year)	−0.01514
$\dot{\varpi}_z$ (mas/year)	−0.01973

Note: Rotations consistent with the notation adopted in this article of anticlockwise rotations assumed positive.

$$\Sigma_{C_iC_{j\,\text{NAD83}}} = (1+s)^2[\delta\mathfrak{R}]\,\Sigma_{C_iC_{j\,\text{IGS08}}}\,[\delta\mathfrak{R}]^T = \Sigma_{C_iC_{j\,\text{IGS08}}} \quad (55)$$

The final general v–c matrix referred to the local geodetic frame (E, N, U) will be computed according to the equation (see Soler and Smith 2010; Soler et al. 2012)

$$\underset{3n\times 3n}{\Sigma_{\text{NAD83, ENU}}} = \begin{bmatrix} \underset{3\times 3}{[R_1]\overline{\Sigma}_{C_1}[R_1]^T} & \underset{3\times 3}{[R_1]\overline{\Sigma}_{C_1C_2}[R_2]^T} & \cdots & \underset{3\times 3}{[R_1]\overline{\Sigma}_{C_1C_n}[R_n]^T} \\ & \underset{3\times 3}{[R_2]\overline{\Sigma}_{C_2}[R_2]^T} & \cdots & \underset{3\times 3}{[R_2]\overline{\Sigma}_{C_2C_n}[R_n]^T} \\ & & \ddots & \vdots \\ \text{sym} & & & \underset{3\times 3}{[R_n]\overline{\Sigma}_{C_n}[R_n]^T} \end{bmatrix}_{\text{IGS08}} \quad (56)$$

where $\overline{\Sigma}_{C_i} = [J]\,\Sigma_{C_{i\,\text{IGS08}}}[J]^T$, $\overline{\Sigma}_{C_iC_j} = [J]\,\Sigma_{C_iC_{j\,\text{IGS08}}}[J]^T$, and

$$\underset{3\times 3}{[R_i]} = \begin{bmatrix} -\sin\lambda_i & \cos\lambda_i & 0 \\ -\sin\varphi_i\cos\lambda_i & -\sin\varphi_i\sin\lambda_i & \cos\varphi_i \\ \cos\varphi_i\cos\lambda_i & \cos\varphi_i\sin\lambda_i & \sin\varphi_i \end{bmatrix} \quad (57)$$

where λ_i and φ_i = east longitude and latitude values for station i, respectively. This rotation matrix is consistent with the transformation between two right-handed local frames, from topocentric (x, y, z) to local geodetic (E, N, U). The authors encourage the future production of SINEX files also with reference to the local geodetic frames (E, N, U). This option will be more intuitive allowing the user to directly check the ASCII files and immediately isolate stations with, e.g., atypically large errors in the ellipsoid height component. The other advantage could be the possibility of directly extracting from the SINEX files the variances and covariances to plot meaningful correlations between the different local components at problematic stations.

Results

This investigation concentrated on the transformation of v–c matrices from a realization of the geocentric frame IGS08, epoch 2005.00, to the geodetic frame NAD 83, epoch 2010.00. The starting SINEX file used in this analysis was referred to the IGS08 frame and resulted from a multiyear solution generated from GPS observations spanning about 18 years (Griffiths et al. 2010). In this research, the original SINEX file that contains all CORS and IGS stations around the world was trimmed in order to select only the subset of CORS stations located on the North American tectonic plate. Eqs. (50) and (56) were processed to determine the accuracy estimates (standard deviations) of the transformed v–c matrix referred to the NAD 83 frame. The dots in Fig. 2(a) show the resultant standard deviations of the positions referred to the IGS08 as extracted from the original SINEX file referred to the local geodetic frame (E, N, U) derived according to Eq. (56). Fig. 2(b) depicts the actual differences of the standard deviations of the coordinates (transformed NAD 83 minus IGS08) along the local (E, N, U) frame. Recall that the epoch of the NAD 83 coordinates is the year 2010.00, whereas the coordinates of IGS08 have an epoch of 2005.00. Note that these differences tend to converge to zero when the number of years of GPS observations increases. Therefore, when the number of years of accumulated GPS observations increases, the velocities of the IGS08 frame become well determined; consequently, their standard deviations decrease, and, as a result, the distortion of the NAD 83 transformed coordinates is reduced. To corroborate this hypothesis, the transformation between the two frames was done at the same epoch, and the difference of accuracies between the IGS08 and NAD 83 frames was negligible (to the order of 10^{-6}m).

