U. S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE
NATIONAL METEOROLOGICAL CENTER

OFFICE NOTE 111

Theory of Smoothing Discrete Functions

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MARCH 1975

Revised December 1975

I. Introduction

The object of this note is to record our derivations of the relationships between weighting factors in smoothing operators and their spectral response. Both periodic and non-periodic functions in an infinite domain are treated. Some basic tools for discrete functions are derived in the next section.

II. Basic Mathematical Tools

We start by developing some basic mathematical tools for periodic discrete functions. Only two of these tools, (3a) and (3e) below, will be used in this paper, but we will develop the others here in anticipation of our use of them elsewhere.

Consider ω and ω'; two functions of a set of integers, m, and a variable, μ, which may be either discrete or continuous.

$$\omega_m(\mu) = \sum_{n=o}^{m} \sigma_n \cos \mu n \tag{1a}$$

$$\omega'_m(\mu) = \sum_{n=o}^{m} \sigma_n \sin \mu n \tag{1b}$$

The discrete function σ_n is introduced as a convenience in the following derivations. It is always associated with a summation, $\sum(\ )$, and has the value ½ at the upper and lower limits as written, and is unity elsewhere. Thus in (1),

$$\sigma_n = \tfrac{1}{2} \quad \text{if } n = 0 \text{ or } n = m$$

$$\sigma_n = 1 \quad \text{if } n \neq 0 \text{ and } n \neq m$$

Now consider a difference of sin μn, taken over an interval in n of unity. We indicate the difference by the operator $\delta_n(\ ) = (\ )_{n+\frac{1}{2}} - (\ )_{n-\frac{1}{2}}$

$$\delta_n \sin \mu n = \sin \mu(n + \tfrac{1}{2}) - \sin \mu(n - \tfrac{1}{2})$$

$$= 2 \sin \tfrac{1}{2}\mu \cdot \cos \mu n$$

Thus, if μ is not a multiple of 2π, i.e., $\mu \neq 2k\pi$ where k is any integer (including zero),

$$\cos \mu n = \tfrac{1}{2} \csc \tfrac{1}{2}\mu \cdot \delta_n \sin \mu n$$

and

$$\omega_m(\mu) = \tfrac{1}{2} \csc \tfrac{1}{2}\mu \sum_{n=o}^{m} \sigma_n \delta_n \sin \mu n$$

The differences under the summation cancel everywhere except at the limits:

$$\omega_m(\mu) = \tfrac{1}{2} \csc \tfrac{1}{2}\mu \cdot$$
$$\cdot \tfrac{1}{2}\begin{bmatrix} \sin \mu(m + \tfrac{1}{2}) + \sin \mu(m - \tfrac{1}{2}) \\ - \sin \mu(\tfrac{1}{2}) - \sin \mu(-\tfrac{1}{2}) \end{bmatrix}$$
$$= \tfrac{1}{2} \cot \tfrac{1}{2}\mu \cdot \sin \mu m \qquad (2a)$$

A similar derivation from (1b) yields

$$\omega'(\mu) = \tfrac{1}{2} \cot \tfrac{1}{2}\mu \cdot (1 - \cos \mu m) \qquad (2b)$$

If $\mu = 2k\pi$, then $\cos \mu n = 1$ and $\sin \mu n = 0$, and

$$\omega_m(2k\pi) = \sum_{n=o}^{m} \sigma_n(1) = m$$

$$\omega'_m(2k\pi) = \sum_{n=o}^{m} \sigma_n(o) = 0$$

Equations (2a) and (2b) are consistent with these in a limiting sense, e.g.,

$$\lim_{\mu \to 2k\pi} \tfrac{1}{2} \cot \tfrac{1}{2}\mu \cdot \sin \mu m = m$$

$$\lim_{\mu \to 2k\pi} \tfrac{1}{2} \cot \tfrac{1}{2}\mu \cdot (1 - \cos \mu m) = 0$$

We will regard, for convenience, (2a) and (2b) to be valid for all μ, with the understanding that indeterminants are to be resolved in a limiting sense.

