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NATIONAL WEATHER SERVICE
NATIONAL METEOROLOGICAL CENTER

OFFICE NOTE 74

Free Gravity Oscillations of the Shuman-Hovermale Model

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1. Introduction

The development of a semi-implicit version of the Shuman-Hovermale (1968) model required as an intermediate step the formulation of a linearized version of the model equations. In the present paper, we present certain results derived by means of those linear equations.

The semi-implicit method is based upon the implicit approximation of those terms in the model equations which enter into a linear computational stability analysis for pure gravitational oscillations about an atmosphere at rest on a nonrotating Earth. The algebraic manipulation of the full equations necessary to arrive at the appropriate linear equations is omitted here for the sake of brevity.

The analyses to be presented below lack generality, but they are believed to be of interest as a first approximation of the structure of the free gravitational modes likely to be found in forecasts made with the Shuman-Hovermale model. In Office Note 47, we compared the free modes exhibited by four-layer models written with Phillips' σ-coordinate and with Shuman-Hovermale's σ-system. The influence of the vertical resolution variation and the significance of the use of a "tropopause" material surface were noted. In this paper, we deal only with the Shuman-Hovermale system but attention is focused on the role played by the special characteristics of this σ-system.

2. Method for Determining Eigenfunctions

The linearized, vertically discretized system of equations governing the proposed semi-implicit version of the Shuman-Hovermale model may be reduced to the following matrix form provided that the Coriolis term is omitted

together with consideration of the sphericity of the Earth,

$$A\,q = 0 \tag{1}$$

The matrix A is the following 7x7 array in which c stands for phase speed of the perturbation wave solution:

$$\begin{matrix}
c(a_{11} - c^2) & c\,a_{12} & c\,a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\
c\,a_{21} & c(a_{22} - c^2) & c\,a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\
c\,a_{31} & c\,a_{32} & c(a_{33} - c^2) & a_{34} & a_{35} & a_{36} & a_{37} \\
c\,a_{41} & c\,a_{42} & c\,a_{43} & a_{44} + \beta c^2 & a_{45} & a_{46} & a_{47} \\
c\,a_{51} & c\,a_{52} & c\,a_{53} & a_{54} - 3c^2 & a_{55} + 6c^2 & a_{56} - 3c^2 & a_{57} \\
c\,a_{61} & c\,a_{62} & c\,a_{63} & a_{64} & a_{65} - 3c^2 & a_{66} + 6c^2 & a_{67} \\
c\,a_{71} & c\,a_{72} & c\,a_{73} & a_{74} & a_{75} & a_{76} & a_{77} + 4c^2
\end{matrix} \tag{2}$$

The vector of unknown amplitudes q stands for

$$q^T = \left(\pi_1,\ \pi_2,\ \pi_3,\ \frac{w_1}{ik},\ \frac{w_2}{ik},\ \frac{w_3}{ik},\ \frac{w_4}{ik}\right) \tag{3}$$

The coefficients a_{ij} appearing in (2) are complicated functions of the basic state thermodynamic structure. The parameter β is a simple function of the basic state pressures

$$\beta \equiv \frac{\tilde{p}_G - p_c - \tilde{p}_T}{p_c} \tag{4}$$

$\tilde{p}_G$ is the basic state surface pressure

$\tilde{p}_T$ is the basic state "tropopause" pressure

p_c is the depth of the boundary layer in pressure.

Before describing the method used to determine the eigenvalues c for which det A vanishes, we may note a special property of the coefficients a_{ij}.

The coefficients

$$a_{ij} \quad (i = 1,2,\ldots,7;\ j = 1,2,3)$$

are determined solely through combinations of the basic state mass distribution without clear dependence upon the limiting properties of the dry-adiabatic lapse rate.

On the other hand, the coefficients

$$a_{ij} \quad (i = 1,2,\ldots,7;\ j = 4,5,6,7)$$

which appear as coefficients of the "vertical" velocity variables are all dependent upon the static stability with reference to dry-adiabatic processes. In particular, if each layer in each of the σ domains were to have a dry adiabatic temperature distribution, these coefficients would vanish identically.

The method used to determine the eigenvalues c was based on the existence of seven distinct, positive real valued roots of the equation

$$\det A = 0 \tag{5}$$

By inserting values for c, in a prescribed range, the determinant was evaluated. Changes in algebraic sign of the determinant between consecutive values of c were used to isolate the roots of (5). An interpolation process was used to locate the root to a satisfactory precision.

Once the roots were determined, the calculated eigenvalue was utilized to determine the eigenvector associated with it. To this end, we used the assumption that π_1, the first component of the eigenvector, had value unity. In all cases, this proved to be satisfactory.

