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A Comparison of Explicit and Implicit Integration Schemes in Four-Dimensional Data Assimilation

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I. INTRODUCTION

One of the more attractive proposals for the four-dimensional assimilation of meteorological data yet to appear in the voluminous literature on this subject is due to Morel and Talagrand (1973). They suggest repeated insertion of observations into a dynamic model equipped with special artificial viscosity terms as the model is integrated forward and backward over the interval encompassing the data. Their experiments, although simulations of the "identical twin" variety, suggest that this approach will be successful. Unfortunately, given the very large models now existing and considering those planned for later in this decade, such an approach appears to be feasible only for large research-oriented institutions with ample megahours of computer time available. For operational use, a much more economical scheme must be designed.

A potential vehicle for such a scheme is the "implicit-backward" integreation method (Kurihara, 1965) that permits a much longer time step to be used, as well as possessing an excellent capability for damping gravitational oscillations (McPherson, 1973). However, it gains its stability with large time steps by effectively reducing the numerical phase velocity of high-frequency gravity waves. Since geostrophic adjustment theory suggests that the rapidity of the adjustment process depends to some extent on the fast-moving gravity waves, adjustment may proceed more slowly in an implicit model. It is therefore possible that the economic advantage of the implicit model might be offset by the disadvantage of slower adjustment. This note describes a series of experiments designed to gain an appreciation of the comparative efficiencies of explicit and implicit integration methods in four-dimensional data assimilation.

II. DESCRIPTION OF THE EXPERIMENTS

Two models were used in the experiments, differing only in the time integration method used. Both were primitive equation barotropic models applied at the 500 mb level. The domain of integration was a quasi-hemispheric rectangle of 27x29 points separated by 762 km at 60N on a polar stereographic projection of the Northern Hemisphere. The finite-difference approximations have been previously described by McPherson (1971, 1973). Both models used temporally-invariant lateral boundary conditions. Identical initial states were used, comprised of 500 mb heights and winds as objectively analyzed on the NMC 53x57 quasi-hemispheric rectangle, and then extracted to the grid used in the experiment. No attempt at initial balancing was made.

The explicit model was integrated in time by the Euler-backward method (Kurihara, 1965) with a time step of 10 minutes. This method has been used by a number of investigators (e.g., Talagrand, 1972) in four-dimensional data assimilation experiments, and is damping with respect to high-frequency oscillations. The implicit model used the implicit-backward method with a time step of 60 minutes. It should be noted that the time advantage of the implicit method over the Euler-backward is approximately 8:1.

The "observational" data to be assimilated were generated by an integration of the implicit model beginning from 00Z April 8, 1973 data. This integration, hereafter denoted the "control run", began from an unbalanced initial state and proceeded to 36 hours. At this point, time was reversed and the model integrated back to the 24th hour. This process was repeated for six cycles, the equivalent of 3 days of integration. The history of the control run is illustrated in Figure 1. The uppermost curve depicts the behavior with time of the total kinetic energy per unit mass. Next is the area-averaged D-value, or departure of the 500 mb height from a standard value. Third is the root-mean-square (RMS) vorticity, and the last curve is the RMS divergence. The curves indicate that the initial oscillations are damped completely by the final 12-hour period. Values of height, and u- and v-wind components at each hour and grid point during this final interval, were taken as the observations to be inserted.

Insertions was performed in the following manner. At each insertion time, the forecast values were replaced by the observations at every grid point in two pie-shaped sectors with apexes at the pole, each of which contains 1/24 of the total grid points. These sectors rotate each hour, so that insertion at hourly intervals results in the forecast value of each grid point being replaced once during the 12-hour period. This roughly simulates the path of an orbiting sounding satellite.

Two basic sets of experiments were performed. In the first, the observations were inserted without errors being added, thus simulating perfect data. In the second set, random errors were added to the height field (with an RMS value of 30 m) and to each wind component (RMS value of 2 m sec^{-1}). In each of these two sets, insertion of height observations only was performed at 1-, 2-, and 4-hour intervals. Also, one experiment in each set was performed in which both heights and winds were inserted at 1-hour intervals. Table I lists the experiments and their characteristics.

All experiments were integrated for the equivalent of 10.5 days. At each hour, RMS height errors, RMS vector wind errors, and RMS vorticity errors were calculated from the difference between the control run and the assimilation run. These values are plotted against time (only to 9.5 days as a matter of drafting convenience) in the Figures 2-6.