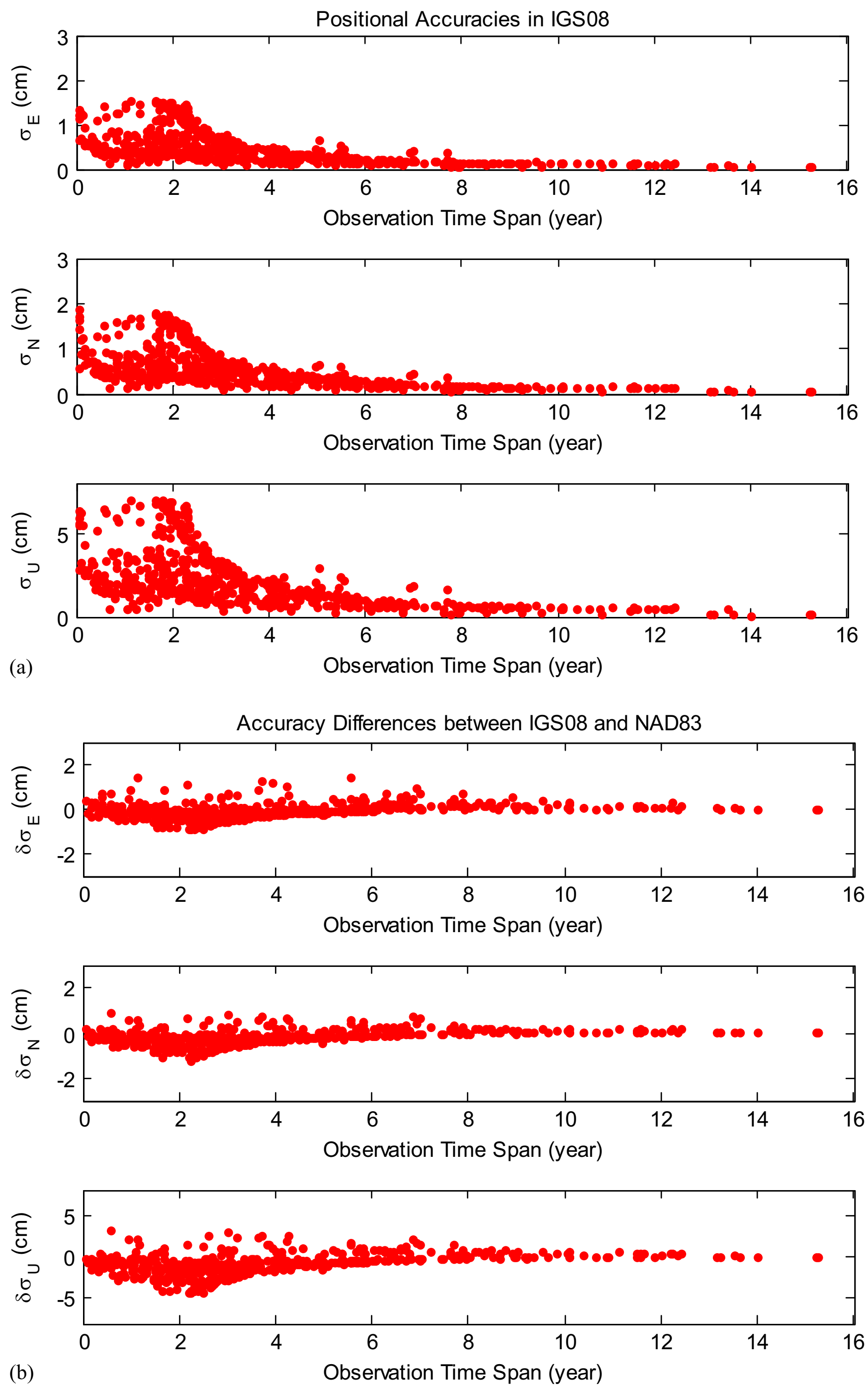

Fig. 2. (a) Positional standard deviations in the IGS08 frame and (b) differences of the standard deviations referred to the original IGS08 frame and the transformed NAD 83 values

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Notice the high correlation in Fig. 2(a) between the uncertainty of the determined coordinates and the cumulative number of GPS years of data used in the solution for each particular station. In other words, as expected, the longer the observation spans, the better the results that one gets for the accuracies of the positions, a well-known proven fact (e.g., Santamaría et al. 2011). Investigations of data collected by GPS commercial receivers, at various time windows, were analyzed using several software packages (e.g., Eckl et al. 2001; Soler et al. 2006, 2012) where a mnemonic rule was introduced alleging that the uncertainties in the determined ellipsoid height are about 3.2 times worse than the horizontal components. Fig. 2(a) shows a rough confirmation of this general rule. These previously published results were restricted to standard static GPS relative methods that were studied for observational windows of only a few hours.

