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The Formulation of the Autobarotropic Layer in the "New LFM" Model

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1. Introduction

The "new LFM" model is being developed in order to evaluate the usefulness of the semi-implicit integration method in multi-level prediction models. The physical framework of the model is to be kept as close to that of the 6-L PE as is possible. One aspect of the physical framework of the 6L PE that it has been found necessary to modify in the new LFM is the formulation of the topmost layer, or cap, of the model.

In the 6L PE, the pressure gradient force $\vec{P}$ in the equations of motion is expressed as

$$\vec{P} = \nabla\phi + c_p\theta\nabla\pi \tag{1}$$

in which ϕ is the geopotential; c_p the specific heat at constant pressure; θ is the potential temperature and π is the Exner function. Similarly, the hydrostatic equation is expressed in the form

$$\frac{\partial\phi}{\partial\sigma} = - c_p\theta\frac{\partial\pi}{\partial\sigma} \tag{2}$$

in which σ is the vertical coordinate.

The uppermost layer in the 6L PE is treated as an autobarotropic fluid, and the initial state is set up as an equilibrium state of rest. The particular choice of autobarotropic fluid--adiabatic, isentropic--reflects the form chosen in equations 1 and 2.

Now, because of the complex form of the relationship between σ and pressure used in the 6L PE*, it was found necessary to deviate from the expressions 1 and 2 when the new LFM was being derived. In the new model, the corresponding expressions are

$$\vec{P} = \nabla\phi + \alpha\nabla p \tag{3}$$

and

$$\frac{\partial\phi}{\partial\sigma} = - \alpha\frac{\partial p}{\partial\sigma} \tag{4}$$

A consequence of this change, which has become obvious only recently, is that the formulation of the autobarotropic layer must be varied to one which is homogeneous and incompressible.

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*A different linear relation is posited in each of the σ domains.

There are two aspects of this change to be considered. First, the technique for initialization of the cap as a quiet layer and, second, the modification of the differential equations for this layer to express the physical modification. In this note, only the first point is taken up.

2. The Initialization of the Quiet Layer

The technique for selecting a geopotential height and pressure for the σ-coordinate surface that forms the base of the autobarotropic layer will follow the outline given by Dr. John Stackpole for the 6L PE. In figure 1, a schematic of the data distribution involved in the process is given.

The parameters which are taken to be known are:

Z_{-1}, the height at the "top" of the model atmosphere, a constant.

$\alpha_{-.5}$, the specific volume of the homogeneous, incompressible fluid.

Z_{100} and α_{100}, the geopotential height and specific volume at 100 mbs.

p_2, Z_2 and α_2, the pressure, geopotential height and specific volume at the tropopause.

The parameters to be determined are p_o and Z_o. Note that these parameters will be constructed to fit model constraints and not to represent observed data. Also, note that the 100 mb surface is assumed to lie in the neighborhood of the σ surface forming the base of the autobarotropic layer.

We begin by making the assumption that the specific volume varies linearly with pressure in the stratosphere. The linear profile is determined through the use of 100 mb and tropopause data. It may be remarked that the use of 100 mb data is reasonable only if p_o is not significantly greater than 100 mb. One obtains

$$\alpha = \hat{\alpha} + B(p-\hat{p}) \tag{5}$$

with

$$\hat{p} \equiv .5(p_{100}+p_2) \tag{6}$$

$$\hat{\alpha} \equiv g\ (Z_{100}-Z_2)/(p_2-p_{100}) \tag{7}$$

$$B \equiv (\alpha_{100}-\alpha_2)/(p_{100}-p_2) \tag{8}$$

One should observe that the autoconvectively stable atmosphere is characterized by $B<0$*. The constant acceleration of gravity is denoted by g_o.

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* The objective analysis scheme requires that the tropopause pressure be greater than 100 mb.

