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A SEMI-IMPLICIT INTEGRATION SCHEME FOR BAROCLINIC MODELS

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1. Introduction

In an effort to reduce the large amount of computing time necessary to carry out an integration of primitive-equation numerical weather prediction models, an effort was initiated at NMC in the summer of 1969 to examine the feasibility of employing a semi-implicit integration method with such models. At that time, Soviet scientists under the direction of Marchuk had successfully applied this technique to a primitive-equation model (9). Robert, in Canada, had adapted this idea for the integration of a spectral model (12), and had begun work on a finite-difference primitive-equation barotropic model. The latter application turned out to be successful also, and was subsequently documented in the literature (Kwizak and Robert, 8).

At Robert's suggestion, the NMC effort began with the development of a primitive-equation barotropic model integrated by the semi-implicit method. The results were encouraging: the method allowed a stable integration with a one-hour time step, whereas an explicit "leap-frog" method allowed only a ten-minute time step. The method requires the solution of a boundary-value problem at each time step, so that the actual time advantage turned out to be something like 4:1 rather than 6:1. These experiments were documented in a paper by McPherson (10). We subsequently combined the method with a staggered spatial lattice, which offers additional computational economy (Gerrity and McPherson, 7).

The potential time advantage offered by the semi-implicit method suggested that its introduction into a multilayer model would effect a significant reduction in computation time. Following a suggestion by Shuman, an effort was begun to extend the technique to a simple baroclinic model as a feasibility study. A series of investigations of alternative methods of implicit integration and alternative specifications of the vertical coordinate followed (1,2,3,4,5,6,11,14). We eventually were led to the formulation presented in this note.

We chose a two-layer representation of the vertical structure, which allows two free modes, one external and one internal. This vertical structure has the advantage of economy over a model with many layers, but suffers some disadvantages as well. For example, only one static stability and one estimate of vertical motion is permitted. It is therefore unreasonable to anticipate that the model can represent baroclinic processes with any skill, although it should capture the barotropic aspects of the mass and motion fields. However, it must be remembered that the primary purpose of this model is to demonstrate the feasibility of the semi-implicit integration method applied to a model which allows both external and internal gravitational oscillations. We therefore determined *a priori* that a successful conclusion of this experiment would consist of a numerically stable calculation using a time step greater than that allowed by an explicit method, with the meteorological aspects of the integration similar to what might be expected from a barotropic model.

The purpose of this note is to demonstrate a stable integration of a baroclinic model carried out in one-hour time steps by employing the semi-implicit method, and to compare that integration with a near-analogous model, but integrated explicitly. An effort is made to construct both versions as nearly alike as possible.

In the next section, the basic differential equations are presented and manipulated to a somewhat more convenient form. Next, the vertical and horizontal structure of the grid lattice is introduced, and difference equations for an explicit "leapfrog" scheme are presented. The fourth section outlines the modification of the difference equations to incorporate the semi-implicit method, and it is followed by a brief description of the lateral boundary conditions. The last two sections deal with the initialization problem for a rather specialized set of data, and the results of an integration of both explicit and semi-implicit versions of the model.

2. Basic Equations

The equations of motion, thermodynamics, and continuity may be written for a general vertical coordinate following Shuman and Hovermale (16) as

$$\frac{\partial u}{\partial t} + m\left(\frac{\partial \phi}{\partial x} + c_p\Theta\frac{\partial \pi}{\partial x}\right) - \hat{f}\, v + m\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \dot{\sigma}\frac{\partial u}{\partial \sigma} = 0 \qquad 2.1$$

$$\frac{\partial v}{\partial t} + m\left(\frac{\partial \phi}{\partial y} + c_p\Theta\frac{\partial \pi}{\partial y}\right) + \hat{f}\, u + m\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \dot{\sigma}\frac{\partial v}{\partial \sigma} = 0 \qquad 2.2$$

$$\frac{\partial \phi}{\partial \sigma} + c_p\Theta\frac{\partial \pi}{\partial \sigma} = 0 \qquad 2.3$$

$$\frac{\partial \Theta}{\partial t} + m\left(u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y}\right) + \dot{\sigma}\frac{\partial \Theta}{\partial \sigma} = 0 \qquad 2.4$$

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \sigma}\right) + \frac{\partial}{\partial \sigma}\left(\dot{\sigma}\frac{\partial p}{\partial \sigma}\right) + m^2\left[\frac{\partial}{\partial x}\left(\frac{u}{m}\,\frac{\partial p}{\partial \sigma}\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\,\frac{\partial p}{\partial \sigma}\right)\right] = 0 \qquad 2.5$$

where the hydrostatic approximation has been employed to obtain eqn. (2.3). The notation in the foregoing set of equations is standard, but it may be helpful to state the following definitions:

$$\hat{f} \equiv f - v\frac{\partial m}{\partial x} + u\frac{\partial m}{\partial y} \qquad 2.6$$

$$\pi \equiv \left(\frac{p}{P_*}\right)^{\kappa} \qquad 2.7$$

$\kappa = R/c_p$, P_* is a reference pressure,

$$\Theta = T\,\pi^{-1} \qquad 2.8$$

Together with a definition of σ, which we will take as

$$\sigma \equiv p/p_* \qquad 2.9$$

where p_* is the surface pressure, and an equation in $\dot\sigma\left(\equiv\frac{d\sigma}{dt}\right)$ which will presently be derived, this is a closed system.

It is convenient, however, to manipulate this system somewhat; in particular, to obtain a continuity equation in the same dependent variable as appears in the nonlinear parts of the pressure-gradient terms of the momentum equations. In order to do this, it is first recognized that, by virtue of the definition of σ (eqn. 2.9),

$$\frac{\partial p}{\partial \sigma} = p_* \ . \qquad 2.10$$

The continuity equation (2.5) then becomes

$$\frac{\partial p_*}{\partial t} + p_*\frac{\partial \dot\sigma}{\partial \sigma} + m^2\left[\frac{\partial}{\partial x}\left(\frac{u}{m}\,p_*\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\,p_*\right)\right] = 0 \ . \qquad 2.11$$

Next, p is replaced in eqn. (2.7) using eqn. (2.9),

$$\pi = \left(\frac{\sigma p_*}{P_*}\right)^{\kappa} , \qquad 2.12$$

so that gradients of π become

$$\frac{\partial \pi}{\partial x} = \frac{\sigma^{\kappa}}{P_*^{\kappa}}\frac{\partial (p_*)^{\kappa}}{\partial x} \qquad 2.13$$

and

$$\frac{\partial \pi}{\partial \sigma} = \kappa\left(\frac{p_*}{P_*}\right)^{\kappa} \sigma^{\kappa-1} \ . \qquad 2.14$$

At this point, it is convenient to introduce a new variable q,

$$q \equiv p_*^{\kappa} \qquad 2.15$$

so that eqns. (2.13) and (2.14) become

$$\frac{\partial \pi}{\partial x} = \left(\frac{\sigma}{P_*}\right)^{\kappa} \frac{\partial q}{\partial x} \qquad 2.16$$

and

$$\frac{\partial \pi}{\partial \sigma} = \kappa\ \sigma^{-1} \left(\frac{\sigma}{P_*}\right)^{\kappa} q \qquad 2.17$$

In order to transform the continuity equation (2.11) into an expression in the new variable q, it will be noted that if eqn. (2.11) is multiplied by $(\kappa p_*^{\kappa-1})$, the terms can be combined to write

$$\frac{\partial q}{\partial t} + \kappa q \frac{\partial \dot{\sigma}}{\partial \sigma} + \kappa q m^2 \left[\frac{\partial}{\partial x}\left(\frac{u}{m}\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\right)\right] + m\left[u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y}\right] = 0 \qquad 2.18$$

Now, one of the principal virtues of the σ-vertical coordinate is that the upper and lower boundaries are material surfaces, so that the boundary conditions are $\dot{\sigma} \equiv 0$. In view of this, and the fact that q is not a function of σ, eqn. (2.18) may be integrated between σ = 1,0 to obtain

$$\frac{\partial q}{\partial t} + \kappa q m^2 \left[\frac{\partial}{\partial x}\left(\frac{\overline{u}^{\sigma}}{m}\right) + \frac{\partial}{\partial y}\left(\frac{\overline{v}^{\sigma}}{m}\right)\right] + m\left[\overline{u}^{\sigma} \frac{\partial q}{\partial x} + \overline{v}^{\sigma} \frac{\partial q}{\partial y}\right] = 0 \qquad 2.19$$

where $\overline{u}^{\sigma} \equiv \int_{\sigma=0}^{\sigma=1} u d\sigma$. Similarly, eqn. (2.18) may be differentiated with respect to σ to obtain

$$\frac{\partial^2 \dot{\sigma}}{\partial \sigma^2} + m^2 \frac{\partial}{\partial \sigma} \left[\frac{\partial}{\partial x}\left(\frac{u}{m}\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\right)\right] + \frac{m}{\kappa q}\left[\left(\frac{\partial u}{\partial \sigma}\right)\frac{\partial q}{\partial x} + \left(\frac{\partial v}{\partial \sigma}\right)\frac{\partial q}{\partial y}\right] = 0 \qquad 2.20$$

which is the desired diagnostic equation in $\dot{\sigma}$.

