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Application of Arakawa's Energy-conserving Layer Model to Operational Numerical Weather Prediction

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In his general circulation modelling, Arakawa has extended Lorenz's 1960 analysis of energy-conserving systems to allow for variable pressure at the bottom of the atmosphere in a "sigma" system.[1] This note analyzes this energy-conserving sigma system with respect to its application to prediction with real data.

Two questions arise immediately in an operational context which are less obvious when modelling the general circulation:

(1) If the input data is the field of geopotential, what pressure level in the atmosphere has a geopotential equal to that associated with a given model layer? Also, how accurate are the potential temperatures deduced from input geopotentials via the hydrostatic difference relations of the model?

(2) Horizontal truncation effects will corrupt computations of the horizontal pressure force in a sigma system. How can these errors be made small when, in the presence of orography, the initial geopotential data is taken from the real atmosphere (where horizontal truncation does not exist)?

The analysis is presented in four sections. In section 1, the continuous equations are analyzed using an arbitrary monotonic function

[1] Lorenz, E., 1960: Energy and numerical weather prediction. Tellus, 12, 364-373.

of pressure $F(p)$ to construct a generalized sigma system. This is explored because functions of p such as p^{κ} and $\ln p$ have the property that sigma surfaces defined by them can become isobaric more rapidly with height than does the usual p- system. A second concept is introduced in section 1, that of an adiabatic reference atmosphere. This allows the geopotential to be expressed as its deviation from this reference atmosphere, and offers the possibility of significantly reducing the orographic truncation effect mentioned in question 2 above. It is shown in section 1 that a well-defined energy integral exists when this device is used.

Arakawa's vertical finite-difference scheme is introduced in section 2. Special attention is given to an accurate assignment of p^{κ} for each layer. This is accomplished by asking the model's initial data to duplicate the atmosphere with respect to the enthalpy and to the potential temperature represented in each layer. It results in a different definition of p^{κ} than that used by Arakawa.

Section 3 contains experimental tests of question 1 above. These show satisfactory values of deduced θ in all cases except for Arakawa's original choice of p^{κ}. A quantitative justification is also made in this section for identifying the velocity and geopotential assigned to each layer with that measured in the atmosphere at the value of p^{κ} assigned to that layer.

Section 4 contains computations of the fictitious horizontal pressure force introduced from horizontal truncation effects in a sigma system when the true geopotential is a function of pressure only. This is a test of answers to question 2 above. Best results are obtained when the adiabatic reference atmosphere is combined with $F = p$. For the extreme case of a surface height-difference of 1800 meters in 200 km, these "best results" correspond to a (vertically-averaged) fictitious horizontal acceleration of 3.6 m sec^{-1} day^{-1}, or a (vertically-averaged) fictitious geostrophic wind of 0.4 m sec^{-1}, or a gravity-wave oscillation of surface pressure with an amplitude of 0.1 mb. These appear to be about 10 percent of the same errors in a simplified NMC model with the same number of vertical levels, but using the total geopotential. The more exotic choices of $F = p^{\kappa}$ or $F = \ln p$ do better than $F = p$ in the upper layers but considerably worse in the lower layers.

1. The differential equations

p = pressure ÷ 100 cb

p_s = surface value of p

p_T = constant p at top of model (1.1)

$F(p)$ = function of p used as unnormalized vertical coordinate (F must, of course, be monotonic)

$$H = F_s - F_T = F(p_s) - F(p_T)$$

The normalized vertical coordinate, σ , ranges from 0 at the ground to 1 at p_T :

$$\sigma = 1 - \frac{F - F_T}{H} \qquad (1.2)$$

or

$$F = F_T + H(1 - \sigma) \qquad (1.3)$$

The following notation is also needed

$$r = H \frac{dp}{dF} = H \div \frac{dF}{dp}$$

$$W = \dot{\sigma} \;,\quad \omega = r W \qquad (1.4)$$

$$\vec{U} = r \vec{V}$$

where $\vec{V}$ is the horizontal velocity. r is a generalized counterpart to the term "$\partial p/\partial\sigma$" used at NMC.

Partial derivatives transform as follows, where $\xi = x, y, \text{or } t$, unsubscripted derivatives are at constant σ, and f is any quantity.

$$\frac{\partial}{\partial\sigma} = -r\frac{\partial}{\partial p} ,$$

$$\left(\frac{\partial f}{\partial\xi}\right)_p = \frac{\partial f}{\partial\xi} + \frac{(1-\sigma)}{H}\frac{\partial f}{\partial\sigma}\frac{\partial H}{\partial\xi} , \tag{1.5}$$

$$r\left(\frac{\partial f}{\partial\xi}\right)_p = \frac{\partial}{\partial\xi}(rf) + \frac{\partial}{\partial\sigma}\left[(1-\sigma)f\frac{dp}{dF}\right]\frac{\partial H}{\partial\xi} .$$

Using the relation

$$\mathring{p} = \frac{r}{H}(1-\sigma)\mathring{H} - rW \tag{1.6}$$

we first derive the <u>continuity</u> <u>equation</u>:

$$\frac{\partial}{\partial\sigma}\left[\omega - (1-\sigma)\frac{dp}{dF}\frac{\partial H}{\partial t}\right] + \nabla\cdot\vec{u} = 0 . \tag{1.7}$$

Or, since

$$\frac{\partial p}{\partial\xi} = \frac{dp}{dF}(1-\sigma)\frac{\partial H}{\partial\xi} , \tag{1.8}$$

we could also write (1.7) as

$$\frac{\partial}{\partial \sigma}\left(\omega - \frac{\partial p}{\partial t}\right) + \nabla \cdot \vec{u} = 0 \tag{1.9}$$

Integration of (1.7) or (1.9) from $\sigma = 0$ to 1, with $\omega = 0$ at these boundaries, produces the familiar surface pressure tendency equation:

$$\left(\frac{dp}{dF}\right)_{\sigma=0} \frac{\partial H}{\partial t} = \frac{\partial p_s}{\partial t} = -\nabla \cdot \int_0^1 r \vec{V} \, d\sigma \tag{1.10}$$

The flux form of the prediction equations is obtained by multiplying (1.7) with f and using the relation

$$\begin{aligned}
\frac{\partial r}{\partial \xi} &= \frac{dp}{dF}\frac{\partial H}{\partial \xi} + H \frac{d^2 p}{dF^2}\frac{\partial F}{\partial \xi} \\
&= \left[\frac{dp}{dF} - (1-\sigma)\frac{\partial}{\partial \sigma}\left(\frac{dp}{dF}\right)\right]\frac{\partial H}{\partial \xi} \\
&= -\frac{\partial}{\partial \sigma}\left[(1-\sigma)\frac{dp}{dF}\right]\frac{\partial H}{\partial \xi}
\end{aligned} \tag{1.11}$$

This flux form is

$$\frac{\partial f r}{\partial t} + \nabla \cdot \vec{u} f + \frac{\partial \omega f}{\partial \sigma} = r \mathring{f} \tag{1.12}$$

The momentum equations for u and v (the components of the velocity, $\vec{V}$) take the form

$$\frac{\partial u\pi}{\partial t} + \nabla\cdot u\vec{U} + \frac{\partial \omega u}{\partial \sigma} = \pi\frac{du}{dt}$$

$$= -\frac{\partial}{\partial x}(\pi\phi) - \frac{\partial}{\partial \sigma}\left[(1-\sigma)\,\phi\,\frac{dp}{dF}\right]\frac{\partial H}{\partial x} + \pi M v + \pi X \tag{1.13}$$

where M represents rotational and curvature effects, and X represents friction. The equation for v is similar. From these, one readily derives the following kinetic energy equation, in which

$$ke = \frac{1}{2}\,(u^2+v^2) \tag{1.14}$$

$$\frac{\partial}{\partial t}(\pi\, ke) + \nabla\cdot\vec{U}(\phi+ke) + \frac{\partial}{\partial\sigma}\,\omega(\phi+ke) - \frac{\partial}{\partial\sigma}\left(\phi\,\frac{\partial p}{\partial t}\right)$$

$$= -\,\dot{p}\,\frac{\partial\phi}{\partial\sigma} + \vec{U}\cdot\vec{X} \tag{1.15}$$

Note that ϕ in (1.13) and (1.15) can be the actual geopotential or it can be the deviation of the actual geopotential from any standard distribution $\bar{\phi}(p)$, since only $\nabla_p\phi$ enters into $d\vec{V}/dt$.

