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A Derivation of the Basic Meteorological Equations

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This is an unreviewed manuscript, primarily
intended for informal exchange of information
among NMC staff members.

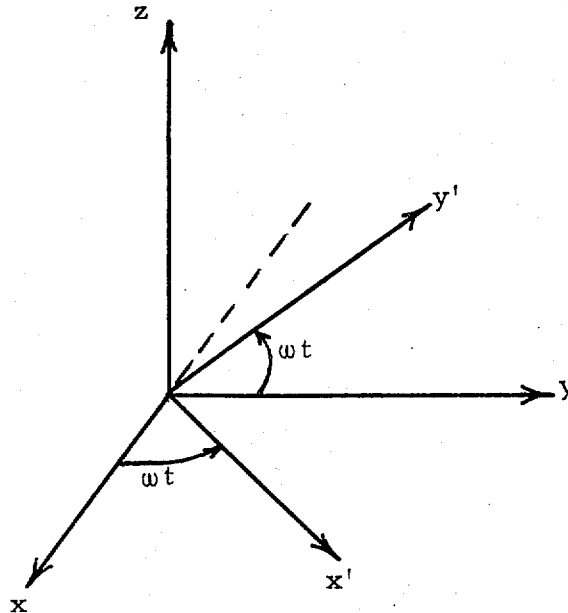
Every once in a while a question arises about whether or not some important terms have been omitted from the basic equations as we use them in meteorology. The questions usually center around metrical terms, terms that involve the shape of the earth, especially its departure from a perfect sphere. The concern is that, although omitted effects may be small, they may be large enough and systematic enough to accumulate in some way into important errors. My aim here is to derive the equations with rigor, so that all terms and effects can be closely inspected. I won't draw any conclusions, except to say that the rigorous derivation doesn't lead to any surprises.

Equations of motion

I start with the vector equations in Newtonian space.

$$\frac{d\vec{V}}{dt} - \vec{G} + \frac{1}{\rho} \nabla p = 0 \quad (1)$$

where t is time, \vec{V} velocity, \vec{G} pure gravity vector, ρ density, and p pressure. Now, consider a frame of reference rotating with the earth. Referring to the figure below, let the origin be at earth's center, the z -axis be directed along the center of earth's rotation, x and y be the pair of other coordinates in Newtonian space, and x' and y' in the rotating frame.



Then x' and y' are related to x and y according to

$$\begin{aligned}x' &= x \cos \omega t + y \sin \omega t \\y' &= -x \sin \omega t + y \cos \omega t\end{aligned}$$

where ω is the magnitude of $\vec{\Omega}$, $\vec{\Omega}$ is the rotation vector of the earth, and t is the time since the x' -axis once coincided with the x -axis. Representing the unit x - and y -vectors as ∇x and ∇y , and likewise for the unit x' - and y' -vectors, by differentiating I get

$$\begin{aligned}\nabla x' &= \nabla x \cos \omega t + \nabla y \sin \omega t \\ \nabla y' &= -\nabla x \sin \omega t + \nabla y \cos \omega t\end{aligned}\tag{2}$$

Consider any vector, \vec{A} , varying in space and time. I expand it in terms of unit vectors and appropriate components,

$$\vec{A} = A_{x'} \nabla x' + A_{y'} \nabla y' + A_z \nabla z$$

and differentiate it,

$$\frac{d\vec{A}}{dt} = \frac{dA_{x'}}{dt} \nabla x' + \frac{dA_{y'}}{dt} \nabla y' + \frac{dA_z}{dt} \nabla z + A_{x'} \frac{d\nabla x'}{dt} + A_{y'} \frac{d\nabla y'}{dt}\tag{3}$$

But, from (2)

$$\frac{d\nabla x'}{dt} = \omega(-\nabla x \sin \omega t + \nabla y \cos \omega t) = \omega \nabla y' = \vec{\Omega} \times \nabla x'$$

and similarly,

$$\frac{d\nabla y'}{dt} = -\omega \nabla x' = \vec{\Omega} \times \nabla y'$$

The sum of the first three terms in the right hand member of (3) are the substantial rate of change of the vector \vec{A} with respect to time, apparent to an observer from a location fixed in the rotating frame. I call such an apparent rate of change $d'\vec{A}/dt$. Thus, since $\vec{\Omega} \times \nabla z = \omega \nabla z \times \nabla z \equiv 0$,

