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Hough Functions to Spherical Harmonics Data Transformation

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ABSTRACT

The NMC global wind height and relative humidity analysis using Hough functions is transformed into an expansion of those fields in spherical harmonics. The computer program uses eigenfrequencies computed by Flattery to generate tidal eigenfunction for twenty-four zonal modes. The resulting spherical harmonic expansion has the same zonal resolution and a meridional truncation of $n = 55$.

1. Introduction

The increasing interest in spectral methods as representation tools in numerical weather prediction resulting from the great increase in computing capacities of new computer systems, has brought the problem of obtaining data for modeling purposes to the forefront. The purpose of this paper is to direct the attention of modelers to a comprehensive source of such data.

At the National Meteorological Center, two analysis schemes are available for initial data. The traditional grid point analysis and a relatively new spectral analysis developed by Flattery using Hough functions. It is possible, of course, to use either scheme to obtain spherical harmonic coefficients but Flattery's method allows a much more compact and elegant transformation, being itself a spectral technique.

The data available from Flattery's procedure consists of wind velocity and heights of isobaric surfaces at 12 atmospheric levels as well as humidity for the six lower levels. The coverage is spherical and the data are available twice daily.

The understanding of the representational aspects using Hough functions is essential to the application described in this paper. For the detailed analysis of the problem, Flattery's original work should be consulted. For our purposes, an outline will be presented.

2. Formulation of the Method

The eigenvalue problem considered by Flattery investigates the free modes of flow on a rotating sphere when the fluid around it is perturbed from a state of no motion. In order to quantitatively describe the possible free modes, the full set of hydrodynamic equations are linearized and a separable solution is introduced:

$$(u_*, v_*)\cos\phi = c_1(u, -iv)e^{i(\ell\lambda+\sigma t)}$$
$$w_* = c_2\, w\, e^{i(\ell\lambda+\sigma t)} \qquad (1)$$
$$z_* = c_3\, z\, e^{i(\ell\lambda+\sigma t)}$$

Here u_*, v_*, w_* are the perturbations of the velocity components in pressure coordinates, $w_* = \frac{dp_*}{dt}$, and z_* is the perturbation of the height of an isobaric surface. u, v, w and z are the corresponding non-dimensional scaled variables where c_1, c_2, c_3 are the nondimensionalizing constants. In the exponential, ℓ and λ are the azimuthal wave number and longitude, while σ and t are the frequency and time. The multiplication of the unscaled horizontal velocity by cosine latitude is necessary in order to render the scaled velocity regular at the poles.

When this form of a solution is introduced into the equations of the horizontal velocity and the tidal potential is neglected, a wind height relationship is obtained:

$$fu + \mu v = - \ell z \tag{2}$$

$$fv + \mu u = - (1-\mu^2)\frac{\partial z}{\partial \mu}$$

here $\mu = \sin\phi$, $f = \frac{\sigma}{2\Omega}$ and Ω is the angular frequency of the Earth's rotation. When the velocity components are introduced into the mass and energy conservation equations, subject to the hydrostatic equation and the gas law and in the absence of external heating, we obtain

$$\left(\frac{\partial}{\partial \mu}\frac{1-\mu^2}{f^2-\mu^2}\frac{\partial}{\partial \mu} - \frac{\ell^2}{(f^2-\mu^2)(1-\mu^2)} - \frac{\ell}{f}\frac{(f^2+\mu^2)}{(f^2-\mu^2)^2}\right)z = \frac{\partial}{\partial p}\left(\frac{1}{s}\frac{\partial z}{\partial p}\right) \tag{3}$$

$$s = - \frac{p_G \alpha_o}{4a^2\Omega^2}\frac{d\ell n\theta_o}{dp}$$

where p_G is the surface pressure, α_o θ_o are the unscaled specific volume and potential temperature, and a is the radius of the Earth.

