# U.S. DEPARTMENT OF COMMERCE NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION NATIONAL WEATHER SER VICE NATIONAL METEOROLOGICAL CENTER 

## OFFICE NOTE 87

On Map Projections for Numerical Weather Prediction

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## 1. Introduction

Standard meteorological texts usually discuss the equations governing the atmosphere in either spherical coordinates or on applane. In the practice of numerical weather prediction, the equations are usually transformed into the map coordinates of a conformal map. This note is an attempt to bring together in one convenient place the definitions of the maps used and to indicate the methodology by which the transformation of the equations may be carried through.

It has become evident that the standard polar stereographic map which has served NWP for so long is no longer an appropriate projection. Attention is shifting toward global and limited area models, for neither is the usage of the polar stereographic map optimal.

The sources of material used in this note are Godske, Bergeron, Bjerknes and Bundgaard (1957), and Deetz and Adams (1945).

## 2. The Lambert Conformal Map

The surface spherical coordinate $\lambda$, longitude, and $\phi$, latitude, may be mapped into the plane polar coordinates, $r$, radial distance, and $\theta$, azimuthal angle by the transformation

$$
\begin{align*}
& r=r_{0}\left(\tan \left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right)^{K}  \tag{la}\\
& \left.\theta=K Q-\lambda_{0}\right) \tag{lb}
\end{align*}
$$

in which $r_{0}$ and $K$ are positive, real valued constants and $\lambda_{0}$ is an arbitrary zero reference for longitude.

If we take $K \leqq l$, the pole $\phi=\pi / 2$ maps into $r=0$ and the pole $\phi=-\pi / 2$ maps into the point at infinity. -The mapping is not "one to one"; since $K<1$, the points in the sector, $2 \pi K<\theta+\lambda_{0}<2 \pi$, do not correspond to points on the sphere.

The mapping is conformal since

$$
\begin{align*}
& \frac{1}{a \cos \phi} \frac{\partial r}{\partial \lambda}=+\frac{r}{a} \frac{\partial \theta}{\partial \phi}=0 \\
& \frac{r}{a \cos \phi} \frac{\partial \theta}{\partial \lambda}=\frac{-1}{a} \frac{\partial r}{\partial \phi}=\frac{r k}{\cos \phi}
\end{align*}
$$

2b
in which a is the radius of the spherical surface.
Since there are two parameters $r_{0}$ and $K$ at our disposal, one may set two compatible constraints upon the mapping. Typically, one may set the scale on the map equal to that on the spherical surface at two latitudes, $\phi_{1}$ and $\phi_{2}$. One requires

$$
\begin{array}{ll}
r d \theta=a \cos \phi_{1} d \lambda & \text { at } \phi=\phi_{1} \\
r d \theta=a \cos \phi_{2} d \lambda & \text { at } \phi=\phi_{2}
\end{array}
$$

These equations give

$$
\begin{align*}
& r_{0} K\left(\tan \left(\frac{\pi}{4}-\frac{\phi_{1}}{2}\right)\right)^{\mathrm{K}}=a \cos \phi_{1} \\
& r_{0} K\left(\tan \left(\frac{\pi}{4}-\frac{\phi_{2}}{2}\right)\right)^{\mathrm{K}}=a \cos \phi_{2}
\end{align*}
$$

One may solve these for $K$ and $r_{o}$ to get,

$$
\begin{equation*}
K=\ln \left(\frac{\cos \phi_{1}}{\cos \phi_{2}}\right) \div \ln \left(\frac{\tan \frac{\pi}{4}-\frac{\phi_{1}}{2}}{\tan \frac{\pi}{4}-\frac{\phi_{2}}{2}}\right) \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{0}=\frac{a \cos \phi_{1}}{K}\left(\tan \frac{\pi}{4}-\frac{\phi_{1}}{2}\right)^{-K} \tag{6}
\end{equation*}
$$

The map scale factor, $m$, at an arbitrary point on the map is defined as the ratio of distance on the map to distance on the sphere. Measuring the distances along the parallel of latitude $\phi$, one finds

$$
\mathrm{m}(\phi)=\frac{\mathrm{d} \theta}{\mathrm{a} \cos \phi d \lambda}=\frac{\cos \phi_{1}}{\cos \phi}\left(\frac{\tan \frac{\pi}{4}-\frac{\phi}{2}}{\tan \frac{\pi}{4}-\frac{\phi_{1}}{2}}\right)^{\mathrm{K}}
$$

or equivalently,

$$
m(\phi)=\left(\frac{\cos \phi_{1}}{\cos \phi}\right)^{1-K}\left(\frac{1+\sin \phi_{1}}{1+\sin \phi}\right)^{K}
$$