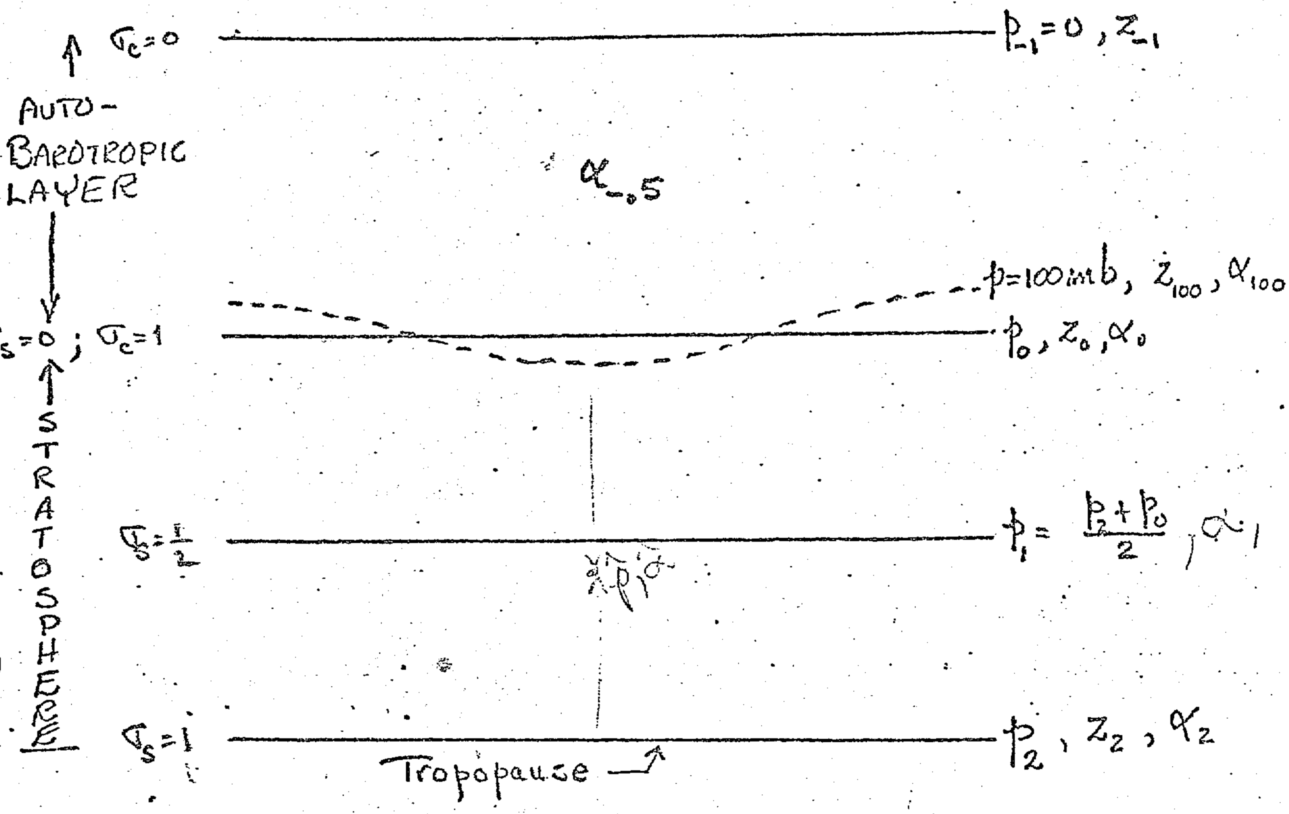

FIGURE 1. Schematic of DATA DISTRIBUTION.

One may now write the hydrostatic equations for the autobarotropic layer and the stratosphere:

$$g_o(Z_{-1}-Z_o) = \alpha_{-.5}\,p_o \tag{9}$$

$$g_o(Z_o-Z_2) = -\alpha_1(p_o-p_{100}) + \alpha_1(p_2-p_{100}) \tag{10}$$

in which

$$\alpha_1 = .5(\alpha_o+\alpha_2) \tag{11}$$

or

$$\alpha_1 = \hat{\alpha} + B\ .5(p_o-p_{100}) \tag{12a}$$

$$\alpha_1 = \hat{\alpha} - .5(\alpha_{100}-\alpha_2)(p_o-p_{100})/(p_2-p_{100}) \tag{12b}$$

One may solve equation 9 for Z_o,

$$Z_o = Z_{-1} - \alpha_{-.5}\ p_o/g_o, \tag{13}$$

and substitute (13) into (10) to get

$$g_o(Z_{-1}-Z_2) - \alpha_{-.5}\ p_o = +\ \alpha_1(p_2-p_{100}) - \alpha_1(p_o-p_{100}) \tag{14a}$$

or

$$g_o(Z_{-1}-Z_2) = (\alpha_{-.5}-\alpha_1)(p_o-p_{100}) + \alpha_{-.5}\ p_{100} + \alpha_1(p_2-p_{100}) \tag{14b}$$

If one defines,

$$r \equiv (p_o-p_{100})/(p_2-p_{100}) \tag{15}$$

equations (12b) and (14b) may be rewritten as,

$$\alpha_1 = \hat{\alpha} - .5(\alpha_{100}-\alpha_2)r \tag{16a}$$

$$a \equiv \left(\frac{g_o(Z_{-1}-Z_2)}{p_2-p_{100}}\right) = (\alpha_{-.5}-\alpha_1)r + \alpha_{-.5}\frac{p_{100}}{p_2-p_{100}} + \alpha_1 \tag{16b}$$