In equation (3), the left hand side involves only meridional operations while the right hand side is a pressure-dependent operator. If we further assume a separation of the form

$$(z, u, v) = \big(H(\mu), \; U(\mu), \; V(\mu)\big) A(p) \tag{4}$$

and a separation parameter $-\beta$, we may write the equation describing the longitudinal dependence of H as Laplace's tidal equation:

$$\frac{d}{d\mu}\left(\frac{(1-\mu^2)}{(f^2-\mu^2)}\frac{dH}{d\mu}\right) - \left(\frac{\ell}{f}\frac{(f^2+\mu^2)}{(f^2-\mu^2)^2} + \frac{\ell^2}{(1-\mu^2)(f^2-\mu^2)} - \beta\right)H = 0 \tag{5}$$

The separation parameter should in principle be determined from the vertical eigenvalue problem. Since we are interested primarily in the horizontal structure of the perturbations, the value of β may be assigned.

Based on the consideration of a homogeneous atmosphere, a meaningful value of β can be estimated.

Considering equation (5), we observe that for a specified value of β the nondimensional frequency f is still undetermined. We now define the Hough functions as those functions H which are regular at the poles. The corresponding values of f will be referred to as their eigenfrequencies.

On the assumption that these eigenfunctions can be computed, the problem of obtaining the vertical functions H(p) still remains. The procedure followed by Flattery consists of an application of vertical empirical orthogonal functions, computed for every analysis.

The numerical solution of equation (5) consists of solving for those functions H which are regular at the poles, together with their corresponding eigenfrequencies. While a direct integration of (5) is possible (the singularities are apparent only), a much more elegant method--introduced by Hough and applied by Flattery--can be used. Only the highlights of this method will be described.

The essence of Hough's approach consists of an application of the theorem by Helmholtz concerning the specification of a vector by its divergent and rotational parts. If χ and ψ are the potential and stream functions, then the assumed form of the separable solution results in the two equations

$$U = \ell\chi - (1-\mu^2)\frac{d\psi}{d\mu} \tag{6}$$

$$V = (1-\mu^2)\frac{\partial\chi}{\partial\mu} - \ell\psi$$

Returning to the horizontal momentum equations, they become

$$\begin{aligned} fu + \mu v + \ell H &= 0 \\ fv + \mu u + (1-\mu^2)\frac{dH}{d\mu} &= 0 \end{aligned} \qquad (7)$$

and the continuity equation becomes:

$$\frac{\ell u}{1-\mu^2} - \frac{dv}{d\mu} + f\beta H = 0 \qquad (8)$$

If the expressions of u and v from equations (6) are introduced into equations (7) and (8), the following equations can be obtained after some manipulation

$$\begin{aligned} - f\nabla^2\psi + \mu\nabla^2\chi + (1-\mu^2)\frac{d\chi}{d\mu} - \ell\psi &= 0 \\ - f\nabla^2\chi + \mu\nabla^2\psi + (1-\mu^2)\frac{d\psi}{d\mu} - \ell\chi &= \nabla^2 H \\ \nabla^2\chi &= f\beta H \end{aligned} \qquad (9)$$

where:

$$\nabla^2 \equiv \frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} - \frac{\ell^2}{1-\mu^2} \qquad (9)$$

The solutions of equation (9) proceeds by assuming the following expansions

$$\begin{aligned} H &= \Sigma\, c_n^{\ell\ell}\, p_n^{\ell}(\mu) \\ \chi &= \Sigma\, E_n^{\ell\ell}\, p_n^{\ell}(\mu) \\ \psi &= -\beta\Sigma\frac{D_n^{\ell}}{n(n+1)}\, p_n^{\ell}(\mu) \end{aligned} \qquad (10)$$

where $p_n^{\ell}(\mu)$ are the associated Legendre polynomials of order ℓ and degree n.

