U.S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE
NATIONAL METEOROLOGICAL CENTER

OFFICE NOTE 82

Derivation of the RSG Criterion With Non-Linear Equations and a Restricted Spectrum

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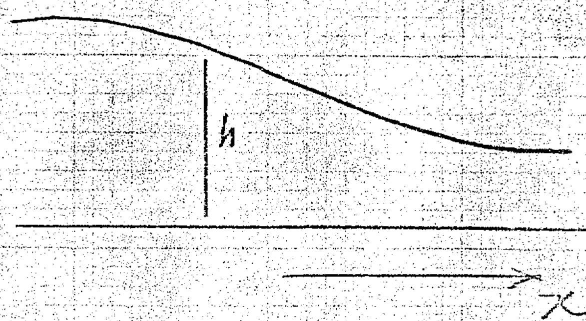
allow water equations ①

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$



efine $c^2 = gh$, then:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + 2c\frac{\partial c}{\partial x} = 0$$

$$2\frac{\partial c}{\partial t} + 2u\frac{\partial c}{\partial x} + c\frac{\partial u}{\partial x} = 0$$

dd and subtract:

$$\frac{\partial(u+2c)}{\partial t} + (u+c)\frac{\partial(u+2c)}{\partial x} = 0$$

$$\frac{\partial(u-2c)}{\partial t} + (u-c)\frac{\partial(u-2c)}{\partial x} = 0$$

Let $c = \bar{c} + c'$ where $\bar{c} = \text{const.}$, then:

$$\frac{\partial(u+2c')}{\partial t} + (u+c'+\bar{c})\frac{\partial(u+2c')}{\partial x} = 0$$

$$\frac{\partial(u-2c')}{\partial t} + (u-c'-\bar{c})\frac{\partial(u-2c')}{\partial x} = 0$$

l at an instant $u = 2c'$ everywhere,
he last equation tells us $u = 2c'$ for
ll time.

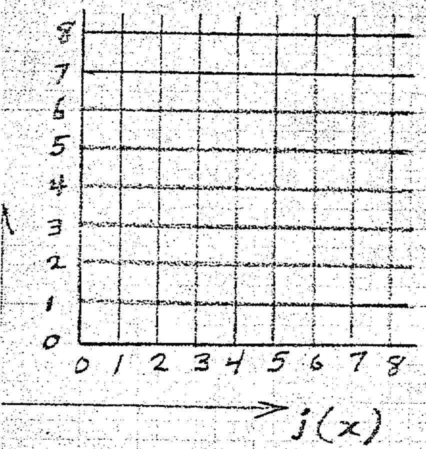
e study the case: $u = 2c'$ then.

$$\frac{\partial c'}{\partial t} + (3c'+\bar{c})\frac{\partial c'}{\partial x} = 0$$

Define: $v = 3c' + \bar{c}$, then $\boxed{\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0}$

lock equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$



inite difference notation

$$u_x = \frac{1}{2\Delta x}\left(u_{j+1} - u_{j-1}\right)$$

$$u_t = \frac{1}{2\Delta t}\left(u_{n+1} - u_{n-1}\right)$$

$$\overline{u}^{x} = \frac{1}{2}\left(u_{j+1} + u_{j-1}\right)$$

$$\overline{u}^{t} = \frac{1}{2}\left(u_{n+1} + u_{n-1}\right)$$

We analyze:

$$u_t + \overline{u}^{x} u_x = 0$$

$+ \overline{\ }^{x} u_x = 0$

et $u = C\cos\frac{\pi}{2}j + S\sin\frac{\pi}{2}j + U\cos\pi j + V$

where C, S, U, V are $f(t)$

j	0	1	2	3	4	5	6	7	8
$\cos\frac{\pi}{2}j$	1	0	-1	0	1	0	-1	0	1
$\sin\frac{\pi}{2}j$	0	1	0	-1	0	1	0	-1	0
$\cos\pi j$	1	-1	1	-1	1	-1	1	-1	1
$\overline{\cos\frac{\pi}{2}j}^x$		0	0	0	0	0	0	0	
$\overline{\sin\frac{\pi}{2}j}^x$		0	0	0	0	0	0	0	
$\overline{\cos\pi j}^x$		1	-1	1	-1	1	-1	1	

