

U. S. DEPARTMENT OF COMMERCE  
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION  
NATIONAL WEATHER SERVICE  
NATIONAL METEOROLOGICAL CENTER

OFFICE NOTE 69

On the Longitudinal Smoothing of the Tendency Fields  
in the Eight-Layer Hemispheric PE Model (HEMPEP/8)

(A Chronicle)

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JANUARY 1972

## The General Problem

In a latitude-longitude coordinate system, with the mathematical pole located at the geographic pole, the convergence of the meridians (as one moves northward) brings the grid points sufficiently close together that the CFL linear stability criterion (1)  $\Delta x/\Delta t > c$  ( $\Delta x = r \cos \delta\lambda$ ) becomes impossible to satisfy without an inordinately short time step  $\Delta t$ . A number of solutions to this problem have been proposed which fall into two broad categories: 1) one may simply delete grid points from the grid as one moves north (Kurihara, (2)) such as to maintain an approximate constancy to the geographic distance between grid points or 2) one may perform some sort of smoothing to either the data fields or the tendency fields to suppress the growth of CFL unstable waves.

There are rather severe difficulties associated with the solutions of category 1 which have been well explored (Shuman (3), Dey (4)); it is some of the less explored problems associated with solution category 2 to which I wish to turn my attention. In particular, we shall investigate some of the apparent consequences of the smoothing of the tendency fields by the original method proposed for the Hemispheric eight-layer PE model and offer some possible solutions to the difficulties encountered.

## A Digression - Fourier Transforms

The most convenient method of talking about smoothing and its effects is in terms of Fourier transforms. This is because "smoothing" is more properly (at least, in Fourier type terms) referred to as convolution. And there are all sorts of nice theorems dealing with convolution. (In this digression, we shall ignore all those problems that excite mathematicians like integrability, and convergence, and existence, and limits, and discontinuities, and pretend that all the needed conditions are met.)

If  $f(x)$   $g(x)$  are functions of (nondimensional space)  $x$ ,  $F(s)$  and  $G(s)$  their transforms in frequency space ( $s=1/x$ ), and convolution is

$$f(x)*g(x) \equiv \int_{-\infty}^{\infty} f(u) g(x-u)du$$

then some useful functions and their transforms are

$$f(x)$$

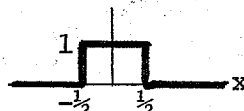
$$F(s)$$

$$f(ax)$$

$$\frac{1}{|a|} F\left(\frac{s}{a}\right)$$

$$f(x)*g(x)$$

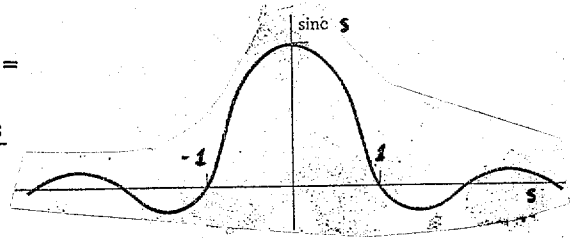
$$F(s) \cdot G(s)$$

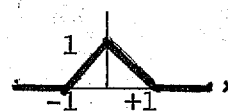
$$\Pi = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$


(rectangle for smoothing)  
(running mean)

$$\text{sinc}(s) =$$

$$\frac{\sin \pi s}{\pi s}$$

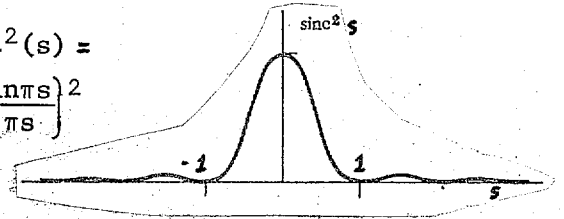


$$\Lambda = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$


(triangle for smoothing)  $(\text{---})^{xx}$

$$\text{sinc}^2(s) =$$

$$\left(\frac{\sin \pi s}{\pi s}\right)^2$$



These various transforms will go either way, i.e.  $x$  and  $s$  can be interchanged as needed for any particular problem. (The notation, nomenclature and lots of results are from R. Bracewell (5).)

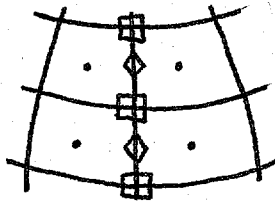
End of introductory digression.

### Our Particular Problem

In terms of the Fourier transforms, let us consider what we are doing in the latitude-longitude grid, to suppress CFL instabilities.

The basic operation we are performing is interpolation; we have tendency computations at the "centers" of grid boxes and we wish to interpolate them to the meridians and thence to the grid points on the latitude circles. See Fig. 1.

Fig. 1:



The tendency values are at the points - we interpolate in longitude to the  $\diamond$  points and then in latitude to the  $\square$  points. (Interpolation is just a variation on convolution--no trouble there.)