Table 1

Experiment	Model	Errors	Integral	Insertion
A.1.a.1	Implicit	No	1	Heights only
A.1.b.1	Explicit	"	1	" "
A.1.a.2	Implicit	"	2	" "
A.1.b.2	Explicit	"	2	" "
A.1.a.3	Implicit	"	4	" "
A.1.b.3	Explicit	"	4	" "
A.1.a.4	Implicit	"	1	Heights & winds
A.1.b.4	Explicit	"	1	" "
A.2.a.1	Implicit	Yes	1	Heights only
A.2.b.1	Explicit	"	1	" "
A.2.a.2	Implicit	"	2	" "
A.2.b.2	Explicit	"	2	" "
A.2.a.3	Implicit	"	4	" "
A.2.b.3	Explicit	"	4	" "
A.2.a.4	Implicit	"	1	Heights & winds
A.2.b.4	Explicit	"	1	" "

III. EXPERIMENTAL RESULTS

The first conclusion one can reach is that insertion of heights only, with or without errors, produces only a very slow reduction in the error--at least for the insertion intervals used. This is in agreement with the results of Rutherford and Asselin (1972) who also conducted experiments with an implicit model. Initially, the RMS height, vector wind, and vorticity errors were 55.4 meters, 7.25 m sec^{-1}, and 17.5×10^{-6} sec^{-1}, respectively. The maximum error reduction (for heights-only insertion) was achieved by the explicit model inserting error-free data at 1-hour intervals (exp. A.1.b.1). In that case, the height-error reduction of 34 meters is only 62 percent of the initial value after 10.5 days. The winds fared even worse: the vector wind error reduction was only 3.4 m sec^{-1} or 47 percent, while the vorticity error was reduced by $6.3 \times 10^{-6} sec^{-1}$, or 37 percent.

Even so, the explicit model did reduce the errors faster than did the implicit model. The corresponding figures are 29 meter height error reduction (53 percent), 2.1 m sec^{-1} vector wind error reduction (29 percent), and 3.1×10^{-6} sec^{-1} vorticity error reduction (18 percent). Figure 2 depicts these errors as functions of time for the two experiments. It will be observed that a significant reduction of the total height error occurs in both cases in the first 12 hours. This is a result of the models striving for adjustment of the unbalanced initial state. As a matter of curiosity, two additional integrations were performed, one with each model, in which no insertion was done. The total reductions of error were the same in both models after 5 days, 11 meters, 0.5 m sec^{-1}, and zero, for the height, vector wind, and vorticity, respectively. Thus, all of the vorticity error reduction and most of the wind error reduction are due to data insertion, but roughly a third of the height error reduction is due to adjustment of the initial imbalance.

Comparing the overall slopes of the error curves in Figure 2, it is apparent that the explicit model reduces the height error only slightly faster than does the implicit model, whereas the difference in slope is larger for the vector wind and vorticity errors. Since the heights are being inserted and the winds allowed to adjust to the heights, it is clear that the adjustment process does proceed more slowly in the implicit model.

Figure 3 shows the result of inserting both heights and winds. In this case, the error reduction is dramatic, approaching zero in both integrations by 6 days. Even so, the implicit model reduces the errors less rapidly than does the explicit model. It might be noted that the explicit curve in Figure 3 compares favorably with a similar experiment performed by Talagrand (1972).

With insertion each hour, the implicit method is receiving corrected data each time step. Morel and Talagrand argue that there must be sufficient time between insertions to allow the shock from the previous insertion to be damped. The explicit model is more in line with this argument since hourly insertion means once every six time steps. Accordingly, experiments were

conducted with 2-hour and 4-hour insertion intervals. The results of the latter are shown in Figure 4. An examination of the curves reveals that the difference in error reduction rates between the two models has decreased, as compared with Figure 2. However, the overall error reduction rates are much smaller with a 4-hour insertion interval. This is again in agreement with Talagrand's findings.

Insertion of "perfect" data is clearly unrealistic. One small step toward greater realism is to add random errors to the observations before insertion. Figures 5 and 6 illustrate the result of this experiment. The former diagram compares the result of implicit integrations inserting height data with and without errors at 1-hour intervals. It will be observed in Figure 5 that the wind and vorticity adjustment proceeds in exactly the same manner with or without errors, but the height error reduction is considerably less when errors are added. Most of the difference appears in the first day, when addition of error-ridden data impedes the initial adjustment. The slopes of the two lines after the first day are nearly identical.

Similar behavior is evident in the height error trace in Figure 6, which displays the explicit experiment. However, the reduction in vector wind error and vorticity is less when errors are added to the height field. This result is more in agreement with previous research (e.g., Talagrand, 1972; Williamson and Kasahara, 1971), whereas the apparent insensitivity of adjustment in the implicit model to errors in the inserted data is surprising.