With equations (2.16) and (2.17) employed as appropriate, the basic system of equations becomes, with

$$\Upsilon \equiv c_p \left(\frac{\sigma}{P_*}\right)^{\kappa}, \text{ and } \gamma \equiv \frac{\kappa \Upsilon}{\sigma} = \frac{\partial \Upsilon}{\partial \sigma},$$

$$\frac{\partial}{\partial t}\left(\frac{u}{m}\right) + \frac{\partial \phi}{\partial x} + \Upsilon \Theta \frac{\partial q}{\partial x} - \left(\frac{v}{m}\right)\hat{f} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\dot{\sigma}}{m}\frac{\partial u}{\partial \sigma} = 0 \qquad 2.21$$

$$\frac{\partial}{\partial t}\left(\frac{v}{m}\right) + \frac{\partial \phi}{\partial y} + \gamma\Theta\frac{\partial q}{\partial y} + \left(\frac{u}{m}\right)\hat{f} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\dot{\sigma}}{m}\frac{\partial v}{\partial \sigma} = 0 \qquad 2.22$$

$$\frac{\partial \phi}{\partial \sigma} + \alpha\Theta q = 0 \qquad 2.23$$

$$\frac{\partial \Theta}{\partial t} + m\left(u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y}\right) + \dot{\sigma}\frac{\partial \Theta}{\partial \sigma} = 0 \qquad 2.24$$

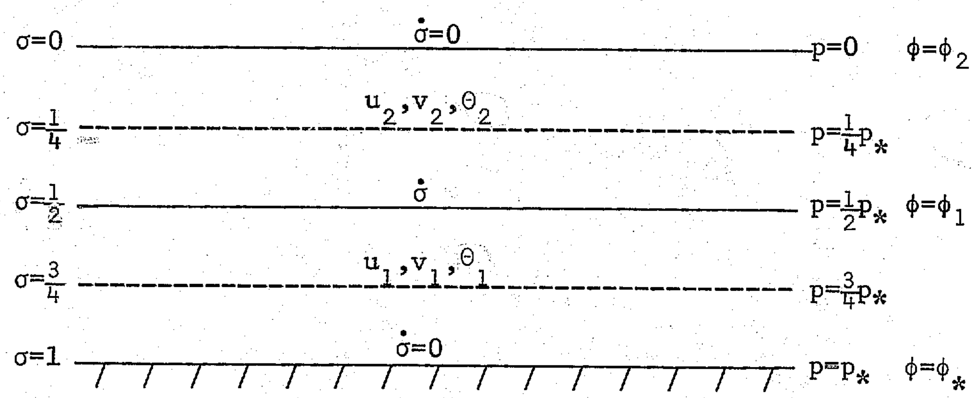
$$\frac{\partial q}{\partial t} + \kappa q m^2\left[\frac{\partial}{\partial x}\overline{\left(\frac{u}{m}\right)}^{\sigma} + \frac{\partial}{\partial y}\overline{\left(\frac{v}{m}\right)}^{\sigma}\right] + m\left[\overline{u}^{\sigma}\frac{\partial q}{\partial x} + \overline{v}^{\sigma}\frac{\partial q}{\partial y}\right] = 0 \qquad 2.25$$

$$\frac{\partial^2 \dot{\sigma}}{\partial \sigma^2} + m^2\frac{\partial}{\partial \sigma}\left[\frac{\partial}{\partial x}\left(\frac{u}{m}\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\right)\right] + \frac{m}{\kappa q}\left[\left(\frac{\partial u}{\partial \sigma}\right)\frac{\partial q}{\partial x} + \left(\frac{\partial v}{\partial \sigma}\right)\frac{\partial q}{\partial y}\right] = 0 \qquad 2.26$$

3. Spatial Differencing and an Explicit Integration Scheme

We now address the question of transforming the differential equations (2.21-2.26) to an analogous set of difference equations. The vertical structure of the model and the spatial differencing system are introduced first, because these will be the same in both the explicit and semi-implicit versions. The difference equations will be written first in conjunction with the usual explicit "leapfrog" time integration scheme. In Section 4, the semi-implicit scheme is introduced.

The vertical structure and arrangement of variables is indicated schematically below



This is a convenient arrangement which has the desirable property of allowing the boundary conditions $\dot{\sigma}=0$ at $\sigma=1,0$ to be incorporated naturally. In developing finite difference equations, we will employ the following standard notation:

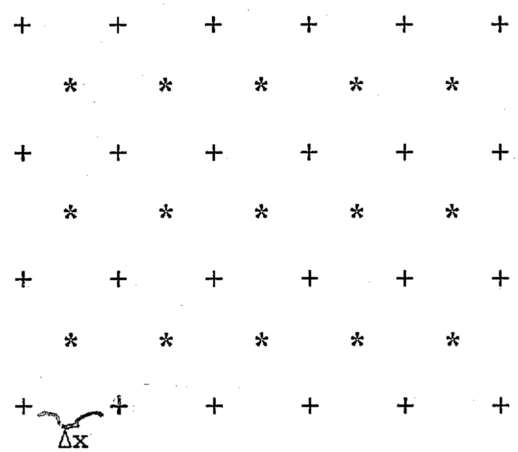
$$\psi_\sigma = [\psi(\sigma + \frac{\Delta\sigma}{2}) - \psi(\sigma - \frac{\Delta\sigma}{2})]/\Delta\sigma \qquad 3.1$$

where, in this model, $\Delta\sigma=\frac{1}{2}$, and

$$\overline{\psi}^\sigma = \frac{1}{2}[\psi(\sigma + \frac{\Delta\sigma}{2}) + \psi(\sigma - \frac{\Delta\sigma}{2})] \qquad 3.2$$

Note that eqn. (3.2) is a redefinition of the $(\overline{}^\sigma)$ notation to reflect a numerical approximation of the integral.

The horizontal differencing system employed here is a variation of the "semimomentum" method of Shuman and Vanderman (15), which possesses desirable stability characteristics. We define a lattice of regular points "+" at each of which all dependent variables are located at each time step. The usual averaging and differencing operators, analogous to eqns. (3.1,3.2) are used.



In addition, wind components u^*, v^* will be defined at the points marked by an asterisk on the schematic; these will subsequently be referred to as "box" winds. The relationship between "box" and gridpoint winds is defined as

$$u = \overline{u^*}^{xy}$$

$$v = \overline{v^*}^{xy} .$$

This arrangement, which has been used successfully by Kwizak and Robert, and by McPherson, in semi-implicit barotropic models, is introduced as a means of avoiding a form of the finite difference Laplacian operator which extends over a $4\Delta x$ interval and requires two sets of boundary conditions. It is not necessary for an explicit integration, nor is it desirable, since it can be shown that the computational stability criterion allows a time step only half of that permitted by straightforward application of the semimomentum scheme. However, since the main thrust in this paper is to employ a semi-implicit integration method, this "partly-staggered" arrangement will be adopted.

The difference equations analogous to (2.21-2.26) may be written as

$$\overline{\left(\frac{u^*}{m}\right)}^{t}_{t} + \overline{\left(\overline{\phi}^{\sigma}\right)_x}^{y} + \gamma\overline{\Theta}^{xy}\,\overline{q}^{y}_{x} = \left(\frac{v^*}{m}\right)\hat{f} - \overline{u}^{xy}\overline{u}^{y}_{x} - \overline{v}^{xy}\overline{u}^{x}_{y} - \frac{1}{2}\overline{\left(\frac{\dot{\sigma}}{m}\right)}^{xy}\left(\frac{\overline{u}_1^{xy}-\overline{u}_2^{xy}}{\Delta\sigma}\right)$$

$$\equiv U \qquad 3.5$$

$$\overline{\left(\frac{v^*}{m}\right)}^{t}_{t} + \overline{\left(\overline{\phi}^{\sigma}\right)_y}^{x} + \gamma\overline{\Theta}^{xy}\,\overline{q}^{x}_{y} = -\left(\frac{u^*}{m}\right)\hat{f} - \overline{u}^{xy}\overline{v}^{y}_{x} - \overline{v}^{xy}\overline{v}^{x}_{y} - \frac{1}{2}\overline{\left(\frac{\dot{\sigma}}{m}\right)}^{xy}\left(\frac{\overline{v}_1^{xy}-\overline{v}_2^{xy}}{\Delta\sigma}\right)$$

$$\equiv V\,, \qquad 3.6$$

$$\hat{f} = \overline{f}^{xy} - \overline{v}^{xy}\,\overline{m}^{y}_{x} + \overline{u}^{xy}\,\overline{m}^{x}_{y}\,,$$

$$\overline{\Theta}^{t}_{t} + \frac{1}{2}\,\overline{\overline{\dot{\sigma}}^{xy}\left(\frac{\overline{\Theta}_1^{xy}-\overline{\Theta}_2^{xy}}{\Delta\sigma}\right)}^{xy} = -\,\overline{\overline{m}^{xy}\left(\overline{u}^{xy}\,\overline{\Theta}^{y}_{x} + \overline{v}^{xy}\,\overline{\Theta}^{x}_{y}\right)}^{xy}$$

$$\equiv N\,. \qquad 3.7$$

Equations (3.5-3.7) apply to each level; a subscript k indicating the level has been suppressed in the interests of notational simplicity. The continuity equation becomes

$$\overline{q}^{t}_{t} + \kappa m^2 q\left[\overline{\left(\overline{\frac{u^*}{m}}^{\sigma}\right)_x}^{y} + \overline{\left(\overline{\frac{v^*}{m}}^{\sigma}\right)_y}^{x}\right] = -\,\overline{\overline{m}^{xy}\left(\overline{u}^{\sigma xy}\,\overline{q}^{y}_{x} + \overline{v}^{\sigma xy}\,\overline{q}^{x}_{y}\right)}^{xy}$$

$$\equiv M, \qquad 3.8$$

a finite difference form which does not, precisely, guarantee conservation of mass. Experience indicates that any violation is negligible, however.