The potential temperature equation is

$$\frac{\partial\theta\pi}{\partial t} + \nabla\cdot\vec{U}\theta + \frac{\partial\omega\theta}{\partial\sigma} = \pi\frac{d\theta}{dt} = \frac{\pi\, q}{C_p\, p^{\kappa}} \quad ,\ \kappa = R/C_p \tag{1.16}$$

Here we note that (1.16) is unchanged if we replace θ by θ minus a constant. The corresponding enthalpy equation is obtained by multiplying

(1.16) by $c_p\, p^{\kappa}$ and defining

$$T \equiv \Pi\,\theta\ , \quad \Pi = p^{\kappa} \tag{1.17}$$

$$\frac{\partial}{\partial t}(\pi c_p T) + \nabla\cdot\vec{u}(\pi c_p T) + \frac{\partial\, \omega c_p T}{\partial \sigma} = \pi q + \pi c_p\, \theta \frac{d\Pi}{dt} \tag{1.18}$$

Adding (1.15) and (1.18) we get

$$\begin{aligned}\frac{\partial}{\partial t}\pi(k_e + c_p T) - \frac{\partial}{\partial \sigma}\left(\phi \frac{\partial p}{\partial t}\right) + \nabla\cdot\vec{u}(k_e + c_p T + \phi) \\ + \frac{\partial}{\partial \sigma}\omega(k_e + c_p T + \phi) = \left[\pi c_p\, \theta \frac{d\Pi}{dt} - \frac{\partial \phi}{\partial \sigma}\frac{dp}{dt}\right] + \vec{u}\cdot\vec{X} + \pi q\end{aligned} \tag{1.19}$$

The square bracket in (1.19) vanishes if we recognize the hydrostatic relation in the form

$$\frac{\partial \phi}{\partial \sigma} = -\pi \frac{\partial \phi}{\partial p} = \pi \frac{RT}{p} = \pi c_p\, \theta \frac{d\Pi}{dp} \tag{1.20}$$

Here we note that if θ and ϕ a is replaced by

$$\begin{aligned}\theta &= \theta' + \bar{\theta} \quad (\bar{\theta} = \text{constant}) \\ \phi &= \phi' + \bar{\phi}(p) \\ \bar{\phi}(p) &= \bar{\phi}_0 - c_p\, \bar{\theta}\, \Pi \quad (\bar{\phi}_0 = \text{constant})\end{aligned} \tag{1.21}$$

so that

$$\frac{d\bar{\phi}}{dp} = - R\,\bar{\theta}\,\frac{\Pi}{p} \tag{1.22}$$

(1.20) becomes identically valid in the primed variables also:

$$\frac{\partial \phi'}{\partial \sigma} = r\,c_p\,\theta'\,\frac{d\Pi}{dp} \tag{1.23}$$

Under these conditions—i.e., the choice of (1.21) as a reference atmosphere —the square bracket in (1.19) still vanishes, and ϕ and θ , whenever they appear in (1.13)-(1.20), can be replaced by ϕ' and θ' if at the same time we also replace T by T' :

$$T' = T - \Pi\bar{\theta} = T - \bar{T}(p) \tag{1.24}$$

The integral energy equation is obtained by integrating (1.19).

$$\frac{\partial}{\partial t}\iiint_0^1 r\,(ke + c_p T)\,d\sigma\, dArea + \iint \phi_{\sigma=0}\,\frac{\partial p_s}{\partial t}\,dArea = \iiint_0^1 r\,(\vec{V}\cdot\vec{X} + q)\,d\sigma\, dArea \tag{1.25}$$

If we have used the total ϕ and T , $\phi_{\sigma=0}$ in the second integral is the actual ground geopotential. Since it is independent of time, the second integral becomes the well-known expression

$$\frac{\partial}{\partial t}\iint \phi_{\sigma=0}\, p_s \, dArea \tag{1.26}$$

If instead we have used ϕ' and T' in place of ϕ and T, $\phi'_{\sigma=0}$ depends on time. The surface integral in (1.25) becomes

$$\iint \phi'_{\sigma=0} \frac{\partial p_s}{\partial t}\, dArea = \frac{\partial}{\partial t}\iint \phi_{\sigma=0}\, p_s\, dArea - \iint \bar{\Phi}(p_s)\frac{\partial p_s}{\partial t}\, dArea$$

$$= \frac{\partial}{\partial t}\iint \left[\phi_{\sigma=0}\, p_s + P(p_s)\right] dArea \tag{1.27}$$

if $P(p)$ is defined as

$$\frac{dP}{dp} = -\bar{\Phi}(p) \tag{1.28}$$

We therefore have a completely closed energy integral even when we use $\phi' = \phi - \bar{\Phi}$.

To examine this further, let horizontal averages and deviations therefrom be denoted by $\langle f \rangle$ and f^*, i.e., $f = \langle f \rangle + f^*$. We first choose the constants $\bar{\Phi}_0$ and $\bar{\Theta}$ in (1.21) to fit observed data as follows

$\bar{\Phi}(p_T) = \langle \phi_T \rangle$ = average (initial) ϕ at p_T.

$\bar{\Phi}(\langle p_s \rangle) = \langle \phi_{\sigma=0} \rangle$ = average surface geopotential

This gives

$$\bar{\theta} = \left[\langle \phi_T \rangle - \langle \phi_{\sigma=0} \rangle\right] \div c_p \left[\langle p_s \rangle^{\kappa} - p_T^{\kappa}\right]$$

$$\bar{\phi}_0 = \left[\langle p_s \rangle^{\kappa} \langle \phi_T \rangle - p_T^{\kappa} \langle \phi_{\sigma=0} \rangle\right] \div \left[\langle p_s \rangle^{\kappa} - p_T^{\kappa}\right]$$

$$\bar{\phi}(p) = \frac{\langle \phi_{\sigma=0} \rangle (p^{\kappa} - p_T^{\kappa}) + \langle \phi_T \rangle (\langle p_s \rangle^{\kappa} - p^{\kappa})}{\langle p_s \rangle^{\kappa} - p_T^{\kappa}} \tag{1.29}$$

We choose P in (1.28) so that $P(\langle p_s \rangle) = 0$:

$$P(p_s) = \frac{\frac{1}{\kappa+1}\left(\langle \phi_{\sigma=0} \rangle - \langle \phi_T \rangle\right)\left(\langle p_s \rangle^{\kappa+1} - p^{\kappa+1}\right) + \left(\langle p_s \rangle^{\kappa} \langle \phi_T \rangle - p_T^{\kappa} \langle \phi_{\sigma=0} \rangle\right)\left(\langle p_s \rangle - p\right)}{\langle p_s \rangle^{\kappa} - p_T^{\kappa}}$$

This gives

$$\langle P(p_s) \rangle = \frac{c_p \bar{\theta}}{(1+\kappa)} \left\langle p_s^{\kappa+1} - \langle p_s \rangle^{\kappa+1} \right\rangle \tag{1.30}$$

The area integral of p_s is independent of time, according to the continuity equation (1.9). Thus, if $p_s = \langle p_s \rangle + p_s^*$

$$\frac{\partial}{\partial t} \langle P(p_s) \rangle = c_p \bar{\theta} \langle p_s \rangle^{\kappa} \left\langle \left[1 + \frac{p_s^*}{\langle p_s \rangle}\right]^{\kappa} \frac{\partial p_s^*}{\partial t} \right\rangle$$

Expanding this, with

$$\left[1 + \frac{p_s^*}{\langle p_s \rangle}\right]^{\kappa} \approx 1 + \kappa \frac{p_s^*}{\langle p_s \rangle} + O(p_s^{*2})$$

we get

$$\frac{\partial}{\partial t} \langle P(p_s) \rangle \approx \frac{R\bar{\theta} \langle p_s \rangle^{\kappa}}{\langle p_s \rangle} \langle p^* \frac{\partial p^*}{\partial t} \rangle \tag{1.31}$$

Thus, the added term $\partial \langle P(p_s) \rangle / \partial t$ appearing in the energy equation if we use ϕ', θ', T' in place of ϕ, θ, T behaves (approximately) like the time rate of change of a positive definite quantity. Secondly, we may note that the combined surface energy integral may be written (approximately) as

$$\left\langle \left[\phi^*_{\sigma=0} + \frac{R\bar{\theta} \langle p_s \rangle^{\kappa}}{\langle p_s \rangle} p_s^* \right] \frac{\partial p_s^*}{\partial t} \right\rangle \tag{1.32}$$

To the extent that the major contribution to $p_s^* = p_s - \langle p_s \rangle$ is due to orography we will have

$$\phi^*_{\sigma=0} \sim -\alpha\, p_s^*$$

where α is a typical surface specific volume. The square bracket in (1.32) will then tend to vanish since $R\bar{\theta} \langle p_s \rangle^{\kappa} \div \langle p_s \rangle$ is only slightly higher than a typical value of α. ($\bar{\theta}$ will be somewhat greater than $\langle \theta_s \rangle$.) In other words, the use of ϕ' , θ' with T' will result in an energy integral in which most of the contribution of $\langle \phi_{\sigma=0}\, \partial p_s / \partial t \rangle$ is cancelled out.