In terms of m as defined in eq. 7 and of the results obtained in eqs. 5 and 6 , one may rewrite eqs. I as

$$
\begin{array}{ll}
r=a m(\phi) \cos \phi / \mathrm{K} & 8 \mathrm{a} \\
\theta=\mathrm{K}\left(\lambda-\lambda_{0}\right) & 8 \mathrm{~b}
\end{array}
$$

In terms of Cartesian coordinates, $(x, y)$, on the plane map; one has

$$
\begin{aligned}
& x=r \cos \theta=a m(\phi) \cos \phi \cos \left[K\left(\lambda-\lambda_{0}\right)\right] / K \\
& y=r \sin \theta=a m(\phi) \cos \phi \sin \left[K\left(\lambda-\lambda_{0}\right)\right] / K
\end{aligned}
$$

The metrics, $h_{x}$ and $h_{y}$, for the coordinate system $x$, $\bar{y}$, are calculable (cf. Morse and Fishbach, p. 24) from the relationships

$$
\begin{aligned}
& 1=h_{x}^{2}\left(\left(\frac{1}{a \cos \phi} \frac{\partial x}{\partial \lambda}\right)^{2}+\left(\frac{1}{a} \frac{\partial x}{\partial \phi}\right)^{2}\right) \\
& 1=h_{y}^{2}\left(\left(\frac{1}{a \cos \phi} \frac{\partial y}{\partial \lambda}\right)^{2}+\left(\frac{1}{a} \frac{\partial y}{\partial \phi}\right)^{2}\right)
\end{aligned}
$$

One may prove that

$$
\begin{equation*}
\frac{\partial}{\partial \phi}(m(\phi) \cos \phi)=-K m(\phi) \tag{11}
\end{equation*}
$$

Evaluation of the partial derivatives in eqs. 10 yields

$$
\begin{equation*}
h_{x}=h_{y}=\frac{1}{m(\phi)} \tag{12}
\end{equation*}
$$

### 2.1 The Quasi-Static Equations in Map Coordinates

The equations of motion in surface spherical coordinates can be written in the following form, which is compatible with the quasi-static approximation and the conservation of total energy (Lorenz, 1967, p. 18),

$$
\begin{align*}
& \frac{d u_{s}}{d t}-\frac{u_{s} v_{S} \sin \phi}{a \cos \phi}-f v_{s}+\frac{\alpha}{a \cos \phi} \frac{\partial p}{\partial \lambda}=F_{\lambda}  \tag{13a}\\
& \frac{d v_{s}}{d t}+\frac{u_{s} u_{s} \sin \phi}{a \cos \phi}+f u_{s}+\frac{\alpha}{a} \frac{\partial p}{\partial \phi}=F_{\phi}
\end{align*}
$$

in which $u_{s}=a \cos \phi \dot{\lambda}$ and $v_{s}=a \dot{\phi}$

If we define the map wind components

$$
\begin{align*}
& u=h_{x} \frac{d x}{d t}=\frac{1}{m} \dot{x} \\
& y=h_{y} \frac{d y}{d t}=\frac{1}{m} \dot{y}
\end{align*}
$$

then the following relationships hold between ( $u_{s}, v_{s}$ ) and ( $u, v$ ), and, for that matter, among the components of any "horizontal" vectors in the two coordinate systems.

$$
\begin{aligned}
& u=-v_{\mathrm{S}} \cos K\left(\lambda-\lambda_{0}\right)-u_{S} \sin K\left(\lambda-\lambda_{0}\right) \\
& \mathrm{v}=-\mathrm{v}_{\mathrm{S}} \sin K\left(\lambda-\lambda_{0}\right)+u_{\mathrm{S}} \cos K\left(\lambda-\lambda_{0}\right)
\end{aligned}
$$

To derive equations in the map coordinater eqs. 13 are multiplied by appropriate combinations of $\sin K\left(\lambda-\lambda_{0} I\right.$ and $\cos K\left(\lambda-\lambda_{0}\right)$, and are then added. The observations that

$$
\overrightarrow{\mathrm{K}} \cdot \vec{v} \dot{v} \nabla \ln _{n} m=-\frac{u_{S}(1-\sin \phi)}{a \cos \phi}=v \frac{\partial m}{\partial x}-u \frac{\partial m}{\partial y}
$$

and

$$
\frac{1}{a} \frac{\partial}{\partial \phi} n \ln m(\phi)=-\frac{1-\sin \phi}{a \cos \phi}
$$

are useful in the manipulation necessary to arrive at,

$$
\begin{align*}
& \frac{d u}{d t}-\left(f+v \frac{\partial m}{\partial x}-u \frac{\partial m}{\partial y}\right) v+\alpha m \frac{\partial p}{\partial x}=F_{x} \\
& \frac{d v}{d t}+\left(f+v \frac{\partial m}{\partial x}-u \frac{\partial m}{\partial y}\right) u+\alpha m \frac{\partial p}{\partial y}=F_{y}
\end{align*}
$$

