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OFFICE NOTE 92

On Vertical Differencing in the σ-System

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The following deals with vertical differencing in the σ-system. Certain features of the continuous fluid are retained in the vertical differencing system presented here. In particular, we will deal with the conservation of mass, total energy, and the first and second moments of the potential temperature. Throughout the derivation we have ignored time and horizontal spatial truncation errors. The derivation follows that of Arakawa, but is more directly applicable to the modeling practices in use at the NMC.

In the usual notation, we can write the equations for an inviscid, adiabatic atmosphere in hydrostatic balance. In the σ-coordinate (where σ is a linear function of pressure) we have

$$\frac{\partial}{\partial t} p_\sigma = - \nabla \cdot p_\sigma \vec{V} - p_\sigma \frac{\partial}{\partial \sigma} \dot{\sigma} \tag{1}$$

$$\frac{\partial}{\partial t}(\theta p_\sigma) = - \nabla \cdot (\theta p_\sigma) \vec{V} - p_\sigma \frac{\partial}{\partial \sigma} (\theta \dot{\sigma}) \tag{2}$$

$$\frac{\partial}{\partial t}(u p_\sigma) = - \nabla \cdot (u p_\sigma) \vec{V} - p_\sigma \frac{\partial \dot{\sigma} u}{\partial \sigma} - p_\sigma \frac{\partial \phi}{\partial x} - c_p \theta p_\sigma \frac{\partial \pi}{\partial x} + f v p_\sigma \tag{3}$$

$$\frac{\partial}{\partial t}(v p_\sigma) = - \nabla \cdot (v p_\sigma) \vec{V} - p_\sigma \frac{\partial \dot{\sigma} v}{\partial \sigma} - p_\sigma \frac{\partial \phi}{\partial y} - c_p \theta p_\sigma \frac{\partial \pi}{\partial y} - f u p_\sigma \tag{4}$$

1. Conservation of Mass

Integrating (1) from the top ($p = \sigma = \dot{\sigma} = 0$) to the bottom ($p = p_*$, $\sigma = 1$, $\dot{\sigma} = 0$) of the continuous atmosphere

$$\int_0^1 \frac{\partial}{\partial t} p_\sigma \, d\sigma = - \int_0^1 \nabla \cdot p_\sigma \vec{V} \, d\sigma - \int_0^1 p_\sigma \frac{\partial \dot{\sigma}}{\partial \sigma} \, d\sigma,$$

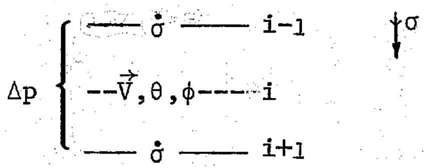
yields

$$\frac{\partial p_*}{\partial t} = - \nabla \cdot \int_\sigma p_\sigma \vec{V} \, d\sigma \tag{5}$$

Now integrating equation (1) over a particular layer in the layered atmosphere yields

$$\frac{\partial}{\partial t}(\Delta p) = - \nabla\cdot(\Delta p)\vec{V}_i - p_\sigma\left(\dot{\sigma}_{i+1} - \dot{\sigma}_{i-1}\right). \tag{6}$$

The convention used here is defined in the figure,



When integrating over all the layers of the atmosphere, the contributions from the last term in (6) must vanish as is the case for the continuous atmosphere. Note that in the case of two adjacent σ-domains (A and B) which are <u>not</u> separated by a material surface, mass will be conserved provided

$$p_{\sigma A}\ \dot{\sigma}_A = p_{\sigma B}\ \dot{\sigma}_B$$

at the interface.

2. <u>Potential Temperature</u>

Multiply (2) by θ^{n-1}.

$$\frac{\partial}{\partial t}\left(\frac{\theta^n}{n}\ p_\sigma\right) = - \nabla\cdot\left(\frac{\theta^n}{n}\ p_\sigma\right)\vec{V} - \frac{\partial}{\partial\sigma}\left(\frac{\theta^n}{n}\ p_\sigma\dot{\sigma}\right).$$

Integrating over σ in the continuous atmosphere yields

$$\frac{\partial}{\partial t}\int_0^1 \frac{\theta^n}{n}\, p_\sigma\, d\sigma = - \nabla\cdot\int_0^1\left(\frac{\theta}{n}\ p_\sigma\right)\vec{V}\ d\sigma\ , \tag{7}$$

since $\dot{\sigma} = 0$ at the top and bottom.