The diagnostic equation in $\dot{\sigma}$ becomes

$$- \dot{\sigma} + \frac{1}{2}\Delta\sigma m^2 \left[\overline{\left(\frac{u_1^*}{m} - \frac{u_2^*}{m}\right)_x}^{y} + \overline{\left(\frac{v_1^*}{m} - \frac{v_2^*}{m}\right)_y}^{x} \right]$$

$$= - \frac{1}{2}\Delta\sigma \overline{\left(\kappa \overline{q}^{xy}\right)^{-1} \left[\left(\overline{u}_1^{xy} - \overline{u}_2^{xy}\right) \overline{q}_x^{y} + \left(\overline{v}_1^{xy} - \overline{v}_2^{xy}\right) \overline{q}_y^{x} \right]}^{xy}$$

$$\equiv S \, , \qquad 3.9$$

where the boundary conditions $\dot{\sigma}=0$ at $\sigma=0,1$ have been accounted for.

Equations (3.5-3.9) have been deliberately arranged as shown and their right sides denoted by U, V, N, M, and S, in order to facilitate the conversion to the semi-implicit method described in the next section. These quantities will be computed exactly as shown, both in the explicit and semi-implicit calculations.

Finally, the hydrostatic equation may be written for each layer as

$$\phi_* - \phi_1 + \Delta\sigma\alpha_1\Theta_1 q = 0 \qquad 3.10$$

and

$$\phi_1 - \phi_2 + \Delta\sigma\alpha_2\Theta_2 q = 0. \qquad 3.11$$

These two equations will undergo considerable reformulation in the semi-implicit model.

Equations (3.5-3.11) constitute a closed set which may be integrated forward in time. The results of such an integration are discussed in the last section of this note.

4. The Semi-Implicit Integration Method

As in the barotropic model, the essence of the semi-implicit integration procedure is to isolate those terms which principally govern gravitational oscillations and average them in time. The remaining terms are evaluated explicitly. To accomplish this, we follow the suggestion of Robert[1] and introduce a type of linearization. In particular, it is assumed

[1] Personal communication.

that the potential temperature is composed of a basic state value $\tilde{\Theta}$ which is a function of the vertical coordinate only, and a deviation Θ' from that basic state,

$$\Theta = \tilde{\Theta} + \Theta' , \tag{4.1}$$

and that the surface pressure variable q is composed of deviations of q' around a mean value $\tilde{q}$,

$$q = \tilde{q} + q' . \tag{4.2}$$

The implications of this linearization are not completely clear, and may give rise to difficulties, particularly in areas of irregular terrain. The principal reason for these assumptions is that they lead to a relatively simple form of the Helmholtz-type equations in which the coefficients are either constants or at most involve variations of the map factor, and in which no first-order differences appear.

We will employ the notation

$$\overline{\psi}^{2t} = \tfrac{1}{2}[\psi(t+\Delta t) + \psi(t-\Delta t)], \tag{4.3}$$

where ψ is an arbitrary dependent variable. It may be shown that the identity

$$\overline{\psi}^{t}_{t} \equiv \tfrac{1}{\Delta t}[\overline{\psi}^{2t} - \psi(t-\Delta t)] \tag{4.4}$$

holds; this will prove to be useful. With the introduction of (4.1-4.4) into the explicit difference equations (3.5-3.9), we may write the semi-implicit difference equations for each layer as

$$\overline{\frac{u_1^*}{m}}^{2t} + \tfrac{1}{2}\Delta t\,\overline{\overline{(\phi_1)}^{y}_{x}}^{2t} + \Delta t\gamma_1\tilde{\Theta}_1\,\overline{\overline{(q)}^{y}_{x}}^{2t} = \frac{u_1^{*\,n-1}}{m} - \tfrac{1}{2}\Delta t\,\overline{(\phi_*)}^{y}_{x} - \Delta t\gamma_1\overline{\Theta_1'}^{xy}\,\overline{q}^{y}_{x} + \Delta t U_1$$

$$\equiv U_1' \tag{4.5}$$

$$\overline{\frac{v_1^*}{m}}^{2t} + \tfrac{1}{2}\Delta t\,\overline{\overline{(\phi_1)}^{x}_{y}}^{2t} + \Delta t\gamma_1\tilde{\Theta}_1\,\overline{\overline{(q)}^{x}_{y}}^{2t} = \frac{v_1^{*\,n-1}}{m} - \tfrac{1}{2}\Delta t\,\overline{(\phi_*)}^{x}_{y} - \Delta t\gamma_1\overline{\Theta_1'}^{xy}\,\overline{q}^{x}_{y} + \Delta t V_1$$

$$\equiv V_1' \tag{4.6}$$

$$\overline{\Theta_1'}^{2t} + \tfrac{1}{2}\Delta t\tilde{\Gamma}\,\overline{\dot{\sigma}}^{2t} = \Theta_1'^{\,n-1} - \tfrac{1}{2}\Delta t\,\overline{\overline{\dot{\sigma}}^{xy}\left(\frac{\overline{\Theta_1'}^{xy} - \overline{\Theta_2'}^{xy}}{\Delta\sigma}\right)}^{xy} + \Delta t N_1 \equiv N_1' \qquad 4.7$$

where $\tilde{\Gamma} \equiv \frac{\partial\tilde{\Theta}}{\partial\sigma}$. In the upper layer,

$$\frac{\overline{u_2^*}^{2t}}{m} + \tfrac{1}{2}\Delta t\,\overline{\overline{(\phi_1+\phi_2)}^{y}_{x}}^{2t} + \Delta t\gamma_2\tilde{\Theta}_2\overline{\overline{(q)}^{y}_{x}}^{2t} = \frac{u_2^{*\,n-1}}{m} - \Delta t\gamma_2\overline{\Theta_2'}^{xy}\,\overline{q}^{y}_{x} + \Delta t U_2$$

$$\equiv U_2' \qquad 4.8$$

$$\frac{\overline{v_2^*}^{2t}}{m} + \tfrac{1}{2}\Delta t\,\overline{\overline{(\phi_1+\phi_2)}^{x}_{y}}^{2t} + \Delta t\gamma_2\tilde{\Theta}_2\overline{\overline{(q)}^{x}_{y}}^{2t} = \frac{v_2^{*\,n-1}}{m} - \Delta t\gamma_2\overline{\Theta_2'}^{xy}\,\overline{q}^{x}_{y} + \Delta t V_2$$

$$\equiv V_2' \qquad 4.9$$

and

$$\overline{\Theta_2'}^{2t} + \tfrac{1}{2}\Delta t\tilde{\Gamma}\,\overline{\dot{\sigma}}^{2t} = \Theta_2'^{\,n-1} - \tfrac{1}{2}\Delta t\,\overline{\overline{\dot{\sigma}}^{xy}\left(\frac{\overline{\Theta_1'}^{xy} - \overline{\Theta_2'}^{xy}}{\Delta\sigma}\right)}^{xy} + \Delta t N_2$$

$$\equiv N_2' \,. \qquad 4.10$$

It will be observed that the temporal-averaging operator has been applied to the geopotential gradient and the linear part of the surface pressure gradient in the momentum equations, and the linear part of the stability term of the thermodynamic equations. These terms are treated implicitly. The terms involving deviations from the basic state are placed on the right sides of the equations along with the advective terms, all of which are to be evaluated explicitly.