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega} \times \vec{A}\tag{4}$$

Now,

$$\vec{V} \equiv \frac{d\vec{R}}{dt}$$

where \vec{R} is the radius vector from earth's center to the mass-point in question. Applying the result (4), I find

$$\vec{V} = \vec{V}' + \vec{\Omega} \times \vec{R}$$

where $\vec{V}' \equiv d'\vec{R}/dt$, the velocity relative to the rotating frame, and $\vec{\Omega} \times \vec{R}$ is the velocity, at the same point, of the rotating frame. Applying (4) again, I find

$$\begin{aligned} \frac{d\vec{V}}{dt} &= \left(\frac{d'}{dt} + \vec{\Omega} \times \right) (\vec{V}' + \vec{\Omega} \times \vec{R}) \\ &= \frac{d'\vec{V}'}{dt} + 2 \vec{\Omega} \times \vec{V}' + \vec{\Omega} \times (\vec{\Omega} \times \vec{R}) \end{aligned}$$

With the last result, (1) becomes

$$\frac{d'\vec{V}'}{dt} + 2 \vec{\Omega} \times \vec{V}' + \vec{\Omega} \times (\vec{\Omega} \times \vec{R}) - \vec{G} + \frac{1}{\rho} \nabla p = 0 \quad (5)$$

Now, $\nabla \times \vec{G} = \nabla \times [\vec{\Omega} \times (\vec{\Omega} \times \vec{R})] = 0$, and I may therefore write

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{R}) - \vec{G} = g \nabla r'$$

where $g = 980 \text{ cm sec}^{-2}$, a constant. The scalar r' , then, is the geopotential height used in meteorology. Assuming mean sea level to be a surface of constant r' , I arbitrarily set $r' = 0$ there, and r' is then specifically geopotential height above mean sea level. Consistent with terminology of meteorology, I call $-g \nabla r'$ the "gravity" vector. "Gravity," then, consists not only of "pure" gravity \vec{G} , the attraction of the earth for air, but also $-\vec{\Omega} \times (\vec{\Omega} \times \vec{R})$, the centrifugal force due to earth's rotation. Substitution into (5) gives

$$\frac{d'\vec{V}'}{dt} + 2 \vec{\Omega} \times \vec{V}' + g \nabla r' + \frac{1}{\rho} \nabla p = 0 \quad (6)$$

I now define unit vectors,

$$\begin{aligned} i &= r \cos \phi \nabla \lambda \\ j &= r \nabla \phi \\ k &= \nabla r \end{aligned}$$

where λ , ϕ , and r are the coordinates, east longitude, latitude, and distance from earth's center, respectively. As I define them here, they are truly spherical coordinates with ϕ being the angle between ∇r and the equatorial plane. In particular, latitude is not that used by the navigator, which is the angle between the vertical, $\nabla r'$, and the equatorial plane. My definition here is consistent with common practice in meteorology, particularly in numerical weather prediction. I similarly define the three components of motion

$$u = r \cos \phi \frac{d'\lambda}{dt}$$

$$v = r \frac{d'\phi}{dt}$$

$$w = \frac{d'r}{dt}$$

I choose the x' -axis to pass through the Greenwich meridian, and for convenience introduce the parameter s , which is a cylindrical coordinate, namely, normal distance from the z -axis. Then

$$x' = s \cos \lambda$$

$$y' = s \sin \lambda$$

$$s = r \cos \phi$$

$$z = r \sin \phi$$

Thus,

$$\nabla x' = -i \sin \lambda + \nabla s \cos \lambda$$

$$\nabla y' = i \cos \lambda + \nabla s \sin \lambda$$

$$\nabla s = -j \sin \phi + k \cos \phi$$

$$\nabla z = j \cos \phi + k \sin \phi$$

These may be inverted by pairs, the first two for i and ∇s , the second two for j and k . Thus,

$$i = -\nabla x' \sin \lambda + \nabla y' \cos \lambda$$

$$\nabla s = \nabla x' \cos \lambda + \nabla y' \sin \lambda$$

$$j = -\nabla s \sin \phi + \nabla z \cos \phi$$

$$k = \nabla s \cos \phi + \nabla z \sin \phi$$

(7)

