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<u>R E V I S E D</u>

Vertical Differencing and Conservation

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Vertical Differencing and Conservation[1]

This note is a presentation of a vertical differencing system which conserves

(1) mass

(2) potential temperature (θ) and θ^2

(3) total energy.

The procedure for the derivation is that of Langlois and Kwok (1969) used in describing Arakawa's system. However, we have used here a two-layer model in pressure coordinates. The usual notation is used throughout.

We have ignored truncation errors in time and horizontal space.

The equations for an inviscid, adiabatic atmosphere in hydrostatic balance are

$$\frac{\partial \omega}{\partial p} = - \nabla \cdot \vec{V} \tag{1}$$

$$\frac{\partial \theta}{\partial t} = - \nabla \cdot \theta \vec{V} - \frac{\partial \omega \theta}{\partial p} \tag{2}$$

$$\frac{\partial u}{\partial t} = - \nabla \cdot u \vec{V} - \frac{\partial \omega u}{\partial p} - \frac{\partial \phi}{\partial x} + fv \tag{3}$$

$$\frac{\partial v}{\partial t} = - \nabla \cdot v \vec{V} - \frac{\partial \omega v}{\partial p} - \frac{\partial \phi}{\partial y} - fu \tag{4}$$

1. Conservation of Mass

Integrate (1) for the continuous atmosphere from $p = 0$ to $p = p_*$, the surface pressure.

$$\int_o^{p_*} \frac{\partial \omega}{\partial p} dp = - \int_o^{p_*} \nabla \cdot \vec{V} \, dp$$

or

$$\omega_* = - \nabla \cdot \int_o^{p_*} \vec{V} \, dp + \vec{V}_* \cdot \nabla \, p_*$$

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[1]An error was made in previous note which led to energy conservation at the expense of mass conservation. The main modification here is in the way of converting the θ-equation to a T-equation.

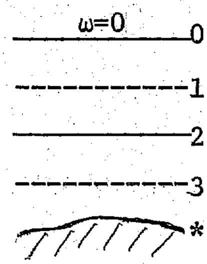
But

$$\omega_* = \frac{\partial p_*}{\partial t} + \vec{V}_* \cdot \nabla p_* \ . \tag{5}$$

Therefore

$$\frac{\partial p_*}{\partial t} = - \nabla \cdot \int_0^{p_*} \vec{V} \, dp \ . \tag{6}$$

Now consider the two-layer atmosphere



For layer 1:

$$\int_0^{p_*} \frac{\partial \omega}{\partial p} \, dp = - \int_0^{p_*} \nabla \cdot \vec{V} \, dp$$

$$\omega_2 = - p_2 \nabla \cdot \vec{V}_1 \tag{7}$$

For layer 3:

$$\int_{p_2}^{p_*} \frac{\partial \omega}{\partial p} \, dp = - \int_{p_2}^{p_*} \nabla \cdot \vec{V} \, dp$$

or

$$\omega_* - \omega_2 = - \nabla \cdot (p_* - p_2) \vec{V}_3 + \vec{V}_* \cdot \nabla p_*$$

or

$$\frac{\partial p_*}{\partial t} = - p_2 \nabla \cdot \vec{V}_1 - \nabla \cdot (p_* - p_2) \vec{V}_3 \tag{8}$$

Thus mass is conserved for a closed system.

2. Potential Temperature

Multiply (2) by θ^{n-1}.

$$\frac{\partial}{\partial t}\left(\frac{1}{n}\theta^n\right) = -\nabla\cdot\left(\frac{1}{n}\theta^n\right)\vec{V} - \frac{\partial}{\partial p}\left(\frac{1}{n}\theta^n\omega\right)$$

Integrate over pressure for the continuous atmosphere

$$\frac{\partial}{\partial t}\int_0^{p_*}\left(\frac{1}{n}\theta^n\right)dp - \underline{\left(\frac{1}{n}\theta_*^n\right)\frac{\partial p_*}{\partial t}} = -\nabla\cdot\int_0^{p_*}\left(\frac{1}{n}\theta^n\vec{V}\right)dp$$

$$+\underline{\frac{1}{n}\theta_*^n\vec{V}_*\cdot\nabla p_*} - \underline{\frac{1}{n}\theta_*^n\omega_*} \qquad (9)$$

The underlined terms vanish due to (5).