Since the eigenvectors q are not readily interpreted, we transformed them into a more usual set. The individual derivative of pressure denoted by

$$\omega = \frac{dp}{dt} \tag{6}$$

may be related to the components of q as follows:

$$\hat{\omega} \equiv \frac{1}{ikc}\omega \tag{7}$$

$$\begin{aligned}
\hat{\omega}_1 &= \pi_1 \\
\hat{\omega}_2 &= w_4\tilde{\pi}_2/c + .5\,\pi_2 \\
\hat{\omega}_3 &= \pi_2 \\
\hat{\omega}_4 &= (\tilde{\pi}_3 - p_c)w_3/c + \pi_3/3 \\
\hat{\omega}_5 &= (\tilde{\pi}_3 - p_c)w_2/c + 2\pi_3/3 \\
\hat{\omega}_6 &= (\tilde{\pi}_3 - p_c)w_1/c + \pi_3 \\
\hat{\omega}_7 &= \pi_1 + \pi_2 + \pi_3
\end{aligned} \tag{8}$$

π_1 is the amplitude of p_θ, where p_θ is the perturbation pressure at the base of the isentropic layer.

π_2 is the amplitude of $p_T - p_\theta$, where p_T is the perturbation pressure at the "tropopause."

π_3 is the amplitude of $p_G - p_T$, where p_G is the perturbation pressure at the ground. The same parameters with curly overbars imply the basic state value. The quantities w_j stand for $\dot{\sigma}_j/(ik)$ with $i = \sqrt{-1}$ and k the horizontal wave number. The subscript 4 is assigned to $\dot{\sigma}$ in the mid-stratosphere. The other subscripts apply for $\dot{\sigma}$ in the troposphere with the $\dot{\sigma}_1$ closest to the ground.

In the subsequent discussion, we shall use the $\hat{\omega}$ depiction of the eigenfunctions. One might equally well have constructed the eigenfunctions in terms of the divergence of the horizontal wind or of the temperature.

3. Results

3.1 Standard Atmosphere 1

The U.S. Standard Atmosphere was used to specify the thermal structure of the basic state. The tropopause level was chosen to be at 260 mb and the surface pressure at 1000 mb. These are not standard, but agree with the levels suggested in Shuman and Hovermale (1968).

Table 1 indicates the temperature and pressure at the midpoints of the seven relevant layers of the model.

p mb	975	835	605	375	220	140	50
T ^{o}K	286.0	277.7	261.3	238.5	216.6	216.6	179.7

Table 1. Standard Atmosphere #1.

Table 2 gives the eigenfunction distribution of $\hat{\omega}$ for each of the seven free modes. The functions are shown graphically in figures 1a and 1b. The phase speed of each free mode is indicated in m sec^{-1}. The $\hat{\omega}$'s apply at the base of each layer in the model.

$c[m\ s^{-1}]$	3.4	11.5	20.9	31.8	94.3	131.6	329.9
$\sigma_I=1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\sigma_s=.5$	-2.4	-2.7	-3.5	-4.9	-.4	1.4	2.5
$\sigma_s=1.0$	3.6	3.2	2.1	-.3	-1.5	1.5	3.8
$\sigma_T=1/3$	-29.9	-20.1	2.6	18.3	-.8	.8	5.6
$\sigma_T=2.3$	117.8	43.7	2.6	17.3	-.4	.2	7.2
$\sigma_T=1.0$	-236.6	19.0	.3	2.9	+.0	-.4	8.8
$\sigma_B=1.0$	1.1	-.7	-.0	-1.0	.2	-.5	9.2

Table 2. Tabulation of eigenfunctions $\hat{\omega}$ for standard atmosphere 1, phase speed c given in m/sec. Level at which $\hat{\omega}$ applies indicated by σ value. $\sigma_I=1$ is base of isentropic layer. $\sigma_s=1$ is at "tropopause." $\sigma_T=1.$ at top of boundary layer. $\sigma_B=1.$ at ground.

It should be noted that the temperature used at 50 mb was obtained by linearly extrapolating the potential temperatures in the stratospheric layers. This is an approximation of the method used in the operational version of the Shuman Hovermale model. In Appendix A, we give the results obtained by inserting at 50 mb the approximately isothermal temperature implied by the actual standard atmosphere. The result only modified the phase speeds of the three fastest modes.

In the course, of effecting this computation, we inadvertently inserted a temperature of 64°K on the 50 mb level. The calculation was affected to the extent that the second fastest mode developed a complex valued phase speed.