No particular difficulty accrues to the interpolation in the north-south direction--the grid rows retain the same geographic separation everywhere on the globe and the analog to a  $(\ )^y$  operation, convolution of the data with a  $\frac{1}{2}\Pi(\frac{x}{2})$  function, seems sufficient to perform the interpolation. We have assumed that the grid row separation has the value unity--we shall maintain this assumption--the grid column separation therefore has the value  $\cos \phi$  ( $\phi$ = latitude) and that is the crux of our problem.

In order to maintain CFL stability, we wish to increase the east-west averaging in terms of grid points such that the effective averaging length (not precisely defined as yet) remains the same with respect to the earth i.e. a grid averaging length which varies as  $\sec \phi$ .

The first of a number of interpolators tested was of the form

$$\frac{1}{2 \sec \phi} \Pi \left( \frac{x}{2 \sec \phi} \right)$$

i.e. a rectangular function which expands as the secant of latitude.

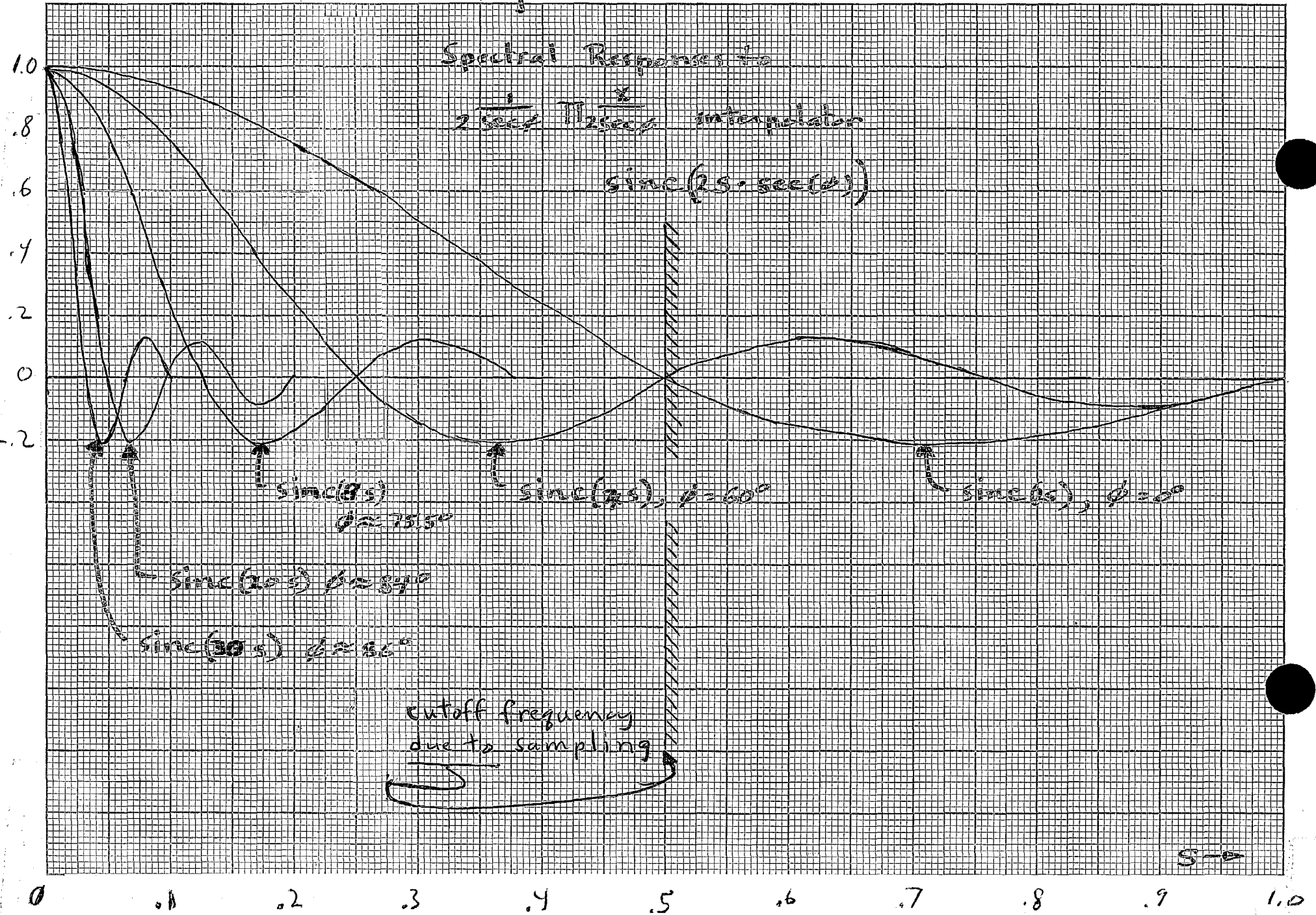
Fig. 2 is a collection of sketches of its response function at various latitudes as a function of grid length frequency. A value of  $s$  of 0.5 represents the two grid length wave, which wave has a shorter geometric length the greater the latitude, and is also the highest frequency wave resolvable by the grid. In this respect, the portions of the curves to the right of  $s = 0.5$  are mythical--there are no waves out there to be filtered.