Apply (2) to layer 1 of the two-layer atmosphere.

$$\int_0^{p_2}\frac{\partial\theta}{\partial t}dp = -\int_0^{p_2}\nabla\cdot\theta\vec{V}\,dp - \int_0^{p_2}\frac{\partial\omega\theta}{\partial p}dp$$

or

$$p_2\frac{\partial\theta_1}{\partial t} = -p_2\nabla\cdot\theta_1\vec{V}_1 - \omega_2\theta_2 \qquad (10)$$

For layer 3:

$$\frac{\partial}{\partial t}\int_{p_2}^{p_*}\theta\,dp - \underline{\theta_*\frac{\partial p_*}{\partial t}} = -\nabla\cdot\int_{p_2}^{p_*}\theta\vec{V}\,dp - \underline{\theta_*\vec{V}_*\cdot\nabla p_*}$$

$$-\left(\underline{\omega_*\theta_*} - \omega_2\theta_2\right)$$

Underlined terms vanish and we get

$$\frac{\partial}{\partial t}\left((p_*-p_2)\theta_3\right) = -\nabla\cdot\left((p_*-p_2)\theta_3\vec{V}_3\right) + \omega_2\theta_2 \qquad (11)$$

Notice that when we add (10) and (11), we get the finite difference form of (9) with $n=1$ no matter how we define θ_2. Thus θ will be conserved for a closed system.

Now multiply (10) by θ_1 and (11) by θ_3 and add result. We get finally

$$\frac{1}{2}\frac{\partial}{\partial t}\left(p_2\theta_1^2 + (p_*-p_2)\theta_3^2\right) = -\frac{1}{2}\, p_2\nabla\cdot\theta_1^2\,\vec{V}_1 - \frac{1}{2}\,\nabla\cdot\left((p_*-p_2)\theta_3^2\,\vec{V}_3\right)$$

$$- p_2\nabla\cdot\vec{V}_1\left(\theta_2(\theta_3-\theta_1) - \frac{1}{2}\,\theta_3^2 + \frac{1}{2}\,\theta_1^2\right).$$

In order that we get a finite difference form of (9) with n=2 so that θ^2 is conserved, the last term above must vanish. Thus,

$$\theta_2 = \frac{1}{2}(\theta_1 + \theta_3) \tag{12}$$

3. Total Energy

The kinetic energy equation is

$$\frac{\partial K}{\partial t} = -\nabla\cdot(K+\phi) - \frac{\partial}{\partial p}(K+\phi) + \omega\frac{\partial\phi}{\partial p}$$

Here $K = \frac{1}{2}(u^2+v^2)$. Integrating over pressure for the continuous atmosphere gives

$$\frac{\partial}{\partial t}\int_0^{p_*} K\,dp = -\nabla\cdot\int_0^{p_*}(K+\phi)dp + \int_0^{p_*}\omega\frac{\partial\phi}{\partial p}\,dp - \phi_*\frac{\partial p_*}{\partial t} \tag{13}$$

Note that $\frac{\partial}{\partial t}\int_0^{p_*}\phi_* dp = \phi_*\frac{\partial p_*}{\partial t}$.

The equation for c_pT can be obtained by using $\theta\pi = T$ and $\pi = (p/p_o)^{\kappa}$ in equation (2). We get

$$c_p\frac{\partial T}{\partial t} = -c_p\nabla\cdot T\,\vec{V} - c_p\pi\frac{\partial\omega\theta}{\partial p}$$

Integrating over pressure in the continuous atmosphere gives finally

$$\frac{\partial}{\partial t}\int_0^{p_*} c_pT\,dp = -\nabla\cdot\int_0^{p_*} c_pT\,\vec{V}\,dp - \int_0^{p_*}\omega\frac{\partial\phi}{\partial p}\,dp \tag{14}$$

Add (13) and (14) to get total energy equation,

$$\frac{\partial}{\partial t}\int_0^{p_*}(K+\phi_*+c_pT)dp = -\nabla\cdot\int_0^{p_*}(K+\phi+c_pT)\vec{V}\,dp \tag{15}$$

In obtaining equations (10) and (11) for the layered atmosphere, we have assumed something about the distribution of θ and $\vec{V}$ within each layer. Let us assume that both θ and $\vec{V}$ are constant with respect to pressure within each layer. Then we define T_1 and T_3 from

$$\int_0^{p_2} T dp = \int_0^{p_2} \theta \pi dp = \frac{\theta_1}{1+\kappa} p_2 \pi_2 \equiv T_1 p_2$$

and

$$\int_{p_2}^{p_*} T dp = \theta_3 \int_{p_2}^{p_*} \pi dp = \frac{\theta_3}{1+\kappa} \left(p_* \pi_* - p_2 \pi_2\right) \equiv T_3 \left(p_* - p_2\right),$$

