U. S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE

OFFICE NOTE 161

Is Locked-in Error due to Peculiar Truncation Errors in the Implied Vorticity Equations?

Frederick G. Shuman
National Meteorological Center

SEPT 1977

This is an unreviewed manuscript, primarily intended for informal exchange of information among NMC staff members.

In designing finite difference forms for the primitive equations of motion, Shuman and Stackpole (1969) indicated the importance of certain peculiar errors in the implied vorticity equation. I am raising in this paper further questions about the implied vorticity equation. How important is its form? Locked-in error might be regarded as a result of poor handling and development of vorticity. Is this error caused by peculiar errors in the implied vorticity equation?

Consider the form proposed by Shuman and Stackpole to avoid divergence-on-divergence interactions in the implied vorticity equation. Generalized for an arbitrary vertical coordinate and for a baroclinic model, their equations (10) may be written

$$\begin{aligned} &\overline{u}^{t}_{t} + \overline{\left(-\overline{v}^{xy}\zeta + \overline{\overline{w}^{xy}\,\overline{u}^{xy}_{z}}^{z} + c_p\,\overline{\theta}^{xy}\,\overline{\pi}^{y}_{x} + \overline{E}^{y}_{x} \right)}^{xy} = 0 \\ &\overline{v}^{t}_{t} + \overline{\left(\overline{u}^{xy}\zeta + \overline{\overline{w}^{xy}\,\overline{v}^{xy}_{z}}^{z} + c_p\,\overline{\theta}^{xy}\,\overline{\pi}^{x}_{y} + \overline{E}^{x}_{y} \right)}^{xy} = 0 \end{aligned} \qquad (1)$$

where

$$\zeta = \overline{v}^{y}_{x} - \overline{u}^{x}_{y} + \overline{f}^{xy}$$

$$E = 1/2\ (u^2 + v^2) + \overline{\phi}^{z}$$

$$\pi = \overline{[\,(p/P)^{2/7}\,]}^{z}$$

and z is the arbitrary vertical coordinate, w its substantial derivative, and ϕ geodynamic height. For illustration I have chosen the model of Shuman and Hovermale (1968), converting their equations to invariant form.

Now

$$\overline{a\,b}^{x} \equiv \overline{a}^{x}\,\overline{b}^{x} + \frac{\Delta^2}{4} a_x\, b_x$$

$$(a\,b)_x \equiv \overline{a}^{x}\, b_x + a_x\, \overline{b}^{x}$$

and likewise for differences and averages of products in the other coordinates. The symbol Δ is the grid interval. Thus, (1) may be written

$$\overline{u_t}^{t} + \overline{\left(-\overline{v}^{xxy}\,\overline{\zeta}^{x} + \overline{\overline{w}^{xxy}\,\overline{u_z}^{xxy}}^{z} + c_p\,\overline{\theta}^{xxy}\,\overline{\pi_x}^{xy} + \overline{E_x}^{xy}\right)}^{y}$$

$$+ \frac{\Delta^2}{4}\overline{\left(-\overline{v_x}^{xy}\,\zeta_x + \overline{\overline{w_x}^{xy}\,\overline{u_{xz}}^{xy}}^{z} + c_p\,\overline{\theta_x}^{xy}\,\overline{\pi_{xx}}^{y}\right)}^{y} = 0$$

$$\overline{v_t}^{t} + \overline{\left(\overline{u}^{xyy}\,\overline{\zeta}^{y} + \overline{\overline{w}^{xyy}\,\overline{v_z}^{xyy}}^{z} + c_p\,\overline{\theta}^{xyy}\,\overline{\pi_y}^{xy} + \overline{E_y}^{xy}\right)}^{x}$$

$$+ \frac{\Delta^2}{4}\overline{\left(\overline{u_y}^{xy}\,\zeta_y + \overline{\overline{w_y}^{xy}\,\overline{v_{yz}}^{xy}}^{z} + c_p\,\overline{\theta_y}^{xy}\,\overline{\pi_{yy}}^{x}\right)}^{x} = 0 \qquad (2)$$

Differencing these, I write the implied vorticity equation (3).

$$\overline{\zeta_t}^{t} + \overline{\left[\overline{u}^{xxyy}\,\overline{\zeta_x}^{y} + \overline{v}^{xxyy}\,\overline{\zeta_y}^{x} + \overline{\zeta}^{xy}\,\overline{\left(\overline{u_x}^{y} + \overline{v_y}^{x}\right)}^{xy}\right]}^{xy}$$

$$+ \overline{\left(\overline{w}^{xxyy}\,\overline{\zeta_z}^{xy} + \overline{w_x}^{xyy}\,\overline{v_z}^{xxyy} - \overline{w_y}^{xxy}\,\overline{u_z}^{xxyy}\right)}^{xyz}$$

$$+ c_p\,\overline{\left(\overline{\theta_x}^{xyy}\,\overline{\pi_y}^{xxy} - \overline{\theta_y}^{xxy}\,\overline{\pi_x}^{xyy}\right)}^{xy}$$