But consider what is happening to the left of the cutoff frequency--at the equator the shortest (in geometric terms) resolvable wave is the 2 grid length wave, a wave length of approximately 560 km. Presumably, we wish to suppress waves shorter than this length further north even though such shorter waves became resolvable as the meridians converge. This wave, the equatorial equivalent two grid wave, falls at the first zero on the  $\text{sinc}(2s \cdot \sec(\phi))$  curves of Fig. 2. It is immediately evident that waves of frequencies higher than the equatorial equivalent two-grid wave are not particularly well-filtered by the rectangular interpolator especially near the pole. Presumably, such higher than desirable frequencies will be generated by the nonlinearities of the equations--what will become of them is a matter of experiment.

Fig 2

Spectral Responses to  
 $\frac{1}{2 \text{ sec}} \Pi_{\frac{x}{2 \text{ sec}}}$  interpolator

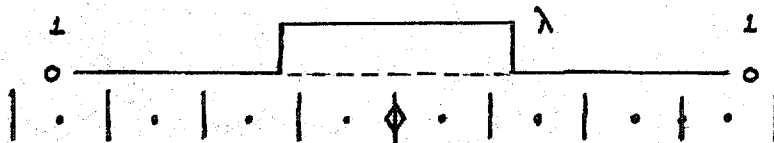
$\text{sinc}(2s \cdot \text{sec}(\phi))$



Another way to write down and compute the form of the interpolating function is to consider a number  $N$ : its integer part shall be the number of tendency values (in the boxes) given unit weight on either side of the meridian being averaged to and its fractional part shall be the weight given the next tendency value out. This is a rectangular interpolator with wings.

The simplest one is:  $N = \sec\phi$ .

At the equator, this average combines the two adjacent boxes, at  $60^\circ\text{N}$  the four neighboring boxes with equal weight, and at  $86.25^\circ\text{N} = 15.29$ , i.e. the interpolator reaches out some 32 grid lengths ( $80^\circ$  longitude over all at the northernmost latitude at which this averaging is performed. Thirty tendencies receive equal unit weight--the two ultimate ones weights of 0.29 each. A geometric interpretation of the interpolator is



and boxes fully under the bar  $\lambda$  get equal weight--those partially covered get that fraction of coverage.

As one might have anticipated from the discussion of the spectral response, difficulties were encountered in the use of the rectangular interpolator. But what was surprising was the nature of the problem and the dilemma that it introduced.

A series of forecasts using a  $2.5^\circ$  barotropic model illustrates the problem. Fig. 3 shows the 72 hr forecast of the  $v$  component of the wind at the grid row just south of the pole for three different interpolators. For the first of these (Fig. 1a), the  $N=1.\sec\phi$  rectangular interpolator previously introduced, it is evident that there is trouble - 300 m/s winds seem a bit excessive. However, somewhat more interesting is the appearance in the data of two periodicities--one with a wave length of  $N$  and another with wave length  $2N$  grid lengths. It seems that the long reach of the interpolator somehow excites waves in the forecasts (after sufficient time--the 48 hr forecasts show none such at all) with a wave length equal to the interpolating length and at least its first harmonic.

What is particularly curious is that these two periodicities fall as nearly as one can tell at the first two zeros of the response function--the filtering interpolator should be eliminating this wave and yet it is

72 hours after 12Z 19 Feb 70

Tropical NEP/ NEP/36

m/s  
300  
200  
100  
0  
-100  
-200  
-300  
20  
10  
0  
-10  
-20  
20  
10  
0  
-10  
-20

Fig 3

72 hour forecast v wind component at first Row south of Pak

a)

b)

c)

N = 5.66  
②  $\phi = 86.25$   $N = 15.29$

N = 0.7511 sec(0)  
②  $\phi = 86.25$   $N = 10.7$

N = 0.5 sec(0.5)  
②  $\phi = 86.25$   $N = 8.15$

X → (grid points)

10 20 30 40 50 60 70 80 90 100 110 120 130 140 150

KE 10 X 10 TO 1/2 INCH 47 1320 MADE IN U.S.A. KEUFFEL & ESSER CO.

somehow exciting it. Presumably this must be related to our filtering the tendencies rather than the quantities themselves.