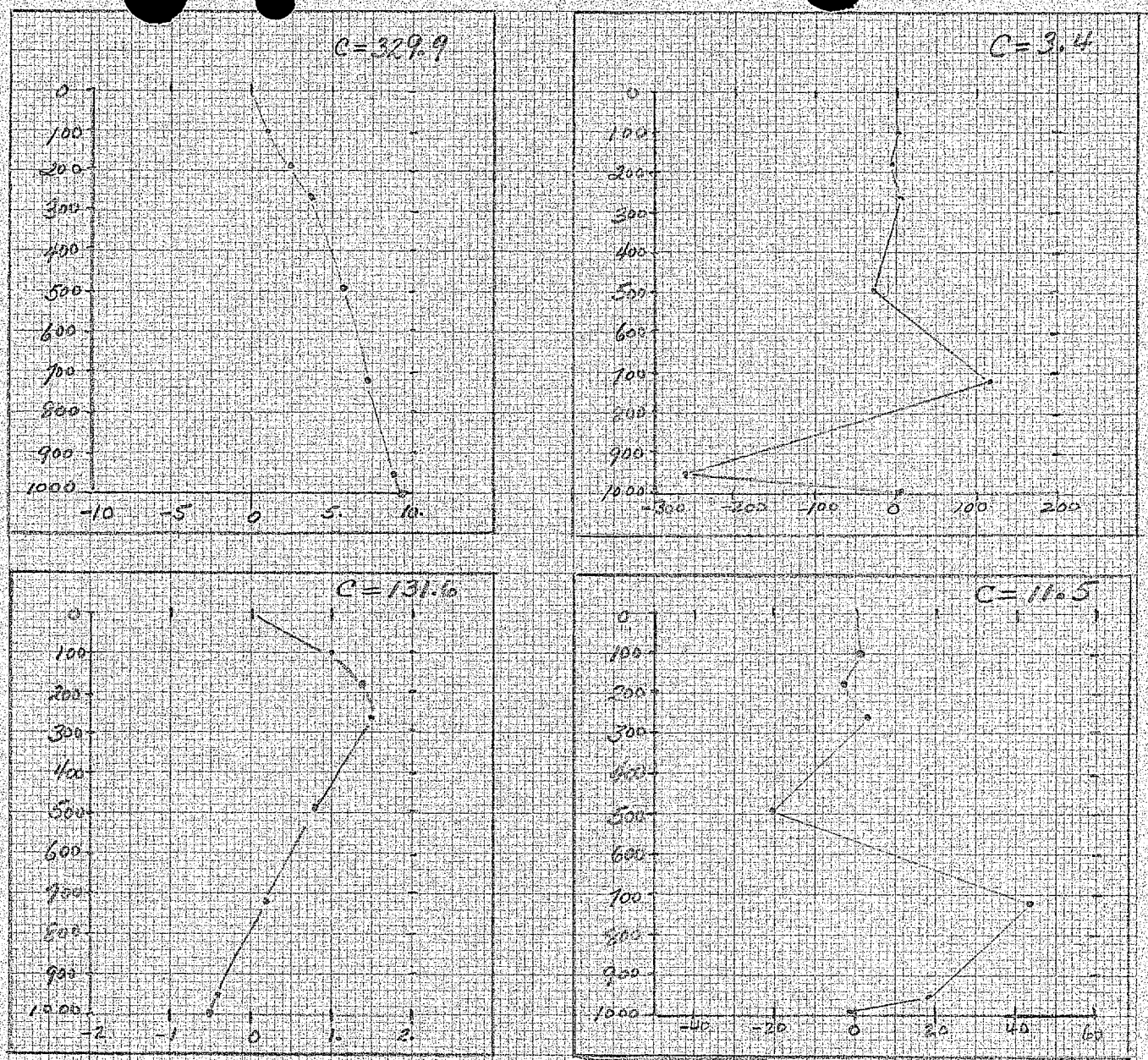

FIG 1a Eigenfunctions $\hat{\omega}$ for standard atmosphere 1
Phase speed c indicated [m s^{-1}]