The vector \vec{V}' expanded is

$$\vec{V}' = ui + vj + wk$$

and therefore,

$$\frac{d'\vec{V}'}{dt} = \frac{d'u}{dt}i + \frac{d'v}{dt}j + \frac{d'w}{dt}k + u \frac{d'i}{dt} + v \frac{d'j}{dt} + w \frac{d'k}{dt}$$

But differentiating (7), we find after a little manipulation,

$$\frac{d'i}{dt} = \frac{u}{r} (j \tan \phi - k)$$

$$\frac{d'j}{dt} = -\frac{u}{r} i \tan \phi - \frac{v}{r} k$$

$$\frac{d'k}{dt} = \frac{u}{r} i + \frac{v}{r} j$$

Thus,

$$\begin{aligned} \frac{d'\vec{V}'}{dt} &= \left(\frac{d'u}{dt} - v \frac{u \tan \phi}{r} + w \frac{u}{r} \right) i \\ &+ \left(\frac{d'v}{dt} + u \frac{u \tan \phi}{r} + w \frac{v}{r} \right) j + \left(\frac{d'w}{dt} - \frac{u^2 + v^2}{r} \right) k \end{aligned} \quad (8)$$

Now

$$\vec{\Omega} = \omega \nabla z = \omega (j \cos \phi + k \sin \phi)$$

and

$$\begin{aligned} \vec{\Omega} \times \vec{V}' &= i \omega (-v \sin \phi + w \cos \phi) \\ &+ j \omega u \sin \phi - k \omega u \cos \phi \end{aligned} \quad (9)$$

With (8) and (9), the component equations of (6) become

$$\begin{aligned} \frac{d'u}{dt} - v \frac{u \sin \phi}{r \cos \phi} + u \frac{w}{r} \\ - 2 \omega v \sin \phi + 2 \omega w \cos \phi \\ + g \frac{\partial r'}{r \cos \phi \partial \lambda} + \frac{1}{\rho} \frac{\partial p}{r \cos \phi \partial \lambda} = 0 \end{aligned} \quad (10a)$$

$$\begin{aligned} \frac{d'v}{dt} + u \frac{u \sin \phi}{r \cos \phi} + v \frac{w}{r} \\ + 2 \omega u \sin \phi \\ + g \frac{\partial r'}{r \partial \phi} + \frac{1}{\rho} \frac{\partial p}{r \partial \phi} = 0 \end{aligned} \quad (10b)$$

$$\begin{aligned}
& \frac{d'w}{dt} - \frac{u^2 + v^2}{r} \\
& - 2 \omega u \cos \phi \\
& + g \frac{\partial r'}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0
\end{aligned} \tag{10c}$$

Now, the unit vector $k = \nabla r$ is not a vertically directed vector, but $\nabla r'$ is. Nor are differentiations with respect to λ and ϕ in (10) along a horizontal surface, which is defined as a surface of constant r' , not constant r . I will now transform the derivatives from λ, ϕ, r -space to λ, ϕ, r' -space. Primed partial derivatives will have the meaning:

$$\frac{\partial'}{\partial \lambda} = \left(\frac{\partial}{\partial \lambda} \right)_{\phi, r'} \tag{11a}$$

$$\frac{\partial'}{\partial \phi} = \left(\frac{\partial}{\partial \phi} \right)_{\lambda, r'} \tag{11b}$$

$$\frac{\partial}{\partial r'} = \left(\frac{\partial}{\partial r'} \right)_{\lambda, \phi} \tag{11c}$$

