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# Conditional Stability Conditions for a Semi-Implicit PE Barotropic Model

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## 1. Introduction

Difficulties encountered with the semi-implicit version of the Shuman-Hovermale model (Gerrity 1973) when orography was incorporated suggested the possibility that linear computational instability might be associated with the large variations of the specific volume on a $\sigma$-coordinate surface in the presence of mountains. Some numerical experiments with a free-surface PE barotropic model with orography were undertaken in order to shed light on the subject. In this note, a stability analysis of the barotropic model is presented which supports the general stability of the semi-implicit method but does indicate the existence of a conditional stability requirement; viz., the "deviation" phase velocity must not be large compared to the "mean" phase velocity.

## 2. Analysis

Consider a one-dimensional free surface, barotropic model atmosphere with a corrugated lower boundary of the fluid. The model equations may be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} \tag{1a}$$

$$\frac{\partial \phi}{\partial t} = -(\phi - \phi_o)\frac{\partial u}{\partial x} \tag{1b}$$

in which $\phi$ is the geopotential of the free surface and $\phi_o$, a function of x alone, is the geopotential of the lower boundary; u is the fluid velocity. The Coriolis acceleration and advection have been omitted.

We assume that solutions to the equations are such that

$$\phi - \phi_o > 0 \tag{2}$$

Numerical solution of the equations (1) may be obtained by the use of finite difference approximations of the differential equations. We shall consider the conditions under which stable numerical solutions of the semi-implicit approximation of equations (1) may be obtained.

The finite difference equations are

$$\overline{u}_t^{\,t} = -\overline{\overline{\phi}^x_x}^{2t} \tag{3a}$$

$$\overline{\phi}_t^{\,t} = -\Phi\, \overline{\overline{u}^x_x}^{2t} - (\phi - \phi_o - \Phi)\,\overline{u}^x_x \tag{3b}$$

in which $\Phi$ is an arbitrary constant. The bar-2t over the space derivatives indicates that these terms are implicitly approximated. The last term in eq. (3b) is approximated explicitly.

In order to analyze the computational stability of the difference equations (3), the problem will be reduced to a linear one with constant coefficients. The explicitly approximated term in (3b) is nonlinear and it is on this term that attention must be focused.

The first case to be examined is that in which $\phi_o$ is zero.