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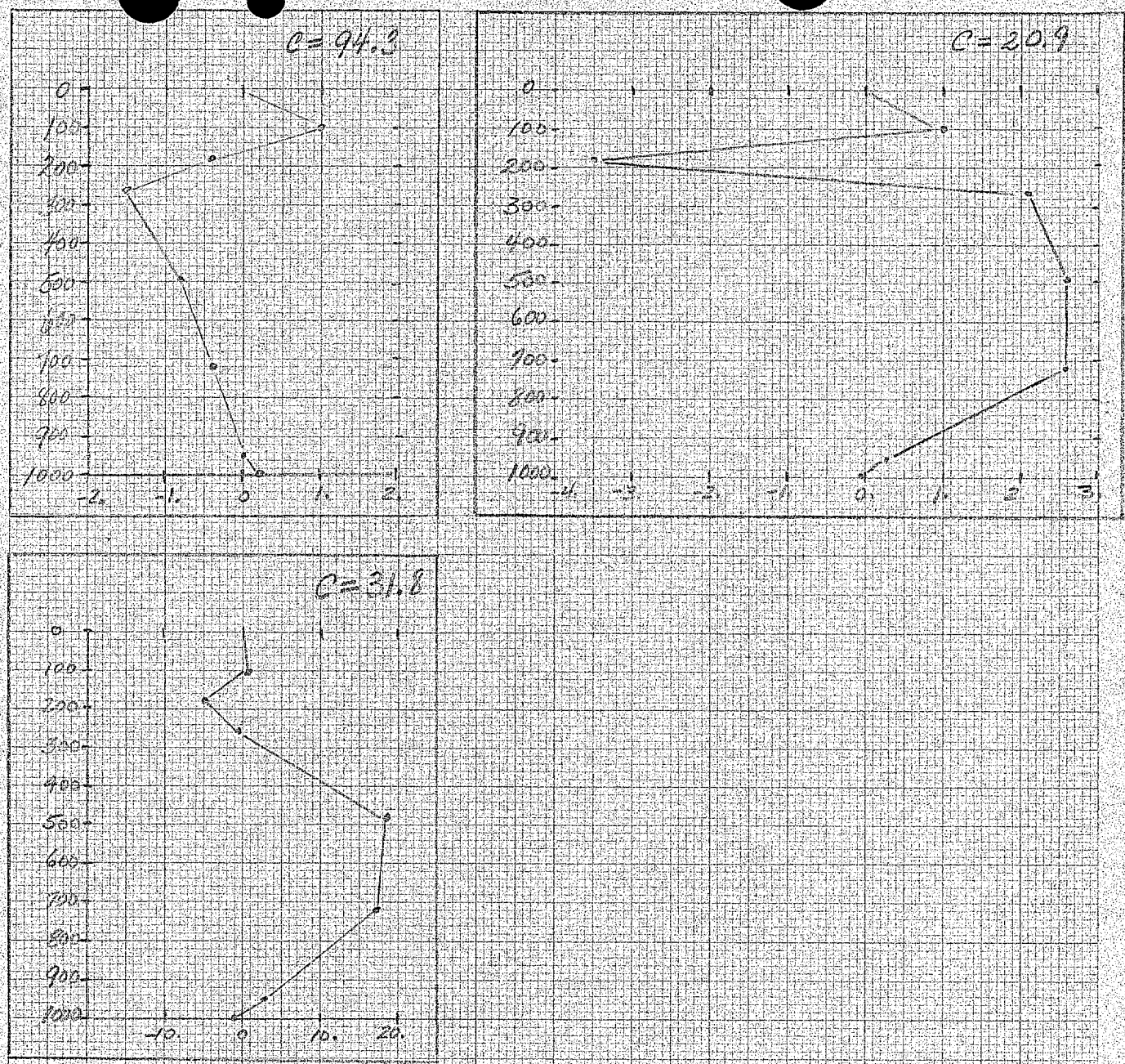

Fig 1b Eigenfunctions $\hat{\omega}$ for standard atmosphere I
Phase speed c indicated [ms^{-1}]

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A check indicated that this was due to the implication of a super-autoconvective distribution of density between the isentropic cap and the upper stratospheric layer. The careful selection of the temperature in the isentropic cap is indicated, otherwise a physical instability may be excited. On the other hand, the insertion of too large a temperature will yield an undesirable increase in the phase speed of the fundamental modes.

3.2 Behavior of Eigenfunction for various stratifications

It may be noted that the eigenfunction corresponding to the phase speed 20.9 m sec^{-1} is somewhat anomalous. Based on the Sturmian theory, one would have anticipated that this eigenfunction would exhibit four internal nodal points. It shows only three.

To investigate this question, we ran a series of idealized stratifications of the basic state through the computer program. We constructed part linear and part isothermal lapse rates with cross overs at various levels of the model. The mode with the anomally was isolated for study.

In figure 2, the results show that in the ISOTHERMAL case the mode has the expected four internal nodes. In the LINEAR lapse rate case, the mode has just three zeroes. It will be seen from the figure that the transition occurs between cases (3) and (4). This implies that one zero is lost when the entire troposphere of the model becomes characterized by a linear lapse rate.