2. The vertical difference equations

In this section, we formulate the vertical finite differencing so as to conserve energy, following in general the technique devised by Arakawa.[1] An important addition to this technique will be introduced, however, in the choice of $\Pi = p^{\kappa}$ assigned to each layer. This addition, which appears in equation (2.26) below, is identical with a suggestion made by J. Brown.[2] The vertical structure is depicted on the following diagram:

Δ_K:

$$\hat{\sigma}_{K+1} = 1 \qquad \hat{W}_{K+1} = 0,\ \hat{\phi}_{K+1},\ \hat{\theta}_{K+1},\ \hat{\Pi}_{K+1};\ \hat{s}_{K+1} = 0,\ \hat{F}_{K+1} = F_T$$

$$\leftarrow \cdots\cdots\ \vec{V}_K,\ \text{etc.}$$

$$\hat{\sigma}_K = 1 - \Delta_K$$

Δ_k:

$$\hat{\sigma}_{k+1} = \hat{\sigma}_k + \Delta_k$$

$$\leftarrow \cdots\cdots\ \vec{V}_k,\ \phi_k,\ \theta_k,\ s_k,\ \Pi_k$$

$$\hat{\sigma}_k \ \leftarrow \cdots\ \hat{W}_k,\ \hat{\phi}_k,\ \hat{\theta}_k,\ \hat{\Pi}_k;\ \hat{s}_k,\ \hat{F}_k$$

Δ_{k-1}:

$$\leftarrow \cdots\cdots\ \vec{V}_{k-1},\ \phi_{k-1},\ \theta_{k-1},\ s_{k-1},\ \Pi_{k-1}$$

$$\hat{\sigma}_{k-1} = \hat{\sigma}_k - \Delta_{k-1}$$

Δ_1:

$$\hat{\sigma}_2 = \Delta_1$$

$$\leftarrow \cdots\cdots\ \vec{V}_1,\ \text{etc}$$

$$\hat{\sigma}_1 = 0 \qquad \hat{W}_1 = 0,\ \hat{\phi}_1,\ \hat{\theta}_1,\ \hat{\Pi}_1;\ \hat{s}_1,\ \hat{F}_1 = F_s$$

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1 A. Arakawa, 1972: Design of the UCLA General Circulation Model. Tech. Rpt. no. 7, Dept. of Meteorology, UCLA.

2 J. Brown, 1974: On vertical differencing in the σ-system. Office Note 92, NMC. See his equation (26).

Variables at the "interfaces" are denoted by carats. These include the coordinate variable itself,

$$\hat{\sigma}_k = \sum_{l=1}^{k-1} \Delta_k \quad , \quad \sum_{l=1}^{K} \Delta_k = 1 , \tag{2.1}$$

and

$$\hat{F}_k = F_T + H(1-\hat{\sigma}_k) \tag{2.2}$$

$$\hat{s}_k = (1-\hat{\sigma}_k)\left(\widehat{\frac{dp}{dF}}\right)_k \tag{2.3}$$

where

$$\left(\widehat{\frac{dp}{dF}}\right)_k \equiv \frac{dp}{dF} \quad \text{at } F = \hat{F}_k \tag{2.4}$$

and

$$\begin{aligned} H &= F_s - F_T = \hat{F}_1 - \hat{F}_{K+1} \\ \hat{F}_{K+1} &= F(p_T) = \text{constant}. \end{aligned} \tag{2.5}$$

The interface variables $\hat{\phi}_k$ and $\hat{\theta}_k$, except in the case of $\hat{\phi}_1$, are only auxiliary variables which are useful in carrying out the details of the Arakawa formulation.

The basic dependent variables, $\vec{V}$, ϕ and θ are defined within the "layer," but F and σ in the layer are never defined. To simplify notation, we will also introduce

$$\vec{U}_k = r_k \vec{V}_k$$
$$\hat{\omega}_k = \hat{r}_k \hat{W}_k \tag{2.6}$$

A specific formula for r_k will be needed. Although defined by (1.4) as $H(dp/dF)$, a more useful expression will be derived soon. A precise form for $\hat{r}_k$ is needed only if it is necessary to convert between $\hat{\omega}_k$ and $\hat{W}_k$ in (2.6). This conversion appears to be needed only if the prediction equations are not put in flux form.

The Arakawa procedure is designed to arrive at layered equations which have energy properties equivalent to the continuous system. In the Arakawa finite-difference formulation, we also end up with deriving the form of the hydrostatic relation which is necessary for energy conservation.

The derivation which follows is written in terms of the total ϕ , T and θ . However, the equations are equally valid for ϕ' , T' and θ' if these are defined as deviations from the adiabatic reference atmosphere (1.21).

The continuity equation (1.7) is first expressed as

$$\hat{\omega}_{k+1} - \hat{\omega}_k - (\hat{s}_{k+1} - \hat{s}_k)\frac{\partial H}{\partial t} = -\nabla \cdot \Delta_k \vec{U}_k \tag{2.7}$$

We sum this from $k = 1$ to K, and note that for $\xi = x, y$ or t,

$$\frac{\partial \hat{p}_k}{\partial \xi} = \left(\widehat{\frac{dp}{dF}}\right)_k \frac{\partial \hat{F}_k}{\partial \xi} = \hat{s}_k \frac{\partial H}{\partial \xi} \tag{2.8}$$

and that $\hat{\omega}_1$, $\hat{\omega}_{K+1}$ and $\hat{s}_{K+1}$ are zero. The result is

$$\hat{s}_1 \frac{\partial H}{\partial t} = \frac{\partial \hat{p}_1}{\partial t} = -\nabla \cdot \sum_{k=1}^{K} \Delta_k r_k \vec{V}_k$$

In order to have this agree with the usual formula for $\partial p_s / \partial t$ we must require that

$$\Delta_k r_k = \hat{p}_k - \hat{p}_{k+1} \tag{2.10}$$

This provides the definition of r_k. Symbolically this is equivalent to

$$r_k = \left(\frac{dp}{dF}\right)_k = \frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{F}_k - \hat{F}_{k+1}} \tag{2.11}$$

However, we do not use this to determine the value of p (e.g., Π) in the layer; condition (2.26) below will be used instead. [If this is surprising, note that (2.11) is useless to determine p_k in the special case of $F = p$, since dp/dF is then not a function of p.]

The flux form of the prediction equations is written for a layer variable ($f = \alpha$ or $\vec{V}$) as

$$\frac{\partial}{\partial t}(\pi_k f_k) + \nabla\cdot\vec{u}_k f_k + \frac{1}{\Delta_k}\left[\hat{\omega}_{k+1}\hat{f}_{k+1} - \hat{\omega}_k \hat{f}_k\right] = \pi_k\left(\frac{df}{dt}\right)_k \tag{2.12}$$

$\hat{f}_k$ is as yet undefined. However, Arakawa shows in the case $F = p$ that if we choose to conserve the function $G(f)$, the choice

$$\hat{f}_k = \frac{\left(f\frac{dG}{df} - G\right)_k - \left(f\frac{dG}{df} - G\right)_{k-1}}{\left(\frac{dG}{df}\right)_k - \left(\frac{dG}{df}\right)_{k-1}} \tag{2.13}$$

will result in an equation conserving G_k. The same formula (2.13) can be shown to hold in this generalized σ-system. It results in the equation

$$\begin{aligned}\frac{\partial}{\partial t}(\pi_k G_k) + \nabla\cdot\vec{u}_k G_k + \frac{1}{\Delta_k}\left(\hat{\omega}_{k+1}\hat{G}_{k+1} - \hat{\omega}_k \hat{G}_k\right) &= \left(\frac{dG}{df}\right)_k \pi_k \left(\frac{df}{dt}\right)_k \\ &= \pi_k\left(\frac{dG}{dt}\right)_k\end{aligned} \tag{2.14}$$

with

$$\hat{G}_k = (\hat{f}_k - f_k)\left(\frac{dG}{df}\right)_k + G_k \tag{2.15}$$

We will probably make use of this only with $G = f^2$, in which

$$\hat{f}_k = \frac{1}{2}(f_k + f_{k-1}) \quad , \quad \hat{G}_k = f_k f_{k-1} \tag{2.16}$$

The horizontal momentum equations are written in flux form. For example, (1.13) is expressed as

$$\frac{\partial r_k u_k}{\partial t} + \nabla\cdot\vec{U}_k u_k + \frac{1}{\Delta_k}\left(\hat{\omega}_{k+1}\hat{u}_{k+1} - \hat{\omega}_k \hat{u}_k\right) = r_k\left(\frac{du}{dt}\right)_k \tag{2.17}$$

$$= -\frac{\partial}{\partial x}(r_k \phi_k) - \frac{1}{\Delta_k}\left(\hat{\phi}_{k+1}\hat{s}_{k+1} - \hat{\phi}_k \hat{s}_k\right)\frac{\partial H}{\partial x} + r_k M_k v_k + r_k X_k$$

with a similar equation for v_k and with $\hat{u}_k$ defined by (2.16). $\hat{\phi}_k$ will be ~~defined~~ considered later, but $\hat{\phi}_1$ equals the ground geopotential.