Now since $\theta\pi = T$, (2) may be written

$$\frac{\partial}{\partial t}\left(c_p T\ p_\sigma\right) = - \nabla\cdot\left(c_p T\ p_\sigma\right)\vec{V} - p_\sigma\frac{\partial}{\partial\sigma}\left(c_p T\ \dot{\sigma}\right)$$

$$+ c_p\theta\, p_\sigma\left(\frac{\partial\pi}{\partial t} + \vec{V}\cdot\nabla\pi + \dot{\sigma}\ \frac{\partial\pi}{\partial\sigma}\right) \tag{8}$$

This will be used later in section 5.

Now integrate (2) over layer i of the layered atmosphere.

$$\frac{\partial}{\partial t}\left(\theta_i \Delta p\right) = -\nabla\cdot\left(\theta_i \Delta p\right)\vec{V}_i - p_\sigma\left(\theta_{i+1}\,\dot{\sigma}_{i+1} - \theta_{i-1}\dot{\sigma}_{i-1}\right) \tag{9}$$

After multiplying by θ_i^{n-1}, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\theta_i^n}{n}\Delta p\right) = -\nabla\cdot\left(\frac{\theta_i^n}{n}\Delta p\right)\vec{V}_i - \theta_i^{n-1}\,p_\sigma\left(\theta_{i+1}\,\dot{\sigma}_{i+1} - \theta_{i-1}\dot{\sigma}_{i-1}\right)$$

$$- \theta_i^n\left(1-\frac{1}{n}\right)\left(\frac{\partial\Delta p}{\partial t} + \nabla\cdot(\Delta p)\vec{V}_i\right).$$

But with the use of (6), this can be rewritten

$$\frac{\partial}{\partial t}\left(\frac{\theta_i^n}{n}\Delta p\right) = -\nabla\cdot\left(\frac{\theta_i^n}{n}\Delta p\right)\vec{V}_i + p_\sigma\,\dot{\sigma}_{i-1}\left(\theta_i^{n-1}\theta_{i-1} - \theta_i^n\left(1-\frac{1}{n}\right)\right)$$

$$- p_\sigma\,\dot{\sigma}_{i+1}\left(\theta_i^{n-1}\,\theta_{i+1} - \theta_i^n\left(1-\frac{1}{n}\right)\right). \tag{10}$$

When summing over all layers, the contributions from the last two terms in (10) should vanish if the properties of the finite difference form are to be similar to those of the continuous atmosphere. When $n = 1$, there is no problem in accomplishing this. When $n = 2$, θ^2 will be conserved provided we set

$$\theta_{i+1} = \frac{1}{2}\left(\theta_{i+2} + \theta_i\right)$$

and (11)

$$\theta_{i-1} = \frac{1}{2}\left(\theta_{i-2} + \theta_i\right),$$

when $\dot{\sigma}_{i+1}$ and $\dot{\sigma}_{i-1}$ do not vanish.

Now

$$\theta_i = T_i/\pi_i, \tag{12}$$

where

$$\pi_i = (p_i/p_o)^{\kappa}$$

$$\kappa = R/c_p,$$

and

$$p_o = 100 \text{ cb.}$$

Thus (10) can be written

$$\frac{\partial}{\partial t}\left(c_p T_i \Delta p\right) = -\nabla\cdot\left(c_p T_i \Delta p\right)\vec{V}_i + c_p \theta_i \Delta p\left[\frac{\partial \pi_i}{\partial t} + \vec{V}_i\cdot\nabla\pi_i\right]$$
$$- c_p \pi_i p_\sigma\left(\theta_{i+1}\ \dot{\sigma}_{i+1} - \theta_{i-1}\ \dot{\sigma}_{i-1}\right). \tag{13}$$

This will be used later in section 5.