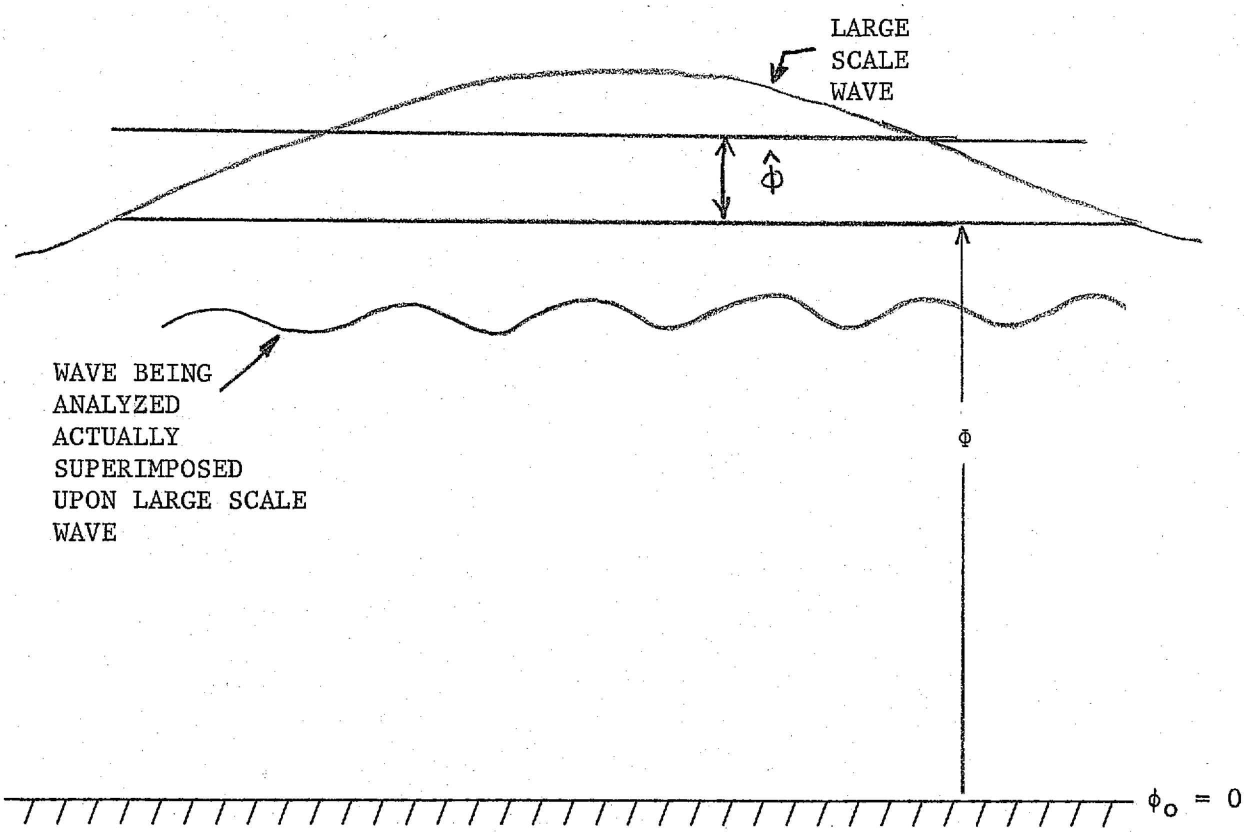


Figure 1. Illustration of situation visualized for stability analysis.

Following Richtmyer's (1962 cf. p. 6) interpretation of the stability analysis problem, we shall consider the stability of a short wave within a large scale wave. Thus, the nonlinear term will not be eliminated completely; instead we write the system of equations

$$\bar{u}_t^t = -\overline{\overline{\phi}_x^x}^{2t} \tag{4a}$$

$$\bar{\phi}_t^t = -\Phi\, \overline{\overline{u}_x^x}^{2t} - \hat{\phi}\, \overline{u}_x^x \tag{4b}$$

with the provision that $\hat{\phi}$ is constant.

Figure 1 illustrates the situation envisaged. Let us next assume that the solutions to (4) may be obtained in the form,

$$\begin{matrix} u \\ \phi \end{matrix} = \begin{matrix} U \\ P \end{matrix} \; \zeta^n e^{ikj\Delta x} \tag{5}$$

where U and P are constants; $\zeta$ is the amplification factor with n the time index; j is the space grid index, $\Delta x$ the space increment and k the wave number.

Introduction of (5) into (4) yields

$$U(\zeta^2-1) + i\hat{k}\Delta t\, P(\zeta^2+1) = 0 \tag{6}$$

$$U\, i\hat{k}\Delta t\left[\Phi(\zeta^2+1) + 2\hat{\phi}\zeta\right] + P(\zeta^2-1) = 0 \tag{7}$$

The wave number $\hat{k}$ absorbs the spatial truncation error.

Solutions of the form (6) will therefore exist only if $\zeta$ satisfies the equation

$$\zeta^4 + 2Q\zeta^3 - 2R\zeta^2 + 2Q\zeta + 1 = 0 \tag{8}$$

in which

$$Q \equiv \left[\frac{(\hat{k}\Delta t)^2\hat{\phi}}{1+(\hat{k}\Delta t)^2\Phi}\right] \tag{9}$$

$$R \equiv \left[\frac{1-(\hat{k}\Delta t)^2\Phi}{1+(\hat{k}\Delta t)^2\Phi}\right] \tag{10}$$

The quartic eq. (8) may be factored into the product of the two quadratics,

$$\left\{ \zeta^2 + \left[Q+\sqrt{Q^2+2(1+R)}\right]\zeta + 1\right\}\left\{ \zeta^2 + \left[Q-\sqrt{Q^2+2(1+R)}\right]\zeta + 1\right\} \tag{11}$$

Inspection of the definition (9) shows that

(a) if the time step is chosen so that $(\hat{k}\Delta t)^2 \Phi \geq 1$, to take full advantage of the "absolute stability" of the implicit method, then

$$-1 \leq R \leq 0$$

(b) if the time step is chosen so that $(\hat{k}\Delta t)^2 \Phi \leq 1$, not fully exploiting the "absolute stability" of the implicit method, then

$$0 \leq R \leq 1$$

In either case, the radicands in eq. (11) are positive and the quadratics have real valued coefficients. The roots $\zeta$ are therefore given by

$$\begin{aligned}
\zeta_1 &= -\frac{Q+\left[Q^2+2(1+R)\right]^{1/2}}{2} + \left\{\left[\frac{Q+\left[Q^2+2(1+R)\right]^{1/2}}{2}\right]^2 - 1\right\}^{1/2} \\
\zeta_2 &= -\frac{Q+\left[Q^2+2(1+R)\right]^{1/2}}{2} - \left\{\left[\frac{Q+\left[Q^2+2(1+R)\right]^{1/2}}{2}\right]^2 - 1\right\}^{1/2} \\
\zeta_3 &= -\frac{Q-\left[Q^2+2(1+R)\right]^{1/2}}{2} + \left\{\left[\frac{Q-\left[Q^2+2(1+R)\right]^{1/2}}{2}\right]^2 - 1\right\}^{1/2} \\
\zeta_4 &= -\frac{Q-\left[Q^2+2(1+R)\right]^{1/2}}{2} - \left\{\left[\frac{Q-\left[Q^2+2(1+R)\right]^{1/2}}{2}\right]^2 - 1\right\}^{1/2}
\end{aligned} \tag{12}$$