The transformation formulas used are

$$\frac{\partial}{\partial \lambda} = \frac{\partial'}{\partial \lambda} - \frac{\partial' r}{\partial \lambda} \frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial'}{\partial \phi} - \frac{\partial' r}{\partial \phi} \frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial r} = \frac{\partial r'}{\partial r} \frac{\partial}{\partial r'}$$

and thus, (10) become

$$\begin{aligned}
& \frac{d'u}{dt} - v \frac{u \sin \phi}{r \cos \phi} + u \frac{w}{r} \\
& - 2 \omega v \sin \phi + 2 \omega w \cos \phi \\
& + \frac{1}{\rho} \frac{\partial' p}{r \cos \phi \partial \lambda} \\
& - \frac{\partial' r}{r \cos \phi \partial \lambda} \frac{\partial r'}{\partial r} \left(g + \frac{1}{\rho} \frac{\partial p}{\partial r'} \right) = 0
\end{aligned} \tag{12a}$$

$$\begin{aligned}
& \frac{d'v}{dt} + u \frac{u \sin \phi}{r \cos \phi} + v \frac{w}{r} \\
& + 2 \omega u \sin \phi \\
& + \frac{1}{\rho} \frac{\partial' p}{r \partial \phi} \\
& - \frac{\partial' r}{r \partial \phi} \frac{\partial r'}{\partial r} \left(g + \frac{1}{\rho} \frac{\partial p}{\partial r'} \right) = 0
\end{aligned} \tag{12b}$$

$$\begin{aligned}
& \frac{d'w}{dt} - \frac{u^2 + v^2}{r} \\
& - 2 \omega u \cos \phi \\
& + \frac{\partial r'}{\partial r} \left(g + \frac{1}{\rho} \frac{\partial p}{\partial r'} \right) = 0
\end{aligned} \tag{12c}$$

We are not quite finished at this point, because w is not generally well related to the "vertical motion" used in meteorology. The substantial derivative applied to a scalar may be expanded in terms of partial derivatives in t, λ, ϕ, r' -space thusly,

$$\frac{d}{dt} = \frac{d'}{dt} = \frac{\partial'}{\partial t} + u \frac{\partial'}{r \cos \phi \partial \lambda} + v \frac{\partial'}{r \partial \phi} + w' \frac{\partial}{\partial r'} \tag{13}$$

where $w' = dr'/dt$, and the prime on the partial in time distinguishes it from the local partial in Newtonian space. Now,

$$w \equiv \frac{dr}{dt} = u \frac{\partial' r}{r \cos \phi \partial \lambda} + v \frac{\partial' r}{r \partial \phi} + w' \frac{\partial r}{\partial r'} \tag{14}$$

The first term in the right-hand-most member is very small, depending only on u and variations of gravity along a latitude circle. The next term, however, is not generally small compared to the last, on the scales with which numerical weather prediction presently deals. The International Ellipsoid of Reference has a flattening of $1/297$, which is a close estimate of $\partial' r / (r \partial \phi)$ at 45° latitude. A northerly wind of 20 m sec^{-1} yields about 7 cm sec^{-1} for that term there, a large vertical motion, but not beyond the order of magnitude of w' on the storm scale and larger. It is w' , of course, that is the "vertical motion" of meteorology. The departure

of $\partial r / \partial r'$ from unity, by the way, can be estimated by comparing the variation of the gravitational force with the constant $g = 980 \text{ cm sec}^{-2}$. The departure from unity turns out to be generally less than 0.3%.

The hydrostatic approximation is obtained by neglecting the first three terms in (12c):

$$g + \frac{1}{\rho} \frac{\partial p}{\partial r'} = 0$$

With this approximation, the last terms in (12a) and (12b) vanish. In meteorology, the terms in (12a) and (12b) in which w appears explicitly are also usually neglected as being very small.