From (1a) and (2a) if μ is continuous

$$\frac{\partial\omega}{\partial\mu} = -\sum_{n=o}^{m} \sigma_n\, n \sin \mu n$$

$$= \tfrac{1}{2} \frac{\partial}{\partial\mu} (\cot \tfrac{1}{2}\mu \cdot \sin \mu m)$$

$$= -\tfrac{1}{4} \csc^2 \tfrac{1}{2}\mu \cdot \sin \mu m + \tfrac{1}{2}m \cot \tfrac{1}{2}\mu \cdot \cos \mu m$$

$$= -\csc \mu \cdot [\omega - m(1 + \cos \mu)] - \omega' m$$

A similar derivation from (1b) and (2b) yields

$$\frac{\partial\omega'}{\partial\mu} = \sum_{n=o}^{m} \sigma_n\, n \cos \mu n = -\omega' \csc \mu + \omega m$$

If μ is discrete, the same formulas hold for the summation, as can easily be shown. Again, any indeterminants are to be resolved in a limiting sense.

In summary, our basic tools are

$$\omega_m(\mu) = \sum_{n=o}^{m} \sigma_n \cos \mu n = \tfrac{1}{2} \cot \tfrac{1}{2}\mu \cdot \sin \mu m \tag{3a}$$

$$\omega'_m(\mu) = \sum_{n=o}^{m} \sigma_n \sin \mu n = \tfrac{1}{2} \cot \tfrac{1}{2}\mu \cdot (1 - \cos \mu m) \tag{3b}$$

$$-\frac{\partial\omega}{\partial\mu} = \sum_{n=o}^{m} \sigma_n\, n \sin \mu n$$

$$= \omega' m + \omega \csc \mu - \tfrac{1}{2}m(1 + \cos \mu) \csc \mu \tag{3c}$$

$$\frac{\partial\omega'}{\partial\mu} = \sum_{n=o}^{m} \sigma_n\, n \cos \mu n = \omega m - \omega' \csc \mu \tag{3d}$$

Now, we will slightly extend the meaning of ω. Consider

$$\omega_{\frac{1}{2}M}(\mu) = \sum_{n=o}^{\frac{1}{2}M} \sigma_n \cos \mu n$$

where M is an integer. If M is even, $\omega_{\frac{1}{2}M}(\mu)$ is given by (1a) and (2a), but if M is odd, the upper limit, $\tfrac{1}{2}M$, does not exist as an integer. In the latter case, it must be understood that the summation extends only to include $\tfrac{1}{2}(M-1)$ and since $n \neq \tfrac{1}{2}M$ at that limit as written, $\sigma_n = 1$ there. Following the derivation of (2a), we find that

$$\omega_{\frac{1}{2}M}(\mu) = \tfrac{1}{2} \csc \tfrac{1}{2}\mu \cdot \sin \tfrac{1}{2}\mu M \text{ ; if M is odd} \tag{3e}$$

Again, all indeterminants are to be resolved in a limiting sense.

III. Smoothing in an Infinite Domain

First we assert

$$f_j = \frac{1}{\pi} \sum_{i=-\infty}^{+\infty} f_i \int_0^{\pi} \cos \lambda(j-i) d\lambda \qquad (4)$$

where i and j are integers, and f is an arbitrary function, not necessarily periodic, given at the set of integer j. This is a true assertion, for the integral vanishes except for j = i; where it is π. Thus there is only one non-zero term in the summation; namely, f_j itself.

Note that (4) is equivalent to the set

$$f_j = \frac{1}{\pi} \int_0^{\pi} [a(\lambda) \cdot \cos \lambda j + b(\lambda) \cdot \sin \lambda j] d\lambda \qquad (5a)$$

$$a(\lambda) = \sum_{j=-\infty}^{+\infty} f_j \cos \lambda j \qquad (5b)$$

$$b(\lambda) = \sum_{j=-\infty}^{+\infty} f_j \sin \lambda j \qquad (5c)$$

if f_j is appropriately restricted for very large j so that the summations in (5b) and (5c) are unambiguous. The equivalence of (4) and (5) can easily be shown by substituting from (5b) and (5c) into (5a). In making such a substitution, the variable of summation, j, must be changed to, say i, for a and b are not functions of j in (5a).

Now, looking at (5), we will construct a function, $\bar{f}_j$, related to f_j, through its spectral components:

$$\bar{f}_j = \frac{1}{\pi} \int_0^{\pi} [\bar{a}(\lambda) \cdot \cos \lambda j + \bar{b}(\lambda) \cdot \sin \lambda j] d\lambda \qquad (6a)$$

$$\bar{a}(\lambda) = w(\lambda) \cdot a(\lambda) = w(\lambda) \cdot \sum_{j=-\infty}^{+\infty} f_j \cos \lambda j \qquad (6b)$$

$$\bar{b}(\lambda) = w(\lambda) \cdot b(\lambda) = w(\lambda) \cdot \sum_{j=-\infty}^{+\infty} f_j \sin \lambda j \qquad (6c)$$

We have multiplied the sine and cosine phase of each component by the same number, $w(\lambda)$, so have not changed their phases. The function $\bar{f}_j$ therefore is in the nature of a smoothed (or unsmoothed) f_j.