We are unable to advance a cogent rationale for this behavior. It has been suggested that the peculiarity must be associated with the passage from differential to finite difference form of the equations. The question is

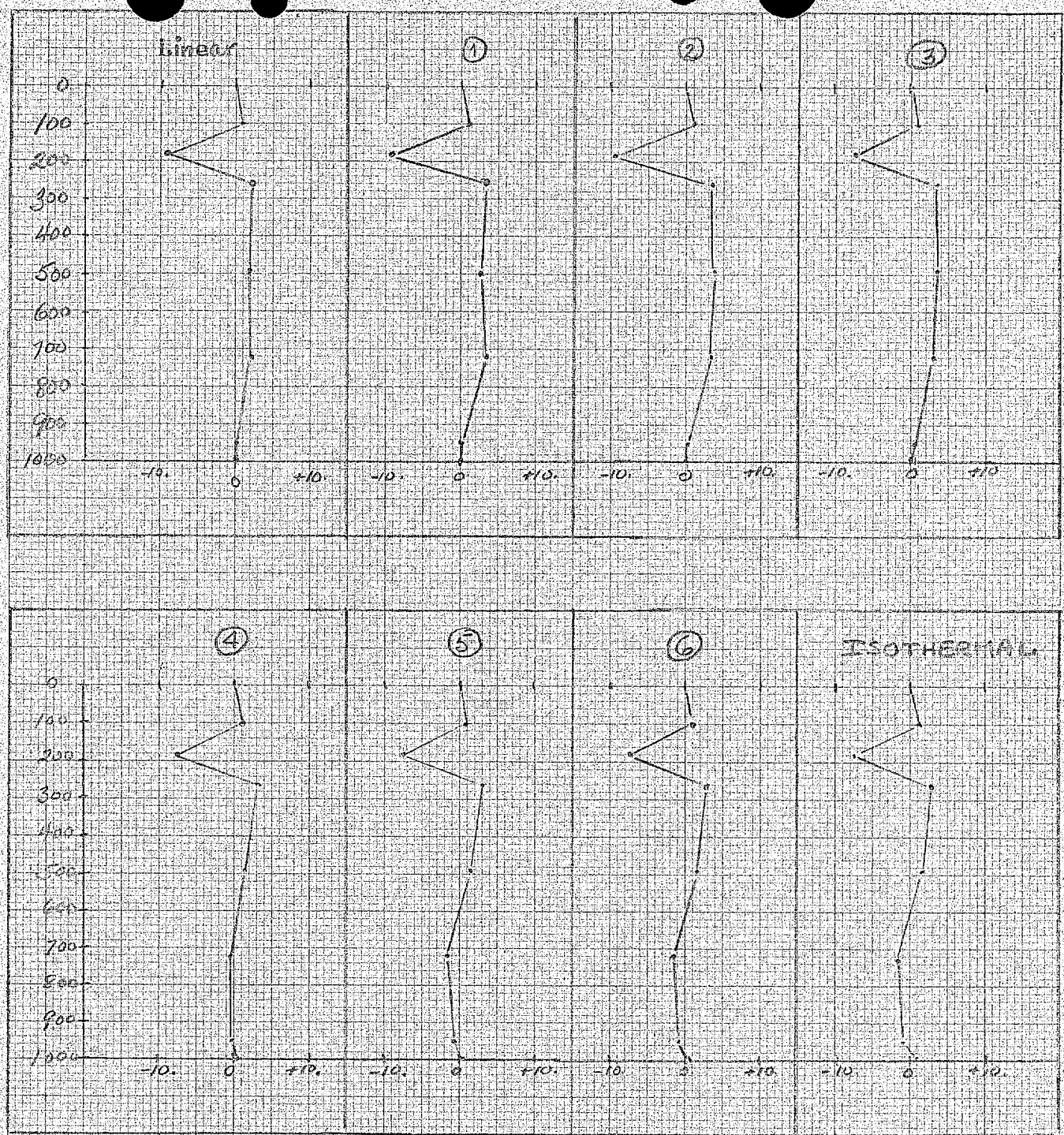

FIG 2 EIGENFUNCTION $\hat{\omega}$ FOR "ANOMALOUS" MODE AS TEMPERATURE LAPSE RATE IS VARIED. LINEAR LAPSE RATE IS 6°K/KM. INDEX (J) GIVES NUMBER OF LAYERS FROM TOP OF MODEL IN WHICH ISOTHERMAL LAPSE EXISTS; BELOW THE LINEAR LAPSE PERSISTS UNTIL ISOTHERMAL CASE.

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a subtle one of minor significance at present. If the eigenfunctions form a complete set for the difference equations, then the anomalous behavior is irrelevant. The further study of this question is beyond the scope of our current investigations.

3.3 Super-Adiabatic Lapse Rates

If the atmosphere is stratified in such a fashion that the lapse rate of temperature exceeds the dry-adiabatic rate, one would expect internal gravity waves to be unstable. (c.f. Haltiner, 1971, p. 30)
As mentioned earlier, only the coefficients of w in equation (1) are explicitly dependent upon the static stability. The lapse rate 15 ^{o}K/KM was used to specify the temperatures at the several layers of the model in order to examine the response of the model. The pressures and temperatures used are given in Table 4. In Table 5 the eigenvalues and eigenfunctions are presented. Since we tried only real values of the phase speed c in the frequency equation (5), we do not obtain all seven modes. Four of the modes are presumably unstable.