Figures 2-6 indicate that the explicit model reduces the errors more rapidly than does the implicit model. In view of the 8:1 time advantage of the latter over the former, it is reasonable to inquire as to how much of that advantage is absorbed by the slower adjustment of the implicit model. From Figures 5 and 6, we have calculated the total reduction of error over 10 days in the two experiments A.2.a.1 and A.2.b.1, and formed a ratio of explicit to implicit error reductions. The result is given in Table 2.

Table 2

Experiment	$\Delta Z_e'$(m)	ratio	$\Delta \vec{V}_e$(m sec^{-1})	ratio	$\Delta \zeta_e$(sec^{-1})	ratio
A.2.a.1	2122		2.0		2.9×10^{-6}	
A.2.b.1	2628	1.26	4.8	1.66	4.8×10^{-6}	1.45

The ratios indicate that the implicit method requires more time than the explicit by a factor of 1.26 to reduce the height error to the same level, 1.66 for the wind error, and 1.45 for the vorticity error. Similar ratios worked out for the other experiments vary somewhat, but the suggestion is clear that the implicit method's slower adjustment absorbs between a factor of 1.5

and 2.0 of the 8:1 time advantage. Therefore, if assimilation is performed with the implicit-backward method, one still has roughly a 4:1 time advantage over the Euler-backward method, for equivalent performance.

IV. CONCLUSIONS

This series of experiments has resulted in the following conclusions:

1. Both explicit and implicit schemes produce a very slow reduction in errors if only heights are inserted, but a much more rapid reduction if both winds and heights are inserted.

2. Inserting observations with random errors added increases the level to which height errors are reduced after 10 days in both models, and also decreases the rate of vector wind and vorticity error reduction in the explicit model; the wind and vorticity error reduction in the implicit model is insensitive to errors in the inserted heights.

3. Reducing the insertion frequency reduces the difference in rate of error reduction between the two models, but also reduces the rates of error reduction themselves.

4. The explicit model reduces errors more rapidly than does the implicit model, especially with respect to vector wind and vorticity errors.

5. The economic advantage of the implicit method over this explicit method more than offsets its slower rate of error reduction.

These results are sufficiently encouraging to warrant continuation of experiments with the implicit method. The next step will be to insert real data instead of model-generated data. Eventually, it will be necessary to extend these investigations to a baroclinic model.

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LIST OF FIGURES

1. The characteristics of the control run. From the top, the curves depict the total kinetic energy per unit mass (KE), the area-averaged departure of the 500 mb height from a standard value ($\overline{Z}_D$), the root-mean-square (RMS) vorticity (ζ_{RMS}), and the RMS divergence ($\Delta \cdot \vec{V}_{RMS}$), plotted as functions of time. The last 12-hour period represents the period selected as the "observational" data.

2. RMS height error (Z_e) vector wind error ($\vec{V}_e$), and vorticity error (ζ_e) as functions of time. The solid curve represents the explicit integration, the dashed curve the implicit integration; 1-hour insertion of error-free heights only.

3. Same as in Figure 2, except 1-hour insertion of error-free heights and winds.

4. Same as Figure 2, except 4-hour insertions of error-free heights only.

5. Same as Figure 2, except the solid curve represents the implicit integration with random errors (RMS 30 m) added to the inserted height observations.

6. Same as Figure 2, except the solid curve represents the explicit integration with random errors added to the inserted heights, and the dashed curve represents the explicit integration without errors added.