The two forms of the continuity equation become

$$\overline{q}^{2t} + \tfrac{1}{2}\Delta t\kappa\tilde{q}m^2\,\overline{\left[\overline{\left(\frac{u_1^*+u_2^*}{m}\right)}^{y}_{x} + \overline{\left(\frac{v_1^*+v_2^*}{m}\right)}^{x}_{y}\right]}^{2t}$$

$$= q^{n-1} - \tfrac{1}{2}\Delta t\kappa q' m^2\left[\overline{\left(\frac{u_1^*+u_2^*}{m}\right)}^{y}_{x} + \overline{\left(\frac{v_1^*+v_2^*}{m}\right)}^{x}_{y}\right] + \Delta t M \equiv M' \qquad 4.11$$

and

$$-\overline{\dot{\sigma}}^{2t} + \tfrac{1}{2}\Delta\sigma m^2\,\overline{\left[\overline{\left(\frac{u_1^*-u_2^*}{m}\right)}^{y}_{x} + \overline{\left(\frac{v_1^*-v_2^*}{m}\right)}^{x}_{y}\right]}^{2t} = S \equiv S' \qquad 4.12$$

In eqn. (4.11), the linear part of the mean divergence term is treated implicitly while the analogous term involving q' is evaluated explicitly. Similarly, the shear-divergence term in the diagnostic equation for $\overset{*}{\sigma}$ is treated implicitly.

There remain to be transformed the two statements of the hydrostatic equation. Introducing (4.1-4.3) into (3.10) and (3.11), we may write

$$- \overline{\phi_1}^{2t} + \Delta\sigma\alpha_1 \tilde{\theta}_1 \overline{q}^{2t} + \Delta\sigma\alpha_1 \tilde{q}\overline{\theta_1'}^{2t} = - \phi_* - \Delta\sigma\alpha_1 \overline{\overline{q'}^{xy}\overline{\theta_1'}^{xy}}^{xy}$$

$$\equiv H_1', \qquad 4.13$$

and

$$\overline{\phi_1}^{2t} - \overline{\phi_2}^{2t} + \Delta\sigma\alpha_2 \tilde{\theta}_2 \overline{q}^{2t} + \Delta\sigma\alpha_2 \tilde{q}\overline{\theta_2'}^{2t} = - \Delta\sigma\alpha_2 \overline{\overline{q'}^{xy}\ \overline{\theta_2'}^{xy}}^{xy}$$

$$\equiv H_2', \qquad 4.14$$

where the "prime" notation is used for consistency. The considerable amount of smoothing on the explicit nonlinear term on the right side of each equation was employed to control noise, but may not be necessary. This particular formulation follows rather naturally from the form of the preceding equations, and in that sense appears to be consistent. It does, however, represent a significant departure from the explicit formulation.

Nevertheless, eqns. (4.5-4.14) constitute a complete set of ten equations in ten unknowns which may be solved numerically, given suitable initial and lateral boundary conditions. It is possible to reduce this set to a pair of Helmholtz-type equations in a pair of

$\overline{q}^{2t}$, $\overline{\overset{*}{\sigma}}^{2t}$, $\overline{\theta_1'}^{2t}$, $\overline{\theta_2'}^{2t}$, $\overline{\phi_1}^{2t}$, or $\overline{\phi_2}^{2t}$. We have elected to pursue a

process of elimination which leads to a pair of Helmholtz-type equations in $\overline{q}^{2t}$ and $\overline{\phi_1}^{2t}$. One of the equations arises basically from eqn. (4.11), and the other from the hydrostatic equation (4.13). Employing the latter, a fundamentally diagnostic equation, in a quasi-prognostic role, may not be the most desirable formulation. It certainly represents a departure from the explicit formulation, and probably accounts for some of the differences between the explicit and implicit forecasts to be discussed in Section 7.

The steps of the process of elimination are as follows:

1. $\overline{\dot{\sigma}}^{2t}$ is replaced in eqns. (4.7) and (4.10) through the use of eqn. (4.12).

2. The potential temperature deviations $\overline{\theta'_1}^{2t}$, $\overline{\theta'_2}^{2t}$ are then eliminated between eqns. (4.13,4.14) and the expressions resulting from step (1). The result of this will be equations for $\overline{\phi}_1^{2t}$ and $\overline{\phi}_2^{2t}$ in terms of $\overline{q}^{2t}$ and the shear divergence.

3. The two equations resulting from step (2) may be multiplied by appropriate factors and subtracted, to yield an expression for $\overline{\phi}_2^{2t}$ in terms of $\overline{\phi}_1^{2t}$ and $\overline{q}^{2t}$:

$$\overline{\phi}_2^{2t} = \left(\frac{\alpha_2+\alpha_1}{\alpha_1}\right)\overline{\phi}_1^{2t} - \tfrac{1}{2}\alpha\left(\tilde{\theta}_1-\tilde{\theta}_2\right)\overline{q}^{2t} + \frac{\alpha_2}{\alpha_1}H_1'' - H_2'', \qquad 4.15$$

where

$$H_1'' \equiv H_1' - \tfrac{1}{2}\alpha_1\tilde{q}N_1' - \tfrac{1}{4}\alpha_1\tilde{q}\Delta t\tilde{\Gamma}S' \qquad 4.16$$

and

$$H_2'' = H_2' - \tfrac{1}{2}\alpha_2\tilde{q}N_2' - \tfrac{1}{4}\alpha_2\tilde{q}\Delta t\tilde{\Gamma}S'. \qquad 4.17$$

In (4.15-4.17), $\Delta\sigma$ has been replaced by its numerical value, $\frac{1}{2}$. Equation (4.15) may then be used to replace $\overline{\phi}_2^{2t}$ in the momentum equations.

4. The vertical average and difference of the divergence may then be formed from the results of step (3), in order to eliminate the mean and shear divergence terms.

5. The mean divergence term in eqn. (4.11) may be elininated using the results of step (4). This will yield one Helmholtz-type equation in $\overline{q}^{2t}$ and $\overline{\phi}_1^{2t}$:

$$\overline{q}^{2t} - \tfrac{1}{4}\left(\frac{m\Delta t}{\Delta x}\right)^2\kappa\tilde{q}[\gamma_1\tilde{\theta}_1 + \gamma_2\tilde{\theta}_2 - \tfrac{1}{4}\alpha_2\left(\tilde{\theta}_1-\tilde{\theta}_2\right)]\nabla^2\overline{q}^{2t}$$

$$- \tfrac{1}{8}\left(\frac{m\Delta t}{\Delta x}\right)^2\kappa\tilde{q}\left(2 + \frac{\alpha_1+\alpha_2}{\alpha_1}\right)\nabla^2\overline{\phi}_1^{2t}$$

$$= M' - \tfrac{1}{2}\Delta t m^2\kappa\tilde{q}\left[\overline{\left(U_1'+U_2'\right)_x}^{y} + \overline{\left(V_1'+V_2'\right)_y}^{x}\right]$$

$$+ \tfrac{1}{8}\left(\frac{m\Delta t}{\Delta x}\right)^2\kappa\tilde{q}\nabla^2\left(\frac{\alpha_2}{\alpha_1}H_1''-H_2''\right) \qquad 4.18$$

6. In order to obtain the second member of the pair of Helmholtz equations, the shear divergence expression obtained in step (4) may be used in the equation for $\overline{\phi}_1^{2t}$ obtained in step (2). This results in

$$- \overline{\phi}_1^{2t} + \left[\frac{1}{64}\left(\frac{m\Delta t}{\Delta x}\right)^2 \tilde{q}\tilde{\Gamma}(\alpha_1+\alpha_2)\right]\nabla^2 \overline{\phi}_1^{2t} + \frac{1}{2}\alpha_1\tilde{\theta}_1\overline{q}^{2t}$$

$$+ \left\{\frac{1}{32}\left(\frac{m\Delta t}{\Delta x}\right)^2 \alpha_1 \tilde{q}\tilde{\Gamma}\left[\gamma_1\tilde{\theta}_1-\gamma_2\tilde{\theta}_2+ \frac{1}{4}\alpha_2\left(\tilde{\theta}_1-\tilde{\theta}_2\right)\right]\right\}\nabla^2 \overline{q}^{2t}$$

$$= H_1'' + \frac{1}{16}\alpha_1\tilde{q}\tilde{\Gamma}m^2\Delta t\left[\overline{(U_1'-U_2')}^y_x + \overline{(V_1'-V_2')}^x_y\right]$$

$$+ \frac{1}{64}\alpha_1\tilde{q}\left(\frac{m\Delta t}{\Delta x}\right)^2\nabla^2 \left(\frac{\alpha_2}{\alpha_1} H_1''-H_2''\right). \qquad 4.19$$