Dot multiplication of these with $\vec{V}_k$, followed by use of (2.10) and (2.8), results in the following kinetic energy equation $(ke_k \equiv \frac{1}{2}\vec{V}_k^2)$:

$$\frac{\partial}{\partial t}(r_k ke_k) + \nabla\cdot\vec{U}_k(ke_k + \phi_k) + \frac{1}{\Delta_k}\left[\hat{\omega}_{k+1}\left(\tfrac{1}{2}\vec{V}_{k+1}\cdot\vec{V}_k + \hat{\phi}_{k+1}\right) - \hat{\omega}_k\left(\tfrac{1}{2}\vec{V}_k\cdot\vec{V}_{k-1} + \hat{\phi}_k\right)\right]$$

$$- \frac{1}{\Delta_k}\left(\hat{\phi}_{k+1}\hat{s}_{k+1} - \hat{\phi}_k\hat{s}_k\right)\frac{\partial H}{\partial t} = \vec{U}_k\cdot\vec{X}_k + \frac{1}{\Delta_k}\left[\hat{\omega}_{k+1}(\hat{\phi}_{k+1} - \phi_k) - \hat{\omega}_k(\hat{\phi}_k - \phi_k)\right]$$

$$+ \frac{1}{\Delta_k}\left(\frac{\partial H}{\partial t} + \vec{V}_k\cdot\nabla H\right)\left[\phi_k(\hat{s}_{k+1} - \hat{s}_k) - (\hat{\phi}_{k+1}\hat{s}_{k+1} - \hat{\phi}_k\hat{s}_k)\right]. \tag{2.18}$$

The potential temperature equation (1.16) is written in flux form as

$$\frac{\partial r_k \theta_k}{\partial t} + \nabla\cdot\vec{U}_k\theta_k + \frac{1}{\Delta_k}\left(\hat{\omega}_{k+1}\hat{\theta}_{k+1} - \hat{\omega}_k\hat{\theta}_k\right) = r_k\left(\frac{g}{c_p p^{\kappa}}\right)_k \tag{2.19}$$

We multiply this by $c_p \Pi_k$ and <u>define</u>

$$T_k = \Pi_k \theta_k \tag{2.20}$$

The result is a finite-difference equivalent of the enthalpy equation (1.18):

$$\frac{\partial}{\partial t}(\pi_k c_p T_k)+\nabla\cdot\vec{U}_k c_p T_k + \frac{c_p}{\Delta_k}(\hat\omega_{k+1}\hat T_{k+1}-\hat\omega_k \hat T_k) = \pi_k q_k$$
$$+ c_p \pi_k \theta_k\left(\frac{\partial \Pi_k}{\partial t}+\vec V_k\cdot\nabla\Pi_k\right) + \frac{c_p}{\Delta_k}\left[\hat\omega_{k+1}(\hat T_{k+1}-\Pi_k\hat\theta_{k+1})-\hat\omega_k(\hat T_k-\Pi_k\hat\theta_k)\right]. \qquad (2.21)$$

$\hat T_k$ is as yet undefined, as is the choice of $\hat\theta_k$ in (2.19).

We add this to the kinetic energy equation (2.18) and use (2.8):

$$\frac{\partial}{\partial t}\left[\pi_k(ke_k+c_pT_k)\right]-\frac{1}{\Delta_k}\left(\hat\phi_{k+1}\frac{\partial \hat p_{k+1}}{\partial t}-\hat\phi_k\frac{\partial \hat p_k}{\partial t}\right)+\nabla\cdot\vec U_k(ke_k+c_pT_k+\phi_k)$$
$$+\frac{1}{\Delta_k}\left[\hat\omega_{k+1}\left(\tfrac{1}{2}\vec V_{k+1}\cdot\vec V_k+c_p\hat T_{k+1}+\hat\phi_{k+1}\right)-\hat\omega_k\left(\tfrac{1}{2}\vec V_k\cdot\vec V_{k-1}+c_p\hat T_k+\hat\phi_k\right)\right]-\pi_k(q_k+\vec V_k\cdot\vec X_k)$$
$$=\frac{1}{\Delta_k}\left\{\hat\omega_{k+1}\left[\hat\phi_{k+1}-\phi_k+c_p(\hat T_{k+1}-\Pi_k\hat\theta_{k+1})\right]-\hat\omega_k\left[\hat\phi_k-\phi_k+c_p(\hat T_k-\Pi_k\hat\theta_k)\right]\right\}$$
$$+c_p\pi_k\theta_k\left(\frac{\partial\Pi_k}{\partial t}+\vec V_b\cdot\nabla\Pi_k\right) \qquad (2.22)$$
$$+\frac{1}{\Delta_k}\left(\frac{\partial H}{\partial t}+\vec V_b\cdot\nabla H\right)\left[\phi_k(\hat\sigma_{k+1}-\hat\sigma_k)-(\hat\phi_{k+1}\hat\sigma_{k+1}-\hat\phi_k\hat\sigma_k)\right]$$

This equation is analogous to the continuous form (1.19) if on the right-hand side the last two terms cancel for $k=1,2,\cdots,K$ and the coefficients of $\hat\omega_{k+1}$ and $\hat\omega_k$ vanish for, respectively, $k=1,2,\cdots,K-1$, and $k=2,3,\cdots,K$. Taking the latter two conditions, we have, for $k=2,3,\cdots,K$:

$$\hat{\phi}_k - \phi_{k-1} + c_p(\hat{T}_k - \Pi_{k-1}\hat{\theta}_k) = 0$$

$$\hat{\phi}_k - \phi_k + c_p(\hat{T}_k - \Pi_k\hat{\theta}_k) = 0 \qquad (2.23)$$

which separate into

$$\phi_k - \phi_{k-1} = c_p(\Pi_{k-1} - \Pi_k)\hat{\theta}_k \; ; \; k = 2, 3, \cdots, K \qquad (2.24a)$$

and

$$\hat{\phi}_k + c_p\hat{T}_k = \frac{1}{2}(\phi_{k-1} + \phi_k) + \frac{c_p}{2}(\Pi_k + \Pi_{k-1})\hat{\theta}_k \; ; \; k = 2, 3, \cdots, K. \qquad (2.24b)$$

The combined energy equation now reduces to

$$\frac{\partial}{\partial t}\left[\Delta_k(ke_k + c_p T_k)\right] - \frac{1}{\Delta_k}\left(\hat{\phi}_{k+1}\frac{\partial\hat{p}_{k+1}}{\partial t} - \hat{\phi}_k\frac{\partial\hat{p}_k}{\partial t}\right) + \nabla\cdot\vec{U}\,(ke_k + c_p T_k + \phi_k)$$

$$+ \frac{1}{2\Delta_k}\left\{\hat{\omega}_{k+1}\left[\vec{V}_{k+1}\cdot\vec{V}_k + \phi_{k+1} + \phi_k + c_p(\Pi_{k+1} + \Pi_k)\hat{\theta}_{k+1}\right] - \hat{\omega}_k\left[\vec{V}_k\cdot\vec{V}_{k-1} + \phi_k + \phi_{k-1} + c_p(\Pi_k + \Pi_{k-1})\hat{\theta}_k\right]\right\}$$

$$= \Delta_k(g_k + \vec{V}_k\cdot\vec{X}_k) + c_p\theta_k\Delta_k\left(\frac{\partial\Pi_k}{\partial t} + \vec{V}_k\cdot\nabla\Pi_k\right) \qquad (2.25)$$

$$+ \frac{1}{\Delta_k}\left(\frac{\partial H}{\partial t} + \vec{V}_k\cdot\nabla H\right)\left[\phi_k(\hat{s}_{k+1} - \hat{s}_k) - (\hat{\phi}_{k+1}\hat{s}_{k+1} - \hat{\phi}_k\hat{s}_k)\right]$$

We can no longer postpone a choice of Π_k. (2.25) identifies the enthalpy in the k-th layer as $(\hat{p}_k - \hat{p}_{k+1})c_p\Pi_k\,\theta_k$. We can therefore arrange that our model have the same initial enthalpy (and therefore total potential energy) as the atmosphere by equating, at $t=0$,

$$\Pi_k \theta_k (\hat{p}_k - \hat{p}_{k+1}) = \int_{\hat{p}_{k+1}}^{\hat{p}_k} T\,dp \equiv \int_{\hat{p}_{k+1}}^{\hat{p}_k} \theta\,\Pi\,dp \quad \text{(observed)}$$