3. Momentum

From equations (3) and (4), we see that the vertical momentum flux terms (second terms on the right hand sides) vanish when integrated over σ in the continuous atmosphere. Now integrating (3) and (4) over layer i of the layered atmosphere yields

$$\frac{\partial}{\partial t}\left(u_i \Delta p\right) = -\nabla\cdot\left(u_i \Delta p\right)\vec{V}_i - \Delta p\,\frac{\partial \phi_i}{\partial x} - c_p\ \theta_i \Delta p\,\frac{\partial \pi_i}{\partial x}$$
$$+ \Delta p f\ v_i - p_\sigma\left(u_{i+1}\ \dot{\sigma}_{i+1} - u_{i-1}\ \dot{\sigma}_{i-1}\right) \tag{14}$$

$$\frac{\partial}{\partial t}\left(v_i \Delta p\right) = -\nabla\cdot\left(v_i \Delta p\right)\vec{V}_i - \Delta p\frac{\partial \phi_i}{\partial y} - c_p\ \theta_i \Delta p\,\frac{\partial \pi_i}{\partial y}$$
$$- \Delta p f\ u_i - p_\sigma\left(v_{i+1}\ \dot{\sigma}_{i+1} - u_{i-1}\ \dot{\sigma}_{i-1}\right). \tag{15}$$

The contributions of the last terms will vanish when we sum over all layers no matter how u, v_{i+1} and u, v_{i-1} are defined. This was also what we found in the case of conserving θ.

4. Kinetic Energy

Multiplying (3) by u and (4) by v and adding yields

$$\frac{\partial}{\partial t}\left(Kp_\sigma\right) = -\nabla\cdot Kp_\sigma\vec{V} - p_\sigma\vec{V}\cdot\nabla\phi - c_p\theta p_\sigma\vec{V}\cdot\nabla\pi - p_\sigma\frac{\partial}{\partial\sigma}\left(\dot{\sigma}K\right),$$

where

$$K = \frac{1}{2}\left(u^2 + v^2\right).$$

This can be rearranged so that

$$\frac{\partial}{\partial t}\left(Kp_\sigma\right) = -\nabla\cdot[(K+\phi)p_\sigma]\vec{V} - c_p\theta p_\sigma\vec{V}\cdot\nabla\pi$$

$$-\phi\left(\frac{\partial p_\sigma}{\partial t} + p_\sigma\frac{\partial\dot{\sigma}}{\partial\sigma}\right) - p_\sigma\frac{\partial}{\partial\sigma}\left(\dot{\sigma}K\right) \qquad (16)$$

For layer i, multiply (14) by u_i and (15) by v_i and add. After some rearranging we obtain

$$\frac{\partial}{\partial t}\left(K_i\Delta p\right) = -\nabla\cdot[(K_i+\phi_i)\Delta p]V_i - c_p\theta_i\Delta p\vec{V}_i\cdot\nabla\pi_i$$

$$-\phi_i\left[\frac{\partial}{\partial t}(\Delta p) + p_\sigma(\dot{\sigma}_{i+1} - \dot{\sigma}_{i-1})\right]$$

$$+p_\sigma[K_i(\dot{\sigma}_{i+1} - \dot{\sigma}_{i-1}) - u_i(u_{i+1}\dot{\sigma}_{i+1} - u_{i-1}\dot{\sigma}_{i-1})$$

$$- v_i(v_{i+1}\dot{\sigma}_{i+1} - v_{i-1}\dot{\sigma}_{i-1})] \qquad (17)$$

The last term must vanish in order that the kinetic energy be properly conserved. This is possible provided we let

$$\vec{V}_{i+1} = \frac{1}{2}\left(\vec{V}_{i+2} + \vec{V}_i\right)$$

and (18)

$$\vec{V}_{i-1} = \frac{1}{2}\left(\vec{V}_{i-2} + \vec{V}_i\right),$$

when $\dot{\sigma}_{i+1}$ and $\dot{\sigma}_{i-1}$ do not vanish.

5. Total Energy

Adding (16) to (8) yields

$$\frac{\partial}{\partial t}[(K + c_pT)p_\sigma] = - \nabla\cdot[(K + \phi + c_pT)p_\sigma]\vec{V}$$

$$+ c_p\theta p_\sigma\left(\frac{\partial\pi}{\partial t} + \dot{\sigma}\frac{\partial\pi}{\partial\sigma}\right) - \phi\left(\frac{\partial p_\sigma}{\partial t} + p_\sigma\frac{\partial\dot{\sigma}}{\partial\sigma}\right) - p_\sigma\frac{\partial}{\partial\sigma}[(K + c_pT)\dot{\sigma}]. \quad (19)$$