The individual derivative may be expressed as

$$
\begin{equation*}
\frac{d}{d t}=\frac{\partial}{\partial t}+m(\phi)\left(u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}\right)+w \frac{\partial}{\partial z} \tag{18}
\end{equation*}
$$

The horizontal divergence and the vertical component of the curl of a vector $\vec{v}$ with components $(u, v)$ are given by,

$$
\begin{align*}
& \nabla \cdot \vec{V}=m^{2}(\phi)\left(\frac{\partial}{\partial x} \frac{u}{m}+\frac{\partial}{\partial y} \frac{v}{m}\right)  \tag{19a}\\
& \vec{K} \cdot \nabla x \vec{v}=m^{2}(\phi)\left(\frac{\partial}{\partial x} \frac{v}{m}-\frac{\partial}{\partial y} \frac{u}{m}\right) \tag{19b}
\end{align*}
$$

In order to express the Coriolis parameter, $f$, and the map factor, $m$, in terms of the coordinates ( $x, y$ ) of a point on the map, one may define

$$
\begin{equation*}
R \equiv\left(\frac{K^{2}\left(x^{2}+y^{2}\right)}{a^{2} \mu^{2}}\right)^{1 / K} \tag{20}
\end{equation*}
$$

with

$$
\begin{equation*}
\mu \equiv\left(\cos \phi_{1}\right)^{1-K}\left(1+\sin \phi_{I}\right)^{K} \tag{21}
\end{equation*}
$$

and then prove that

$$
\begin{equation*}
\sin \phi=\frac{1-R}{I+R} \tag{22}
\end{equation*}
$$

and

$$
\begin{equation*}
\cos \phi=\left(\frac{2}{d+R}\right)^{\frac{1}{2}} \tag{23}
\end{equation*}
$$

It then follows that

$$
\mathrm{f}=2 \omega\left(\frac{1-\mathrm{R}}{1+\mathrm{R}}\right)
$$

and

$$
m=\frac{\mu(1+R)}{2} R^{\frac{K-1}{2}}
$$

## 3. The Polar Stereographic Map

A special case of the Lambert conformal map is one in which the transformation is modified to

$$
\begin{align*}
& r=r_{0}\left(\tan \left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right) \\
& \theta=\lambda
\end{align*}
$$

This is now a one-to-one, conformal mapping, but one has only one parameter at one's disposal. The scale may be made "true" at one latitude, say $\phi_{0}$.

$$
r d \theta=a \cos \phi d \lambda \quad \text { at } \phi=\phi_{0}
$$

One gets

$$
r_{0} \frac{\cos \phi_{0}}{1+\sin \phi_{0}}=a \cos \phi_{0}
$$

so

$$
r_{0}=a\left(1 \neq \sin \phi_{0}\right)
$$

The transformation equations may be rewritten as

$$
\begin{aligned}
& r=a \cos \phi \frac{1+\sin \phi_{0}}{1+\sin \phi}=a m(\phi) \cos \phi \\
& \theta=\lambda
\end{aligned}
$$

in which the map factor $m(\phi)$ has been introduced.
4. The Mercator Map

Another special case of the Lambert type conformal mapping uses Cartesian map coordinates

$$
\begin{aligned}
& x=K \lambda \\
& y=F(\phi) \quad \text { with } \quad y=0 \quad \text { at } \phi=0
\end{aligned}
$$

To make the map conformal, one must satisfy

$$
\begin{equation*}
\frac{1}{a} \frac{\partial y}{\partial \phi}=\frac{1}{a \cos \phi} \frac{\partial x}{\partial \lambda}=\frac{K}{a \cos \phi} \tag{29}
\end{equation*}
$$

The functional form of $F$ which satisfies this equation and the condition $y=0$ at $\phi=0$, is

$$
\begin{aligned}
& F=K \int_{0}^{\phi} \sec \zeta d \zeta=\left.K \ln \tan \left(\frac{\pi}{4}+\frac{\zeta}{2}\right)\right|_{0} ^{\phi} \\
& F=K \ln \left(\frac{\tan \frac{\pi}{4}+\frac{\phi}{2}}{\tan -\frac{\pi}{4}}\right)=K \ln \tan \left(\frac{\pi}{4}+\frac{\phi}{2}\right)
\end{aligned}
$$