There are other small aspects of (12) that are not always realized. The velocity components u and v are not quite horizontal, but their variation from the horizontal is less than $1/297$ rad. An estimate of their departure from horizontal components is $1 - \cos(1/297)$, less than 0.0006%. The partial derivatives (11a) and (11b) divided by $r \cos \phi$ and r , respectively, are not quite partials with respect to distance in the horizontal. These appear explicitly in (12a) and (12b) operating on p , but are also implicit in the first terms of those equations, through (13). The variation of the differentiated variable is taken on a horizontal surface, a geopotential surface, but the variation of distance is taken on a coincident sphere concentric with earth's center. Departures from "horizontal" derivatives can be estimated again by $1 - \cos(1/297)$, less than 0.0006%. Similarly the partial (11c) is not taken along the vertical, but has a similarly small departure from a vertical derivative with respect to r' .

The equation of continuity

I start with the equation in the form

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{V} = 0$$

so that the problem centers around divergence of velocity. I first note that $\nabla \cdot (\vec{\Omega} \times \vec{R}) = 0$, and therefore $\nabla \cdot \vec{V} = \nabla \cdot \vec{V}'$, and

$$\frac{1}{\rho} \frac{d\rho}{dt} + \nabla \cdot \vec{V}' = 0$$

Now,

$$\begin{aligned}\nabla \cdot \vec{V}' &= \nabla \cdot (ui + vj + wk) \\ &= i \cdot \nabla u + j \cdot \nabla v + k \cdot \nabla w \\ &\quad + u \nabla \cdot i + v \nabla \cdot j + w \nabla \cdot k\end{aligned}$$

But, taking the divergence of (7), we find after a little manipulation,

$$\nabla \cdot i = 0$$

$$\nabla \cdot j = \frac{-\sin \phi}{r \cos \phi}$$

$$\nabla \cdot k = \frac{2}{r}$$

Thus,

$$\nabla \cdot \vec{V}' = \frac{\partial u}{r \cos \phi \partial \lambda} + \frac{\partial v}{r \partial \phi} + \frac{\partial w}{\partial r} - \frac{v \sin \phi}{r \cos \phi} + \frac{2w}{r} \quad (15)$$

Now consider the quantity

$$\frac{\partial w}{\partial r} + \frac{w}{r} \equiv \frac{\partial rw}{r \partial r} = \frac{\partial r'}{\partial r} \frac{\partial rw}{r \partial r'}$$

From (14), after some manipulation, we find

$$\begin{aligned}\frac{\partial rw}{r \partial r} &= \frac{\partial r'}{\partial r} \frac{\partial}{\partial r'} \left(u \frac{\partial' r}{\cos \phi \partial \lambda} + v \frac{\partial' r}{\partial \phi} + rw' \frac{\partial r'}{\partial r'} \right) \\ &= \frac{\partial u}{\partial r} \frac{\partial' r}{r \cos \phi \partial \lambda} + \frac{\partial v}{\partial r} \frac{\partial' r}{r \partial \phi} \\ &\quad + \frac{\partial w'}{\partial r'} + \frac{w'}{r} \frac{\partial r}{\partial r'} + \frac{\partial r'}{\partial r} \frac{d}{dt} \frac{\partial r}{\partial r'}\end{aligned}$$

and the first two terms in the right hand member of (15) when transformed to λ, ϕ, r' -space are

$$\begin{aligned}&\frac{\partial u}{r \cos \phi \partial \lambda} + \frac{\partial v}{r \partial \phi} \\ &= \frac{\partial' u}{r \cos \phi \partial \lambda} + \frac{\partial' v}{r \partial \phi} \\ &\quad - \frac{\partial u}{\partial r} \frac{\partial' r}{r \cos \phi \partial \lambda} - \frac{\partial v}{\partial r} \frac{\partial' r}{r \partial \phi}\end{aligned}$$

With the last two results, (15) becomes

$$\begin{aligned}\vec{\nabla} \cdot \vec{V}' = & \frac{\partial' u}{r \cos \phi \partial \lambda} + \frac{\partial' v}{r \partial \phi} + \frac{\partial w'}{\partial r'} - \frac{v \sin \phi}{r \cos \phi} \\ & + \frac{w}{r} + \frac{w'}{r} \frac{\partial r}{\partial r'} + \frac{\partial r'}{\partial r} \frac{d}{dt} \frac{\partial r}{\partial r'}\end{aligned}$$