Substitution of (16a) into (16b) gives, after some manipulation,

$$r^2 - 2b\ r + c = 0 \tag{17}$$

with

$$b = \left[\frac{1}{2} - \frac{(\alpha_{-.5}-\hat{\alpha})}{(\alpha_{100}-\alpha_2)}\right] \tag{17a}$$

$$c = \frac{2}{\alpha_{100}-\alpha_2}\left[\hat{\alpha} + \frac{p_{100}}{p_2-p_{100}}\,\alpha_{-.5} - a\right] \tag{17b}$$

The expression for c may be rewritten using the definitions of $\hat{\alpha}$ and a,

$$c = \frac{-2}{(\alpha_{100}-\alpha_2)} \quad \frac{p_{100}}{(p_2-p_{100})} \left[\frac{g_o(Z_{-1}-Z_{100})}{p_{100}} - \alpha_{-.5}\right] \tag{18}$$

If we define

$$\alpha_e \equiv \frac{g_o(Z_{-1}-Z_{100})}{p_{100}} \tag{19}$$

the expression for c becomes

$$c = \frac{-2\, p_{100}(\alpha_e-\alpha_{-.5})}{(\alpha_{100}-\alpha_2)(p_2-p_{100})} \tag{20}$$

Note that α_e is an estimate of the specific volume in the uppermost atmospheric layer.

The quadratic equation (17) has two roots,

$$r_{\pm} = \left(\frac{1}{2} - \frac{\alpha_{-.5}-\hat{\alpha}}{\alpha_{100}-\alpha_2}\right) \pm \left[\left(\frac{1}{2} - \frac{\alpha_{-.5}-\hat{\alpha}}{\alpha_{100}-\alpha_2}\right)^2 + \frac{2\, p_{100}(\alpha_e-\alpha_{-.5})}{(\alpha_{100}-\alpha_2)(p_2-p_{100})}\right]^{\frac{1}{2}} \tag{21}$$

The appropriate root is that given by the negative sign. For suppose that $\alpha_e \simeq \alpha_{-.5}$, than the negative sign gives

$$\frac{p_o-p_{100}}{p_2-p_{100}} = 0$$

or

$$p_o = p_{100}; \tag{22}$$

whereas the positive sign gives

$$p_o = p_2 - \frac{p_2-p_{100}}{(\alpha_{100}-\alpha_2)}(\alpha_{-.5}-\hat{\alpha}). \tag{23}$$

But our assumption that $\alpha_e \simeq \frac{1}{2}\,\alpha_{-.5}$ implies that

$$\alpha_{-.5} \simeq \alpha_{100} \quad \text{and} \quad \hat{\alpha} \simeq \frac{1}{2}(\alpha_{100}+\alpha_2) \tag{24}$$

so that

$$p_o = \frac{1}{2}(p_2 + p_{100}) \tag{25}$$

which is too large a value of p_o.

In this procedure, the values of $\alpha_{-.5}$ and Z_{-1} were assumed to be known constants. The solution for r,

$$r = \left(\frac{1}{2} - \frac{\alpha_{-.5} - \hat{\alpha}}{\alpha_{100} - \alpha_2}\right)\left(1 - (1 + M)^{\frac{1}{2}}\right) \tag{26}$$

with

$$M = \frac{8\, p_{100}(\alpha_{100} - \alpha_2)(\alpha_e - \alpha_{-.5})}{(p_2 - p_{100})(\alpha_{100} - \alpha_2 - 2\,\alpha_{-.5} + 2\,\hat{\alpha})^2} \tag{27}$$

can yield a non-meaningful result if

$$M < -1 \tag{28}$$

For an autoconvectively stable atmosphere, M will be non-positive only if

$$\alpha_e \leq \alpha_{-.5} \tag{29}$$

Replacement of α_e by use of equation (19) in equation (29) gives

$$\frac{g_o(Z_{-1} - Z_{100})}{p_{100}} \leq \alpha_{-.5} \tag{30}$$

Thus, one can insure real valued solutions for r in equation (26) provided that

$$Z_{-1} > Z_{100} + \frac{p_{100}}{g_o}\alpha_{-.5} \tag{31}$$

This suggests taking

$$\alpha_{-.5} = \max\left(\alpha_{100}\right) \tag{32}$$

and

$$Z_{-1} = \max\left(Z_{100}\right) + \frac{p_{100}}{g_o}\max\left(\alpha_{100}\right) \tag{33}$$

where the max( ) means the maximum value over the entire region of integration.

The proof that the horizontal pressure gradient force will vanish within the autobarotropic layer, when p_o is determined via equation (26) and Z_o is obtained through equation (13), follows directly from the model's use of the hydrostatic equation (9) and the horizontal pressure gradient formulation

$$\vec{P} = \frac{g_o}{2} \nabla (Z_{-1} + Z_o) + \alpha_{-.5} \frac{1}{2} \nabla p_o .$$