Equations (4.18) and (4.19) constitute a set of two Helmholtz equations in $\overline{q}^2$ and $\overline{\phi}_1^{2t}$ which may be solved numerically. However, the solution is facilitated somewhat if the equations are manipulated to another form. Specifically, if we adopt the notation

$$a_1 = -\left\{\frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)^2 \kappa\tilde{q}\left[\gamma_1\tilde{\theta}_1+ \gamma_2\tilde{\theta}_2- \frac{1}{4}\alpha_2\left(\tilde{\theta}_1-\tilde{\theta}_2\right)\right]\right\}^{-1} \qquad 4.20$$

$$a_2 = - a_1\left[\frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^2 \kappa\tilde{q} \left(2 + \frac{\alpha_1+\alpha_2}{\alpha_1}\right)\right] \qquad 4.21$$

$$b_1 = - \left\{\frac{1}{64}\left(\frac{\Delta t}{\Delta x}\right)^2 \tilde{q}\tilde{\Gamma}(\alpha_1+\alpha_2)\right\}^{-1} \qquad 4.22$$

$$b_2 = b_1\left\{\frac{1}{32}\left(\frac{\Delta t}{\Delta x}\right)^2\alpha_1\tilde{q}\left[\gamma_1\tilde{\theta}_1- \gamma_2\tilde{\theta}_2+ \frac{1}{4}\alpha_2\left(\tilde{\theta}_1-\tilde{\theta}_2\right)\right]\right\} \qquad 4.23$$

$$c = \frac{1}{2}b_1\alpha_1\tilde{\theta}_1 \qquad 4.24$$

$$F_1 = a_1\left\{M' - \frac{1}{2}\Delta t m^2\kappa\tilde{q}\left[\overline{(U_1'+U_2')}^y_x + \overline{(V_1'+V_2')}^x_y\right]\right.$$

$$\left. + \frac{1}{8}\left(\frac{m\Delta t}{\Delta x}\right)^2 \kappa\tilde{q}\nabla^2\left(\frac{\alpha_2}{\alpha_1} H_1''-H_2''\right)\right\} \qquad 4.25$$

and

$$F_2 = b_1\{H_1^{\wedge\prime} + \frac{1}{16}\alpha_1\tilde{q}\tilde{\Gamma}m^2\Delta t[\overline{(U_1^{\wedge}-U_2^{\prime})}^{y}_{x} + \overline{(V_1^{\wedge}-V_2^{\prime})}^{x}_{y}]$$

$$+ \frac{1}{64}\left(\frac{m\Delta t}{\Delta x}\right)^2\alpha_1\tilde{q}\nabla^2\left(\frac{\alpha_2}{\alpha_1}H_1^{\wedge\prime}-H_2^{\prime\prime}\right)\} \qquad 4.26$$

then eqns. (4.18,4.19) may be transformed to the pair

$$\nabla^2\,\overline{q}^{2t} + \left(\frac{\varepsilon_1}{m^2}\right)\overline{q}^{2t} + \left(\frac{\varepsilon_2}{m^2}\right)\overline{\phi}_1^{2t} = m^{-2}G_1 \qquad 4.27$$

$$\nabla^2\,\overline{\phi}_1^{2t} + \left(\frac{\varepsilon_3}{m^2}\right)\overline{\phi}_1^{2t} + \left(\frac{\varepsilon_4}{m^2}\right)\overline{q}^{2t} = m^{-2}G_2 \qquad 4.28$$

where:

$$\varepsilon_1 = a_1 D\ , \quad D \equiv [1 + a_2\left(\frac{c-a_1b_2}{a_1-a_2c}\right)]^{-1} \qquad 4.29$$

$$\varepsilon_2 = \left(\frac{a_1a_2b_1}{a_1-a_2c}\right)D \qquad 4.30$$

$$\varepsilon_3 = -\left(\frac{a_1b_1}{a_1-a_2c}\right)D \qquad 4.31$$

$$\varepsilon_4 = \left(\frac{c-a_1b_2}{a_1-a_2c}\right)D \qquad 4.32$$

$$G_1 = [F_1 - a_2\left(\frac{a_1F_2-cF_1}{a_1-a_2c}\right)]D \qquad 4.33$$

and

$$G_2 = [\frac{a_1F_2-a_1b_2F_1}{a_1-a_2c}]D\ . \qquad 4.34$$

In these equations, the symbol ∇^2 is defined as

$$(\nabla^2\psi)_{ij} \equiv \psi_{i+1,j+1} + \psi_{i+1,j-1} + \psi_{i-1,j-1} + \psi_{i-1,j+1} - 4\psi_{ij}. \qquad 4.35$$

The coefficients a_1, a_2, b_1, b_2, c, and ϵ_1 may be computed only once, given the definition of the basic state. The forcing functions G_1, G_2 are computed each time step from known fields. Relaxation is then used to solve (4.27) and (4.28) for $\overline{q}^{2t}$ and $\overline{\phi}_1^{2t}$.

Once this is accomplished, a time step is completed by calculating:

$\overline{\phi}_2^{2t}$ from eqn. (4.15)

$\overline{u^*}^{2t}$, $\overline{v^*}^{2t}$ from eqns. (4.5, 4.6, 4.8, 4.9)

$\overline{\dot{\sigma}}^{2t}$ from eqn. (4.12)

$\overline{\Theta^*}^{2t}$ from eqns. (4.7, 4.10).

Values of the variables at $t = (n+1)\Delta t$ are then recovered from their temporally-averaged values by appealing to eqn. (4.3).

A sample integration of this model, both explicit and semi-implicit, is presented in the last section of this note.

5. Lateral Boundary Conditions

Temporally-invariant lateral boundary conditions were employed in both explicit and semi-implicit integrations. This was accomplished in the semi-implicit case simply by allowing the values of the dependent variables at the outer row/column of grid points, and the values of the "box" winds in the outer row/column of grid squares, to retain their initial values throughout the integration. No filtering or diffusion near the boundaries was found to be necessary.

The form of the Laplacian operator (eqn. 4.35), while it does not require two sets of boundary conditions, nevertheless yields independent solutions on two distinct sublattices of the basic array. Such an arrangement is susceptible to separation of the two solutions. One might therefore consider filtering the fields which result from the relaxation, following Shuman (13), but such filtering has not been done here.

A modification was necessary in the case of the explicit integration in order to suppress unwanted noise. In this case, the outer row/column of grid point values was discarded. The outer row/column of "box" winds and the penultimate row/column of grid point values were held constant at their initial values. In addition, at the penultimate row/column of "boxes" and at the antepenultimate row/column of grid points, strong diffusion in the form of the first step of the Lax-Wendroff integration system is applied to all prognostic quantities. This is accomplished by replacing the time derivatives by

$$\frac{\partial \psi_{ij}}{\partial t} \simeq \frac{1}{\Delta t}[\psi_{ij}^{n+1} - \frac{1}{4}(\psi_{i+ij}^{n} + \psi_{i-ij}^{n} + \psi_{ij+1}^{n} + \psi_{ij-1}^{n})]. \qquad 5.1$$

The lateral boundary condition formulation represents another possible source of differences between the explicit and implicit integrations. It is reasonable to anticipate that, as long as the boundaries are positioned in quiescent zones and the initial data possesses good balance, this source may not be as important as that associated with the differing treatments of the hydrostatic equation. Evidence from the experimental integrations tends to confirm this.

6. Initialization

In view of the unrealistic nature of a two-layer representation of the atmosphere, and the anticipated resulting inability to describe little beyond barotropic meteorological behavior, we elected to construct a particularly simple set of initial data. First, we assumed that the surface pressure field is constant at 1000 mb, and the terrain is everywhere at mean sea level. This means that the $\sigma = \frac{1}{2}$ level is at 500 mb. We then obtained 500 mb D-values for 00Z 10 February 1970 from an NMC B3 tape, and constructed the 500 mb height field. Using this field, and a 1000 mb sea level pressure, we computed a mean temperature for the 1000-500 mb layer and converted it to potential temperature.

We next computed the areal average of the layer-mean potential temperature in the lower layer, $\tilde{\Theta}_1$. Using this and an assumed basic-state static stability $\tilde{\Gamma}$ of $60^{o}/\Delta\sigma$, we then constructed the areal mean of the potential temperature in the upper layer, $\tilde{\Theta}_2$. This value and the areal mean of the 500 mb geopotential were then used to calculate hydrostatically the areal mean of the geopotential at the "top" of the model atmosphere. We then assumed that the geopotential at the top of the model, ϕ_2, was equal to its areal average. The actual potential temperature in the upper layer $(\tilde{\Theta}_2+\Theta_2')$ was then calculated hydrostatically from ϕ_1 and ϕ_2. We thus arrive at a set of initial data which features a flat surface pressure and a flat "top" of the atmosphere. The thermal gradient is directed poleward in the lower layer, and equatorward in the upper layer.

This allows the use of a simplified form of the nonlinear balance equation to obtain nondivergent winds, since the gradients of both surface pressure and geopotential at the top vanish. The principal balance is between the Coriolis acceleration and half of the 500 mb geopotential gradient, since that gradient is a layer-mean. This means that the resulting winds will be the same in both layers, and approximately of half the magnitude that would be calculated at the 500 mb level.

The final set of initial data is thus nondivergent and hydrostatic, with no vertical wind shear, but featuring a marked thermal stratification. It is therefore expected that the external gravity mode, although minimized initially, will be free to develop, and that the one internal mode allowed in this model will be generated as a result of the thermal stratification. With respect to the meteorological modes, the distribution of geopotential at midtroposphere is realistic, but the advecting winds are underestimated. The 500 mb height forecasts would be expected to be quite slow with respect to the translation of flow patterns.