In meteorology, the last three terms are usually neglected. On the scales numerical weather prediction deals with, the divergence $\vec{\nabla} \cdot \vec{V}'$ is of the order 10^{-5} sec^{-1} , roughly 1000 times larger than those terms. For example, for $w' = 10 \text{ cm sec}^{-1}$,

$$\frac{w}{r} \sim \frac{w'}{r} \frac{\partial r}{\partial r'} \sim 2 \times 10^{-8} \text{ sec}^{-1}$$

To evaluate the last term, I note that

$$\frac{\partial r}{\partial r'} \cong \frac{g}{g'} \sim 1$$

where g' is the magnitude of the gravity vector including the centrifugal force of earth's rotation. An approximation giving the variation of g' with latitude is

$$g' \cong g(1 - 0.003 \cos 2\phi)$$

and with altitude in cm,

$$g' \cong g(1 - 3 \times 10^{-9} r')$$

Now,

$$\frac{\partial r'}{\partial r} \frac{d}{dt} \frac{\partial r}{\partial r'} \cong \frac{-1}{g'} \left(u \frac{\partial' g'}{r \cos \phi \partial \lambda} + v \frac{\partial' g'}{r \partial \phi} + w' \frac{\partial g'}{\partial r'} \right)$$

The first term on the right hand side is very small, depending only on u and the variation of g' along a latitude circle. With $v = 20 \text{ m sec}^{-1}$ and $w' = 10 \text{ cm sec}^{-1}$, the last two terms are

$$\frac{v}{g'} \frac{\partial' g'}{r \partial \phi} \sim 2 \times 10^{-8} \text{ sec}^{-1}$$

$$\frac{w'}{g'} \frac{\partial g'}{\partial r'} \sim 3 \times 10^{-9} \text{ sec}^{-1}$$

Other basic equations

The other basic meteorological equations that contain derivatives, and therefore depend on choice of coordinates and the shape of the earth are

$$\frac{d\theta}{dt} = 0$$

$$\frac{dq}{dt} = 0$$

where θ is potential temperature and q is specific humidity. In meteorology, in effect, the substantial derivatives are expanded with (13), and therefore no direct error whatsoever is involved.

Supplement to NMC Office Note 139

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JULY 1979

In Office Note 139, top of p. 8, I perhaps too summarily disposed of

$$0 = \frac{\partial r'}{\partial r} \left[g + \frac{1}{\rho} \frac{\partial p}{\partial r'} \right] = g \frac{\partial r'}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} \quad (16)$$

without showing how closely it approximates the "true" hydrostatic condition.

To show this, I invent a set of orthogonal coordinates, one of which is r' . For illustration purposes, I neglect the variation of r' with λ , taking into account only the oblateness of geopotential surfaces. I adopt λ , then, for another of the orthogonal set, and use ϕ' to indicate the third coordinate of the orthogonal set. The set consists, then, of λ , ϕ' , r' .

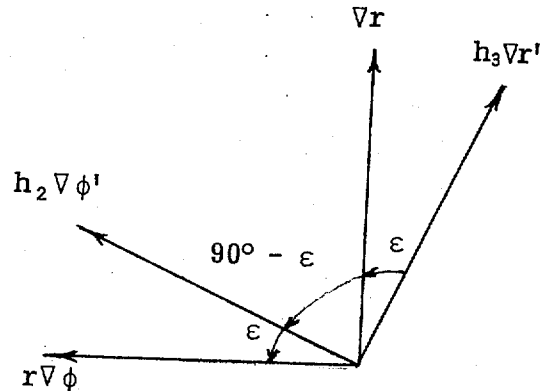
Let h_2 , h_3 be so defined that $h_2 \nabla \phi'$, $h_3 \nabla r'$ are unit vectors. Now, the gravity vector, including centrifugal force due to earth's rotation, is

$$-g' h_3 \nabla r' = -g \nabla r'$$

so that

$$h_3 = \frac{g}{g'} \quad (17)$$

From the figure, drawn in a $\lambda = \text{constant}$ plane, I note that



$$h_3 \nabla r' \cdot \nabla r = h_2 \nabla \phi' \cdot r \nabla \phi = \cos \epsilon \quad (18a)$$

$$h_2 \nabla \phi' \cdot \nabla r = -h_3 \nabla r' \cdot r \nabla \phi = \sin \epsilon \quad (18b)$$