p mb	975	835	605	375	220	140	50
T ^{o}K	284.2	265.6	230.7	187.1	148.2	121.6	77.5

Table 4. Super-adiabatic basic state.

c m sec^{-1}	9.5	34.5	298.8
$\sigma_I = 1.$	1.0	1.0	1.0
$\sigma_s = 1/2$	-1.1	-.8	2.7
$\sigma_s = 1.$	.3	-2.2	4.5
$\sigma_T = 1/3$	-.4	.4	7.2
$\sigma_T = 2/3$	.1	.3	10.0
$\sigma_T = 1.$	-.1	.1	13.0
$\sigma_B = 1.$	0.	-0.	13.7

Table 5. Eigenfunctions $\hat{\omega}$ obtained for superadiabatic lapse rate. See Table 2 caption for other notations.

Two of the retained modes are readily identified--one is the external mode, the other is the highest internal mode. The latter is markedly different in the amplitude of its structure from previously given forms. It now possesses a maximum amplitude at upper levels. The third of the retained modes is anomalous.

It had been anticipated that the results of this calculation would be more readily interpreted than is now apparent. They are presented here for comment by others.

3.4 Isothermal Stratification and Tropopause Level

The isothermal lapse rate is a particularly simple one for analysis. We calculated the eigenvalues and eigenfunction for a set of isothermal basic states with temperature in the range, 250 to 290 ^{o}K. This calculation was

repeated with a modification of the vertical resolution of the model. The pressures at the midpoint of the regular and modified layers of the model are given in Table 6.

Regular	975	835	605	375	220	140	50
Modified	975	873	720	566.7	392.5	197.5	50

Table 6. Pressure in mb at midpoint of model layers. Modified model has "tropopause" at 490 mb; regular model has tropopause at 260 mb.

The eigenfunctions do not vary their shape as the temperature is varied. Different eigenfunctions are obtained for the different resolutions. The eigenfunctions are shown in Figures 3a and 3b. The phase speeds of each mode are given as a function of the isothermal temperature for both modified and regular tropopause pressures. The regular tropopause is at 260 mb in the basic state; the modified level is at 490 mb.

With regard to the phase speeds, one observes the anticipated increase of phase speed with increased isothermal temperature. The large value of the zero, or external, mode phase speed is surprising (cf. Office Note 47). It is likely that this is accountable for by the presence of the isentropic layer in the present analysis. The possibility that such large phase speeds must be accounted for in the linear computational stability criterion cannot be dismissed without examination of the thermal structures used in the model integrations, especially in low latitudes.

