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The Potential Vorticity Theorem in General σ- Coordinates

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1. Introduction

The principle of the conservation of potential vorticity formed the basis for early work in numerical weather prediction (Charney and Phillips, 1953). During the past decade, modeling research has departed from those early roots. It occurred to me that I have no recollection of ever seeing the potential vorticity theorem stated in the general σ coordinate system (Phillips, 1957; Shuman and Hovermale, 1968).

This may seem to be an academic point, but I think it is worthwhile to stress that the theorem is still valid for the primitive equation formulation of adiabatic, inviscid and hydrostatic motion. Indeed, the potential vorticity has served an excellent tracer for particle trajectories, and may serve a useful function in efforts to diagnose the occasional errors found in numerical forecasts.

2. Potential Vorticity Theorem

The theorem of Ertel (cf. Greenspan, 1968) may be stated as the fact that in adiabatic, inviscid flow the quantity,

$$\rho^{-1}\ [\nabla_3 \times \vec{V} + 2\vec{\Omega}] \cdot \nabla_3 \theta, \qquad (1)$$

in which ρ is density, $\vec{\Omega}$ the Earth's rotation vector, and θ is potential temperature, is conserved following the motion of a particle (Haltiner, 1971). Most meteorological uses of the theorem have been based on hydrostatically balanced flow. The simplest derivation of the theorem may be given using isentropic coordinates (Thompson, 1961).

The equations of motion, continuity and thermodynamics take the simple forms

$$\frac{\partial}{\partial t}\vec{V} + \eta_\theta \vec{k} \times \vec{V} + \vec{\nabla}_\theta \frac{\vec{V}\cdot\vec{V}}{2} + \vec{\nabla}_\theta \Psi = 0 \qquad (2)$$

$$\frac{\partial}{\partial t}\frac{\partial p}{\partial \theta} + \vec{\nabla}_\theta \cdot \left(\vec{V}\frac{\partial p}{\partial \theta}\right) = 0 \qquad (3)$$

$$\frac{d\theta}{dt} = 0 \qquad (4)$$

in which η_θ is the absolute vorticity in the θ surface and Ψ is the Montgomery streamfunction, The vorticity equation is obtained from (2) and the individual derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}_\theta \tag{5}$$

is used in (3) to obtain

$$\frac{d}{dt}\eta_\theta = -\eta_\theta\vec{\nabla}_\theta\cdot\vec{V} \tag{6}$$

$$\frac{d}{dt}\frac{\partial p}{\partial\theta} = -\frac{\partial p}{\partial\theta}\vec{\nabla}_\theta\cdot\vec{V} \tag{7}$$

Combining equations (6) and (7) yields

$$\frac{d}{dt}\left\{\eta_\theta\left(\frac{\partial p}{\partial\theta}\right)^{-1}\right\} = 0 \tag{8}$$

This is the potential vorticity theorem in a hydrostatically balanced flow. Note that the absolute vorticity is measured in the θ coordinate surface.

The transformation of equation (8) into a general σ coordinate is relatively straightforward. One obtains

$$\frac{d}{dt}\left(\left(\frac{\partial p}{\partial\sigma}\right)^{-1}\left(\eta_\sigma\frac{\partial\theta}{\partial\sigma} + \vec{k}\times\frac{\partial\vec{V}}{\partial\sigma}\cdot\vec{\nabla}_\sigma\theta\right)\right) = 0 \tag{9}$$

Note that η_σ is the absolute vorticity in the relevant σ coordinate surface and the $\vec{k}$ is the unit vertical vector.

If Q is any quantity, one may prove through the use of the continuity equation,

$$\frac{\partial}{\partial t}\frac{\partial p}{\partial\sigma} + \vec{\nabla}_\sigma\cdot\left(\frac{\partial p}{\partial\sigma}\vec{V}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\partial p}{\partial\sigma}\dot{\sigma}\right) = 0 \tag{10}$$

that

$$\left(\frac{\partial p}{\partial\sigma}\right)^{-1}\left\{\frac{\partial}{\partial t}Q + \vec{\nabla}_\sigma\cdot(\vec{V}Q) + \frac{\partial}{\partial\sigma}(\dot{\sigma}Q)\right\} \equiv \frac{d}{dt}\left(\left(\frac{\partial p}{\partial\sigma}\right)^{-1}Q\right) \tag{11}$$

Consequently, it follows that (9) is equivalently expressed in the form,

$$\frac{\partial \xi}{\partial t} + \vec{\nabla}_\sigma \cdot \{\xi \vec{V}\} + \frac{\partial}{\partial \sigma}(\xi \dot{\sigma}) = 0 \qquad (12)$$

if one defines

$$\xi \equiv \eta_\sigma \frac{\partial \theta}{\partial \sigma} + \vec{k} \times \frac{\partial \vec{V}}{\partial \sigma} \cdot \vec{\nabla}_\sigma \theta \qquad (13)$$

From the form (12), it is readily seen that ξ is an integral invariant of the motion when the hypothesis of inviscid, adiabatic flow is appropriate.

3. Remarks

The derivation given above is algebraically simple. It is, however, a difficult task to prove the theorem beginning with the basic equations in σ coordinates. Although that derivation has been carried through, it will not be repeated here. It may be noted, however, that the proof required the use of many vector identities. The probability is high that the finite difference versions of these identities do not hold for any of the widely used difference schemes. Consequently, one may anticipate the violation of this fundamental meteorological principle in many prediction models.

It is particularly suggestive to consider the sensitivity of the potential vorticity to the parameter, $\frac{\partial p}{\partial \sigma}$.

In the Shuman-Hovermale σ-system, the distribution of $\frac{\partial p}{\partial \sigma}$ will show considerable horizontal variability in the neighborhood of the high mountains and in regions with steeply sloping "tropopause." It appears to be worthwhile to examine how closely the potential vorticity is conserved following the motion of particles moving through such environs. Such calculations could be made with the aid of the trajectory calculation scheme developed by R. Reap of TDL.

4. References

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