If we not substitute from (6b) and (6c) into (6a), we may write the result

$$\bar{f}_j = \frac{1}{\pi} \sum_{i=-\infty}^{+\infty} f_i \int_0^\pi w(\lambda) \cdot \cos \lambda(j - i) d\lambda \tag{7}$$

If we next invent a variable, i',

$$i' = i - j$$

and substitute into (7), we may write the result as the set

$$\bar{f}_j = \sum_{i=-\infty}^{\infty} f_{j+i} W_i \tag{8a}$$

$$W_i = \frac{1}{\pi} \int_0^\pi w(\lambda) \cdot \cos \lambda i \cdot d\lambda \tag{8b}$$

But (8b) has the form of (5a), where $a(\lambda) = w(\lambda)$ and $b(\lambda) = 0$, and therefore,

$$w(\lambda) = \sum_{i=-\infty}^{\infty} W_i \cos \lambda i$$

Equation (8b) shows W_i to be an even function of i, that is, $W_{-i} = W_{+i}$, therefore

$$w(\lambda) = 2 \sum_{i=0}^{\infty} \sigma_i W_i \cos \lambda i \tag{8c}$$

As with f_j in (5a), W_i in (8c) must be appropriately restricted for very large i.

Equation (8a) may be regarded as describing a "smoothing" process, with i being the distance from the central point, j, and W_i being the corresponding weight given each point in the "smoothing." Given a set of such weights, W_i, the response may be determined by (8c). If, on the other hand, we were given a desired response, $w(\lambda)$, the "smoothing" weights, W_i, could be determined from (8b).

IV. Smoothing Periodic Functions

If f_j cycles in J points, that is, if

$$f_{j-kJ} = f_j$$

where k is any integer, we assert

$$f_j = \frac{2}{J} \sum_{i=o}^{J} \sigma_i f_i \sum_{\ell=o}^{\frac{1}{2}J} \sigma_\ell \cos \frac{2\pi\ell(j-i)}{J} \tag{9a}$$

where ℓ is an integer, and may be related to λ in (4):

$$\lambda_\ell = \frac{2\pi\ell}{J} \tag{9b}$$

In (9) the upper limit, $\frac{1}{2}J$, on the summation over ℓ does not exist as an integer if J is odd. In that case it is to be understood that the summation extends only to include $\ell = \frac{1}{2}(J-1)$, where since $\ell \neq \frac{1}{2}J$, $\sigma_\ell = 1$.

Equation (9) is true for reasons similar to those for (4). The summation over ℓ, according to (3a), is if J is even:

$$\omega_{\frac{1}{2}J}\left(\frac{2\pi(j-i)}{J}\right) = \frac{1}{2} \cot \frac{\pi(j-i)}{J} \cdot \sin\left(\frac{2\pi(j-i)}{J} \cdot \frac{J}{2}\right)$$

which vanishes unless

$$\frac{2\pi(j-i)}{J} = 2k\pi$$

i.e., $i = j - kJ$. In that case its limiting value is $\frac{1}{2}J$. If J is odd, we use (3e) for the summation over ℓ:

$$\omega_{\frac{1}{2}J}\left(\frac{2\pi(j-i)}{J}\right) = \frac{1}{2} \csc \frac{\pi(j-i)}{J} \cdot \sin\left(\frac{2\pi(j-i)}{J} \cdot \frac{J}{2}\right)$$

which also vanishes unless $i = j - kJ$, in which case its limiting value is also $\frac{1}{2}J$.