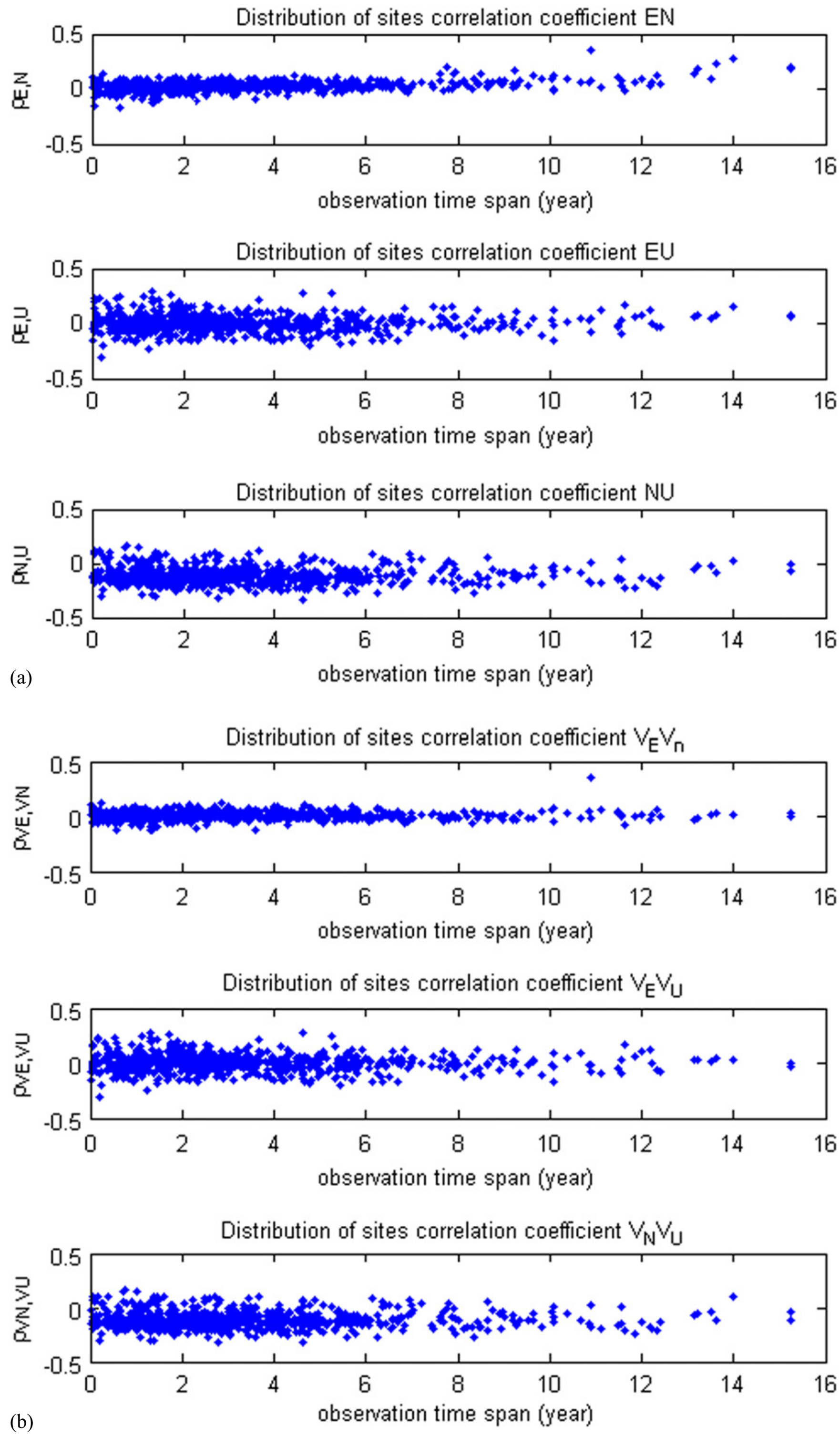
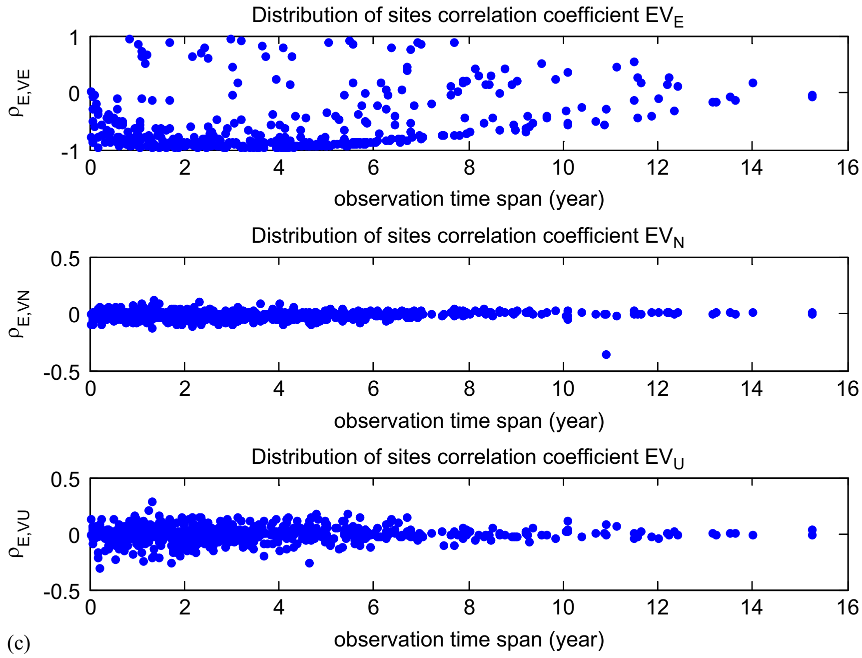
