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A Note on the Thermodynamics of

the NMC PE Model

J. P Gerrity
Development Division

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1. Introduction

The NMC model described in the paper by Shuman and Hovermale (1968) treated the water vapor content of the atmosphere in a highly parameterized manner. The total precipitable water of an atmospheric column was carried as the sole dependent variable reflecting the water content of the atmosphere. The advection of water substance and the condensation process were calculated by using idealized (parameterized) distributions relating water vapor concentration at a particular level to the total precipitable water.

In view of the crude approximations involved in such a treatment of water vapor, it is not surprising that the thermodynamics of the model were not discussed in detail.

More recently, a layered distribution of water vapor has been incorporated into the NMC model. Since the specific humidity is now available to the model, it seemed appropriate to reconsider the formulation of thermodynamical processes in the NMC model. In particular, the omission of the thermodynamical properties of unsaturated water vapor in the ideal gas law and the first law of thermodynamics may be corrected without major modifications in the numerical formulation of the equations.

2. The characteristic equation

It is well known that water vapor behaves in accordance with the ideal gas law provided that the vapor pressure is below saturation. If we consider a unit mass of moist air with specific humidity, q, the use of Dalton's law permits one to express the characteristic equation for the moist air as

$$p = R_d \rho T(1 + 0.61 q) \tag{1}$$

R_d is the gas constant for pure dry air; p is the total pressure of the mixture; ρ is the density of the moist air mixture; T is the temperature of the mixture. The constant 0.61 is calculated from the approximate molecular weights of water vapor and pure dry air.

3. Potential temperature and the hydrostatic equation

In σ coordinates, the hydrostatic equation is expressible in the form,

$$\frac{\partial \phi}{\partial \sigma} = -\alpha \frac{\partial p}{\partial \sigma} \tag{2}$$

ϕ is the geopotential; α is the specific volume of the moist air.

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From the definition of α and eq. (1), one obtains

$$\alpha = R_d \frac{T}{p} (1 + .61\ q) \tag{3}$$

Thus eq. (2) becomes upon use of (3):

$$\frac{\partial \phi}{\partial \sigma} = - R_d T (1 + .61\ q) \frac{\partial \ln p}{\partial \sigma} \tag{4}$$

Now the Exner function, π, and potential temperature, θ, are defined as

$$\theta \pi \equiv T \tag{5}$$

$$\pi \equiv \left(\frac{p}{P}\right)^{\frac{R_d}{c_{p_d}}} \tag{6}$$

in which c_{p_d} is the specific heat at constant pressure of pure dry air and P is a constant with dimensions of pressure.

It is easy to see that

$$\frac{\partial \ln \pi}{\partial \sigma} = \frac{R_d}{c_{p_d}} \frac{\partial \ln p}{\partial \sigma} \tag{7}$$

If (7) is used in (4), one has

$$\frac{\partial \phi}{\partial \sigma} = - c_{p_d} (1 + .61\ q) \frac{T}{\pi} \frac{\partial \pi}{\partial \sigma} \tag{8}$$

and with (5),

$$\frac{\partial \phi}{\partial \sigma} = - c_{p_d} (1 + .61\ q) \theta \frac{\partial \pi}{\partial \sigma} \tag{9}$$

Thus the hydrostatic equation (9) involves a correction term compared to the equation

$$\frac{\partial \phi}{\partial \sigma} = - c_{p_d} \theta \frac{\partial \pi}{\partial \sigma} \tag{10}$$

now used in the NMC model.

The omission of this term may cause only a small error but it is systematic. The correction does not seem to involve any difficulty.