The parameter $K$ may be chosen to make the map true to scale at a latitude $\phi_{0}$,

$$
\begin{aligned}
& \mathrm{dx}=\operatorname{Kd\lambda }=\mathrm{a} \cos \phi d \lambda \quad \text { at } \phi=\phi_{0} \\
& \mathrm{~K}=\mathrm{a} \cos \phi_{0}
\end{aligned}
$$

Thus the Mercator map, true at $\phi= \pm \phi_{0}$, is given by the transformation

$$
\begin{aligned}
& x=a \cos \phi_{0} \lambda \\
& y=a \cos \phi_{0} \ln \tan \left(\frac{\pi}{4}+\frac{\phi}{2}\right)
\end{aligned}
$$

The map scale factor

$$
m(\phi)=\frac{\cos \phi_{o}}{\cos \phi}
$$

may be introduced into 32 to get

$$
\begin{aligned}
& x=a m(\phi) \cos \phi \lambda \lambda \\
& y=a m(\phi) \cos \phi \ln \tan \left(\frac{\pi}{4}+\frac{\phi}{2}\right)
\end{aligned}
$$

## 5. Jambert Osculating Conic Map

A final, conformal conic map of the Lambert type is one which is true at just one latitude, $\phi_{0}$. The transformation equations are

$$
\begin{aligned}
& r=r_{0}\left(\tan \left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right)^{\sin \phi_{0}} \\
& \theta=\left(\sin \phi_{0}\right)\left(\lambda-\lambda_{0}\right)
\end{aligned}
$$

From the conformality equationscoone must have

$$
\begin{equation*}
\frac{r}{a \cos \phi} \frac{\partial \theta}{\partial \lambda}=-\frac{1}{a} \frac{\partial r}{\partial \phi} \tag{36}
\end{equation*}
$$

But, $\quad \frac{-1}{a r} \frac{\partial r}{\partial \phi}=\frac{\sin \phi_{0}}{a \cos \phi}$
and

$$
\frac{1}{a \cos \phi} \frac{\partial \theta}{\partial \lambda}=\frac{\sin \phi_{0}}{a \cos \phi}
$$

so conformality is assured. The latitude at which the scale is true is the latitude of osculation, $\phi_{0}$. To prove this, we calculate

$$
r d \theta=r_{0}\left(\tan \frac{\pi}{4_{4}}-\frac{\phi_{\phi}}{2}\right)^{\sin \phi_{0}} \sin \phi_{0} d \lambda
$$

and show that this is equal to a cos $\phi_{0} d \lambda$ provided that one sets,

$$
r_{0}=a \tan \left(\frac{\pi}{4}-\frac{\phi_{0}}{2}\right)^{-\sin \phi_{0}} \dot{\operatorname{tn} \phi_{0}}
$$

One may use the identity

$$
\tan \left(\frac{\pi}{4}-\frac{\phi}{2}\right) \equiv \frac{\cos \phi}{I+\sin \phi}
$$

and substitute 37 into 35 a to get

$$
\begin{equation*}
r=\frac{a \cos \phi}{\sin \phi_{0}} \frac{\cos \phi_{0}}{\cos \phi}\left(\frac{\cos \phi}{\cos \phi_{0}}\right) \sin \phi_{0}\left(\frac{1+\sin \phi_{0}}{1+\sin \phi}\right)^{\sin \phi_{0}} \tag{38}
\end{equation*}
$$

or

$$
\begin{equation*}
r=\frac{a \cos \phi}{\sin \phi_{o}} m(\phi) \tag{39}
\end{equation*}
$$

in which

$$
\begin{equation*}
m(\phi)=\left(\frac{\cos \phi_{0}}{\cos \phi}\right)^{1-\sin \phi_{0}}\left(\frac{1+\sin \phi_{0}}{1+\sin \phi}\right)^{\sin \phi_{0}} \tag{40}
\end{equation*}
$$

That $m(\phi)$ is the map scale factor is seen by forming the ratio,

$$
\frac{r d \theta}{a \cos \phi d \lambda^{i}}=\frac{a \cos \phi \sin \phi_{0} d \lambda}{a \sin \phi_{0} \cos \phi d \lambda}=m(\phi)
$$