The initialization scheme proposed is adequate for the main thrust of this experiment, i.e. to demonstrate that semi-implicit method behaves stably for relatively long time steps in the presence of high-frequency gravitational oscillations. With more effort, a less specialized set of data might be constructed. However, it appears more profitable to defer this effort until work is begun on a baroclinic model with more vertical resolution.

7. Results of an Experimental Integration

Both explicit and semi-implicit models were integrated to 48 hours using the initial data described in the previous sections. Figure 1 displays the initial distribution of the height of the $\sigma = \frac{1}{2}$ (500 mb, initially) level. The 48-hour forecasts of the height field are shown in Figures 2 (implicit) and 3 (explicit), and of the sea level pressure field in Figures 4 and 5. The explicit integration required about 33 minutes on the CDC 6600 computer, while the semi-implicit integration required about 9 minutes to reach 48 hours in one-hour steps. Both times are greater than they should be, by approximately a factor of two: the explicit version because the "staggered" arrangement of variables necessitates the use of a five-minute time step, whereas ten minutes could be used in a straightforward semimomentum scheme; and the semi-implicit because the relaxation procedure employed was not optimized. Convergence required an average of 50 scans per time step, at least twice the number to be anticipated with an efficient relaxation scheme. Nevertheless, the rough estimate of the time advantage of the semi-implicit method over an explicit method of 3.5:1 is in fair agreement with results reported by Kwizak and Robert.

In spite of the differences in formulation between the explicit and semi-implicit versions, a superficial comparison of Figures 2 and 3 indicates good agreement in the height forecasts. All major features are located in approximately the same positions on both forecasts. The noticeable tendency to amplify (compared to the initial data) short meteorological waves is evident in both. In general, the patterns are very similar.

On closer inspection, however, significant differences may be observed, primarily in local detail. The system southeast of Kamchatka, for example, is forecast to be more intense and to move more rapidly by the semi-implicit model. The tendency to increase the height gradient south of the system is present in both, but is especially noticeable in the semi-implicit version. Downstream, the series of minor troughs and ridges is further advanced in the semi-implicit version. There is no apparent reason why the semi-implicit integration method should exhibit faster movement of short waves, and indeed, this tendency is not observed with respect to other systems. For example, the shortwave trough over Siberia is treated almost identically by the two methods, but the system over eastern North America is moved more rapidly in the explicit version.

In view of the fact that the behavior of the two forecasts around the boundaries is almost identical, it would appear that the reason for these differences cannot be attributed to the differing lateral boundary condition formulation. Rather, a process of elimination rather quickly leads to the suspicion that the treatment of the hydrostatic equation is responsible.

These differences are primarily of small horizontal scale. Careful analysis reveals that there is also a very large scale difference pattern. In general, the heights on the left half of the semi-implicit forecast are lower than the corresponding values in the explicit forecast, and the reverse is true on the right-hand half. A similar tendency may be noted in the sea level pressure forecasts, Figures 4 and 5. All of the centers are identifiable and located in nearly the same positions, but their intensities differ considerably. Again, pressures on the left-hand half of the semi-implicit forecast tend to be lower than the explicit forecast by 5-10 mb, while the reverse is true on the right-hand half. This suggests the presence of a very large scale external gravity oscillation, which of course is treated differently by the two models.

It is tempting to perform an extensive noise analysis of these two models, and to duplicate the Canadian result of very close agreement between explicit and semi-implicit integrations. Such a process, however, would be both expensive and time-consuming, and in the end would not yield benefits in proportion to its cost.

A more profitable allocation of resources would appear to lie in the direct application of the semi-implicit integration method to a model with considerably greater vertical resolution. This experiment has successfully demonstrated the feasibility of implicit treatment of both external and internal gravity modes, and has resulted in a stable integration to 48 hours with roughly a 3.5:1 advantage in computing time over an explicit method. The next logical step is to introduce the method into a model comparable with currently operational primitive equation models, and preliminary efforts in that direction are under way.

References

1. Gerrity, J. P., and R. D. McPherson: "An Analysis of the Computational Stability Criteria for Explicit and Implicit Integration Schemes Using a Two-Layer Model in Phillips' σ-Coordinate," NMC Office Note 45, October 1970.

2. Gerrity, J. P., R. D. McPherson, and P. D. Polger: "An Analysis of the Free Modes of One, Two, and Four Layer Models Based on Sigma Coordinates," NMC Office Note 47, January 1971.

3. Gerrity, J. P., and R. D. McPherson: "An Analysis of Linear Computational Stability of Explicit and Implicit Integration Schemes for a Two-Layer Model Using Shuman's σ-Coordinate," NMC Office Note 49, January 1971.

4. Gerrity, J. P., R. D. McPherson, and P. D. Polger: "Linear Computational Stability Criteria for a Modified Semi-Implicit Integration Technique for Gravity Oscillations in a Two-Layer Model in Phillips' σ-Coordinate," NMC Office Note 52, January 1971.

5. Gerrity, J. P., R. D. McPherson, and P. D. Polger: "Requiem for an Integration Procedure," NMC Office Note 53, March 1971.

6. Gerrity, J. P., and R. D. McPherson, "Numerical Instability of a Mixed Implicit-Explicit Integration Procedure," NMC Office Note 55, March 1971.

7. Gerrity, J. P., and R. D. McPherson, "On an Efficient Scheme for the Numerical Integration of a Primitive Equation Barotropic Model," Journal of Applied Meteorology, Vol. 10, No. 3, pp. 353-363, June 1971.

8. Kwizak, M., and A. Robert: "Implicit Integration of a Grid-Point Model," Monthly Weather Review, Vol. 99, No. 1, pp. 32-36, January 1971.

9. Marchuk, G.: Numerical Methods for Solving Weather-Forecasting and Climate-Theory Problems - Part I. Translated from Russian by American Meteorological Society for Air Force Cambridge Research Laboratory, Contract AF 19(628)-3880, September 1965.

10. McPherson, R.: "Note on the Semi-Implicit Integration of a Primitive-Equation Limited-Area Prediction Model on an Offset Grid," Monthly Weather Review, Vol. 99, No. 3, pp. 242-247, March 1971.

11. McPherson, R. D., and P. D. Polger: "The Red Herring Affair: Modified Semi-Implicit Methods Revisited," NMC Office Note 64, November 1971.

12. Robert, A.: "The Integration of a Spectral Model of the Atmosphere by the Implicit Method," Proc. WMO/IUGG Symp. on Num. Wea. Pred. in Tokyo, November 1968, Section VII, pp. 19-24.

13. Shuman, F. G.: "Numerical Methods in Weather Prediction, Part I: The Balance Equation," Monthly Weather Review, Vol. 55, No. 10, pp. 329-332, October 1957.

14. Shuman, F. G.: "Resuscitation of an Integration Procedure," NMC Office Note 54, March 1971.

15. Shuman, F. G., and L. W. Vanderman: "Difference System and Boundary Conditions for the Primitive-Equation Barotropic Forecast," Monthly Weather Review, Vol. 94, No. 5, pp. 329-335, May 1966.

16. Shuman, F. G., and J. P. Hovermale: "An Operational Six-Layer Primitive Equation Model," Journal of Applied Meteorology, Vol. 7, No. 4, pp. 525-547, August 1968.