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It may be noted that the horizontal pressure gradient force in the equations of motion should be modified in a similar fashion, viz.,

$$\nabla_\sigma \phi + c_{p_d} \theta (1 + .61\ q)\ \nabla_\sigma \pi \tag{11}$$

in lieu of the present

$$\nabla_\sigma \phi + c_{p_d} \theta\ \nabla_\sigma \pi \tag{12}$$

4. The first law of thermodynamics

If we let ε be the specific entropy of unsaturated moist air, then one has

$$\varepsilon = \varepsilon_o + [q(c_{p_v} - c_{p_d}) + c_{p_d}]\ \ln T$$
$$- [q(R_v - R_d) + R_d]\ \ln p \tag{13}$$

in which ε_o is a constant, c_{p_v} and R_v are the specific heat at constant pressure for water vapor, and R_v is the gas constant for water vapor. According to Richardson (1922)

$$c_{p_v} = 20.08 \times 10^6 \qquad R_v = 4.616 \times 10^6$$

$$c_{p_d} = 9.92 \times 10^6 \qquad R_d = 2.870 \times 10^6$$

in cgs units.

If we define

$$\theta^* = T\left(\frac{p}{P}\right)^{-\left(\frac{R_v - R_d}{c_{p_v} - c_{p_d}}\right)} \tag{14}$$

one may derive from (13) and the previous definition of potential temperature (eq. 5),

$$\frac{d\varepsilon}{dt} = c_{p_d}\frac{d\ \ln \theta}{dt} + \left(c_{p_v} - c_{p_d}\right)q\ \frac{d\ \ln \theta^*}{dt} \tag{15}$$

Note that $\frac{dq}{dt} = 0$ in the unsaturated case.

Let us denote the change in specific entropy following the motion of a moist parcel by

$$\frac{d\varepsilon}{dt} = \frac{\overset{\circ}{Q}}{T} \tag{16}$$

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In which $\dot{Q}$ is the rate at which heat is being added to the parcel. The eq. 15 may then be written as

$$\frac{d\theta}{dt} = \frac{\theta}{c_{P_d}T}\,\dot{Q} - \frac{c_{P_V} - c_{P_d}}{c_{P_d}}\,q\,\frac{\theta}{\theta^*}\,\frac{d\theta^*}{dt} \tag{17}$$

The second term on the right hand side of (17) is omitted in the present model formulation.

By use of the definitions of θ and θ^*, one may show that

$$\frac{\theta}{\theta^*}\frac{d\theta^*}{dt} = \frac{d\theta}{dt} + \left[\frac{R_d}{c_{P_d}} - \frac{R_v - R_d}{c_{P_v} - c_{P_d}}\right]\theta\frac{d\ln p}{dt} \tag{18}$$

So eq. (17) becomes,

$$\left[1 + \frac{c_{P_V} - c_{P_d}}{c_{P_d}}\,q\right]\frac{d\theta}{dt} = \frac{\theta}{c_{P_d}\,T}\,\dot{Q} - \frac{R_d c_{P_V} - R_v c_{P_d}}{(c_{P_d})^2}\,q\,\theta\,\frac{d\ln p}{dt} \tag{19}$$

Using the cited values for the thermodynamical constants, one has

$$(1 + 1.02\,q)\frac{d\theta}{dt} \simeq \frac{\theta}{c_{P_d}T}\,\dot{Q} - (.12)q\,\theta\,\frac{d\ln p}{dt} \tag{20}$$

The second term on the right hand side of (20) is usually less than

$$7.2 \times 10^{-6}\ \text{deg/sec}$$

or about .6 deg/day, and is probably neglected with justification.

It would appear to be adequate to introduce the following form for the first law of thermodynamics in the unsaturated state:

$$\frac{d\theta}{dt} \simeq \frac{\theta}{c_{P_d}(1+q)T}\,\dot{Q} \tag{21}$$

5. Concluding remarks

The modified equations suggested for use in the NMC model are

$$\frac{\partial\phi}{\partial\sigma} = - c_{P_d}[1 + .61\,q]\theta\,\frac{\partial\pi}{\partial\sigma}$$

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Pressure gradient $= \nabla_\sigma \phi + c_{p_d}[1 + .61\ q]\theta\ \nabla_\sigma \pi$

and

$$\frac{d\theta}{dt} = \frac{\theta}{c_{p_d}(1+q)T}\ \dot{Q}$$

There is no apparent reason why these terms might not be incorporated in the NMC model.