## 6. The Pseudo-Spherical Coordinate Map

The partial differential equations written using surface spherical coordinates may be replaced by a set of finite difference equations. This is done in certain models now under development at NMC (Vanderman, 1972) by the construction of a finite difference gridpoint array with the property that points are equally spaced in both latitude and longitude. If this array of points is plotted on a plane surface, the Cartesian coordinates $x, y$ may be defined by means of the transformation

$$
\begin{array}{ll}
x=a \lambda & 42 a \\
y=a \phi & 42 b
\end{array}
$$

Considered as a mapping transformation, eq. 42 is neither conformal nor equivalent. The scale factor is different in the two directions

$$
\begin{array}{ll}
m_{x}=1 / \cos \phi & 43 \mathrm{a} \\
m_{y}=1 & 43 \mathrm{~b}
\end{array}
$$

One may rewrite the eqs. 42 as

$$
x=a m(\phi) \cos \phi_{i} \lambda \quad 44 a
$$

$y=a m(\phi) \cos \phi \phi$
44 b
if $\quad m(\phi) \equiv m_{x}(\phi)=I / \cos \phi$

To evaluate the metrics $h_{X}$ and $h_{Y}$, one has

$$
\begin{aligned}
& I=h_{x}^{2}\left(\left(\frac{1}{a \cos \phi} \frac{\partial x}{\partial \lambda}\right)^{2}+\left(\frac{1}{a} \frac{\partial x}{\partial \phi}\right)^{2}\right) \\
& I=h_{y}^{2}\left(\left(\frac{1}{a \cos \phi} \frac{\partial y}{\partial \lambda}\right)^{2}+\left(\frac{1}{a} \frac{\partial y}{\partial \phi}\right)^{2}\right) \\
& h_{x}=\cos \phi \equiv \frac{1}{m} \\
& h_{y}=1
\end{aligned}
$$

$$
46 a
$$

$$
46 b
$$

The velocity components on the pseudo-map are

$$
\begin{align*}
& u=h_{x} \frac{d x}{d t}=a \cos \phi \dot{\lambda} \equiv u_{s} \\
& v=h_{y} \frac{d y}{d t}=a \dot{\equiv} \quad \equiv v_{s}
\end{align*}
$$

The equations of motion transform into

$$
\begin{align*}
& \frac{\partial u}{\partial t}+m u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}+w^{\frac{\partial u}{\partial z}}-v\left(\frac{u}{m} \frac{\partial m}{\partial y}+f\right)+m \alpha \frac{\partial p}{\partial x}=F_{x} \\
& \frac{\partial v}{\partial t}+m u \frac{\partial v}{\partial x}+v \frac{\partial v}{\partial y}+w \frac{\partial v}{\partial z}+u\left(\frac{u}{m} \frac{\partial m}{\partial y}+f\right)+\alpha \frac{\partial p}{\partial y}=F_{y}
\end{align*}
$$

48b

Shuman (1970, p. 569) notes that troublesome pair of metric terms may be coalesced into one, if one introduces the wind functions

$$
\begin{array}{rlr}
u & =u \cos \theta-v \sin \theta & 49 a \\
v & =u \sin \theta+v \cos \theta & 49 b \\
\text { in which } \quad \theta & =\sin \phi_{0}\left(\lambda-\lambda_{0}\right) & 50
\end{array}
$$

is the azimuthal coordinate of the Lambert Conic Map (cf. sections5) osculating the sphere at $\phi_{0}$. With 49 cidefined the equations transform to

$$
\begin{align*}
& \frac{\partial}{\partial t} U+m u \frac{\partial U}{\partial x}+v \frac{\partial U}{\partial y}+w \frac{\partial U}{\partial z}-V f+\alpha \cos \theta m \frac{\partial p}{\partial x}-\alpha \sin \theta \frac{\partial p}{\partial y}=0 \\
& \frac{\partial}{\partial t} V+m u \frac{\partial V}{\partial x}+v \frac{\partial V}{\partial y}+w \frac{\partial V}{\partial z}+U f+\alpha \sin \theta m \frac{\partial p}{\partial x}+\alpha \cos \theta \frac{\partial p}{\partial y}=0
\end{align*}
$$

in which we have omitted the frictional force, $F$, which transforms as the pressure gradient terms.

Shuman proposes that eqs. 49 be modified at each grid point so that $\phi_{0}$ and $\lambda_{0}$ take on the values at the local grid point. The equations given by Shuman, as the set below his eq. 20, are correct only for a local region about the central point where $\theta \simeq 0$ and $U \simeq u$ and $V \simeq v$.

It will be noted that the two poles become lines under this pseudo mapping. The process for treating these singular points may be considered a form of boundary condition specification.

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