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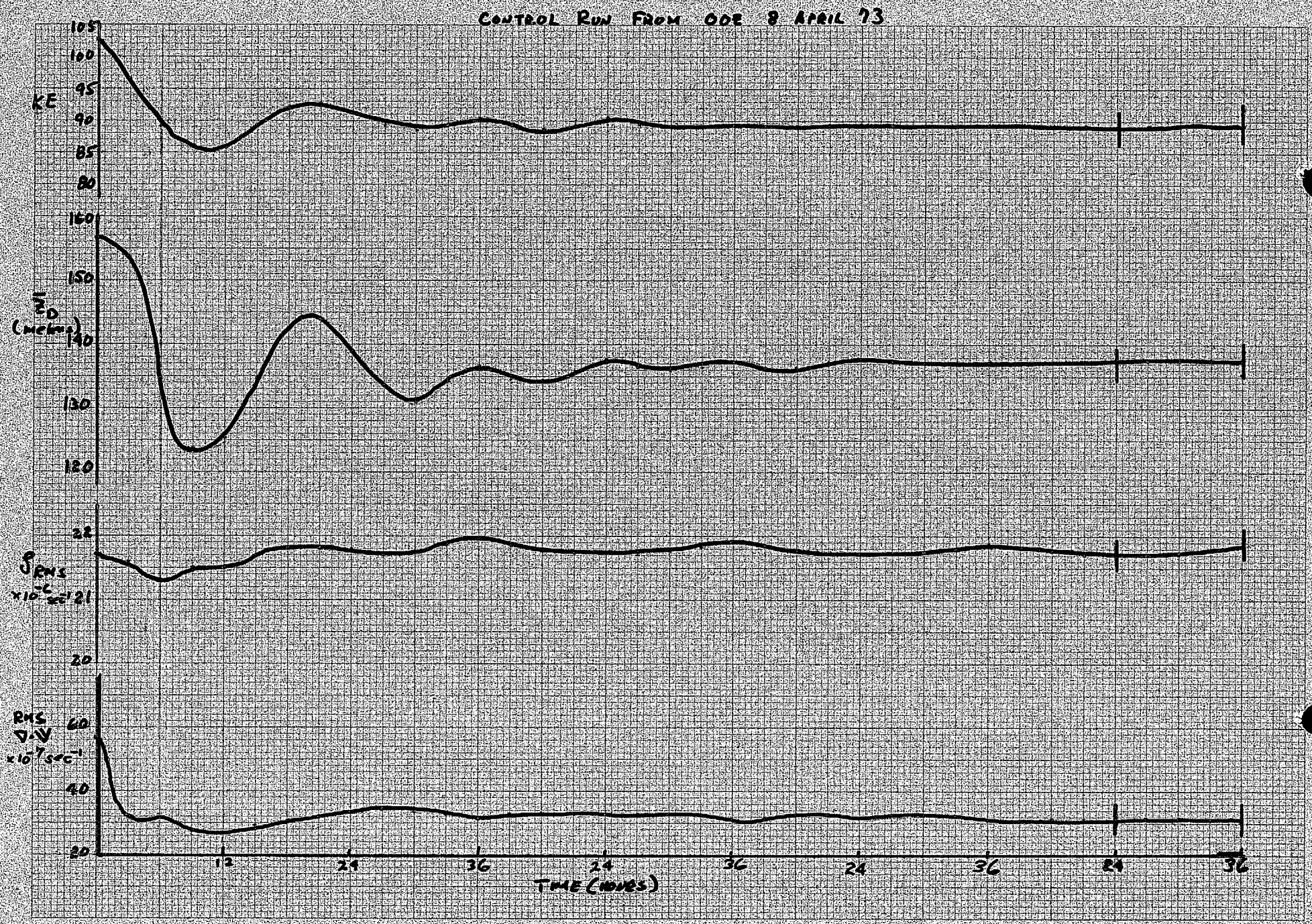

Figure 1.