That the excited waves are related to the length of the interpolator is borne out by the results of using a variation on the rectangular interpolator--one defined by

$$N = \sec\phi / \sec\phi_0$$

where  $\phi_0$  is some latitude, north of which more than just the adjacent boxes are combined in the same integer and fractional sense as before, while south of  $\phi_0$  only the two contiguous boxes are averaged together. This will have the effect of reducing the averaging length with the amount of reduction dependent upon the base latitude,  $\phi_0$ .

In terms of the response function, this reduction of the filter length will expand the sinc(s) curves, moving the zero points to the right in Fig. 2. Fig. 1b shows the same v wind for a 72 hr forecast with  $\phi_0 = 45^\circ$  giving

$$N = 0.7071 \sec\phi$$

The winds no longer have excessive magnitude but the spacial variation is there again, and again shows periods of N and 2N wavelengths--quite undesirable, and again at the zero points of the shifted response function.

It is important to note that these short wavelengths and excessive velocity winds were limited to just the two or three grid rows nearest to the pole. This of course suggests that a further shrinking of the averaging length, by selecting a larger value of  $\phi_0$ , should have a beneficial effect upon the polar winds. A forecast was attempted using  $N=0.5 \sec\phi$  (equivalent to  $\phi_0=60^\circ$ ) and we were impaled on the other horn of our cow (her name is Dilemma) - the CFL criteria was violated and the forecast went unstable independently of any peculiarly polar problems.

Back to the drawing board.

From the board we produced a trapezoidal sort of averaging device:

$$N = 0.5 \sec\phi + 0.5$$

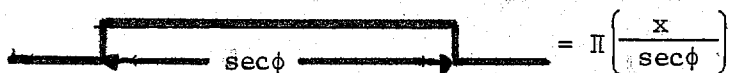
with the same integer and fractional interpretation as before.

In terms of the Fourier forms introduced earlier, this trapezoidal interpolator has the form

$$\frac{1}{\sec\phi} \Pi\left(\frac{x}{\sec\phi}\right) * \frac{1}{2} \Pi\left(\frac{x}{2}\right)$$

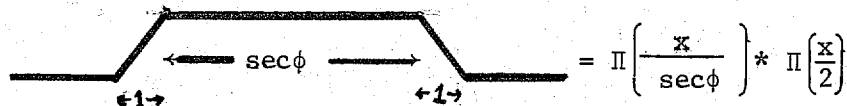


i.e. the convolution of a rectangle function which expands as the secant of latitude with a simple two point rectangle function. The effect of the convolution is to change the interpolator function from this



$$= \Pi \left( \frac{x}{\sec \phi} \right)$$

to this



$$= \Pi \left( \frac{x}{\sec \phi} \right) * \Pi \left( \frac{x}{2} \right)$$

putting triangle wings on the original rectangle. Note that the rectangle portion of the trapezoid (the portion that expands) is half the width of the rectangle previously used. This is the same width as the  $N=0.5 \sec \phi$  interpolator that did not suppress the CFL instabilities--however, as we shall see the addition of the trapezoidal sides allowed for a successful forecast. But before that, let's take a look at the spectral characteristics of our compound interpolator.

Since convolution in one domain is equivalent to multiplication in the other, we can use our rules from our digression on transforms to immediately write down the spectral character of the interpolator as