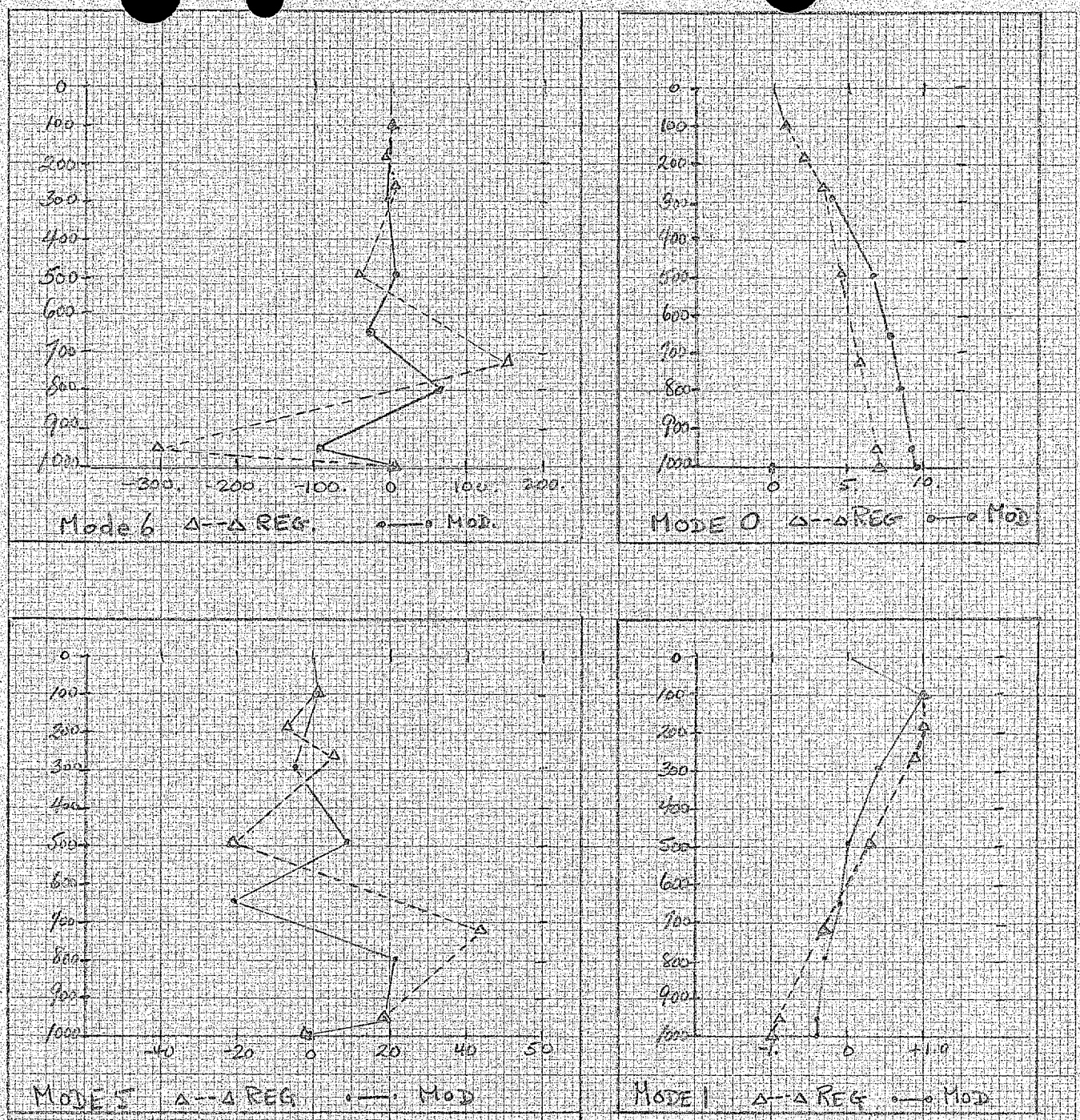

FIG 3a EIGENFUNCTIONS $\hat{\omega}$ FOR ISOTHERMAL BASIC STATE. REGULAR CURVES FOR TROPOPAUSE AT 260 mb. MODIFIED CURVES FOR TROPOPAUSE AT 490 mb.

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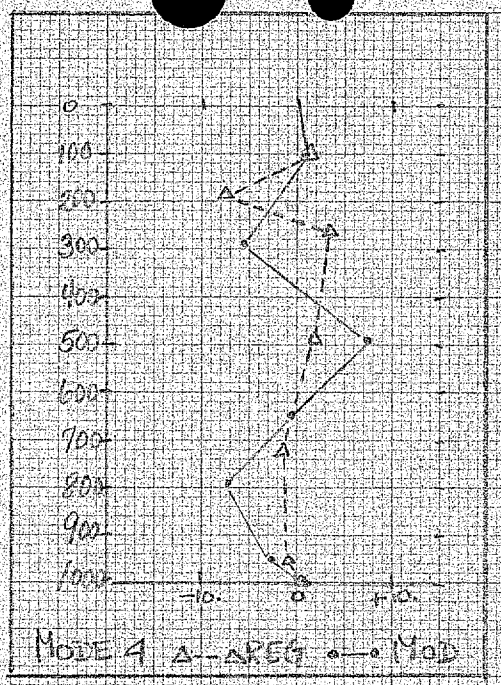