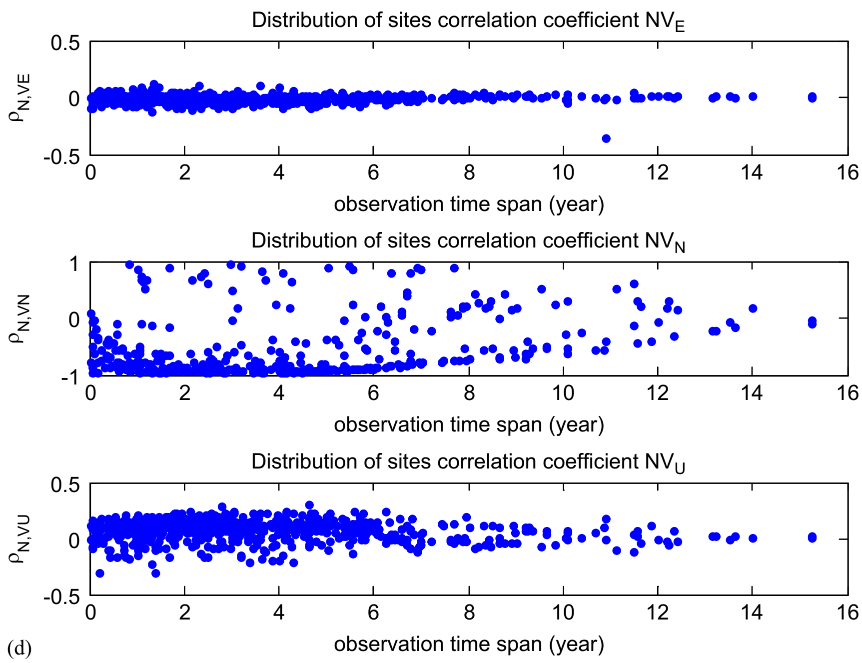