$$\text{sinc}(s \sec(\phi)) \cdot \text{sinc}(2 s)$$

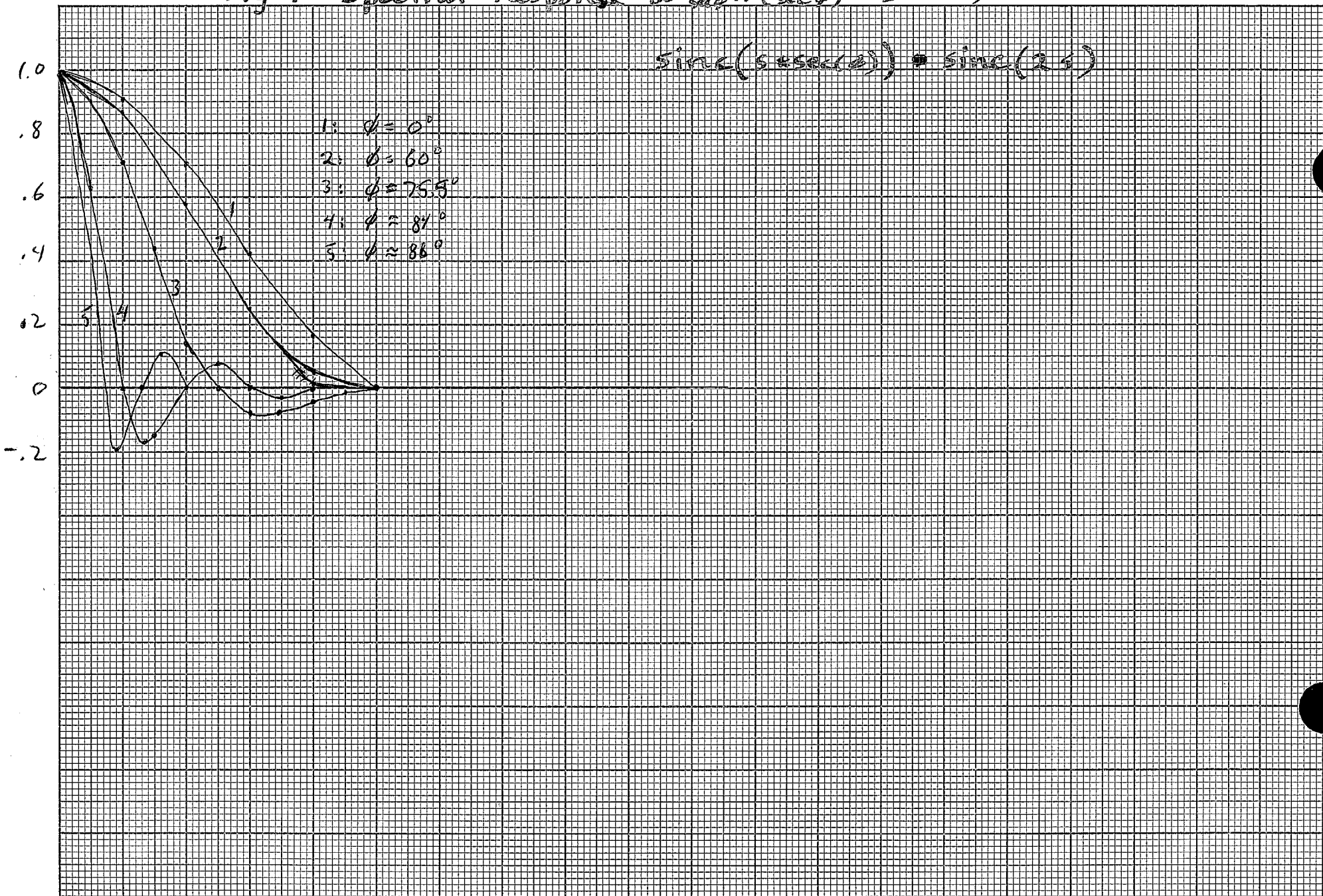
Fig. 4 contains some sketches of the responses for selected values of  $\phi$ . I have not bothered to extend the curves beyond the cutoff frequency which, please note, has been approached by the zeros of the response function--this is a simple reflection of diminishing the length of the interpolator: the response function widens in the frequency domain. We can note in the figure in comparison to Fig. 2 that the responses at the higher latitudes are little changed other than being shifted to higher frequencies while the responses from  $60^\circ\text{S}$  are appreciably altered--principally by the elimination of the negative lobe.

The very encouraging result of the 72 hr forecast with the trapezoidal interpolator is the third portion of Fig. 1: no trace of 8 or 16 grid length waves are to be seen and the wind values are quite proper. That this improvement is due to the form of the interpolator and not due to the slightly smaller extreme averaging length ( $N=8.15$  vs  $N=10.7$  for the  $0.7071 \sec \phi$  case) is evidenced by a run made with  $N=0.543 \sec \phi$  - the  $\phi_0$  selected so that the maximum, near polar, averaging length would be the same, 8.15, as the

Fig 4 Spectral Response to  $\frac{1}{\sec \phi} \Pi\left(\frac{x}{\sec \phi}\right) * \frac{1}{2} \Pi\left(\frac{x}{2}\right)$

$$\text{sinc}(s \sec \phi) * \text{sinc}(2s)$$

- 1:  $\phi = 0^\circ$
- 2:  $\phi = 60^\circ$
- 3:  $\phi = 75.5^\circ$
- 4:  $\phi \approx 84^\circ$
- 5:  $\phi \approx 86^\circ$



0 .1 .2 .3 .4 .5 s → cycles/grd length

trapezoidal interpolation has there. This run succumbed to CFL instabilities within the 6th hour of its forecast. The explanation is simple - the trapezoidal interpolator is wider farther south than the rectangular one and the width is needed.

A certain degree of enthusiasm might well be justified by the results of this series of experiments--and it would seem we are comfortably positioned on Dilemma's head, firmly grasping her horns, forging ahead. Dilemma promptly grew a third horn (labeled "Baroclinic Model") midway between the others.

Incorporating the trapezoidal interpolator into the eight-layer model did not prevent the model from blowing up (numbers too large for the computer, etc.) after all of two hours of forecast time. The instabilities were most pronounced near the pole. As an uneconomic means of getting something out, the time step was reduced to 7.5 min with the result that the forecast survived to about hour 37 with the severe instabilities first appearing near the pole. Fig. 5 illustrates the situation with the near polar v wind component in the topmost layer of the model at hour 36. Although 8.15 or 16.3 grid length waves are not as clear cut as were the average-excited waves with the rectangle smoother in the barotropic model, things are most assuredly not what they ought to be. Why?