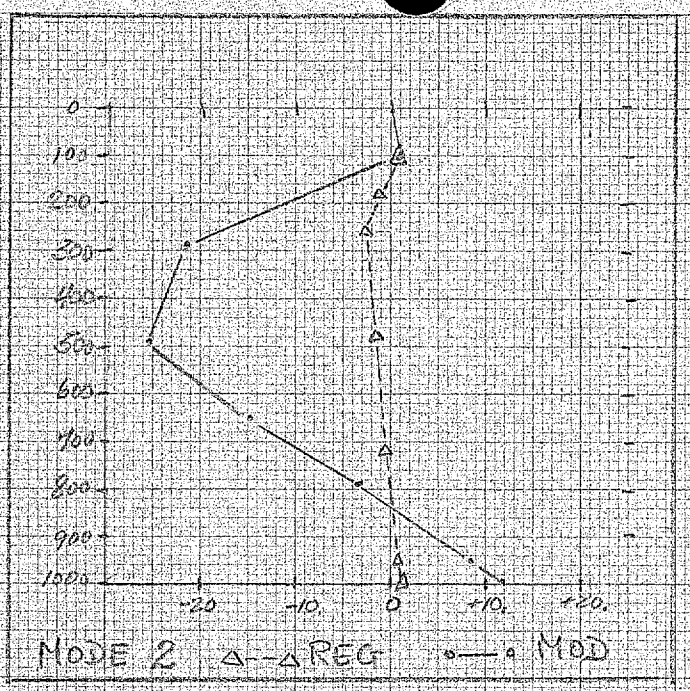

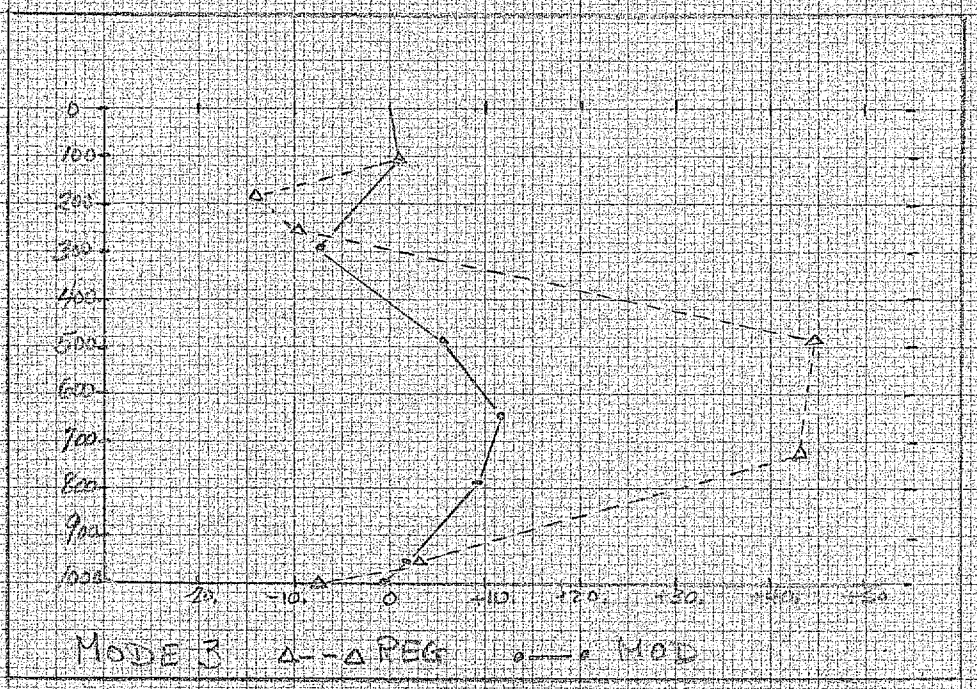

FIG 3b EIGENFUNCTIONS $\hat{\omega}$ FOR ISOTHERMAL BASIC STATE. REGULAR CURVES FOR TROPOPAUSE AT 200 mb. MODIFIED CURVES FOR TROPOPAUSE AT 490 mb.

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One may also note that the influence of a "lower tropopause" is to increase the phase speed of modes 0 and 1 and decrease those of the higher modes.

TEMP	MODE:	6	5	4	3	2	1	0
250	REG	5.4	18.4	25.6	50.1	125	178	346
	MOD	3.9	9.8	20.4	34.5	110	200	373
270	REG	5.7	19.1	26.6	52.0	130	185	359
	MOD	4.1	10.2	21.2	35.8	114	208	388
290	REG	5.9	19.8	27.5	53.9	135	192	372
	MOD	4.2	10.6	22.0	37.1	118	216	402

Table 7: Phase speeds of different modes of isothermal basic state.

The form of the eigenfunctions when plotted versus pressure, as in Figs. 3a and 3b, is considerably different for the two choices of "tropopause" pressure. One may regard this as a vertical truncation error. Since in practice, the tropopause pressure does vary (although generally by a smaller amount), one may anticipate the development of computational noise analogous to that noted when horizontal truncation error varies sharply in response to abrupt variations in mesh size.

A final point may be noted with reference to Figs. 3a and 3b. The possibility of utilizing normal mode analysis as a diagnostic tool (cf. Dickinson and Williamson, 1971 and 1972) may be impossible in the Shuman Hovermale σ system and even in the Phillips σ system, owing to the significant variation in the eigenfunctions. Further investigation of this question is required.

4. References

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Appendix A. Eigenfunctions for standard atmosphere with 50 mb temperature obtained from U.S. standard atmosphere.

p mb	T °K
50	217.2
140	216.6
220	216.6
375	238.5
605	261.3
835	277.7
975	286.0

Table A1.

c m sec^{-1}	3.4	11.5	21.5	31.8	103.2	164.9	333.1
$\sigma_I = 1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\sigma_s = .5$	-5.0	-5.6	-7.3	-10.2	-2.7	1.0	2.3
$\sigma_s = 1$	6.9	6.1	3.9	- .5	-5.1	.8	3.4
$\sigma_T = 1/3$	-61.4	-42.8	5.1	35.0	-3.0	.4	4.9
$\sigma_T = 2/3$	240.0	91.5	5.0	33.1	-1.2	- .0	6.4
$\sigma_T = 1$	-537.3	40.2	.6	5.5	.5	- .5	7.7
$\sigma_B = 1$	2.2	-1.4	- .1	-1.8	.8	- .6	8.0

Table A2.