But with the hydrostatic equation,

$$\frac{\partial\phi}{\partial\pi} = - c_p\theta, \quad (20)$$

it can be shown that

$$c_p\theta p_\sigma\frac{\partial\pi}{\partial t} - \phi\frac{\partial p_\sigma}{\partial t} = - \frac{\partial}{\partial\sigma}\left(\phi\frac{\partial p}{\partial t}\right) \quad (21)$$

and

$$c_p\theta p_\sigma\dot{\sigma}\frac{\partial\pi}{\partial\sigma} - \phi p_\sigma\frac{\partial\dot{\sigma}}{\partial\sigma} = - p_\sigma\frac{\partial}{\partial\sigma}(\dot{\sigma}\phi). \quad (22)$$

Therefore (19) may be written

$$\frac{\partial}{\partial t}[(K + c_pT)p_\sigma] + \frac{\partial}{\partial\sigma}\left(\phi\frac{\partial p}{\partial t}\right) = - \nabla\cdot[(K + \phi + c_pT)p_\sigma]\vec{V}$$

$$- \frac{\partial}{\partial\sigma}[(K + \phi + c_pT)\dot{\sigma}] \quad (23)$$

Integrating over σ yields,

$$\frac{\partial}{\partial t}\int_\sigma(K + c_pT + \phi_*)p_\sigma d\sigma = - \nabla\cdot\int_\sigma(K + \phi + c_pT)p_\sigma\vec{V}\,d\sigma\ . \quad (24)$$

Here the asterisk subscript refers to the ground value. Thus the total energy is conserved for a closed, adiabatic system.

For layer i of the layered atmosphere, add (13) and (17)

$$\frac{\partial}{\partial t}[(K_i + c_pT_i)\Delta p] = -\nabla\cdot[(K_i + \phi_i + c_pT_i)\Delta p]\vec{V}_i$$
$$+ c_p\theta_i\Delta p\frac{\partial\pi_i}{\partial t} - c_p\pi_i p_\sigma(\theta_{i+1}\dot{\sigma}_{i+1} - \theta_{i-1}\dot{\sigma}_{i-1})$$
$$- \phi_i[\frac{\partial}{\partial t}(\Delta p) + p_\sigma(\dot{\sigma}_{i+1} - \dot{\sigma}_{i-1})] \quad (25)$$
$$- p_\sigma[K_i(\dot{\sigma}_{i+1} - \dot{\sigma}_{i-1}) - u_i(u_{i+1}\dot{\sigma}_{i+1} - u_{i-1}\dot{\sigma}_{i-1}) - v_i(v_{i+1}\dot{\sigma}_{i+1} - v_{i-1}\dot{\sigma}_{i-1})].$$

Let us now assume that θ and $\vec{V}$ are constants in each layer[1]. Then

$$\int_{\sigma_{i-1}}^{\sigma_{i+1}} Tp_\sigma d\sigma \equiv T_i\Delta p$$
$$= \int_{\sigma_{i-1}}^{\sigma_{i+1}} \theta\pi p_\sigma d\sigma = \theta_i \int_{\sigma_{i-1}}^{\sigma_{i+1}} \pi p_\sigma d\sigma$$
$$= \theta_i(\pi_{i+1}p_{i+1} - \pi_{i-1}p_{i-1})/(1 + \kappa),$$

or

$$\pi_i \equiv \frac{\pi_{i+1}p_{i+1} - \pi_{i-1}p_{i-1}}{(1 + \kappa)\ \Delta p}. \quad (26)$$

Therefore

$$\frac{\partial\pi_i}{\partial t} = \frac{\pi_{i+1} - \pi_i}{\Delta p}\frac{\partial p_{i+1}}{\partial t} + \frac{\pi_i - \pi_{i-1}}{\Delta p}\frac{\partial p_{i-1}}{\partial t}. \quad (27)$$

Now when (26) and (27) are substituted into (25), we obtain

$$\frac{\partial}{\partial t}[(K_i + c_pT_i)\Delta p] = -\nabla\cdot[(K_i + \phi_i + c_pT_i)\Delta p]\vec{V}_i$$
$$- \frac{\partial p_{i+1}}{\partial t}[\phi_i - c_p\theta_i(\pi_{i+1} - \pi_i)] + \frac{\partial p_{i-1}}{\partial t}[\phi_i + c_p\theta_i(\pi_i - \pi_{i-1})]$$
$$- \dot{\sigma}_{i+1}p_\sigma[\phi_i + c_p\pi_i\theta_{i+1}] + \dot{\sigma}_{i-1}p_\sigma[\phi_i + c_p\pi_i\theta_{i-1}] \quad (28)$$
$$- p_\sigma[K_i(\dot{\sigma}_{i+1} - \dot{\sigma}_{i-1}) - u_i(u_{i+1}\dot{\sigma}_{i+1} - u_{i-1}\dot{\sigma}_{i-1}) - v_i(v_{i+1}\dot{\sigma}_{i+1} - v_{i-1}\dot{\sigma}_{i-1})].$$