When equations (10)are introduced into (9) and the orthogonality of the Legendre polynomials is invoked, the following relations among the coefficients are obtained:

$$c_n^{\ell} = -\frac{n(n+1)}{f\beta} E_n^{\ell}$$

$$(n+1)^2 R_n^{\ell} D_{n-1}^{\ell} - M_n^{\ell} c_n^{\ell} + n^2 R_{n+1}^{\ell} D_{n+1}^{\ell} = 0 \qquad (11)$$

$$(n+1)^2 R_n^{\ell} C_{n-1}^{\ell} - N_n^{\ell} D_n^{\ell} + n^2 R_{n+1}^{\ell} c_{n+1}^{\ell} = 0$$

where:

$$N_n^{\ell} = n(n+1) - \frac{\ell}{f}, \; M_n^{\ell} = f^2 N_n^{\ell} - \frac{n^2(n+1)^2}{\beta} .$$

At this point we distinguish between two types of solutions, those that remain oscillatory when the Earth's rotation is removed ($\Omega \to 0$), referred to as gravitational modes, while those that turn into steady motion are termed rotational modes. In the analysis of atmospheric data, only the rotational modes are included.

Equations (11) can be manipulated in such a way that the coefficients c_n^{ℓ} and D_n^{ℓ} would be eliminated if we assume that the series (10) converge and the two arbitrary constants required by the second order equations are specified. The resulting expressions involve continued fractions and are very suitable for machine calculations. The numerical solution given by Flattery covers 576 functions: for each ℓ(=1...24), 24 eigenfrequencies were computed together with their eigen heights and winds; for the case ℓ=0, a special treatment is necessary; for this case and $\beta>0$, no Hough functions exist, i.e., no solutions to Laplace's equation exist. However, a set of functions can be found that satisfy the governing equations (7) and (8). If we set $\ell=f=0$ in the above equations, we obtain

$$U^o = -\frac{(1-\mu^2)}{\mu} \frac{dH^o}{d\mu}, \; V = 0. \qquad (12)$$

if we further specify

$$U_m^o = (1-\mu^2) \; p_m^o \qquad (13)$$

we can find $H_m^{\,o}$ such that:

$$H_m^{\,o} = \sum_{j=0}^{\infty} c_{m,j}^{\,o}\ p_j^{\,o} \qquad (14)$$

It should be noted that in equation (13), care has been taken to satisfy the boundary conditions of U.

In order to accomplish the transformation from Hough functions and wind functions to the spherical harmonic domain, it is necessary to rewrite equation (13) in the form of a series. This can be conveniently accomplished by recursion relations between Legendre polynomials.

In summary, for each ℓ (ℓ=0...24) 24 modes for each of the functions $H_m^{\,\ell}$, $U_m^{\,\ell}$, $V_m^{\,\ell}$ are computed, totalling 1800 functions corresponding to one eigenvalue β. The functions $U_m^{\,\ell}$, $V_m^{\,\ell}$ are obtained from a substitution of equations (10) into equations (6).

3. Application

The results of the Hough analysis are given through the expansion coefficients and the empirical orthogonal functions. We have

$$z_j = \sum_{\ell=0}^{24} \sum_{m=1}^{24} \sum_{k=1}^{7} \left(a_{m,k}^{\,\ell} \cos\ell\lambda + b_{m,k}^{\,\ell} \sin\ell\lambda\right) H_m^{\,\ell}\ A_j^{\,k} + S_j$$

$$u_j = \sum_{\ell} \sum_{m} \sum_{k} \left(c_{m,k}^{\,\ell} \cos\ell\lambda + d_{m,k}^{\,\ell} \sin\ell\lambda\right) U_m^{\,\ell}\ A_j^{\,k} \qquad (15)$$

$$v_j = \sum_{\ell} \sum_{m} \sum_{k} \left(c_{m,k}^{\,\ell} \cos\ell\lambda - d_{m,k}^{\,\ell} \sin\ell\lambda\right) V_m^{\,\ell}\ A_j^{\,k}$$

where j=1...12 is the pressure level index, s_j is the standard height of jth level, $A_j^{\,k}$ are the empirical orthogonal functions and $H_m^{\,\ell}\ U_m^{\,\ell}\ V_m^{\,\ell}$ are the height and wind eignefunctions. It should be mentioned that when $a_{m,k}^{\ell} = c_{m,k}^{\ell}$ and $b_{m,k}^{\ell} = d_{m,k}^{\ell}$, the wind height relation implied by equations (2) is satisfied. When the wind and height are computed by different coefficients, different relative weights are assigned to them during the analysis.