To obtain separate information about Π_k and θ_k in addition to their product we may also ask that the integrals of θ correspond in (2.19).

$$(\hat{p}_k - \hat{p}_{k+1})\,\theta_k = \int_{\hat{p}_{k+1}}^{\hat{p}_k} \theta\,dp \quad \text{(observed)}$$

Recognizing that the pressure intervals are presumably close together, we simply assume that θ may be taken outside the integrals in both of these equations, whence

$$\Pi_k = \frac{\hat{p}_k^{\,1+\kappa} - \hat{p}_{k+1}^{\,1+\kappa}}{(1+\kappa)(\hat{p}_k - \hat{p}_{k+1})}$$

$$\theta_{k\,\text{(initial)}} = \frac{1}{(\hat{p}_k - \hat{p}_{k+1})} \int_{\hat{p}_{k+1}}^{\hat{p}_k} \theta\,dp \quad \text{(observed)} \tag{2.26}$$

An alternate assumption is to assume constant T instead of constant θ in the integrals. This yields

$$\Pi_k = \frac{(1-\kappa)(\hat{p}_k - \hat{p}_{k+1})}{\hat{p}_k^{\,1-\kappa} - \hat{p}_{k+1}^{\,1-\kappa}}$$

$$\theta_{k\,\text{(initial)}} = \frac{1}{\Pi_k(\hat{p}_k - \hat{p}_{k+1})} \int_{\hat{p}_{k+1}}^{\hat{p}_k} T\,dp \quad \text{(observed)}$$

Other more complicated assumptions involving the variation of θ or T within the layers are conceivable, but (in order to be realistic) would introduce into the above simple definition of Π_k as a function of $\hat{p}_k$ and $\hat{p}_{k+1}$ a new dependence on θ_k at adjacent layers, and would thereby probably eliminate any possibility of cancelling the last two terms in (2.25) for all distributions of θ_k. [In his general circulation model using $F = p$, Arakawa simply puts $\Pi_k = [\frac{1}{2}(\hat{p}_k + \hat{p}_{k+1})]^{\kappa}$. This is demonstrably a poor choice for real data; see the end of section 3.]

One final point before returning to (2.25) and (2.24) should be noted. This is that although we have now defined Π_k and we shall shortly complete the hydrostatic system for determining values of ϕ_k from known values of $\hat{\phi}_1$ and θ_k, the values of Π within the layer at which $\phi = \phi_k$ is still unresolved. This question becomes important when initial input data are not in the form of layer integrals of θ, but consist instead--as is part of NMC practice--of ϕ-values in order that geostrophic-like relations can be used in objective analysis. This point will be discussed in section 3.

In order to return to (2.25) we invoke (2.8) in (2.26) to arrive at

$$s_k \frac{\partial \Pi_k}{\partial \xi} = \frac{\hat{p}_k - \hat{p}_{k+1}}{\Delta_k} \frac{\partial \Pi_k}{\partial \xi}$$

$$= \frac{1}{\Delta_k}\left[(\hat{s}_k \hat{\Pi}_k - \hat{s}_{k+1}\hat{\Pi}_{k+1}) - \Pi_k(\hat{s}_k - \hat{s}_{k+1})\right]\frac{\partial H}{\partial \xi} \tag{2.27}$$

The last two terms in (2.25) cancel if we require that, for $k = 1$, $2, \ldots, K$,

$$\hat{\phi}_{k+1}\hat{s}_{k+1} - \hat{\phi}_k\hat{s}_k = \phi_k(\hat{s}_{k+1} - \hat{s}_k) - c_p\theta_k\left[\hat{s}_{k+1}\hat{\Pi}_{k+1} - \hat{s}_k\hat{\Pi}_k - \Pi_k(\hat{s}_{k+1} - \hat{s}_k)\right] \qquad (2.28)$$

$\hat{s}_{K+1}$ is zero [see (2.3)] and

$$\hat{s}_1 = \left(\frac{d\hat{p}}{dF}\right)_1 = \frac{1}{\left(\frac{d\hat{F}}{dp}\right)_1} \qquad (2.29)$$

Equation (2.28) therefore gives no information about $\hat{\phi}_{K+1}$, but we do identify $\hat{\phi}_1$ with the surface geopotential:

$$\hat{\phi}_1 = \phi(p = p_s) \qquad (2.30)$$

[or $\phi(p_s) - \bar{\phi}(p_s)$ if we are using ϕ' in place of ϕ.] Summation of (2.28) from $k = 1$ to K followed by use of (2.24) produces an equation relating ϕ_1 and $\hat{\phi}_1$:

$$\frac{\hat{s}_1(\phi_1 - \hat{\phi}_1)}{c_p} = -\sum_2^K \hat{s}_k(\Pi_{k-1} - \Pi_k)\hat{\theta}_k - \sum_1^K \theta_k\left[\hat{s}_{k+1}(\hat{\Pi}_{k+1} - \Pi_k) - \hat{s}_k(\hat{\Pi}_k - \Pi_k)\right] \qquad (2.31)$$

This plus the ($K-1$) equations (2.24a) gives K relations between the K values of ϕ_k and the K values of θ_k once $\hat{\theta}_k$ has been defined in (2.15).

The totality of hydrostatic considerations can be summarized as follows:

a. Knowing H [predicted by (2.9)], $\hat{F}_k$ is determined by (2.2).

b. This determines $\hat{\pi}_k = [p(\hat{F}_k)]^{\kappa}$

c. π_k comes from (2.26)

d. Predicted values of θ_k enable $\hat{\theta}_k$ to be determined according to the choice for $\hat{\theta}_k$ made in (2.13).

e. If ϕ is being used, (2.31) determines ϕ_1, since $\hat{\phi}_1$ is the known surface value of geopotential. If ϕ' is being used, $\hat{p}_1 = p(\hat{F}_1)$ is first determined from (2.2) and used to compute $\bar{\phi}(\hat{p}_1)$ in (1.21) or (1.29). $\hat{\phi}_1'$ is equal to $\hat{\phi}_1 - \overline{\phi}(\hat{p}_1)$. (2.31) then determines ϕ_1' directly.

f. (2.24a) enables ϕ_k (or ϕ_k') to be determined for $k = 2, 3, \cdots, K$.

3. Temperatures as given by known geopotentials

In numerical prediction as practiced as NMC, however, the initial input data is primarily the geopotential distribution rather than (for example) $\hat{\phi}_1$ plus a complete analysis of $T(p)$. Nonetheless, an important part of the forecast itself is the temperature or potential temperature. It is therefore a legitimate question to ask what values of θ_k are determined by the above equations. For example, given ϕ_k as input at $t=0$, are the inferred θ_k's reasonable, even at $t=0$? To examine this we first define

$$\tau_1 = \frac{\phi_1 - \hat{\phi}_1}{c_p(\hat{\pi}_1 - \pi_1)} \qquad (k=1)$$
$$\tau_k = \frac{\phi_k - \phi_{k-1}}{c_p(\pi_{k-1} - \pi_k)} \qquad (k=2,3,\cdots,K) \tag{3.1}$$

$$\nu_1 = 1$$
$$\nu_k = (-1)^k \hat{s}_k(\pi_k + \pi_{k-1} - 2\hat{\pi}_k) \div \hat{s}_1(\hat{\pi}_1 - \pi_1) \tag{3.2}$$
$$\nu_{K+1} = 0 \quad (\hat{s}_{K+1} = 0)$$

τ_1 and τ_k have the properties of potential temperatures at some location between the π_k and ϕ_k levels appearing in their definition.

We specialize (2.16) now to

$$\hat{\theta}_k \equiv \frac{1}{2}(\theta_k + \theta_{k-1}), \quad k = 2, \cdots, K \tag{3.3}$$

Equations (2.24a) and (2.31) then reduce to

$$\tau_1 = \theta_1 - \frac{1}{2} \sum_{2}^{K} \nu_k (-1)^k (\theta_k - \theta_{k-1}) \tag{3.4}$$

$$\theta_k = 2\tau_k - \theta_{k-1} \quad , \quad k = 2, \cdots, K. \tag{3.5}$$

We note that identity of all θ_k produces $\tau_k \equiv \theta$ for all k. This is an exact solution of the hydrostatic relation for an adiabatic atmosphere <u>if we identify ϕ_k in each layer as the geopotential at the pressure corresponding to π_k</u>. <u>This assumption is made from now on.</u> Its accuracy will be examined later [see equation (3.14)].