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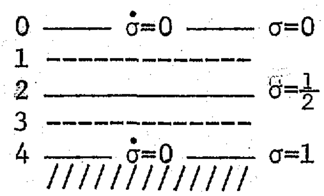
[1]Other assumptions can be introduced here which will lead to other forms of the hydrostatic equations.

In summing over all layers of the atmosphere and adding $\phi_* \frac{\partial p_*}{\partial t}$ to both sides of the result, we obtain the conditions for total energy conservation by allowing the contributions from the last four terms of (28) to vanish (together with $\phi_* \frac{\partial p_*}{\partial t}$). In the following, we will present these conditions for several example atmospheres.

(a) A 2-layer model--

Consider



$p=0$

Let $\sigma = p/p_4$

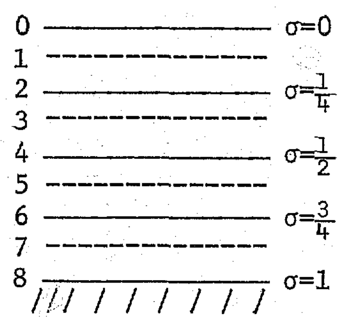
From (28) we obtain the conditions:

$$\phi_1 - \phi_3 = c_p\theta_2(\pi_3 - \pi_1)$$

$$(\phi_1 - \phi_4) + (\phi_3 - \phi_4) = 2c_p\theta_3(\pi_4 - \pi_3) + c_p\theta_3(\pi_3 - \pi_2) + c_p\theta_1(\pi_2 - \pi_1)$$

Here $\theta_2 = \frac{1}{2}(\theta_1 + \theta_3)$.

(b) A 4-layer model--



$\sigma=p/p_*$

One obtains:

$$\phi_1 - \phi_3 = c_p\theta_2(\pi_3 - \pi_1)$$

$$\phi_3 - \phi_5 = c_p\theta_4(\pi_5 - \pi_3)$$

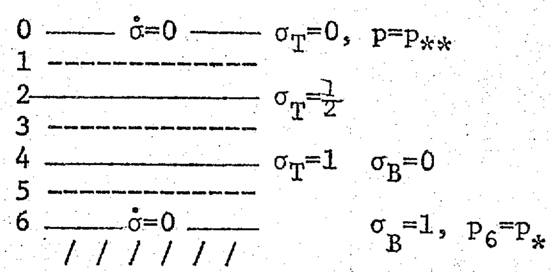
$$\phi_5 - \phi_7 = c_p\theta_6(\pi_7 - \pi_5)$$

$$(\phi_1 - \phi_8) + (\phi_3 - \phi_8) + (\phi_5 - \phi_8) + (\phi_7 - \phi_8) = \Phi(1,8) + \Phi(3,8) + \Phi(5,8) + \Phi(7,8)$$

Here we have introduced the notation:

$$\left.\begin{aligned}
\Phi(1,8) &= c_p\theta_7(\pi_8-\pi_6) + c_p\theta_5(\pi_6-\pi_4) + c_p\theta_3(\pi_4-\pi_2) + c_p\theta_1(\pi_2-\pi_1) \\
\Phi(3,8) &= c_p\theta_7(\pi_8-\pi_6) + c_p\theta_5(\pi_6-\pi_4) + c_p\theta_3(\pi_4-\pi_3) \\
\Phi(5,8) &= c_p\theta_7(\pi_8-\pi_6) + c_p\theta_5(\pi_6-\pi_5) \\
\Phi(7,8) &= c_p\theta_7(\pi_8-\pi_7).
\end{aligned}\right\} \quad (29)$$