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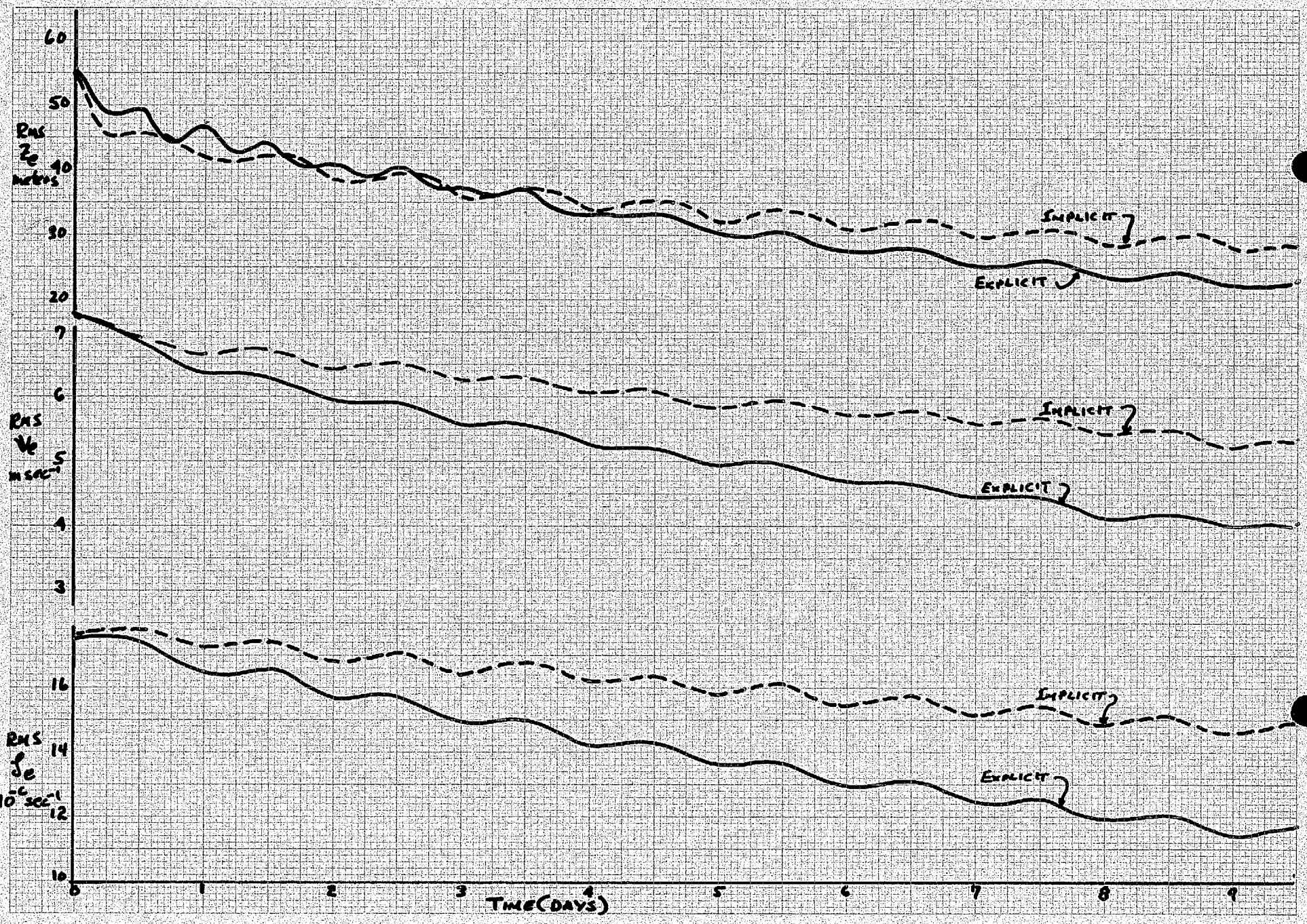

FIGURE 2.