Replacement of R and Q by $m^2$ and $\ell^2$ gives

$$\frac{1}{1+m^2} < 1 - \frac{m^2}{\ell^2(1+m^2)} \tag{22}$$

which is satisfied for all m provided that $\ell^2 \geq 1$ , i.e. $\underline{\Phi} \geq \hat{\phi}$

Condition (20) is also satisfied if $\ell^2 > 1$, thus the system (3) (with $\phi_0 = 0$) is absolutely stable provided that

$$\underline{\Phi} > \hat{\phi} \tag{23}$$

Conversely, if

$$\frac{Q + \left[ Q^2 + 2(1+R) \right]^{1/2}}{2} > 1 \tag{24}$$

then the magnitude of $\zeta_2$ will exceed unity. The inequality (24) is certainly satisfied if $Q \geq 2$ or

$$\frac{m^2}{1+m^2} > 2\ell^2 \tag{25}$$

Thus, if $\ell^2 < 1/2$ there exists a condition on m, in order for the system (3) to be computationally stable; viz.

$$m^2 < \frac{2\ell^2}{1-2\ell^2} \tag{26}$$

Using (16) and (17) in (26) with the notation

$$\bar{c} = \sqrt{\underline{\Phi}}$$
$$\hat{c} = \sqrt{\hat{\phi}} \tag{27}$$

the condition on $\Delta t$ becomes

$$\hat{k}\Delta t < \frac{\sqrt{2}}{(\hat{c}^2 - 2\bar{c}^2)^{1/2}} \tag{28}$$

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Notice that as $\bar{c} \to 0$ the condition (27) is weaker than the usual explicit condition. This fact is due to the implicit approximation of eq. (3a) and is related to the mixed implicit-explicit scheme reported earlier by Shuman (1971) and Gerrity and McPherson (1971), and subsequently referred to as the "time-averaged pressure gradient scheme."

If the explicit term in eq. (4b) is neglected, Q = 0, and one may demonstrate that

$$|S_k| = 1 \qquad \text{for } k = 1, 2, 3, 4$$

provided that

$$R \leq 1 \tag{13}$$

We noted above that (13) will be satisfied for all choices of time step. Consequently, one may conclude that the implicit system is absolutely stable.

Suppose now that

$$\Phi = \ell^2 \hat{\phi} \tag{14}$$

and that the time step is taken to be

$$\Delta t = \frac{m}{\hat{k}\sqrt{\Phi}} \tag{15}$$

then

$$(\hat{k}\Delta t)^2 \Phi = m^2 \tag{16}$$

and

$$(\hat{k}\Delta t)^2 \hat{\phi} = \frac{m^2}{\ell^2} \tag{17}$$

The choice $m = \sqrt{\ell}$ will yield

$$(\hat{k}\Delta t)^2 \hat{\phi} = 1 \tag{18}$$

the ordinary limit for computational stability of an explicit difference approximation. If m is taken to be larger than $\sqrt{\ell}$, the explicit term might cause the solution of eqs. (3) to be computationally unstable.

However, it is readily demonstrated that the roots (12) will have magnitude unity if

$$\frac{Q + \left[Q^2 + 2(1+R)\right]^{1/2}}{2} < 1 \tag{19}$$

Provided that

$$2 - Q > 0 \tag{20}$$

the inequality (19) may be expressed as

$$(1+R) < 2(1-Q) \tag{21}$$

We have seen therefore that the scheme (3) can possess linearly unstable solutions, but only if the coefficient of the explicit term in eq. (3b) is more than twice as large as the coefficient of the implicit term in the same equation.

Consideration of the case when $\phi_o$ is nonzero does not add appreciably to the foregoing analysis unless one considers the unlikely situation of flow over a very deep canyon.

In that situation, $\phi_o$ can be large negative in the vicinity of the depression, thereby giving rise to an instance in which the unstable solution just analyzed may arise. We therefore concluded that to excite a computational instability in the semi-implicit, free-surface barotropic model, it is necessary to have

(a) $\Phi$ small compared with $(\phi - \Phi)$, and

(b) $\hat{k}\Delta t > \dfrac{\sqrt{2}}{[\phi - 3\Phi]^{\frac{1}{2}}}$

## 3. Conclusion

Although it has been possible to demonstrate cases in which computational instability may occur in a semi-implicit model, the event is associated only with very large deviations. The difficulty experienced with the semi-implicit baroclinic model may be partially due to this aspect of the numerical technique, but it would be rash to draw that conclusion without more relevant evidence than that deduced here.

## 4. References


Gerrity and McPherson (1971), NMC Office Note 55.

Gerrity, McPherson, and Scolnik (1973), NOAA Tech. Memo. NWS NMC-53.

Richtmyer (1962), NCAR Tech. Notes 63-2.

Shuman (1971), NMC Office Note 54.