Equations (3.4) and (3.5) can be solved for θ_1 in terms of τ :

$$\begin{aligned} \theta_1 &= \sum_{k=1}^{K} \psi_k \nu_k \div \sum_{k=1}^{K} \nu_k \\ \psi_{k=1} &= \tau_1 \quad , \quad \psi_{k=2} = \tau_2 \\ \psi_k &= (-1)^k (\tau_k - \tau_{k-1}) + \psi_{k-1} , \quad k = 3, 4, \cdots, K \end{aligned} \tag{3.6}$$

Θ_k for $k = 2, ---, K$ then follows from (3.5). [Note that the proportionality of ν_k to ($\pi_k + \pi_{k-1} - 2\hat{\pi}_k$) reflects a second-order truncation error in (3.6).]

As a numerical test of this system, the equations have been applied to the following ϕ distribution:

$$\phi\,(m^2 sec^{-2}) = 1110\left\{0.95 + Z\left[72.43 + Z\,(-6.9 + Z)\right]\right\} \tag{3.7}$$

$$Z = -\ln p$$

$$\Theta = \frac{T}{\pi} = \frac{1}{R\pi}\frac{d\phi}{dZ} = \frac{1110}{R\pi}\left\{72.43 + Z\left[2(-6.9) + 3Z\right]\right\} \tag{3.8}$$

These formulae duplicate reasonably well the usual standard atmosphere up to $p = 0.1$ (i.e., 100 cb). $\hat{p}_1$ and $\hat{p}_T$ were set at 1 and 0.1, R and c_p for simplicity were set equal to 287 and 1000, and the interval from $Z = 0$ to $Z = -\ln(0.1)$ was divided up into 10 equal increments in Z. This defines $\hat{p}_k$ and $\hat{\pi}_k$, and π_k followed from (2.26). "Input" values of ϕ_k were then obtained by applying (3.7) at $Z_k = -(\ln \pi_k) \div \varkappa$, and $\hat{\phi}_1$ is equal to (1110) (.95) = 1054.5 m^2 sec^{-2} ($\hat{Z}_1 = -\ln 1 = 0$). Computations of Θ_k from (3.6) and (3.5) were made with $F = p$, $F = p^{\varkappa}$ and $F = -\ln p$. These were compared with " Θ true"—the values of Θ given by (3.8) at Z_k.

Table I. Excess (x) of finite-difference Θ over true Θ for three choices of F(p), for the NMC scheme, and for Arakawa's choice of π_k. Values of ν (etc.) are also shown for the three revised Arakawa schemes as is Θ_k (true) for those three schemes.

k	$\hat{\pi}_k$	π_k	Θ_k (true)	τ_k	ν_k (F=p)	ν_k (F=π)	ν_k (F= $-\ell$n p)
1	1.0	.968900	283.203	281.641	1.0	1.0	1.0
2	.936052	.906941	290.326	286.642	.092733	.088489	.085916
3	.876193	.848944	298.551	294.293	-.066367	-.060322	-.056785
4	.820163	.794655	308.095	303.151	.046945	.040386	.036913
5	.767715	.743839	319.199	313.443	-.032605	-.026559	-.023537
6	.718621	.696272	322.151	325.438	.022186	.016977	.014598
7	.672667	.651746	347.272	339.433	-.014469	-.010418	-.008682
8	.629650	.610068	364.925	355.773	.008939	.005981	.004823
9	.589386	.571056	385.523	374.847	-.004887	-.003087	-.002379
10	.551696	.534538	409.534	397.092	.002058	.001158	.000900

x = Θ (model) $-$ Θ (true):

k	(F=p)	(F=π)	(F= $-\ell$n p)	(NMC)	(Arakawa π_k)
1	-.149	-.362	-.463	.056	-6.866
2	-.096	.117	.218	.064	7.921
3	-.195	-.408	-.509	.084	-10.887
4	-.149	.064	.165	.092	10.535
5	-.259	-.472	-.573	.110	-10.933
6	-.215	-.002	.099	.131	10.448
7	-.342	-.555	-.656	.140	-10.994
8	-.309	-.096	.005	.182	10.347
9	-.445	-.658	-.883	.204	-11.106
10	-.428	-.215	-.114	.233	10.217

The results are summarized in Table I., with $x_k = \theta_k - \theta$ true. All results for these three choices of F are satisfactory although $F = p$ has less of the slight alternating character in x apparent in $F = \pi$ or $F = -\ln p$. In all three " F " cases, θ_1 was a degree or so greater than τ_1. This shows that (3.6) amounts to a recognition that the mean potential temperature τ_1 from the ground ($\hat{\pi}_1$) to π_1 is less than θ_1, i.e., (3.6) produces an extrapolation below π_1 of the stable θ distribution characteristic of (3.8).

The next to the last column in Table II, labelled x_{NMC}, is based on the type of hydrostatic equation employed in the current NMC models, with ϕ located at the interfaces. In terms of our present notation, this is

$$\theta_k = \frac{\hat{\phi}_{k+1} - \hat{\phi}_k}{c_p(\hat{\pi}_k - \hat{\pi}_{k+1})} \tag{3.9}$$

and x_{NMC} is this value of θ minus θ from (3.8) where

$$\pi_k = \frac{1}{2}(\hat{\pi}_k + \hat{\pi}_{k+1}), \quad z_k = -(\ln \pi_k) \div \varkappa \tag{3.10}$$

is used in accordance with NMC practice. These results are slightly better than for the revised Arakawa system, which shows a small tendency to oscillate. However, all four methods give satisfactory results.

A much poorer field of deduced θ_k appears if we instead of (2.26) follow Arakawa's choice of Π_k (he uses $F = p$):

$$\Pi_k = \left[\tfrac{1}{2}\left(\hat{p}_k + \hat{p}_{k+1}\right)\right]^{\kappa} \tag{3.11}$$

This gives

$$\frac{\partial \Pi_k}{\partial \xi} = \frac{\kappa \Pi_k}{\hat{p}_k + \hat{p}_{k+1}} \left(\hat{s}_k + \hat{s}_{k+1}\right) \frac{\partial H}{\partial \xi} \tag{3.12}$$

and, in place of (2.28),

$$\hat{\phi}_{k+1}\hat{s}_{k+1} - \hat{\phi}_k \hat{s}_k = \phi_k\left(\hat{s}_{k+1} - \hat{s}_k\right) + \frac{c_p \theta_k \kappa \Pi_k}{\left(\hat{p}_k + \hat{p}_{k+1}\right)} \left(\hat{s}_k + \hat{s}_{k+1}\right)\left(\hat{p}_k - \hat{p}_{k+1}\right) \tag{3.13}$$

When carried through the steps analogous to (2.31) and (3.1)-(3.6) and applied to the above test case [using the simple choice (3.3) for $\hat{\theta}_k$] the inferred θ_k values result in the last column of ν_k shown in Table I. [The NMC model and the original Arakawa model have, of course, their own values of Θ_{true} (not shown) since they use different values of Π_k than do the first three tests.] A large oscillation is present--large enough in fact to produce alternating changes in sign of $\theta_k - \theta_{k-1}$. This choice of Π_k is clearly unsatisfactory.

A final point concerns the "location" of the ϕ_k -values--i.e., at what value of p are they located? The implicit assumption we have made so far is to identify them with the π_k levels. An alternate approach, however, would be to follow through to completion the view introduced earlier of having $\partial\theta/\partial\sigma$ equal to zero in each layer, which led to (2.26). This would lead to the relation $\phi_k - \phi_{k-1} = c_p[\theta_k(\hat{\pi}_k - \tilde{\pi}_k) + \theta_{k-1}(\tilde{\pi}_{k-1} - \hat{\pi}_k)]$ where $\phi_k = \phi(\tilde{\pi}_k)$. Comparing this to (2.24), with $\hat{\theta}_k = \frac{1}{2}(\theta_k + \theta_{k-1})$ leads to the conclusion that $\tilde{\pi}_k - \pi_k = \tilde{\pi}_{k-1} - \pi_{k-1}$ for $k = 2, \cdots, K$. The uniform value of $\tilde{\pi}_k - \pi_k$ can be fixed by equating $(\hat{\pi}_1 - \pi_1)\tau_1 = (\phi_1 - \hat{\phi}_1)c_p^{-1}$ to $(\hat{\pi}_1 - \tilde{\pi}_1)\theta_1$, i.e., by assuming $\theta \equiv \theta_1$ in the lowest layer. The result is

$$\tilde{\pi}_k - \pi_k = (\hat{\pi}_1 - \pi_1)\left(1 - \frac{\tau_1}{\theta_1}\right), \quad k = 1, \cdots, K \tag{3.14}$$

For the three cases $F = p, \pi$, and $-\ln p$ in Table I., $\tilde{\pi}_k - \pi_k$ has the values .000155, .000132 and .000121 corresponding to a fraction .0050, .0042 and .0039 of $(\hat{\pi}_1 - \pi_1)$. This would correspond to a negligible change in ϕ of $c_p\theta \times 10^{-4}$, or about 4 meters in height. We therefore conclude that it is sufficiently accurate to assume

$$\phi_k = \phi(\pi_k) \tag{3.15}$$

when input data consists of ϕ values rather than the temperature integrals in (2.26). As discussed below, this also serves to identify the levels at which $\vec{V}_k$ is analyzed at $t=0$, and will enable forecast values of $\vec{V}_k$,

ϕ_k and θ_k to be converted to pressure surfaces for presentation of output. [In fact, if ϕ_k -values are to be used as input data instead of the θ integrals in (2.26), a decision about the pressure levels corresponding to them must be made <u>before</u> ϕ_k is defined. Since (3.14) is useless until all $\tau_k \propto (\phi_k - \phi_{k-1})$ are known, an approximation like (3.1) is unavoidable.]