From these, I draw

$$h_3 \frac{\partial r'}{\partial r} = \frac{\partial r}{\partial r'} = h_2 \frac{\partial \phi'}{r \partial \phi} = r \frac{\partial \phi}{h_2 \partial \phi'} = \cos \epsilon \quad (19a)$$

$$h_2 \frac{\partial \phi'}{\partial r} = \frac{\partial r}{h_2 \partial \phi'} = -h_3 \frac{\partial r'}{r \partial \phi} = r \frac{\partial' \phi}{h_3 \partial r'} = \sin \epsilon \quad (19b)$$

where all the partials have conventional meanings, except

$$\frac{\partial'}{\partial r'} = \left(\frac{\partial}{\partial r'} \right)_{\lambda, \phi'}$$

I invent this symbolism here because I have already made the following definition (11c),

$$\frac{\partial}{\partial r'} = \left(\frac{\partial}{\partial r'} \right)_{\lambda, \phi}$$

Subscripts on the partials here, and elsewhere, denote the variables held constant in differentiation.

Equations (17) and (19a) give

$$g \frac{\partial r'}{\partial r} = g' \cos \epsilon \quad (20)$$

I expand ∇p ,

$$\nabla p = \left(\frac{\partial p}{\partial \lambda} \right)_{\phi', r'} \nabla \lambda + \frac{\partial p}{\partial \phi'} \nabla \phi' + \frac{\partial' p}{\partial r'} \nabla r'$$

and take its dot product with ∇r ,

$$\nabla p \cdot \nabla r = \frac{\partial p}{h_2 \partial \phi'} h_2 \nabla \phi' \cdot \nabla r + \frac{\partial' p}{h_3 \partial r'} h_3 \nabla r' \cdot \nabla r$$

Or, by (18),

$$\frac{\partial p}{\partial r} = \frac{\partial' p}{h_3 \partial r'} \cos \epsilon + \frac{\partial p}{h_2 \partial \phi'} \sin \epsilon \quad (21)$$

Equations (20) and (21) give

$$g \frac{\partial r'}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = \left(g' + \frac{\partial' p}{h_3 \partial r'} \right) \cos \epsilon + \frac{\partial p}{h_2 \partial \phi'} \sin \epsilon$$

But the "true" hydrostatic condition involves the derivative of p in the true vertical, and thus is expressed by

$$0 = g' + \frac{\partial' p}{h_3 \partial r'}$$

Therefore, instead of zero, the expression in (16) is

$$g \frac{\partial r'}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{\partial p}{h_2 \partial \phi'} \sin \epsilon \quad (22)$$

But $\epsilon < 1/297$ rad, so even with a geostrophic component of 100 m sec^{-1} ,

$$\frac{1}{\rho} \frac{\partial p}{h_2 \partial \phi'} \sin \epsilon \sim \frac{1}{297} \text{ cm sec}^{-2}$$

which is to be compared with $g = 980 \text{ cm sec}^{-2}$. Neglect of this small term in (12c) is therefore justified.

The next question is, what about neglecting the hydrostatic terms in (12b)? From the second of the three transformation formulas following (11), and from (19),

$$\frac{\partial' r}{r \partial \phi} = - \frac{h_3}{\cos \epsilon} \frac{\partial r'}{r \partial \phi} = \tan \epsilon$$

From this and (22), the hydrostatic terms in (12b) are therefore

$$\frac{\partial' r}{r \partial \phi} \frac{\partial r'}{\partial r} \left(g + \frac{1}{\rho} \frac{\partial p}{\partial r'} \right) = \frac{1}{\rho} \frac{\partial p}{h_2 \partial \phi'} \sin \epsilon \tan \epsilon$$

which is about 10^{-5} times $-f u_g$ where u_g is the geostrophic wind component, and about 10^{-4} times characteristic values of the acceleration terms in (12b). Thus its neglect is also justified in (12b).