$$+ \frac{\Delta^2}{4}\overline{\left[\overline{u_{xy}}^{xy}\,\overline{\zeta_y}^{x} + \overline{v_{xy}}^{xy}\,\overline{\zeta_x}^{y} + \zeta_{xy}\,\overline{\left(\overline{u_y}^{x} + \overline{v_x}^{y}\right)}^{xy}\right]}^{xy}$$

$$+ \frac{\Delta^2}{4}\overline{\left[\overline{w_{xy}}^{xy}\,\overline{\left(\overline{v_y}^{x} - \overline{u_x}^{y}\right)_z}^{xy} + \overline{w_y}^{xxy}\,\overline{v_{xyz}}^{xy} - \overline{w_x}^{xyy}\,\overline{u_{xyz}}^{xy}\right]}^{xyz}$$

$$+ \frac{\Delta^2}{4}\,c_p\,\overline{\left[\overline{\theta_{xy}}^{xy}\left(\overline{\pi_{yy}}^{xx} - \overline{\pi_{xx}}^{yy}\right) + \overline{\theta_y}^{xxy}\,\overline{\pi_{xyy}}^{x} - \overline{\theta_x}^{xyy}\,\overline{\pi_{xxy}}^{y}\right]}^{xy} = 0 \qquad (3)$$

Now, the first four terms in (3) correspond in form to terms in the differential vorticity equation, but the last three don't. Note, for example, the dependence on deformation shown in the fifth and sixth terms! For a given scale, the size of these terms decrease with decreasing mesh size, so one would expect errors arising from them to be at least somewhat suppressed in higher resolution models. I believe this fits our experience at NMC with locked-in error.

The extraneous terms in (3), however, may be relatively easily eliminated entirely by integrating the equations of motion in their form (2), but omitting the last term in each.

<u>References</u>

Shuman, F. G., and J. B. Hovermale, 1968: An operational six-layer primitive equation model. <u>J. Appl. Meteor.</u>, 7, 525-547.

Shuman, F. G., and J. D. Stackpole, 1969: The currently operational NMC model, and results of a recent simple numerical experiment. <u>Proc. of the WMO/IUGG Symposium on Numerical Weather Prediction in Tokyo, Nov. 26-Dec. 4, 1968.</u> JMA, Tokyo, II-85 to II-98.

Supplement to NMC Office Note 161

Frederick G. Shuman

The equations of motion as written:

$$\bar{u}^{t}_{t} + \overline{\left(\bar{u}^{xxy}\, \bar{u}^{xy}_{x} + \bar{v}^{xxy}\, \bar{u}^{xx}_{y} + \overline{\bar{w}^{xxy}\, \bar{u}^{xxy}_{z}}^{z} + c_p\, \bar{\theta}^{xxy}\, \bar{\pi}^{xy}_{x} + \bar{\phi}^{xy}_{x} \right)}^{y} = 0$$

$$\bar{v}^{t}_{t} + \overline{\left(\bar{u}^{xyy}\, \bar{v}^{yy}_{x} + \bar{v}^{xyy}\, \bar{v}^{xy}_{y} + \overline{\bar{w}^{xyy}\, \bar{v}^{xyy}_{z}}^{z} + c_p\, \bar{\theta}^{xyy}\, \bar{\pi}^{xy}_{y} + \bar{\phi}^{xy}_{y} \right)}^{x} = 0 \qquad (4)$$

have the "clean" implied invariant forms

$$\bar{u}^{t}_{t} + \overline{\left(-\bar{v}^{xxy}\, \bar{\zeta}^{x} + \overline{\bar{w}^{xxy}\, \bar{u}^{xxy}_{z}}^{z} + c_p\, \bar{\theta}^{xxy}\, \bar{\pi}^{xy}_{x} + E'_{x} \right)}^{y} = 0$$

$$\bar{v}^{t}_{t} + \overline{\left(\bar{u}^{xyy}\, \bar{\zeta}^{y} + \overline{\bar{w}^{xyy}\, \bar{u}^{xyy}_{z}}^{z} + c_p\, \bar{\theta}^{xyy}\, \bar{\pi}^{xy}_{y} + E'_{y} \right)}^{x} = 0 \qquad (5)$$

where

$$E' = \tfrac{1}{2}\left[\left(\bar{u}^{xy}\right)^2 + \left(\bar{v}^{xy}\right)^2 \right] + \bar{\phi}^{xyz}$$

Except for the difference between E and E', (5) are identical to (2) if the last term in each of (2) were omitted. Since E' in (5) (and E in (2)) drops out in the derivation of the vorticity equation, (2) with their last terms omitted and (5) have identical finite difference vorticity equations.

This means that explicitly invariant forms of the equations of motion need not be used in order to avoid either divergence-on-divergence interactions or extraneous terms in the implied vorticity equation like the last 3 in (3). Equations (4) accomplish both purposes. (9-29-77)