The point may be raised that the original definition of $\mathcal{r}_k$ in the continuity equation assumed that $\vec{V}_k$ was the mass-representative velocity in that layer, since (2.9) and (2.10) give

$$\frac{\partial \hat{p}_1}{\partial t} = -\nabla \cdot \sum_{k=1}^{K} (\hat{p}_k - \hat{p}_{k+1}) \vec{V}_k$$

In other words, we implicitly assumed then that $\vec{V}_k$ was defined at

$$\underline{p}_k = \frac{1}{2} (\hat{p}_k + \hat{p}_{k+1})$$

whereas we now have ϕ_k and $\vec{V}_k$ defined at

$$p_k = \pi_k^{\frac{1}{\kappa}} = \left[\frac{\hat{p}_k^{(1+\kappa)} - \hat{p}_{k+1}^{(1+\kappa)}}{(1+\kappa)(\hat{p}_k - \hat{p}_{k+1})} \right]^{\frac{1}{\kappa}}$$

However, this difference is small. Expanding p_k we find

$$\frac{p_k - \underline{p}_k}{\underline{p}_k} \approx -\frac{1}{6}(1-\kappa) \left[\frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{p}_k + \hat{p}_{k+1}} \right]^2$$

which, in the case of equal intervals $\Delta \hat{z}_k$ is approximately equal to

$$-\frac{1}{24}(1-\kappa)(\Delta z)^2$$

This has a value of -.0016 in the 10-level model tested here.

4. The Horizontal pressure force

We examine here the artificial value of $\nabla_p \phi$ introduced by the sigma system when ϕ is only a function of pressure, using again the test distribution ($Z = -\ln p$):

$$\phi(m^2 sec^{-2}) = 1110\left\{0.95 + Z\left[72.43 + Z(-6.9 + Z)\right]\right\} \qquad (4.1)$$

We consider two columns, A and B, located at different values of x (say), and having different values of surface pressure:

$$\text{Point A: } \hat{p}_1 = 1,\ \hat{Z}_1 = 0,\ \hat{\phi}_1 = 1054.5$$

$$\text{Point B: } \hat{p}_1 = 0.8,\ \hat{Z}_1 = 0.2231436,\ \hat{\phi}_1 = 18{,}625.727 \qquad (4.2)$$

This corresponds to a difference in surface elevation of 1791.15 meters, and furnishes a significant test of this effect. A "mid-point" 0 is defined as having the average surface pressure of A and B:

$$\text{Point 0: } \hat{p}_1 = 0.9,\ \hat{Z}_1 = 0.1053605,\ \hat{\phi}_1 = 9{,}441.478 \qquad (4.3)$$

Column 0 will be considered both as the mid-point between A and B—at which we will compute the finite-difference equivalent of $(\partial\phi/\partial x)_p$ —and it will also be used to define the sigma-levels and the adiabatic reference atmosphere.

A 10-layer model is used again, with $p_T = \hat{p}_{K+1} = 0.1$, with interfaces located at equal intervals in Z at point 0. (This interval is small enough that $\hat{p}_2$ at column A differs by at most 0.8 percent from surface pressure $\hat{p}_1$ at column B.) The first five columns in Table II. show the vertical distribution of interface pressure at column 0, while the last three columns show the values of $\hat{\sigma}_k$ at these 11 levels corresponding to three choices for $F(p)$ in the relation $(1-\hat{\sigma}_k)=(\hat{F}_k-F_T)/(F_S-F_T)$. The reference adiabatic atmospheric used data from column 0 in the formula (1.29), in that $\langle p_s \rangle$, p_T , $\langle \phi_s \rangle$ and $\langle \phi_T \rangle$ in (1.29) are identified with the values of $\hat{p}_1$, $\hat{p}_{K+1}\ (=0.1)$, $\hat{\phi}_1$, and $\hat{\phi}_{K+1}$ at column 0:

$$\bar{\phi}\,(m^2\,sec^{-2}) = 329{,}451.809 - 329{,}834.79\ \Pi \tag{4.4}$$

$$\bar{\theta} = 329.83479$$

Table II. Choice of σ-levels for three choices of F(p).

k	$\hat{z}_k$	$\hat{p}_k$	$(\hat{p}_k - \hat{p}_{k+1})$	$\hat{\pi}_k$	F=p $\hat{\sigma}_k$	F=π $\hat{\sigma}_k$	F= −ℓn p $\hat{\sigma}_k$
1	.1053605	.9	.177533	.970214	0	0	0
2	.3250830	.7224670	.142512	.910921	.221916	.130659	0.1
3	.5448054	.5799546	.114401	.855252	.400057	.253333	0.2
4	.7645279	.4655537	.091835	.802949	.543058	.368589	0.3
5	.9842503	.3737193	.073719	.753912	.657851	.476648	0.4
6	1.2039728	.3000000	.059178	.707838	.750000	.578178	0.5
7	1.4236953	.2408225	.047504	.664580	.823972	.673502	0.6
8	1.6434177	.1933182	.038133	.623965	.883352	.763002	0.7
9	1.8631402	.1551846	.030612	.585833	.931019	.847031	0.8
10 (K)	2.0828626	.1245731	.024573	.550030	.969284	.925927	0.9
11	2.3025851	.1000000		.516416	1.000000	1.000000	1.0

In the Arakawa scheme, the pressure force is

$$\left[(\hat{p}_k-\hat{p}_{k+1})\left(\frac{\partial\phi}{\partial y}\right)_p\right] = \frac{\partial}{\partial y}\left[(\hat{p}_k-\hat{p}_{k+1})\phi_k\right] + \Big\{\phi_k(\hat{a}_{k+1}-\hat{a}_k)+c_p\theta_k\Big[(\hat{a}_k\hat{\Pi}_k-\hat{a}_{k+1}\hat{\Pi}_{k+1}) + (\hat{a}_{k+1}-\hat{a}_k)\Pi_k\Big]\Big\}\frac{\partial H}{\partial y} \tag{4.5}$$

The NMC scheme in our notation is

$$\left(\frac{\partial \phi}{\partial x}\right)_p = \frac{\partial}{\partial x}\left(\frac{\hat{\phi}_k + \hat{\phi}_{k+1}}{2}\right) + c_p \left\{ \theta_k \right\} \frac{\partial}{\partial x}\left(\frac{\hat{\Pi}_k + \hat{\Pi}_{k+1}}{2}\right) \tag{4.6}$$

For comparison with the Arakawa form, we put (4.6) into "flux form" by multiplying it with $(\hat{p}_k - \hat{p}_{k+1})$ and using the definition $\hat{p}_k = p_T + (1-\hat{\sigma}_k)(\hat{p}_1 - p_T)$. (We use a simple NMC-like model, without the "tropopause" surface.) The result is

$$\left[(\hat{p}_k - \hat{p}_{k+1})\left(\frac{\partial \phi}{\partial x}\right)_p\right] = \frac{\partial}{\partial x}\left[(\hat{p}_k - \hat{p}_{k+1})\left(\frac{\hat{\phi}_k + \hat{\phi}_{k+1}}{2}\right)\right]$$
$$+ \left\{(\hat{\phi}_k + \hat{\phi}_{k+1})(\hat{\sigma}_k - \hat{\sigma}_{k+1}) + R\theta_k(\hat{p}_k - \hat{p}_{k+1})\left[(1-\hat{\sigma}_{k+1})\frac{\hat{\Pi}_{k+1}}{\hat{p}_{k+1}} + (1-\hat{\sigma}_k)\frac{\hat{\Pi}_k}{\hat{p}_k}\right]\right\}\frac{1}{2}\frac{\partial \hat{p}_1}{\partial x} \tag{4.7}$$