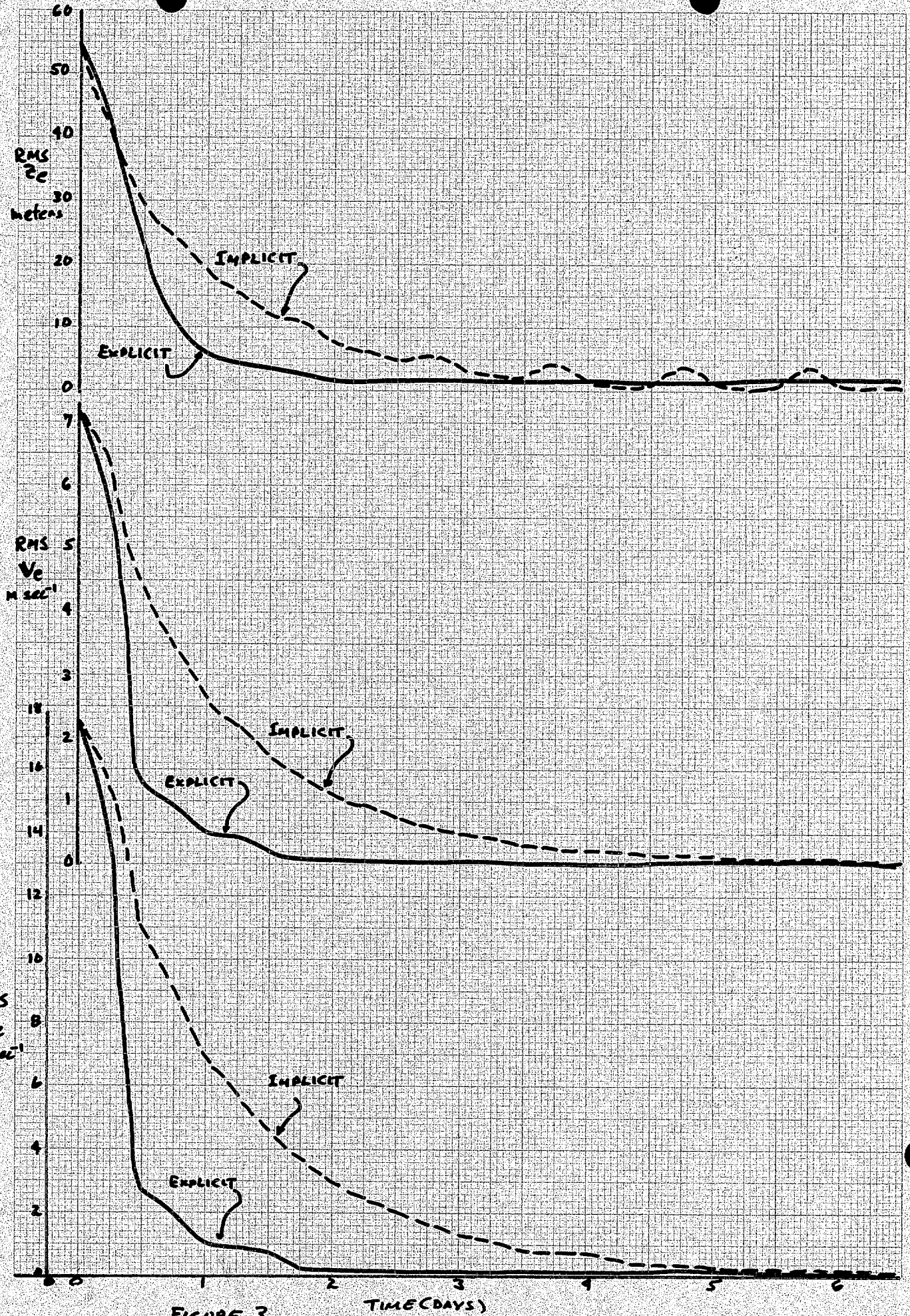

FIGURE 3.

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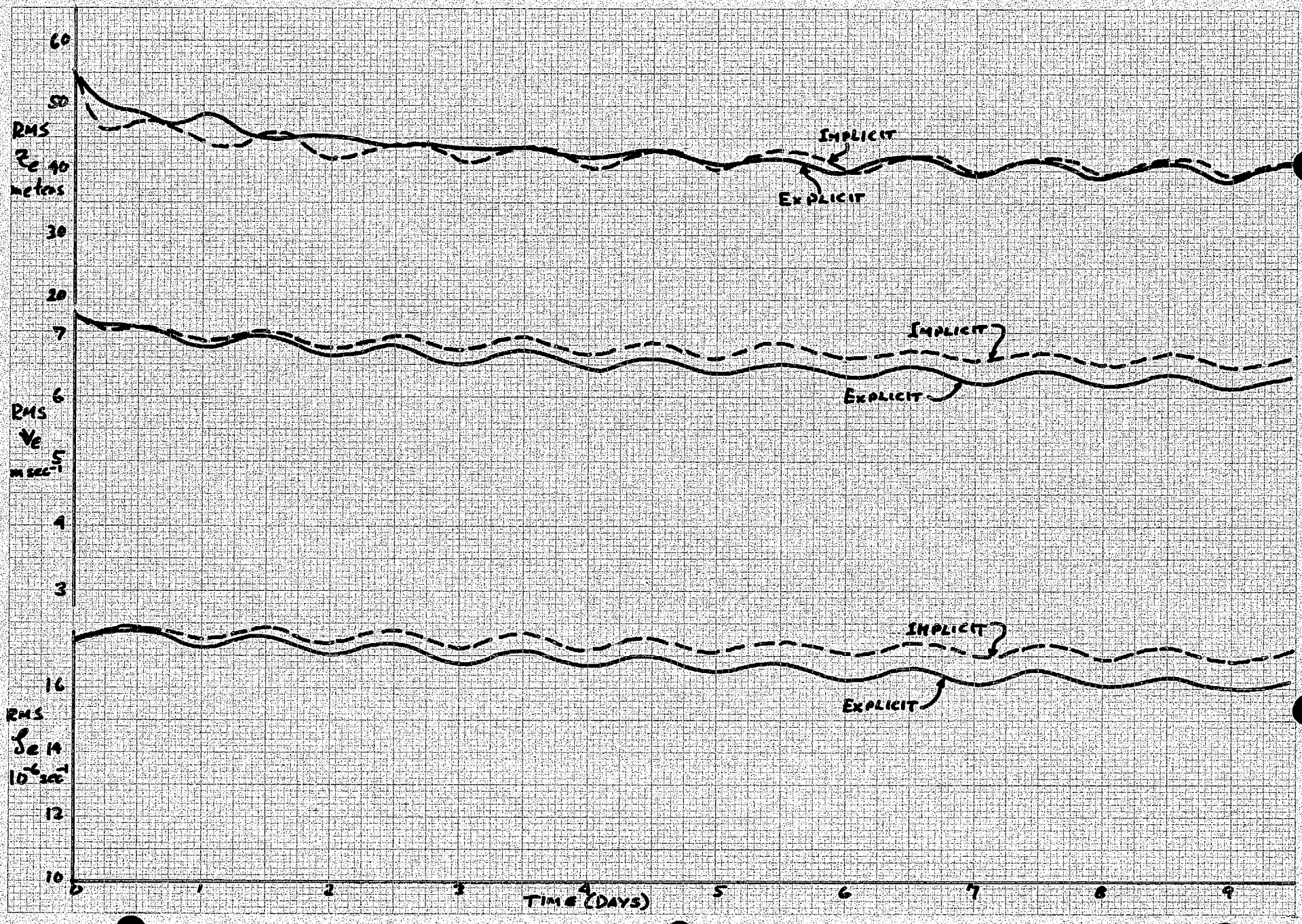