(c) Vanderman's 3-Layer Model



Here $\sigma_T = (p - p_{**})/(p_4 - p_{**})$,

$\sigma_B = (p - p_4)/(p_* - p_4)$, $p_* - p_4 = 10$ cb,

and

$$\dot{\sigma}_{4T}\, p_{\sigma T} = \dot{\sigma}_{4B}\, p_{\sigma B} .$$

One obtains:

$$\phi_1 - \phi_3 = c_p\theta_2(\pi_3 - \pi_1)$$

$$\phi_3 - \phi_5 = c_p\theta_4(\pi_5 - \pi_3)$$

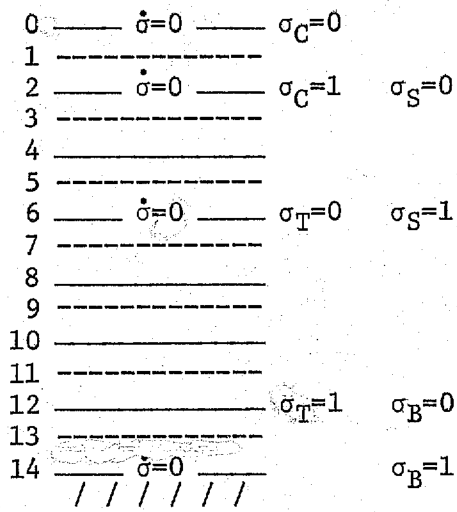
$$(\phi_1 - \phi_6) + (\phi_3 - \phi_6) = \Phi(1,6) + \Phi(3,6)$$

Note here that

$$\theta_2 = \tfrac{1}{2}(\theta_1 + \theta_3)$$

$$\theta_4 = \tfrac{1}{2}(\theta_3 + \theta_5).$$

(d) Shuman-Hovermale 6-Layer Model--



Here

$$\dot{\sigma}_{12T}\, p_{\sigma T} = \dot{\sigma}_{12B}\, p_{\sigma B}$$

and $$p_{14} - p_{12} = 5 \text{ cb.}$$

One obtains the following conditions:

$$\phi_3 - \phi_5 = c_p\theta_4(\pi_5 - \pi_3) = D$$

$$\phi_7 - \phi_9 = c_p\theta_8(\pi_9 - \pi_7)$$

$$\phi_9 - \phi_{11} = c_p\theta_{10}(\pi_{11} - \pi_9)$$

$$\phi_{11} - \phi_{13} = c_p\theta_{12}(\pi_{13} - \pi_{11})$$

$$2\phi_1 - \phi_3 - \phi_5 = \Phi(1,3) + \Phi(1,5) = A$$

$$2(\phi_7 + \phi_9 + \phi_{11}) - 3(\phi_3 + \phi_5) = \Phi(3,7) + \Phi(5,7) + \Phi(3,9) + \Phi(5,9) + \Phi(3,11) + \Phi(5,11) = B$$

$$\phi_{11} + \phi_9 + \phi_7 = 3\phi_{14} + \Phi(11,14) + \Phi(9,14) + \Phi(7,14) = C$$

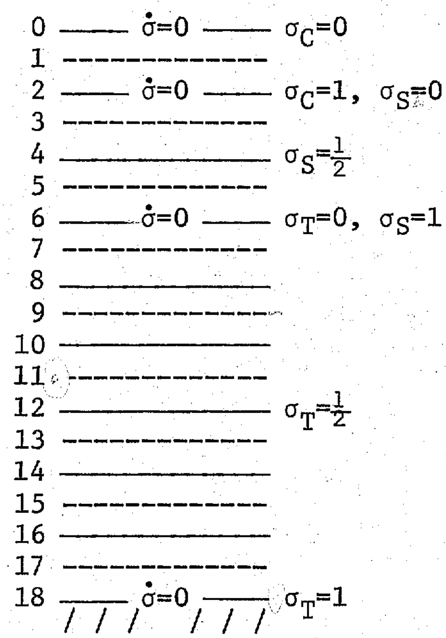
Note that:

$$\phi_1 = \left(\tfrac{3}{2}A - \tfrac{1}{2}B + C\right)/3$$

and

$$\phi_3 = \left(-A + D + 2\phi_1\right)/2 \;.$$

(c) <u>Stackpole's 8-layer Model</u>--



Summing (29) over these layers and setting the contributions from last four terms to zero yields:

$$\phi_3 - \phi_5 = c_p\theta_4(\pi_5 - \pi_3) = D$$

$$\phi_7 - \phi_9 = c_p\theta_8(\pi_9 - \pi_7)$$

$$\phi_9 - \phi_{11} = c_p\theta_{10}(\pi_{11} - \pi_9)$$

$$\phi_{11} - \phi_{13} = c_p\theta_{12}(\pi_{13} - \pi_{11})$$

$$\phi_{13} - \phi_{15} = c_p\theta_{14}(\pi_{15} - \pi_{13})$$

$$\phi_{15} - \phi_{17} = c_p\theta_{16}(\pi_{17} - \pi_{15})$$

$$2\phi_1 - \phi_3 - \phi_5 = \Phi(1,3) + \Phi(1,5) = A$$

$$-2(\phi_7 + \phi_9 + \phi_{11} + \phi_{13} + \phi_{15} + \phi_{17}) + 6(\phi_3 + \phi_5) = \Phi(3,7) + \Phi(5,7) + \Phi(3,9) + \Phi(5,9) + \Phi(3,11) + \Phi(5,11) + \Phi(3,13) + \Phi(5,13) + \Phi(3,15) + \Phi(5,15) + \Phi(3,17) + \Phi(5,17) = B$$

$$\phi_7 + \phi_9 + \phi_{11} + \phi_{13} + \phi_{15} + \phi_{17} = 6\phi_{18} + \Phi(7,18) + \Phi(9,18) + \Phi(11,18) + \Phi(13,18) + \Phi(15,18) + \Phi(17,18) = C$$

Note that

$$\phi_1 = (3A - \tfrac{1}{2}B + C)/6$$

and $$\phi_3 = (-A + D + 2\phi_1)/2.$$

6. Summary

In order to incorporate the conservation principle as presented here into the vertical differencing system of a model, there are three major ingredients necessary:

(a) The vertical differencing in the continuity, momentum, and thermodynamic equations must be included. See equations (6), (9) with (11), and (14) and (15) with (18). Use (6) to change (9), (14) and (15) to advective form.

(b) Equation (26) is to be used to obtain the values of π for the middle of the layers. These, of course, are to be used in evaluating the pressure gradient terms of the momentum equations and in the hydrostatic calculations for obtaining the geopotentials.

(c) By applying (28) to each layer of the model, summing and setting the contributions of the last four terms plus $\phi_* \frac{\partial p_*}{\partial t}$ to zero, one obtains the proper hydrostatic relationships. Note that the resulting geopotentials are valid in the middle of the layers, rather than at the boundaries of the layers as in the Shuman-Hovermale system.

How does one best utilize the objectively analyzed data on the constant pressure surfaces? The answer to this question will not be an easy one to find. However, let us assume that we are going to be working within the present NMC initialization framework. That is, we will use the analyzed geopotentials together with the analyzed rotational wind component and the 12-hr forecast divergent component.

Suggested procedure:

• Interpolate the geopotential, wind and temperature fields from pressure to the mid-points of the σ-layers. The present method of interpolation should be acceptable. However, interpolate temperatures also and interpolate all variables to the pressure values defined by (26).

• Since the vertical truncation error will vary with the particular vertical resolution of the model, it is suggested that each hydrostatic relationship be normalized to remove the area bias. For example, the geopotential height of level 1 in the Shuman-Hovermale model is

$$\phi_1 = \left(\tfrac{3}{2}A - \tfrac{1}{2}B + C\right)/3, \tag{29}$$

where A, B, & C are functions of the potential temperatures of the various layers.

Compute N from

$$N = \frac{1}{S}\int_S [\phi_1 - (\tfrac{3}{2}A - \tfrac{1}{2}B + C)/3]dS,$$

where S is the total area of integration and ϕ_1, A, B, and C are from the analyzed gridpoint values which have been interpolated to the σ-layers. Thus, rather than use (29) in the model integration, we use instead

$$\phi_1 = (\tfrac{3}{2}A - \tfrac{1}{2}B + C)/3 + N. \tag{30}$$

Normalizing each hydrostatic equation accordingly will leave the horizontal geopotential gradients unchanged and yield a gross atmospheric stability equivalent to that of the objectively analyzed temperatures.

• After normalizing the hydrostatic relationships, it is suggested that the initial model temperatures be obtained from these relationships given the analyzed geopotential heights.

• As in Office Note #89, it is recommended that we measure the model forecast skill using the tendency method for all parameters relative to the skill of the present practice of using the model forecast parameters directly.