Again, in (9) as in (4), unless $j = kJ$, where k is any integer, there is only one term in the summation over i, namely f_{j-kJ}, and (9) is merely a reassertion of the periodicity of f:

$$f_j = f_{j-kJ} \text{ .}$$

If $j = kJ$, there are two terms each equal to $J/2$, but they are the terms for $i = 0$ and $i = J$, for which $\sigma_i = \frac{1}{2}$. Note that (9) is equivalent to the system

$$f_j = \sum_{\ell=o}^{\frac{1}{2}J} \sigma_\ell \left(a_\ell \cos \lambda_\ell j + b_\ell \sin \lambda_\ell j\right) \tag{10a}$$

$$a_\ell = \frac{2}{J} \sum_{j=o}^{J} \sigma_j f_j \cos \lambda_\ell j \tag{10b}$$

$$b_\ell = \frac{2}{J} \sum_{j=o}^{J} \sigma_j f_j \sin \lambda_\ell j \tag{10c}$$

Now, we construct a function, $\bar{f}_j$, related to f_j through its spectral components:

$$\bar{f}_j = \sum_{\ell=o}^{\frac{1}{2}J} \sigma_\ell \left(\bar{a}_\ell \cos \lambda_\ell j + \bar{b}_\ell \sin \lambda_\ell j\right) \tag{11a}$$

$$\bar{a}_\ell = w_\ell a_\ell = w_\ell \cdot \frac{2}{J} \sum_{j=o}^{J} \sigma_j f_j \cos \lambda_\ell j \tag{11b}$$

$$\bar{b}_\ell = w_\ell b_\ell = w_\ell \cdot \frac{2}{J} \sum_{j=o}^{J} \sigma_j f_j \sin \lambda_\ell j \tag{11c}$$

If we substitute from (11b) and (11c) into (11a), we may write the result,

$$\bar{f}_j = \frac{2}{J} \sum_{i=o}^{J} \sigma_i f_i \sum_{\ell=o}^{\frac{1}{2}J} \sigma_\ell w_\ell \cos \lambda_\ell (j-i) \tag{12}$$

We next invent a variable, i',

$$i' = i - j$$

and substitute into (12), at the same time noting that any limits which differ by J may be used in the summation over i because of the periodicity of f_j. We write the result as

$$\overline{f}_j = \sum_{i=-\frac{1}{2}J}^{\frac{1}{2}J} \sigma_i f_{j\pm i} W_i \tag{13a}$$

$$W_i = \frac{2}{J} \sum_{\ell=o}^{\frac{1}{2}J} \sigma_\ell w_\ell \cos \lambda_\ell i \tag{13b}$$

But (13b) has the form of (10a) with $a_\ell = \frac{2}{J} w_\ell$ and $b_\ell = 0$, and therefore,

$$w_\ell = \sum_{i=o}^{J} \sigma_i W_i \cos \lambda_\ell i$$

Because W_i, according to (13b) is periodic and even in i,

$$w_\ell = 2 \sum_{i=o}^{\frac{1}{2}J} \sigma_i W_i \cos \lambda_\ell i \tag{13c}$$

Equation (13a) describes a "smoothing" process, with "smoothing" weights and responses related to each other by (13b) and (13c).

V. Maxima of $|\omega'|$

Now, consider some of the properties of $\omega_m'(\mu)$. The fundamental definition of ω' is

$$\omega_m'(\mu) = \sum_{n=o}^{m} \sigma_n \sin \mu n$$

and therefore,

$$\omega_m'(k\pi) = 0; \text{ k is any integer}$$

Furthermore, ω' is periodic in μ, with period 2π:

$$\omega_m'(\mu + 2k\pi) = \omega_m'(\mu)$$

and it is odd in μ:

$$\omega_m'(-\mu) = - \omega_m'(\mu)$$

Thus, ω' at $\mu = k\pi$ is not only zero, it changes sign there and therefore is neither a maximum nor minimum there. For the same reasons, we also conclude that in examining ω' for its largest absolute value, we may limit our attention to the range

$$0 < \mu < \pi .$$

We can, however, quickly limit our attention to an even smaller range. We have

$$\omega' = \cot \tfrac{1}{2}\mu \cdot \sin^2 \tfrac{1}{2}\mu m$$

made up of the two factors, $\cot \frac{1}{2}\mu$ and $\sin^2 \frac{1}{2}\mu m$. The factor $\sin^2 \frac{1}{2}\mu m$ is positive definite, and $0 < \cot \frac{1}{2}\mu < \infty$ in the range $0 < \mu \leq \pi$, and therefore ω' is everywhere positive in our range. Now,

$$\left.\begin{array}{l} 0 < \cot \tfrac{1}{2}\mu < \cot \dfrac{\pi}{2m} \\ 0 \leq \sin^2 \tfrac{1}{2}\mu m \leq 1 \end{array}\right\} \text{ if } \frac{\pi}{m} < \mu < \pi$$