$\overline{\cos\frac{\pi}{2}j}^x = \overline{\sin\frac{\pi}{2}j}^x = 0$

$\overline{\cos\pi j}^x = -\cos\pi j$

	0	1	2	3	4	5	6	7	8
$\frac{\pi}{2} j$	1	0	-1	0	1	0	-1	0	1
$\frac{\pi}{2} j$	0	1	0	-1	0	1	0	-1	0
πj	1	-1	1	-1	1	-1	1	-1	1
$\frac{\pi}{2} j)_x \cdot \Delta x$		-1	0	1	0	-1	0	1	
$\frac{\pi}{2} j)_x \cdot \Delta x$		0	-1	0	1	0	-1	0	
$\pi j)_x \cdot \Delta x$		0	0	0	0	0	0	0	

$$\therefore \left(\cos \frac{\pi}{2} j\right)_x = -\frac{1}{\Delta x} \sin \frac{\pi}{2} j$$

$$\left(\sin \frac{\pi}{2} j\right)_x = \frac{1}{\Delta x} \cos \frac{\pi}{2} j$$

$$\left(\cos \pi j\right)_x = 0$$

j	0	1	2	3	4	5	6	7	8
$\frac{\pi}{2} j$	1	0	−1	0	1	0	−1	0	1
$\frac{\pi}{2} j$	0	1	0	−1	0	1	0	−1	0
πj	1	−1	1	−1	1	−1	1	−1	1
$\frac{\pi}{2} j \cdot \cos \pi j$	1	0	−1	0	1	0	−1	0	1
$\frac{\pi}{2} j \cdot \cos \pi j$	0	−1	0	1	0	−1	0	1	0

$$\therefore \quad \cos \frac{\pi}{2} j \cdot \cos \pi j = \cos \frac{\pi}{2} j$$

$$\sin \frac{\pi}{2} j \cdot \cos \pi j = - \sin \frac{\pi}{2} j$$

$+ \overline{u}^x u_x = 0$ (6)

$= C \cos\frac{\pi}{2}j + S\sin\frac{\pi}{2}j + U\cos\pi j + V$

$\overline{\cos\frac{\pi}{2}j}^x = \overline{\sin\frac{\pi}{2}j}^x = 0$

$\overline{\cos\pi j}^x = -\cos\pi j$

$(\cos\frac{\pi}{2}j)_x = -\frac{1}{\Delta x}\sin\frac{\pi}{2}j$

$(\sin\frac{\pi}{2}j)_x = \frac{1}{\Delta x}\cos\frac{\pi}{2}j$

$(\cos\pi j)_x = 0$

$\cos\frac{\pi}{2}j\cdot\cos\pi j = \cos\frac{\pi}{2}j$

$\sin\frac{\pi}{2}j\cdot\cos\pi j = -\sin\frac{\pi}{2}j$

$\overline{u}^x = -U\cos\pi j + V$

$u_x = \frac{1}{\Delta x}\left(-C\sin\frac{\pi}{2}j + S\cos\frac{\pi}{2}j\right)$

$C_t\cos\frac{\pi}{2}j + S_t\sin\frac{\pi}{2}j + U_t\cos\pi j + V_t$

$$= -\frac{1}{\Delta x}\begin{pmatrix}-UC\sin\frac{\pi}{2}j - US\cos\frac{\pi}{2}j \\ -VC\sin\frac{\pi}{2}j + VS\cos\frac{\pi}{2}j\end{pmatrix}$$

$$= \frac{1}{\Delta x}\left[C(U+V)\sin\frac{\pi}{2}j + S(U-V)\cos\frac{\pi}{2}j\right]$$

$\therefore\ C_t = \frac{1}{\Delta x}S(U-V)$

$S_t = \frac{1}{\Delta x}C(U+V)$

$U_t = V_t = 0$

$U_t = V_t = 0$ (7)

$\therefore\ U = a_0 + b_0 \cos \pi n$

$V = a_1 + b_1 \cos \pi n$

Recall $\quad u_t + \bar{u}^x u_x$

$$\bar{u}^x = -U \cos \pi j + V$$

We therefore have a special case of

$$u_t + (U_0 + U_1 \cos \pi j + U_2 \cos \pi n + U_3 \cos \pi j \cdot \cos \pi n) u_x = 0$$

Difference by $(\,)_t$ and then by $(\,)_x$ and eliminate u_{xt}:

$$u_{tt} - [(U_0^2 - 2 U_0 U_3 \cos \pi j \cdot \cos \pi n + U_3)^2$$
$$- (U_1^2 - 2 U_1 U_2 \cos \pi j \cdot \cos \pi n + U_2^2)\, u_{xx} = 0$$

at even $j+n$:

$$u_{tt} - [(U_0 - U_3)^2 - (U_1 - U_2)^2]\, u_{xx} = 0$$

Stability criterion:

$$0 \leq \left(\frac{\Delta t}{\Delta x}\right)^2 [(U_0 - U_3)^2 - (U_1 - U_2)^2] < 1$$

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Time \ Space	Low frequency	High frequency
Low frequency	Stable	Unstable
High frequency (Computational mode)	Unstable	Stable