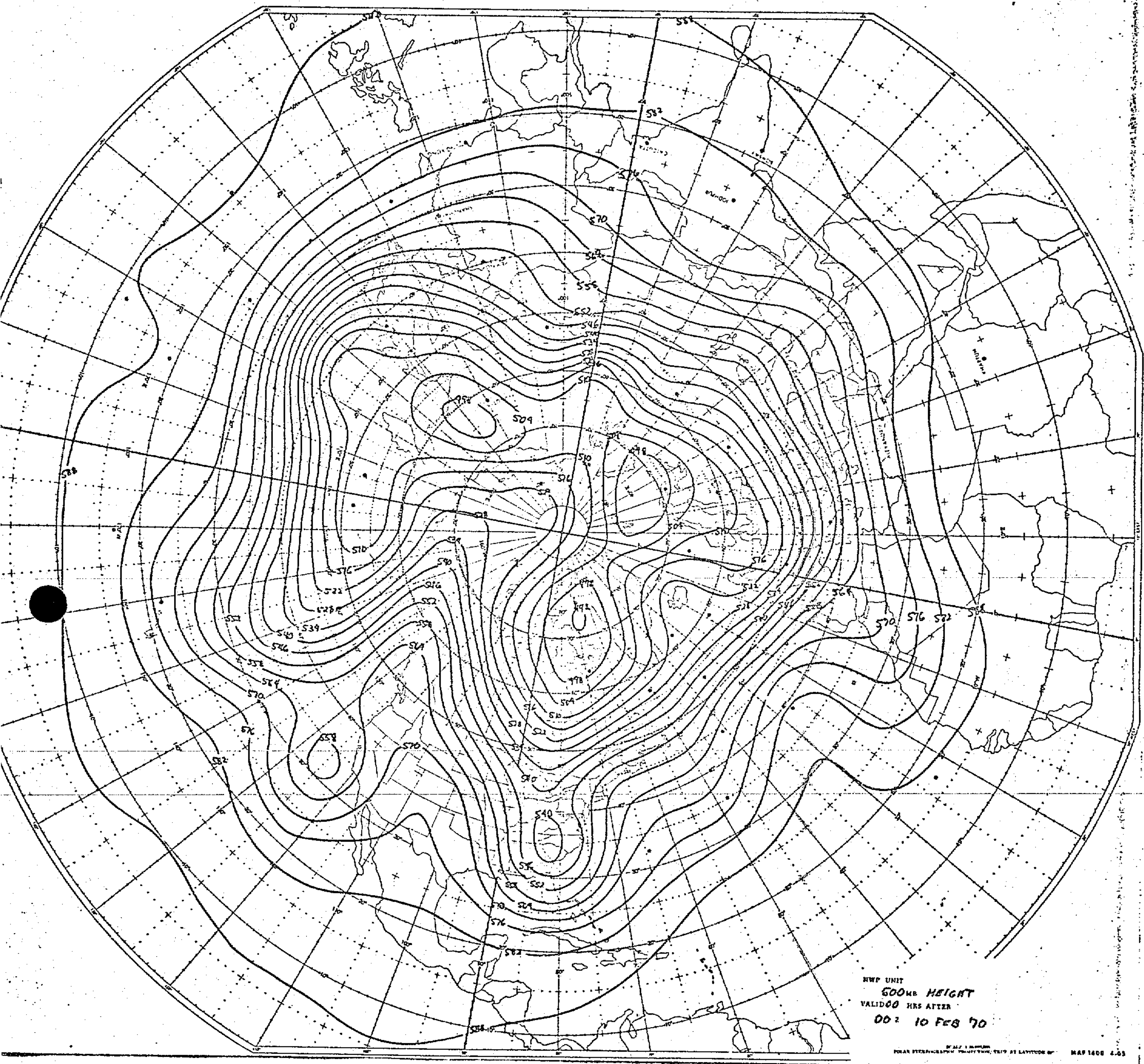


FIGURE 1. Initial distribution of 500mb height.

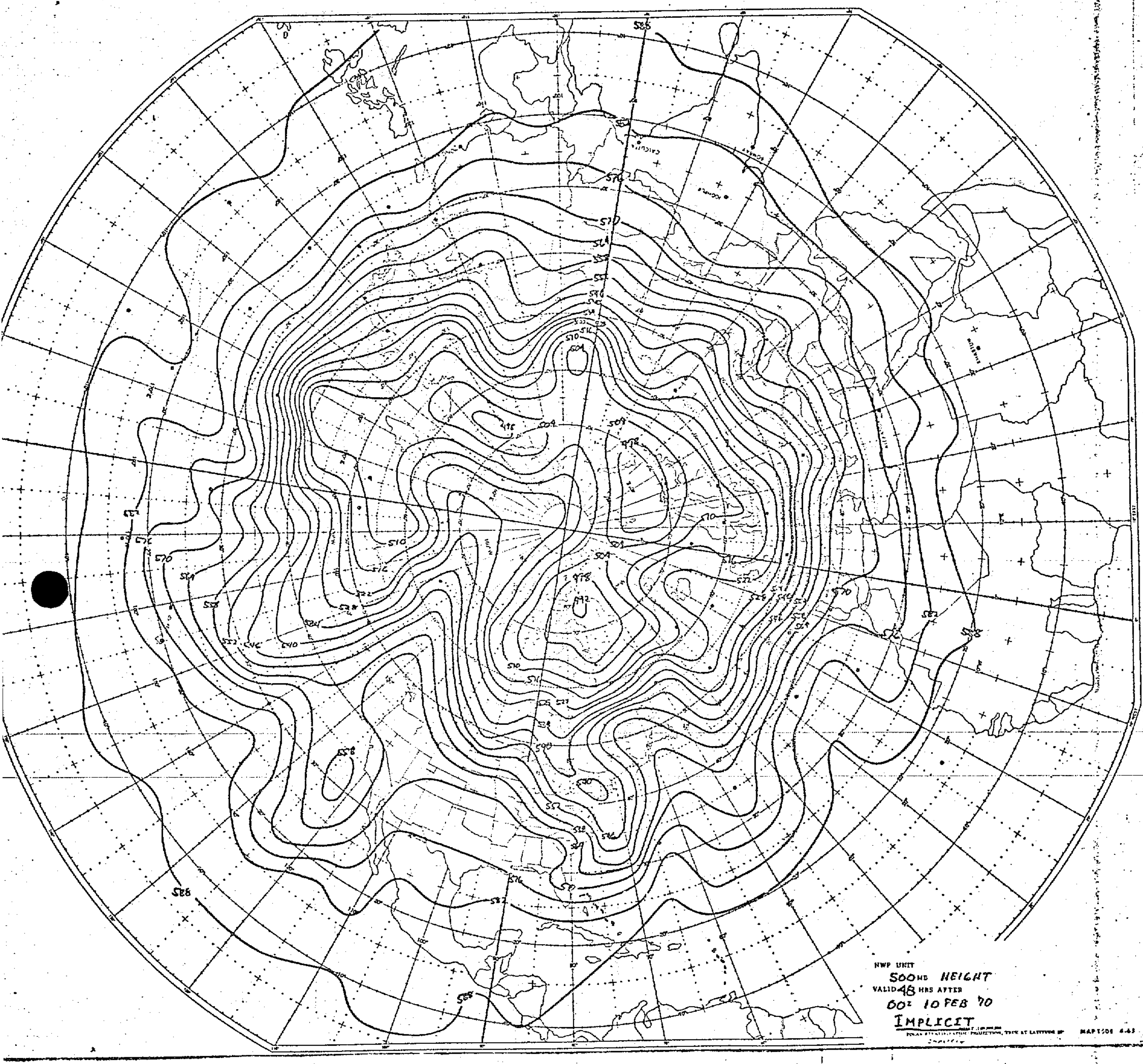


FIGURE 2. Semi-implicit 48-hr forecast of height of $\sigma = \frac{1}{2}$ surface, approximately 500mb.

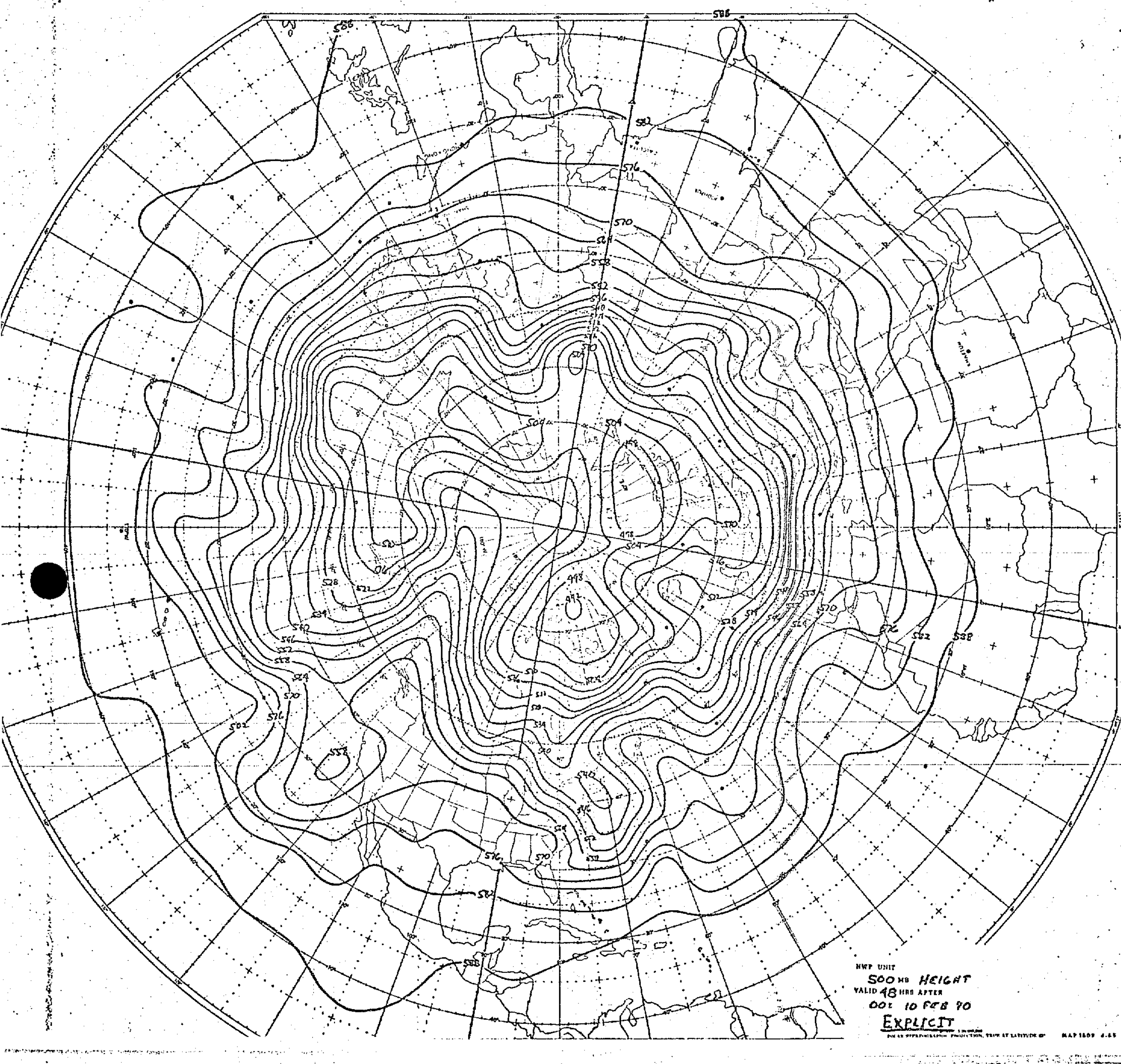


FIGURE 3. Explicit 48-hr forecast of height of $\sigma = \frac{1}{2}$ surface, approximately 500 mb.