Fig. 3. (a) Correlations between local geodetic components ($\rho_{E,N}; \rho_{E,U}; \rho_{N,U}$); (b) correlations between velocity components ($\rho_{v_E\ v_N}; \rho_{v_E\ v_U}; \rho_{v_N\ v_U}$); (c) correlations between E-component and velocity components ($\rho_{E,\ v_E}; \rho_{E,\ v_N}; \rho_{E,\ v_U}$); (d) correlations between N-component and velocity components ($\rho_{N,\ v_E}; \rho_{N,\ v_N}; \rho_{N,\ v_U}$); and (e) correlations between U-component and velocity components ($\rho_{U,\ v_E}; \rho_{U,\ v_N}; \rho_{U,\ v_U}$)



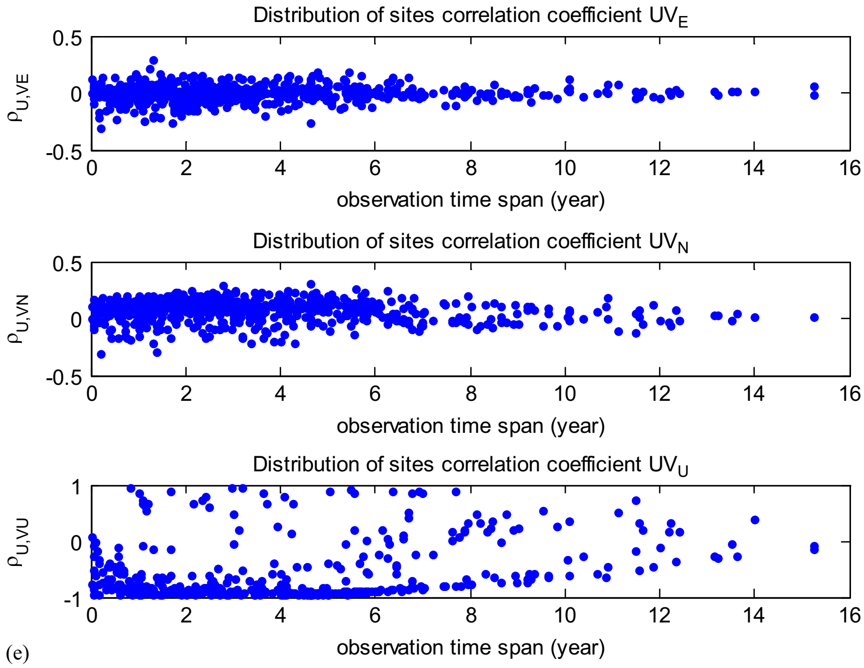
(c)



(d)

Fig. 3. (*Continued.*)

Now, the correlations between positions and velocities at each station will be addressed. The SINEX file has a full v–c matrix; however, the covariances between coordinates or velocities and the cross-covariances between coordinates and velocities are rarely used in practice. In this particular exercise, it was decided to bring them under scrutiny by computing the correlations between coordinates and velocities at each station to get further insight into the behavior of the variations of positions and velocities in multiyear solutions. The first immediate conclusion of the correlation analysis as shown in Figs. 3(a–e) is that minimum correlations ($-0.2 \leq \rho \geq 0.2$) always affect the combination of errors of positions and velocities along the east and north components [$\rho_{E,N}$, Fig. 3(a)]. Minimum correlations ($\rho_{v_E\ v_N}, \rho_{E,\ v_N}, \rho_{N,\ v_E}$) are between the velocities along the east and north component [Fig. 3(b)], and the east–north velocity [Fig. 3(c)] and north–east velocity (Fig. 3d). This is followed by correlations between -0.5 and $+0.5$ for the combination of horizontal components ($\rho_{E,U}, \rho_{N,U}, \rho_{v_E,v_U}, \rho_{E,v_U}, \rho_{N,v_U}, \rho_{U,v_E}, \rho_{U,v_N}$). Nevertheless, what is clear from the correlation plots is that the errors between the position and the velocity along the same components ($\rho_{E,\ v_E}, \rho_{N,\ v_N}, \rho_{U,\ v_U}$) produce maximum correlations. For example, as expected, an error along the local east will be highly correlated with the value of the velocity along the east component [Fig. 3(c)]. Similar reasoning applies to the correlation between the north position and velocity [Fig. 3(d)], and up position and velocity (Fig. 3e). However, more detailed research should be devoted to this topic to clarify the behavior of these high correlations and shed light on the interaction of the different position–velocity statistics at each station as compared with the rest of the sites in the network, especially in the context of the positional discontinuities introduced by equipment changes.



(e)

Fig. 3. (*Continued.*)

Conclusions

This investigation described a rigorous transformation of the v–c matrices of positions (the SINEX file) between two frames that are related by a standard set of geodetic Helmert transformation parameters with their own particular uncertainties (its stochastic model). In this specific case, the two frames used were the geocentric IGS08 GPS-defined reference frame and the frame defining the NAD 83 geodetic datum. The theory was explained in detail emphasizing the peculiarities implicit in the computation of geodetic coordinates at any prespecified epoch t_D for currently defined geodetic datums as a consequence of taking into account the effect of tectonic plate rotations. The general theory presented here could be applied to other specific situations when a plate-fixed datum implemented primarily for surveying and mapping applications is defined.

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