and therefore

$$0 \leq \omega' < \cot \frac{\pi}{2m} \text{ if } \frac{\pi}{m} < \mu < \pi$$

But

$$\omega'\left(\frac{\pi}{m}\right) = \cot \frac{\pi}{2m}$$

Therefore, the largest maximum is not in the range $\frac{\pi}{m} < \mu < \pi$, and we now limit our attention to the range

$$0 < \mu \leq \frac{\pi}{m}$$

The necessary condition for maxima and minima is

$$0 = \frac{\partial\omega'}{\partial\mu} = \frac{\sin \frac{1}{2}\mu m}{\sin \frac{1}{2}\mu} \left(m \cos \tfrac{1}{2}\mu \cdot \cos \tfrac{1}{2}\mu m - \frac{\sin \frac{1}{2}\mu m}{2 \sin \frac{1}{2}\mu} \right)$$

Note that this condition is not satisfied for $\mu = \frac{\pi}{m}$:

$$\frac{\partial\omega'}{\partial\mu}\left(\frac{\pi}{m}\right) = -\tfrac{1}{2} \csc^2 \frac{\pi}{2m}$$

and its slope there is negative, consistent with a maximum closer to the origin. We therefore further limit our attention to the range

$$0 < \mu < \frac{\pi}{m}$$

In that range, our necessary condition may be written

$$0 = \tan \tfrac{1}{2}\mu m - m \sin \mu$$

We call the right-hand member F:

$$F = \tan \tfrac{1}{2}\mu m - m \sin \mu$$

and consider it along with its first two derivatives with respect to μ:

$$F' = m(\tfrac{1}{2}\sec^2 \tfrac{1}{2}\mu m - \cos\mu)$$

$$F'' = m(\tfrac{1}{2}m \sec^2 \tfrac{1}{2}\mu m \cdot \tan \tfrac{1}{2}\mu m + \sin\mu)$$

Now

$$\lim_{\mu\to 0} F = 0$$

and

$$\lim_{\mu\to 0} F' = -\tfrac{1}{2}m < 0$$

which shows $F < 0$ near the origin. But

$$\lim_{\mu\to\frac{\pi}{m}} F = +\infty$$

which shows $F = 0$ has at least one root in the range $0 < \mu < \frac{\pi}{m}$. But in that range, $F'' < 0$, and therefore $F = 0$ has only one root there.

The range in which the largest maximum occurs can be further limited. Because for $0 < \mu$,

$$m \sin\mu < m\mu,$$

therefore

$$\tan \tfrac{1}{2}\mu m < m\mu$$

or,

$$\frac{\tan \frac{1}{2}\mu m}{\frac{1}{2}m\mu} < 2$$

Because the left-hand member is unity in the neighborhood of the origin, and monotonically increases to infinity at $\mu = \frac{\pi}{m}$,

$$\mu m < (\mu m)_c$$

where $(\mu m)_c$ satisfies

$$\tan \tfrac{1}{2}(\mu m)_c = (\mu m)_c \qquad 0 < (\mu m)_c < \pi$$

which has a unique solution:

$$(\mu m)_c = 0.74202\pi$$

On the other hand, because for $0 \leq \pi \leq \frac{\pi}{m}$,

$$\sin m\mu \leq m \sin \mu$$

therefore

$$\sin m\mu \leq \tan \tfrac{1}{2}\mu m$$

or,

$$\cos m\mu \leq 0$$

and, therefore

$$m\mu \leq \tfrac{1}{2}\pi$$

To summarize, for the largest maximum of the absolute value of $\omega'(\mu)$, μ satisfies the set:

$$\tan \tfrac{1}{2}\mu m - m \sin \mu = 0$$

$$0.5\pi \leq \mu m < 0.74202\pi$$

$$1 \leq m < \infty$$

VI. General Discrete Operators

Our previous use here of the term "smoothing operators" has meant those linear operators that do not change phase relationships. For illustration, in constructing $\bar{f}_j$ in (6) and (11), we multiplied $a(\lambda)$ and $b(\lambda)$ by the same variable, $w(\lambda)$, to get $\bar{a}(\lambda)$ and $\bar{b}(\lambda)$, respectively. We have shown that an operator that does not change phase relationships is symmetrical, or "even," i.e., $W_{-i} = W_i$. In this section, which has been added to an earlier version of this Office Note, we will develop the theory for general discrete operator.