In order to express equations (15) in spherical harmonics, it is possible to leave the vertical dependence in. As most modelers would probably require isobaric data, the vertical index k will be first summed out.

Let

$$a_m^{\ell,j} = \sum_{k=1}^{7} a_{m,k}^{\ell} A_j^k, \quad b_m^{\ell,j} = \sum_{k=1}^{7} b_{m,k}^{\ell} A_j^k$$
$$c_m^{\ell,j} = \sum_{k=1}^{7} c_{m,k}^{\ell} A_j^k, \quad d_m^{\ell,j} = \sum_{k=1}^{7} d_{m,k}^{\ell} A_j^k \tag{16}$$

then equations (15) may be written

$$z_j = \sum_{\ell=0}^{24} \sum_{m=1}^{24} \left(a_m^{\ell,j}\cos\ell\lambda + b_m^{\ell,j}\sin\ell\lambda\right)H_m^{\ell} + S_j$$
$$u_j = \Sigma\ \Sigma \left(c_m^{\ell,j}\cos\ell\lambda + d_m^{\ell,j}\sin\ell\lambda\right)U_m^{\ell} \tag{17}$$
$$v_j = \Sigma\ \Sigma \left(c_m^{\ell,j}\cos\ell\lambda - d_m^{\ell,j}\sin\ell\lambda\right)V_m^{\ell}\ .$$

The major part of this transformation resides in the generation of the series

$$H_m^{\ell} = \sum_{n=\ell} c_n^{\ell} P_n^{\ell}$$
$$U_m^{\ell} = \sum_{n=\ell} U_n^{\ell} P_n^{\ell} \tag{18}$$
$$V_m^{\ell} = \sum_{n=\ell} V_n^{\ell} P_n^{\ell}$$

In generating these series, care must be taken to capture the even odd relations between H_m^ℓ and the wind functions. The missing upper limit of summation merely indicates the variable length of these series. The longest series ranges up to n=55 and gives a measure of the truncation involved in the Hough functions analysis.

In practice, the generation of the coefficients in equations (18) is performed for the even and odd Hough function separately. This enables a computation of spherical harmonic coefficients of a given parity only. If equations (18) are substituted into equations (17) and the even odd character is identified, we complete the transformation:

$$z_j = \sum_{\ell=0}^{24} \left[\sum_{m=1}^{12} \left(a_m^{\ell,j}\cos\ell\lambda + b_m^{\ell,j}\sin\ell\lambda\right) \sum_{n=\ell}^{55} c_{E,n}^{\ell,m} P_n^\ell \right.$$

$$\left. + \sum_{m=13}^{24} \left(a_m^{\ell,j}\cos\ell\lambda + b_m^{\ell,j}\sin\ell\lambda\right) \sum_{n=\ell}^{55} c_{d,n}^{\ell,m} P_n^\ell \right]$$

$$= \sum_{\ell=0}^{24} \sum_{n=\ell}^{55} \left(\alpha_n^{\ell,j} \cos\ell\lambda + \beta_n^{\ell,j}\sin\ell\lambda\right) P_n^\ell$$

where

$$\alpha_n^{\ell,j} = \sum_{m=1}^{12} a_m^{\ell,j} c_{E,n}^{\ell,m} + \sum_{m=13}^{24} a_m^{\ell,j} c_{d,n}^{\ell,m}$$

$$\beta_n^{\ell,j} = \sum b_m^{\ell,j} c_{E,n}^{\ell,m} + \sum_{m=13}^{24} b_m^{\ell,j} c_{d,n}^{\ell,m}$$

Here the subscript E represents even functions while d represents odd ones. Similar manipulation on u_j, v_j and the relative humidity complete the transformation.