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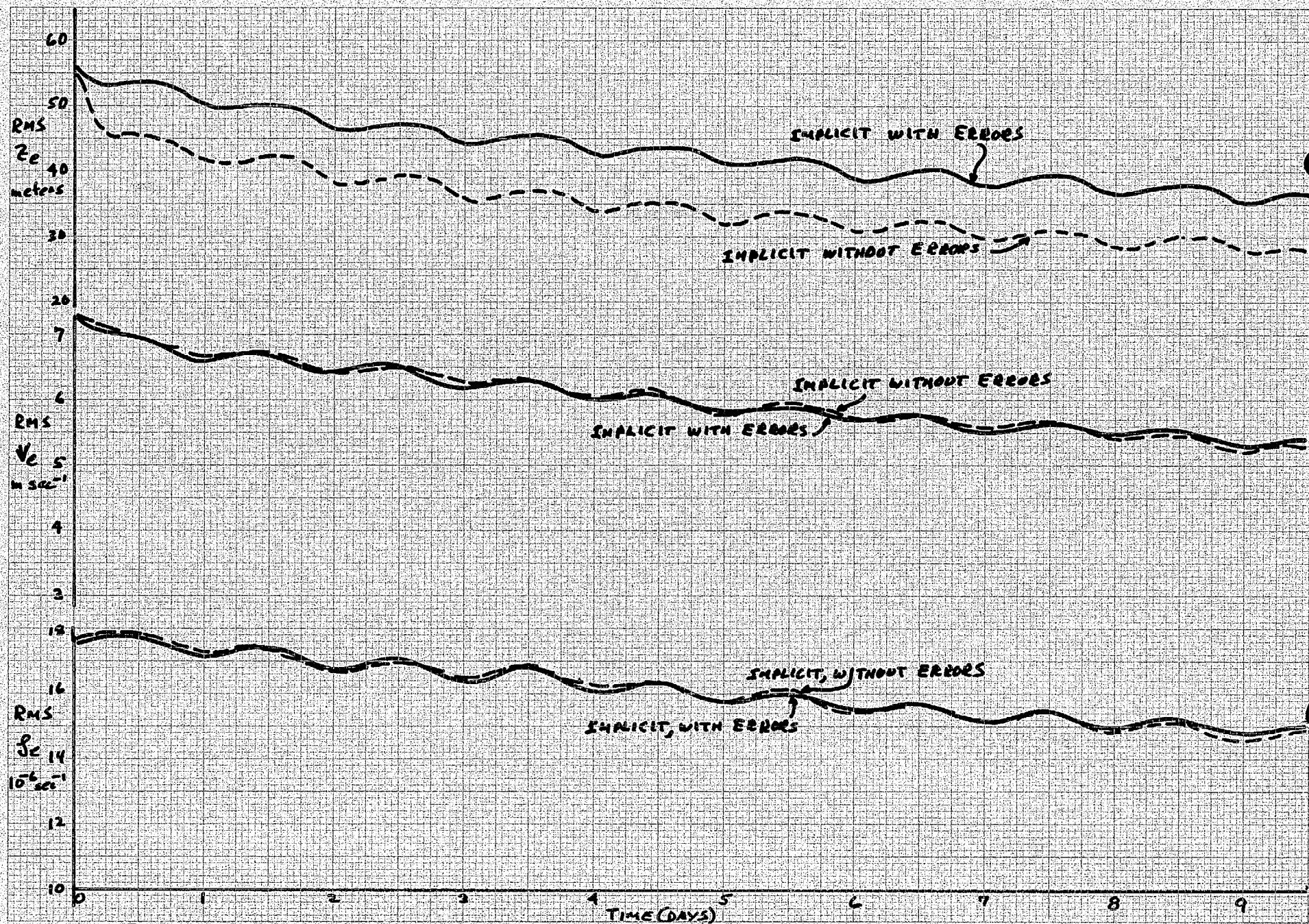

FIGURE 5.