Parenthetically, we should point out that "smoothing," as we have previously defined it, does not necessarily leave the mean value of the field unchanged. For instance, a second-difference operator, such as $W_i = 1, -2, 1$, for $i = -1, 0, +1$, respectively, does not change phase relationships, and is therefore called a smoothing operator, but reduces the mean value of the field to zero. More commonly, the term "smoothing" implies that the mean value of the field ($\lambda = 0$) is unaffected, as well as phase relationships. This simply implies that

$$\sum_{i=-\infty}^{+\infty} W_i = 1.$$

In order to avoid confusion in this section, we will call operators "even" if $W_{-i} = W_{+i}$; and "odd" if $W_{-i} = -W_{+i}$. "Symmetrical" operators are "even."

Now, according to (5), a general discrete function, f_j, may be written

$$f_j = \frac{1}{\pi} \int_0^\pi [a(\lambda) \cdot \cos \lambda j + b(\lambda) \cdot \sin \lambda j] d\lambda \tag{14a}$$

$$a(\lambda) = \sum_{i=-\infty}^{+\infty} f_i \cos \lambda i \tag{14b}$$

$$b(\lambda) = \sum_{i=-\infty}^{+\infty} f_i \sin \lambda i \tag{14c}$$

We apply to f_j a general discrete operator, with weights W_i, writing the result $\overline{f}_j$:

$$\overline{f}_j = \sum_{i=-\infty}^{+\infty} f_{j+i} W_i \tag{15a}$$

The function $\overline{f}_j$ may be written in spectral space:

$$\overline{f}_j = \frac{1}{\pi} \int_0^\pi [\overline{a}(\lambda) \cdot \cos \lambda j + \overline{b}(\lambda) \cdot \sin \lambda j] d\lambda \tag{15b}$$

and so may W_i:

$$W_i = \frac{1}{\pi} \int_0^\pi [\alpha(\lambda) \cdot \cos \lambda i + \beta(\lambda) \cdot \sin \lambda i] d\lambda \tag{16}$$

$$\alpha(\lambda) = \sum_{i=-\infty}^{\infty} W_i \cos \lambda i \tag{17a}$$

$$\beta(\lambda) = \sum_{i=-\infty}^{\infty} W_i \sin \lambda i \tag{17b}$$

If $\beta = 0$, then $W_{-i} = W_i$ and the operator is even. On the other hand, if $\alpha = 0$, then $W_{-i} = -W_i$ and the operator is odd.

Substitution from (16) into (15a) yields

$$\overline{f}_j = \frac{1}{\pi} \sum_{i=-\infty}^{+\infty} f_{j+i} \int_0^\pi [\alpha(\lambda) \cdot \cos \lambda i + \beta(\lambda) \cdot \sin \lambda i] d\lambda$$

Inventing a new variable of summation

$$i' = j + i$$

and substituting, after some manipulation we get

$$\overline{f}_j = \frac{1}{\pi}\int_0^{\pi} \sum_{i=-\infty}^{+\infty} f_i \begin{bmatrix} [\alpha(\lambda)\cdot \cos \lambda i + \beta(\lambda)\cdot \sin \lambda i] \cos \lambda j \\ +[\alpha(\lambda)\cdot \sin \lambda i - \beta(\lambda)\cdot \cos \lambda i] \sin \lambda j \end{bmatrix} d\lambda$$

Comparing this with (15b), we find

$$\bar{a}(\lambda) = \sum_{i=-\infty}^{+\infty} f_i \, [\alpha(\lambda) \cos \lambda i + \beta(\lambda) \sin \lambda i]$$

$$\bar{b}(\lambda) = \sum_{i=-\infty}^{+\infty} f_i \, [\alpha(\lambda) \sin \lambda i - \beta(\lambda) \cos \lambda i]$$

Therefore, because of (14b) and (14c),

$$\bar{a}(\lambda) = \alpha(\lambda)\cdot a(\lambda) + \beta(\lambda)\cdot b(\lambda) \tag{18a}$$

$$\bar{b}(\lambda) = -\beta(\lambda)\cdot a(\lambda) + \alpha(\lambda)\cdot b(\lambda) \tag{18b}$$