or

$$\pi_1 \equiv \pi_2 / (1+\kappa)$$

$$\pi_3 \equiv \frac{p_* \pi_* - p_2 \pi_2}{(1+\kappa)(p_* - p_2)}.$$

Therefore, from (10) and (11) we get after multiplying through by c_p:

$$\frac{\partial}{\partial t}\left(p_2 c_p T_1\right) = - \nabla \cdot p_2 c_p T_1 \vec{V}_1 - c_p \omega_2 \theta_2 \pi_1 \tag{16}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left((p_* - p_2) c_p T_3\right) &= - \nabla \cdot \left((p_* - p_2)\ c_p T_3 \vec{V}_3\right) \\ &+ c_p (\pi_* - \pi_3)\ \theta_3 \left(\frac{\partial p_*}{\partial t} + \vec{V}_3 \cdot \nabla p_*\right) + \omega_2 \theta_2 c_p \pi_3 . \end{aligned} \tag{17}$$

Add (16) and (17), and use (7) and (8). Then

$$\begin{aligned} \frac{\partial}{\partial t}\left(p_2 c_p T_1 + (p_* - p_2) c_p T_3\right) &= - \nabla \cdot p_2 c_p T_1 \vec{V}_1 - \nabla \cdot \left((p_* - p_2) c_p T_3 \vec{V}_3\right) \\ &- \left(c_p (\pi_3 - \pi_1) \theta_2 + c_p (\pi_* - \pi_3) \theta_3\right) p_2 \nabla \cdot \vec{V}_1 \\ &- c_p (\pi_* - \pi_3) \theta_3 (p_* - p_2) \nabla \cdot \vec{V}_3 \end{aligned} \tag{18}$$

Now we form the momentum equations for the layered atmosphere. For the u-equations, we integrate (3) from $p=0$ to $p=p_2$ and then from $p=p_2$ to $p=p_*$. We get finally

$$\frac{\partial}{\partial t}(p_2 u_1) = -\nabla\cdot(p_2 u_1 \vec{V}_1) - \omega_2 u_2 - \frac{\partial}{\partial x}(p_2 \phi_1) + f v_1 p_2 \tag{19}$$

$$\frac{\partial}{\partial t}\left((p_*-p_2)u_3\right) = -\nabla\cdot\left((p_*-p_2)u_3\vec{V}_3\right) + \omega_2 u_2 - \frac{\partial}{\partial x}\left((p_*-p_2)\phi_3\right) + f v_3(p_*-p_2) + \phi_*\frac{\partial p_*}{\partial x} \tag{20}$$

The v-equations are

$$\frac{\partial}{\partial t}(p_2 v_1) = -\nabla\cdot(p_2 v_1 \vec{V}_1) - \omega_2 v_2 - \frac{\partial}{\partial y}(p_2 \phi_1) - f u_1 p_2 \tag{21}$$

$$\frac{\partial}{\partial t}\left((p_*-p_2)v_3\right) = -\nabla\cdot\left((p_*-p_2)v_3\vec{V}_3\right) + \omega_2 v_2 - \frac{\partial}{\partial y}\left((p_*-p_2)\phi_3\right) - f u_3(p_*-p_2) + \phi_*\frac{\partial p_*}{\partial y} \tag{22}$$

From these equations we get the kinetic energy equation finally:

$$\frac{\partial}{\partial t}\left\{p_2 K_1 + (p_*-p_2)K_3\right\} = -\nabla\cdot(p_2 K_1 \vec{V}_1) - \nabla\cdot\left((p_*-p_2)K_3\vec{V}_3\right) - \vec{V}_1\cdot\nabla(p_2\phi_1) - \vec{V}_3\cdot\nabla\left((p_*-p_2)\phi_3\right) + \phi_*\vec{V}_3\cdot\nabla p_* - p_2\nabla\cdot\vec{V}_1\left\{K_1 - K_3 - (u_1u_2+v_1v_2-u_2u_3-v_2v_3)\right\}. \tag{23}$$

The last term vanishes provided

$$\vec{V}_2 = \tfrac{1}{2}(\vec{V}_1 + \vec{V}_3) \tag{24}$$

Now the total energy equation is obtained by adding (18) and (23). We get finally

$$\frac{\partial}{\partial t}\Big(p_2E_1 + (p_*- p_2)E_3 + \phi_*p_*\Big) = - \nabla\cdot\Big(p_2(E_1+ \phi_1)\vec{V}_1\Big)$$

$$- \nabla\cdot\Big((p_*- p_2)(E_3+ \phi_3)\vec{V}_3\Big)$$

$$+ \underline{\Big(\phi_1- \phi_* - c_p(\pi_3- \pi_1)\theta_2 - c_p(\pi_*- \pi_3)\theta_3\Big)p_2\nabla\cdot\vec{V}_1} \qquad (25)$$

$$+ \underline{\Big(\phi_3- \phi_* - c_p(\pi_*- \pi_3)\theta_3\Big)(p_*- p_2)\nabla\cdot\vec{V}_3}.$$

Here $E = K + c_pT$.