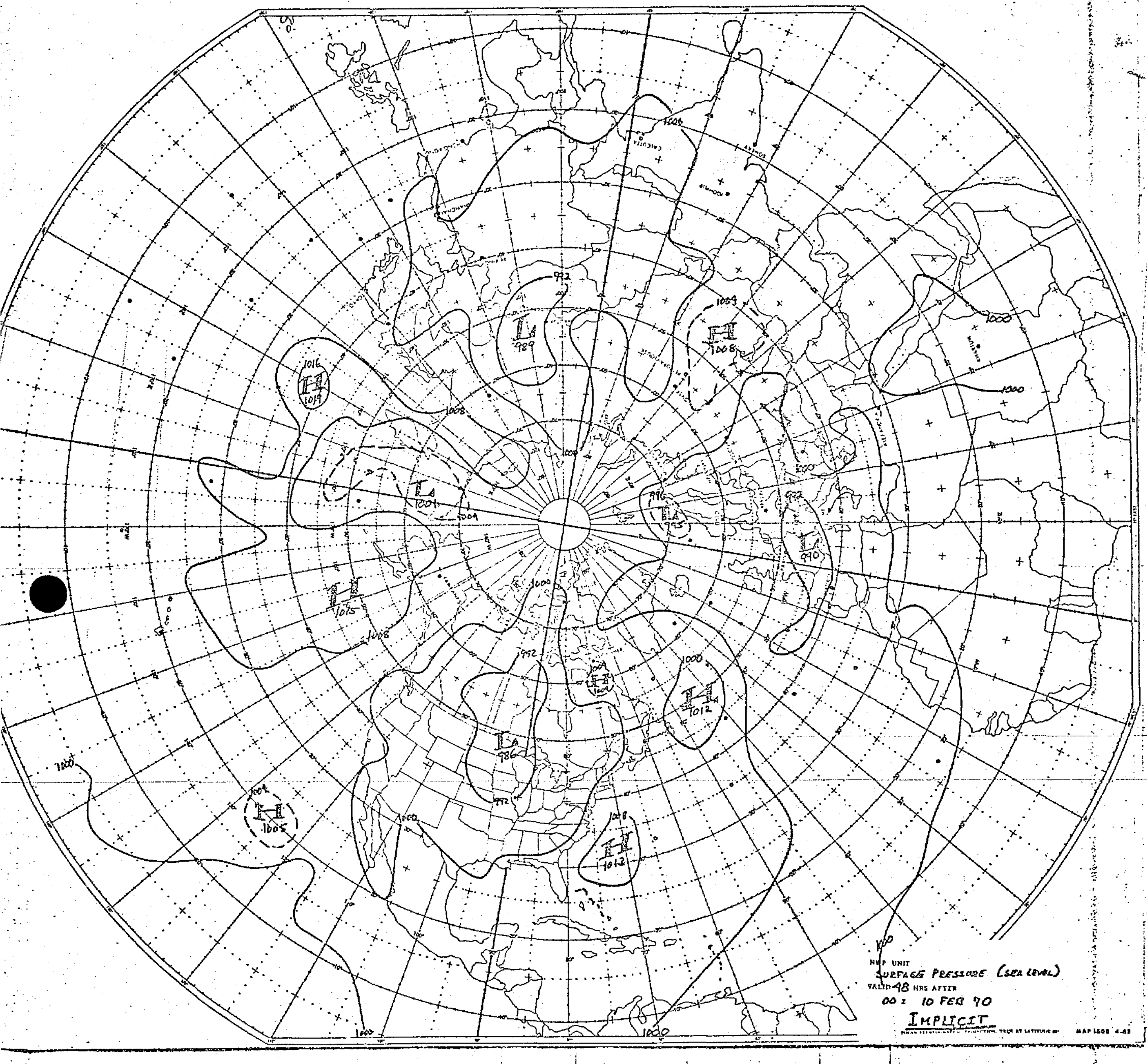


FIGURE 4. Implicit 48-hr forecast of sea level pressure.

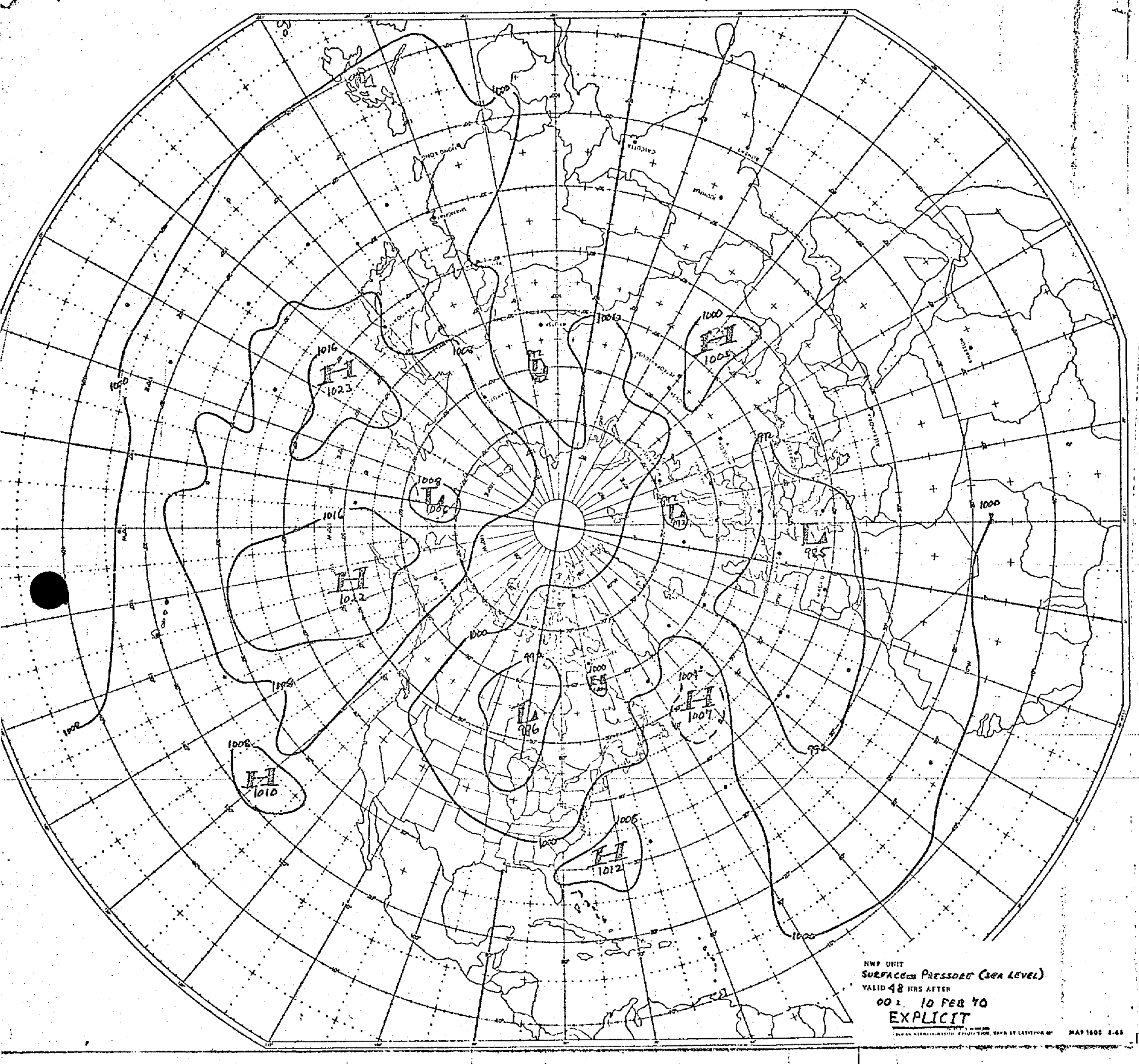


FIGURE 5. Explicit 48-hr forecast of sea level pressure.