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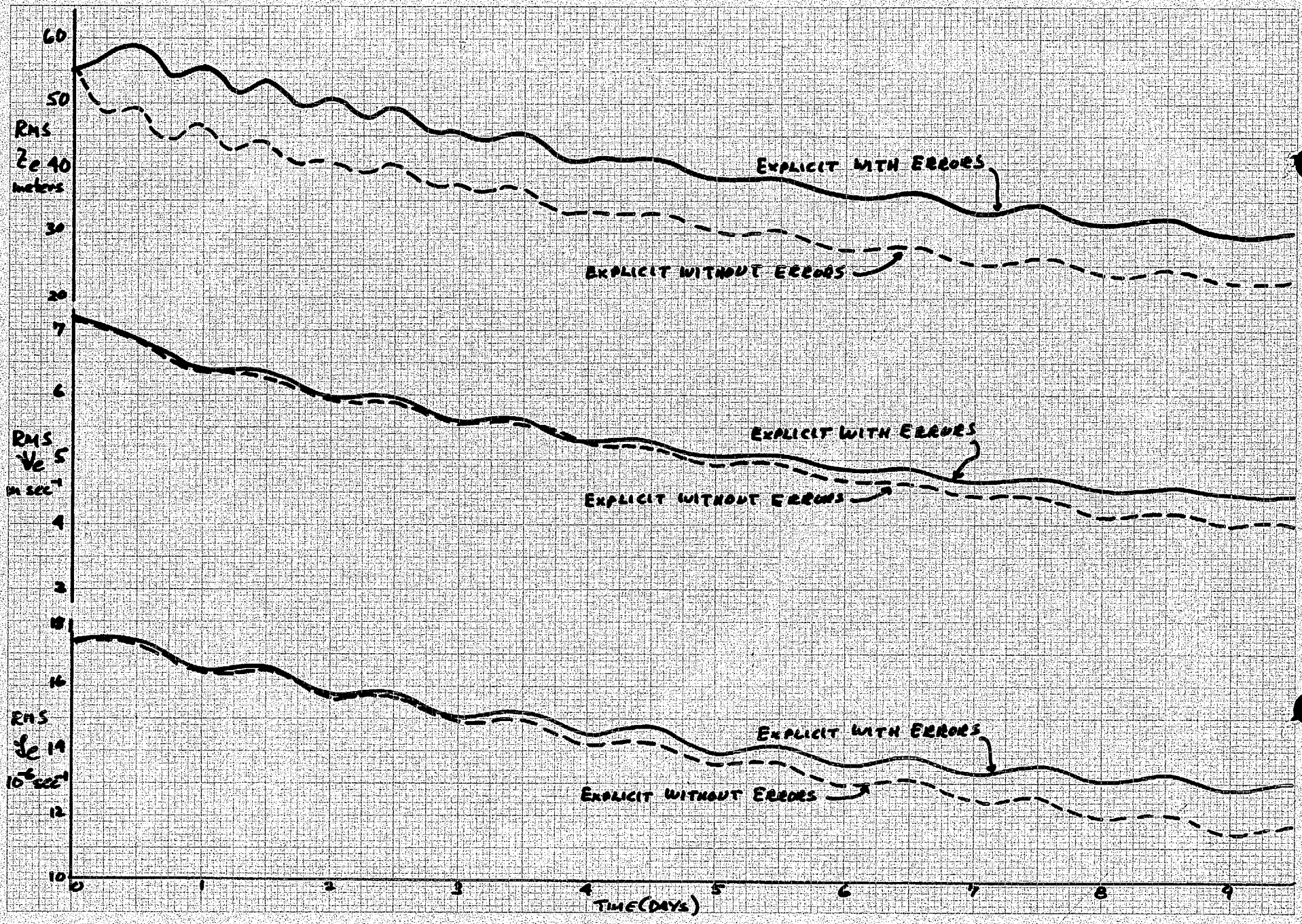

FIGURE 6.