In order that (25) represent a finite-difference form of (15), the underlined terms must vanish. This is the case provided

$$\phi_1 - \phi_* - c_p(\pi_3- \pi_1)\frac{\theta_1+ \theta_3}{2} - c_p(\pi_*- \pi_3)\theta_3 = 0 \qquad (26)$$

$$\phi_3 - \phi_* - c_p(\pi_*- \pi_3)\theta_3 = 0 \qquad (27)$$

These are the forms of the hydrostatic equations which must be used in order that we properly conserve the total energy.

The final set of equations for the two-layer model is:

$$\omega_2 = - p_2\nabla\cdot\vec{V}_1$$

$$\frac{\partial p_*}{\partial t} = - p_2\nabla\cdot\vec{V}_1 - \nabla\cdot(p_* - p_2)\vec{V}_3$$

$$\frac{\partial\theta_1}{\partial t} = - \nabla\cdot\theta_1\vec{V}_1 - \tfrac{1}{2}\,\omega_2(\theta_1+\theta_3)/p_2$$

$$\frac{\partial u_1}{\partial t} = - \nabla\cdot u_1\vec{V}_1 - \frac{\partial\phi_1}{\partial x} + fv_1 - \tfrac{1}{2}\,\omega_2(u_1+u_3)/p_2$$

$$\frac{\partial v_1}{\partial t} = - \nabla\cdot v_1\vec{V}_1 - \frac{\partial\phi_1}{\partial y} - fu_1 - \tfrac{1}{2}\,\omega_2(v_1+v_3)/p_2$$

$$\frac{\partial\theta_3}{\partial t} = - \nabla\cdot\theta_3\vec{V}_3 + \tfrac{1}{2}\,\omega_2(\theta_1+\theta_3)/(p_*-p_2)$$

$$\frac{\partial u_3}{\partial t} = -\nabla\cdot u_3\vec{V}_3 - \frac{\partial \phi_3}{\partial x} - \underline{\frac{(\phi_3-\phi_*)}{(p_*-p_2)}\frac{\partial p_*}{\partial x}} + fv_3 + \tfrac{1}{2}\,\omega_2(u_1+u_3)/(p_*-p_2)$$

$$\frac{\partial v_3}{\partial t} = -\nabla\cdot v_3\vec{V}_3 - \frac{\partial \phi_3}{\partial y} - \underline{\frac{(\phi_3-\phi_*)}{(p_*-p_2)}\frac{\partial p_*}{\partial y}} - fu_3 + \tfrac{1}{2}\,\omega_2(v_1+v_3)/(p_*-p_2)$$

$$\phi_1 = \phi_3 + c_p(\pi_3 - \pi_1)(\theta_1 + \theta_3)/2$$

$$\phi_3 = \phi_* + c_p(\pi_* - \pi_3)\theta_3$$

Here

$$\pi_1 = \pi_2/(1+\kappa)$$

$$\pi_3 = \frac{p_*\pi_* - p_2\pi_2}{(1+\kappa)(p_*-p_2)}.$$

Notice the underlined terms in the momentum equations of layer 3. We see that even in p-coordinates a nonlinear pressure gradient term exists--but only in the layer above the ground. This term can be significant.

A problem exists when the hydrostatic equations are applied to real data. For example, given the fields of surface pressure and temperatures, the resulting geopotential heights are unrealistic. This problem can be circumvented by applying the change predicted by the model to the initial analyzed fields to get the predicted fields. This has been referred to in NMC as the "tendency method" and is applied to the surface pressure field.

Why not also apply the "tendency method" to the variables in the NMC operational models? This could add to the forecast skill, particularly in the short range. Verification of such a method relative to the present operational method might lead one to blend the two methods in time, such that the tendency method would dominate in the early periods and gradually give way to the present operational method in the longer ranges. Furthermore, statistical evaluations might guide us in how to blend the methods in both space and time.

The vertical differencing system which has been derived here can also be obtained for the NMC models in σ-coordinates. It would be useful to evaluate the forecast skills of such a system with those in general use here at NMC.

Reference

Langlois, W. E., and H. C. W. Kwok, 1969: Description of the Mintz-Arakawa Numerical General Circulation Model. UCLA Technical Report No. 3, 95 pp.