Thus, given the set of weights, W_i, associated with an operator, its response is determined by (17) and (18). If on the other hand, the response [$\bar{a}(\lambda)$ and $\bar{b}(\lambda)$ or equivalent information] is given, the weights are determined by (16) and the solution of (18) for α and β:

$$\alpha(\lambda) = \frac{\bar{a}\ a + \bar{b}\ b}{a^2 + b^2} \tag{19a}$$

$$\beta(\lambda) = \frac{\bar{a}\ b - a\ \bar{b}}{a^2 + b^2} \tag{19b}$$

A general discrete operator can be separated into two parts, an even part and an odd part. To show this, we take a general discrete operator with weights W_i, and invent two new operators, with weights A_i and B_i, and with each related to W_i:

$$A_i = \tfrac{1}{2}(W_i + W_{-i}) \tag{20a}$$

$$B_i = \tfrac{1}{2}(W_i - W_{-i}) \tag{20b}$$

Note that W_i is the sum of A_i and B_i:

$$W_i = A_i + B_i \tag{21}$$

And at the central point

$$A_o = W_o$$

$$B_o = 0$$

Also note that A_i is even, and B_i is odd:

$$A_{-i} = \tfrac{1}{2}(W_{-i} + W_{+i}) = +A_i \quad (22a)$$

$$B_{-i} = \tfrac{1}{2}(W_{-i} - W_{+i}) = -B_i \quad (22b)$$

Noting (22), and substituting from (21) into (17), we get

$$\alpha(\lambda) = 2\sum_{i=o}^{\infty} \sigma_i A_i \cos \lambda i \quad (23a)$$

$$\beta(\lambda) = 2\sum_{i=1}^{\infty} B_i \sin \lambda i \quad (23b)$$

Now consider two general discrete operators, with weights W_i^o and W_i', which operate successively on a field f_j defined by (14). The result of the operation with W_i^o by itself is

$$\overline{f}_j^o = \frac{1}{\pi}\int_o^{\pi} [\overline{a}^o(\lambda)\cdot \cos \lambda j + \overline{b}^o(\lambda)\cdot \sin \lambda j]d\lambda$$

$$\overline{a}^o = +\alpha^o a + \beta^o b \quad (24a)$$

$$\overline{b}^o = -\beta^o a + \alpha^o b \quad (24b)$$

Next W_i' operates on $\overline{f}_j^o$, and we call the result $\overline{f}_j$:

$$\overline{f}_j = \frac{1}{\pi}\int_o^{\pi} [\overline{a}\ (\lambda)\cdot \cos \lambda j + \overline{b}(\lambda)\cdot \sin \lambda j]d\lambda$$

$$\overline{a} = \alpha'\overline{a}^o + \beta'\overline{b}^o \quad (25a)$$

$$\overline{b} = -\beta'\overline{a}^o + \alpha'\overline{b}^o \quad (25b)$$

Substituting from (24) into (25), we find

$$\overline{a} = \alpha\ a + \beta\ b \quad (26a)$$

$$\overline{b} = -\beta a + \alpha\ \beta \quad (26b)$$

where

$$\alpha = \alpha^o\alpha' - \beta^o\beta' \quad (27a)$$

$$\beta = \alpha^o\beta' + \alpha'\beta^o \quad (27b)$$

Now, looking at (27), a sufficient condition for the combined response of W_i° and W_i' to be even and positive for all λ is

$$\alpha' = +\alpha^{\circ} \quad \text{and} \tag{28a}$$

$$\beta' = -\beta^{\circ} \tag{28b}$$

for then

$$\alpha(\alpha^{\circ},\beta^{\circ},\alpha',\beta') = \alpha^{\circ 2} + \beta^{\circ 2} \tag{29a}$$

$$\beta(\alpha^{\circ},\beta^{\circ},\alpha',\beta') = 0 \tag{29b}$$

Equations (29) imply a relationship between W_i° and W_i', for according to (16),

$$W_i^{\circ} = \frac{1}{\pi}\int_0^{\pi} [\alpha^{\circ}(\lambda)\cdot \cos \lambda i + \beta^{\circ}(\lambda)\cdot \sin \lambda i]d\lambda$$

$$W_i' = \frac{1}{\pi}\int_0^{\pi} [\alpha^{\circ}(\lambda)\cdot \cos \lambda i - \beta^{\circ}(\lambda)\cdot \sin \lambda i]d\lambda$$

Changing the sign of i everywhere in the last equation, we find

$$W'_{-i} = W^{\circ}_{+i} \tag{30}$$

which is a sufficient condition on the weights for an even and globally positive response.