About all that one can do is speculate that the additional layers afford more degrees of freedom in the model (vis a vis the barotropic well-behaved case) and hence more modes of response to the filter. And the numerics proceed to respond. Badly. Dilemma is still tossing her head about.

Back to the drawing board.

And off the board we pull a triangular averager with the form

$$w = 1 - \frac{x}{L \sec(\phi)} \quad x = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2} \leq L \sec\phi$$

x is in grid length units, the values x takes on being the location of the tendencies and the L in the denominator allows us to make the triangle as wide or narrow as we please.

This interpolator was tested first on the barotropic model (with L = 1) and as might have been anticipated, worked just fine - there was no trace of interpolator excited polar waves.

Not without some anxiety, the triangular interpolator was introduced into the baroclinic model (with L=1). With a 10 min time step, the model

m/s

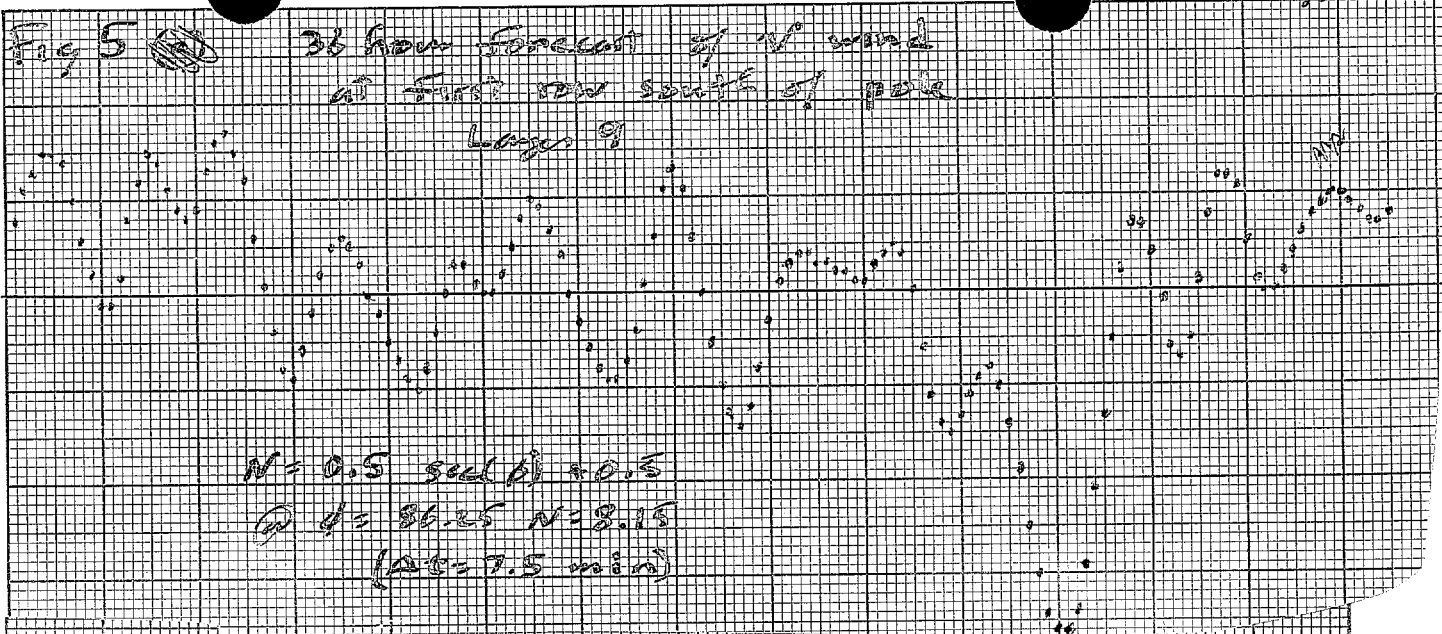
36 hours after 12 Z 18 Feb 70

Barclimic  
NEP/36

Fig 5

36 hour forecast of wind  
at first row south of pole  
Layer 9

20  
10  
0  
-10  
-20  
-30



$N = 0.5$   $gull \pi = 0.5$   
 $\phi = 86.25$   $N = 8.15$   
 $(\Delta t = 7.5 \text{ min})$

x -> (gridpoints)

10 20 30 40 50 60 70 80 90 100 110 120 130 140 150

succumbed to linear instabilities before the third hour was reached but they were in no way polar related.