Following accepted practice, the curly brackets in (4.5) and (4.7) are taken as the average of their values at the two points (in our case A and B) that are used to evaluate $\partial/\partial x$. Let us define

$$\Delta\phi = \Delta x \left(\frac{\partial \phi}{\partial x}\right)_p$$

as the apparent difference in geopotential along a pressure surface from A to B in the sigma system when $\phi = \phi(p)$. Equations (4.5) and (4.7) both have the same form now,[1]

$$(\hat{p}_k - \hat{p}_{k+1})\Delta\phi = \underline{I} + \underline{II}$$

[1] The following exact equivalent of (4.5) could also have been used for comparison with (4.6):

$$\left(\frac{\partial \phi}{\partial x}\right)_p = \frac{\partial \phi_k}{\partial x} + c_p \theta_k \frac{\partial \Pi_k}{\partial x}$$

Term I is the change in $(\hat{p}_k - \hat{p}_{k+1})\phi_k$ from point A to B and term II is an averaged coefficient times the change in H (or $\hat{p}_1$) from point A to point B.[1]

Table III. shows the results for three versions of the revised Arakawa scheme and for the NMC scheme, using the total geopotential (4.1) as input. [In order to duplicate operational practice, θ_k in (4.5) and in (4.7) was computed from the ϕ_k according to the hydrostatic equation for each model, as was done in the single column computations of section 3.] The general nature of the results is similar, in fact those for $F = p$ are very similar to the NMC scheme. The results for $F = \pi$ and $-\ln p$ are negative instead of the generally positive values found for $F = p$ and for NMC. Compared to those for $F = p$, their magnitude is definitely larger in the lower part of the atmosphere, but does tend to be smaller in the upper layers (presumably because there is less variation of pressure along a sigma surface in these coordinates).

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[1] Some complicated numerical evaluations have been reported by J. Gary (Estimation of truncation error in transformed coordinate, primitive equation atmospheric models. J. Atmos. Sci., 30, 223-233, 1973). The computations presented here differ from his in concentrating on the most important case $\phi = \phi(p)$, in using expressions appropriate to the flux form of the horizontal equations of motion, in using coefficients of the last terms in (4.5) and (4.7) which are exactly consistent with the finite-difference hydrostatic equations of the model, in using the more accurate two-point averaged values of these coefficients, and finally, in considering a reference atmosphere which is completely compatible with the entire set of equations.

Table III. "Artificial" values of $(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$ using total geopotential of (4.1) in three versions of the revised Arakawa system ($F=p$, π and $-\ell n\ p$) and in the NMC system. k-values define sigma levels from Table II. Units are $m^2\ sec^{-2}$.

	NMC			F=p		
k	I	II	$(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$	I	II	$(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$
1	2200.117	−2185.345	14.772	2216.059	−2198.740	17.319
2	1001.643	−990.674	10.969	1014.570	−1003.743	10.827
3	204.609	−196.594	8.015	215.294	−205.408	9.886
4	−309.651	315.319	5.668	−300.678	307.422	6.744
5	−626.900	630.748	3.848	−619.234	624.548	5.314
6	−809.050	811.367	2.317	−802.346	805.743	3.396
7	−900.094	901.255	1.161	−894.105	896.579	2.474
8	−931.503	931.735	0.232	−926.035	927.391	1.356
9	−925.426	924.850	−0.576	−920.322	920.964	0.642
10	−896.984	895.958	−1.026	−892.115	892.253	0.138
			45.380			58.096

	$F=\pi$			$F=-\ell n\ p$		
k	I	II	$(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$	I	II	$(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$
1	2034.144	−2065.811	−31.667	1926.002	−1988.717	−62.715
2	737.104	−759.866	−22.762	588.494	−625.783	−37.289
3	−49.963	39.455	−10.508	−171.996	156.771	−15.225
4	−493.505	486.056	−7.449	−563.003	554.075	−8.928
5	−712.452	709.777	−2.675	−724.292	721.846	−2.446
6	−787.356	785.468	−1.888	−748.790	747.315	−1.475
7	−774.508	774.230	−0.278	−697.283	697.267	−0.016
8	−711.723	711.354	−0.369	−608.185	608.061	−0.124
9	−623.607	623.714	0.107	−505.391	505.771	0.380
10	−526.540	526.500	−0.040	−403.661	403.520	−0.141
			−77.529			−127.979

The same computations have been made for the three versions of the revised Arakawa model, using the difference between the geopotential (4.1) and the adiabatic reference atmosphere (4.4). These results are in Table IV. They show a reduction in $(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$ in all three cases from those of Table III., but most satisfactorily so in the case $F = p$. An oscillation is present again in all three cases with minimum values at even k.

Table IV. also contains values of $\Delta\phi$, obtained from dividing $(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$ by the $(\hat{p}_k - \hat{p}_{k+1})$ values listed for column 0 in Table II. Suppose columns A and B are separated by 200 km. (This corresponds to the large ground slope on the Pacific Coast.) A value of $\Delta\phi$ of 10 m^2 sec^{-2} corresponds then to an acceleration of 5 x 10^{-5} m sec^{-2} $\sim$ 4.3 m sec^{-1} day^{-1} or to a geostrophic wind of 0.5 m sec^{-1} in middle latitudes. This value of $\Delta\phi$ seems small enough to be tolerated, and the results for $F = p$ in Table IV. are therefore acceptable. However, the results for $F = \pi$ and $F = -\ln p$ in Table IV. are too large, as are all of those in Table III.

Table IV. "Artificial" values of $(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$ and $\Delta\phi$ using deviation of geopotential in (4.1) for three versions of the revised Arakawa system. Units are $m^2\ sec^{-2}$.

	F=p			
k	I	II	$(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$	$\Delta\phi$
1	-399.783	400.093	0.310	1.7
2	-149.280	148.758	-0.522	-3.7
3	-1.307	2.319	1.012	8.8
4	77.938	-77.412	0.526	5.7
5	111.844	-110.796	1.048	14.2
6	116.770	-116.030	0.740	12.5
7	103.524	-102.667	0.857	18.0
8	79.588	-79.112	0.476	12.5
9	49.815	-49.481	0.334	10.9
10	17.276	-17.211	0.065	2.6
			4.846	

	F=π				F= -ℓn p			
	I	II	$(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$	$\Delta\phi$	I	II	$(\hat{p}_k - \hat{p}_{k+1})\Delta\phi$	$\Delta\phi$
1	-382.066	396.121	14.055	79.2	-371.509	393.289	21.780	122.7
2	-117.477	122.088	4.611	32.4	-100.238	106.756	6.518	45.7
3	25.874	-19.371	6.503	56.8	36.308	-30.042	6.266	54.8
4	89.654	-88.382	1.272	13.9	92.769	-96.985	-4.216	-45.9
5	108.749	-106.836	1.913	25.8	104.381	-102.657	1.724	23.4
6	101.811	-101.535	0.276	4.7	92.780	-92.783	-0.003	-0.1
7	81.916	-81.305	0.611	12.9	71.319	-70.832	0.487	10.3
8	57.171	-57.105	0.066	1.7	47.626	-47.663	-0.037	-1.0
9	32.358	-32.206	0.152	5.0	25.814	-25.694	0.120	3.9
10	10.173	-10.169	0.004	0.2	7.802	-7.799	0.003	0.1
			29.463				32.642	

The time derivative of the surface pressure tendency is

$$\frac{\partial^2 \hat{p}_1}{\partial t^2} = -\frac{\partial}{\partial x} \sum_{k=1}^{K} \frac{\partial}{\partial t}\left[(\hat{p}_k - \hat{p}_{k+1}) u_k\right]$$

If the atmosphere is resting at $t=0$, this is equal to

$$\left.\frac{\partial^2 \hat{p}_1}{\partial t^2}\right|_{t=0} = \frac{\partial}{\partial x} \sum_{k=1}^{K} \frac{\partial}{\partial x}\left[(\hat{p}_k - \hat{p}_{k+1}) \Delta\phi\right] = \frac{\partial^2}{\partial x^2} \sum_{k=1}^{K}\left[(\hat{p}_k - \hat{p}_{k+1}) \Delta\phi\right]$$

Suppose conditions at columns A and B are repeated in x, and that the motion erroneously created at $t=0$ acts subsequently like a gravity wave with effective phase speed C. The amplitude $\delta\hat{p}_1$ of the resulting oscillation in surface pressure is then

$$\delta\hat{p}_1 \sim \frac{1}{C^2} \sum_{k=1}^{K}\left[(\hat{p}_k - \hat{p}_{k+1}) \Delta\phi\right]$$

Values of these sums are tabulated under the columns in Tables III. and IV. In many of the cases, the values of $\Delta\phi$ are fairly uniform with k, suggesting that a large value of $C \sim$ 200 or 300 m sec^{-1} is appropriate. If we take $C \sim$ 250 m sec ^{1}, we get $\delta\hat{p}_1$ about 0.1 mb for the best case ($F=p$ in Table IV.) and $\delta\hat{p}_1$ about 2 mb for the worst case ($F=-\ln p$ in Table III.).