Upon reflection, this was not so surprising. Fig. 6 illustrates the spectral response function of the triangular interpolator

$$W = \frac{1}{L \sec \phi} \Lambda \left\{ \frac{x}{L \sec \phi} \right\}$$

which is

$$R = \text{sinc}^2(L \sec \phi s)$$

for  $L=1$  and the same selection of latitude values used previously. Comparison of Fig. 6 with Fig. 2 and Fig. 4 suggests that we should better use a triangular interpolator with  $L=2$ . This will shrink the response functions, moving their zero points to the left by a factor of 0.5. Also the Fourier sketches show that the spectral response of a triangular filter is such that the triangle must be twice as wide as a rectangle in order for the response function to drop to its first zero at the same frequency as the rectangular response function. An earlier run of the baroclinic model with a rectangular interpolator with half width equal to  $\sec(\phi)$  had run to 42 hours with 10 min time steps before blowing out at the pole. This was sufficiently encouraging to try the triangle again with  $L=2$ .

In spite of the near polar reach of the triangle being some 62 grid points ( $155^\circ$  of longitude), and using a 10 min time step, no polar problems were encountered at all--the forecast ran just fine to 48 hours. See Fig. 7 for the near polar v wind component at 48 hours.

It would seem that the difficulties with the averager-interpolator have been resolved and we can go on to meteorology.

The solution was not free--the use of the variable weight interpolator, i.e. the triangle, increased the running time of the model some 40 to 50%--these problems will be neglected for the time being. At least the model runs.

An engineering approach to helping the running time is to reduce the triangle width--a run with  $L=1.5$  ran some 10% faster than  $L=2.0$  and behaved just as well.

As to why the triangular interpolator worked where the others didn't, about all one can speculate is that the absence of negative (phase changing) lobes in the triangle's response function made the difference.

But it works and that's enough for now.

Fig 6

Spectral Response to  
triangular  $\frac{1}{\text{sinc}} \left( \frac{\pi}{\text{sinc}} \right)$  filter

$$\text{sinc}^2(s \text{ sec } \theta)$$

- 1:  $\theta = 0^\circ$
- 2:  $\theta = 60^\circ$
- 3:  $\theta = 75.5^\circ$
- 4:  $\theta = 84^\circ$

sampling  
cutoff  
frequency

0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0  $s \rightarrow$

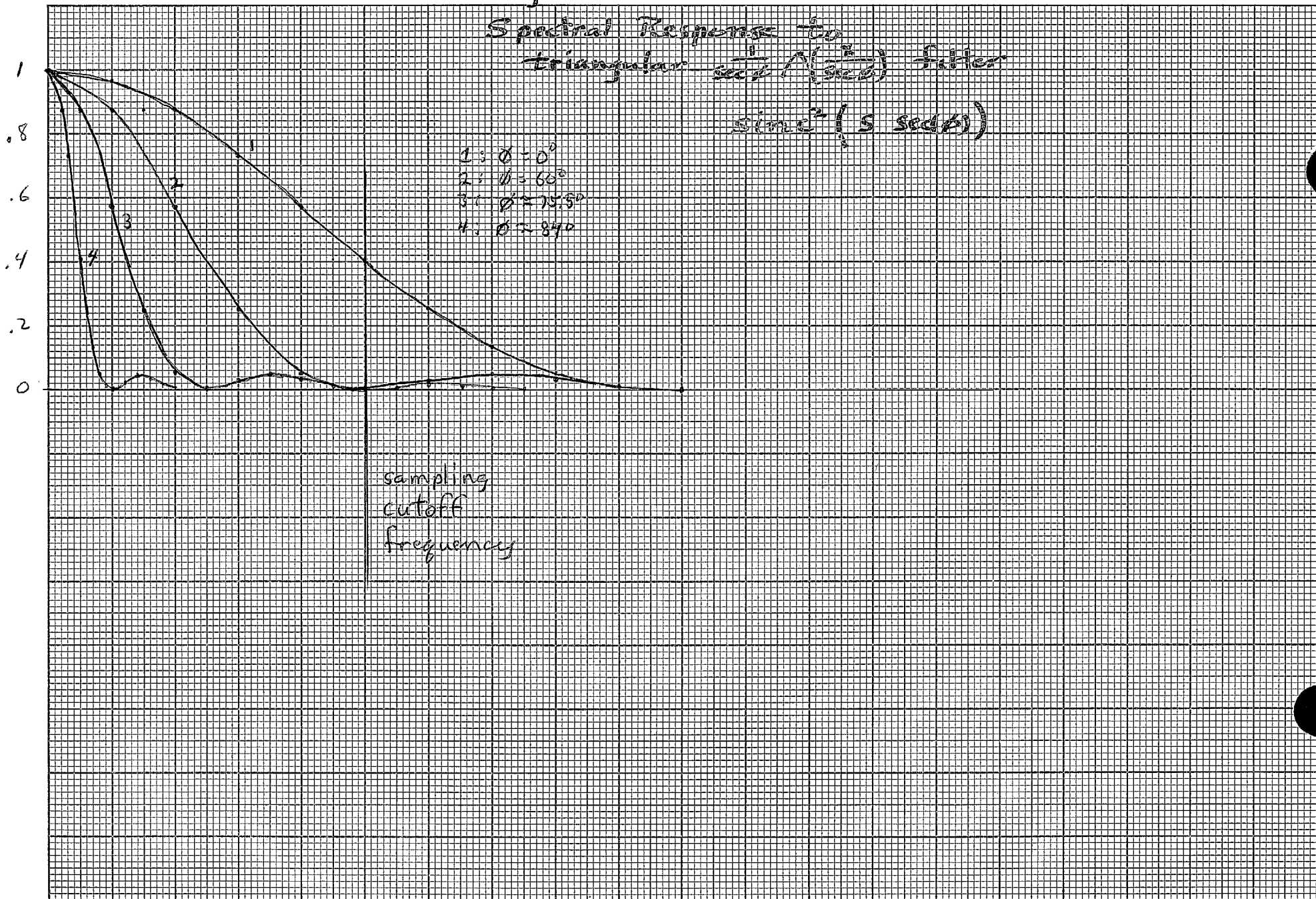
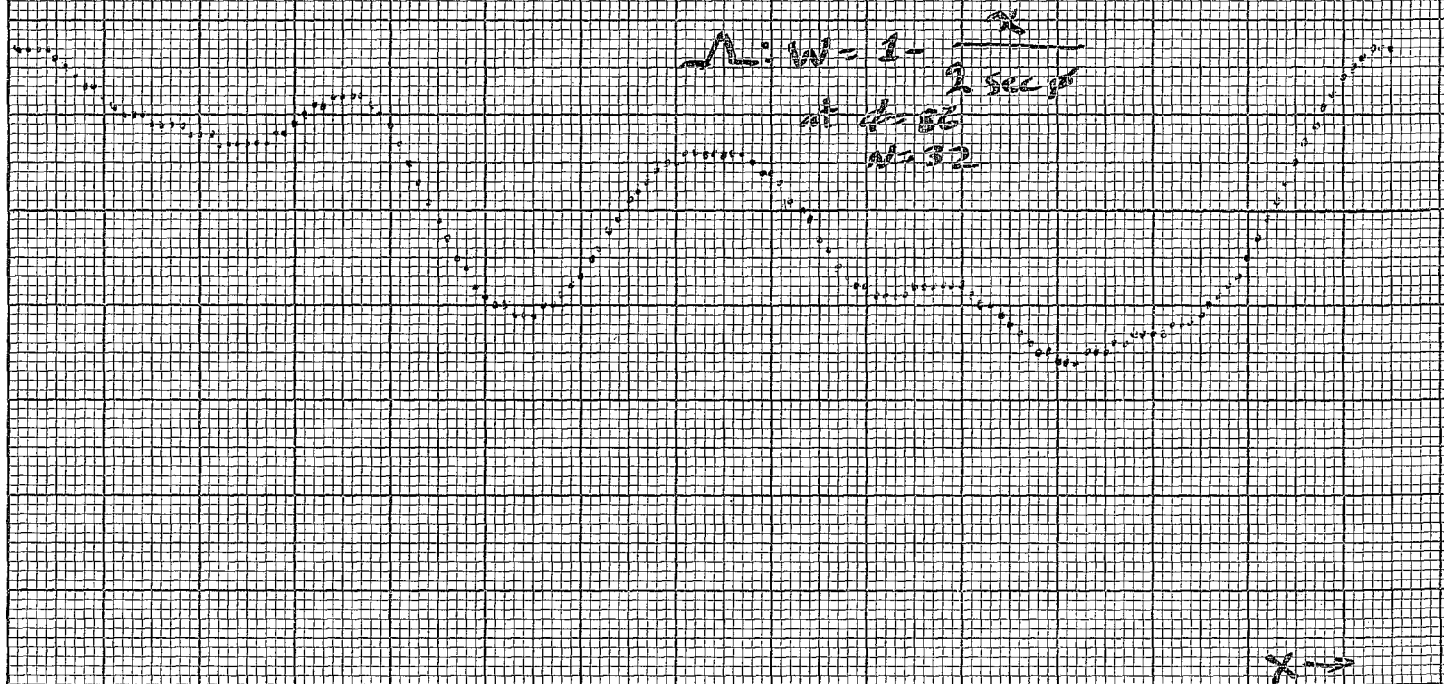




Fig 7 48 hours after 12Z 18 FEB 70  
Baroclinic wave  
V wind component, First row  
south of pole, Layer 9.

20  
10  
0  
-10  
-20



50 60 70 80 90 100 110 120 130 140 150  
grid points

### References

- (1) Courant, Friedrichs & Levy, 1928: Math. Ann., 100: 32-74.
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