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SOME COMMENTS ON ROBERT'S TIME FILTER

FOR TIME INTEGRATION

by

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OFFICE NOTE 62

1. Introduction

In a recent paper, Asselin [1] has presented an analysis of numerical time integration methods subjected to a temporal filter as originally proposed by Robert [2]. Asselin's paper is quite compact in its description of the results of his analysis, which makes its interpretation somewhat difficult.

Prior to the appearance of Asselin's paper, Vanderman [3] had undertaken a set of numerical experiments utilizing Robert's temporal filter in conjunction with the explicit, leapfrog method of time integration. To assist in the understanding of Vanderman's results, a simple stability analysis of the method as applied to a one-dimensional wave equation was undertaken. Preliminary results of this effort were documented in NMC Office Notes 51 and 60. A final analysis is now complete and this paper is intended to present those results.

Certain points in our analysis are merely specialized results of Asselin's more complete study; there are included, however, certain points of clarification which may serve as a supplement to his work.

2. The Analysis

We consider the numerical formulation of the analytic problem posed by the differential equation,

$$\frac{du}{dt} = i\omega u \quad (1)$$

with the initial condition,

$$u = 1 \text{ at } t = 0 \quad (2)$$

Note that $i \equiv \sqrt{-1}$ and ω is a real constant. The finite difference approximation of the differential equation combines the centered, second order accurate, leapfrog method and Robert's temporal filter. This method may be expressed in two equations

$$u_*^{(n+1)} = u^{(n-1)} + 2i\omega\Delta t u_*^{(n)} \quad (3a)$$

$$u^{(n)} = \alpha u_*^{(n)} + \frac{1-\alpha}{2} (u^{(n-1)} + u_*^{(n+1)}) \quad (3b)$$

in which the indices indicate the time level, Δt is the time step and α is a constant in the range, (0 to 1). If (3a) and (3b) are manipulated to eliminate reference to quantities with an asterisk, one may derive the

expression

$$u^{(n+1)} - (1 - \alpha + 2ib)u^{(n)} - (\alpha - i[1-\alpha]b)u^{(n-1)} = 0 \quad (4)$$

in which we have used the definition,

$$b = \omega \Delta t \quad (5)$$

Solutions to equation (4) may be obtained in the form

$$u^{(n)} = K \zeta^n \quad (6)$$

in which K is a complex constant, n is an integer exponent and ζ is a complex quantity satisfying the quadratic expression

$$\zeta^2 - 2B\zeta - C = 0 \quad (7)$$

derived by inserting (6) into equation (4) and defining

$$B = \frac{1-\alpha}{2} + ib \quad (8a)$$

and

$$C = \alpha - (1-\alpha)ib. \quad (8b)$$

Clearly, there are two roots of (7) and therefore two solutions of the form (6),

$$u^{(n)} = K_1 \zeta_+^n + K_2 \zeta_-^n \quad (9)$$

with

$$\zeta_{\pm} = \left[\frac{1-\alpha}{2} + ib \right] \pm \left[\left(\frac{1-\alpha}{2} \right)^2 - b^2 \right]^{1/2} \quad (10)$$

One may prove that a necessary and sufficient condition for

$$|\zeta_{\pm}| \leq 1 \quad (11)$$

is the requirement[†],

$$b = \omega \Delta t < \sqrt{\frac{1+\alpha}{3-\alpha}} \quad (12)$$

Expression (12) is the linear computational stability criterion.

[†]An incorrect condition was given in Office Note 60, because we neglected the converse of equation (13) in Office Note 60.

The initial value problem posed in equations (1) and (2) cannot be solved numerically by means of equations (3) alone. One must specify a starting procedure. The extra condition which arises out of the starting procedure permits one to fix the extra solution given in (9). We shall now indicate a specific starting method.

$$\text{Define } \hat{u} = 1 + 0i \quad (13)$$

We then have from a forward approximation of equations (3):

$$u_*^{(1)} = \hat{u} + i\omega\Delta t \hat{u} = (1 + i\omega\Delta t) \quad (14a)$$

$$u^{(0)} = \alpha\hat{u} + \frac{1-\alpha}{2} (\hat{u} + u_*^{(1)}) = 1 + \frac{1-\alpha}{2} i\omega\Delta t \quad (14b)$$

Again using equations (3)

$$u_*^{(2)} = u^{(0)} + 2i\omega\Delta t u_*^{(1)} \quad (15a)$$

$$u^{(1)} = \alpha u_*^{(1)} + \frac{1-\alpha}{2} (u^{(0)} + u_*^{(2)}) \quad (15b)$$

Using (14a) and (14b), one has

$$u^{(1)} = \left\{ 1 - (1-\alpha) (\omega\Delta t)^2 \right\} + \left\{ 1 + \frac{1}{2}(1-\alpha)^2 \right\} i\omega\Delta t \quad (16)$$

Now K_1 and K_2 in equation (9) may be determined from (14b), (16) and (10). One may calculate

$$K_2 = X^{-1} \left\{ \frac{1}{2} (X + (1-\alpha) (\omega\Delta t)^2 - (1+\alpha)) + \frac{1}{4} (1-\alpha) (X - (1-\alpha)) i\omega\Delta t \right\} \quad (17)$$

and

$$K_1 = 1 + \frac{1-\alpha}{2} i\omega\Delta t - K_2 \quad (18)$$

in which

$$X \equiv \left\{ (1+\alpha)^2 - (2\omega\Delta t)^2 \right\}^{\frac{1}{2}} \quad (19)$$

For particular choices of α and $\omega\Delta t$, the solution of the initial value problem may be tabulated.

3. Discussion of Results

We have seen that the solution of the initial value problem consists of two components. The first component, $K_1 \zeta_1^n$, is usually termed the physical mode since the locus of this solution is similar to that of the analytic solution of the differential equation. The second component of the numerical solution, $K_2 \zeta_2^n$, is referred to as the "computational mode" because its behavior is quite different from that of the analytic solution.

In figure 1, we show the locus of the numerical solution of the initial value problem for two values, .999 and .75, of α for a wave with analytic frequency ω equal to $2\pi/(\mathcal{R}\Delta t)$. The bulk of the amplitude is in the physical mode as shown in the diagrams. The computational mode has been magnified by one order of magnitude in the diagrams. The points are labeled with integers indicating the number of time steps taken. Analytically, the wave should go through one complete cycle in the ten steps shown. In both cases, the wave rotates more rapidly than it would analytically. Note that the erratic behavior of the computational mode is the result of a regular negative rotation plus a 180 degree shift at even time steps. When α is set at .75, one observes a dampening of both modes.

The numerical integration technique discussed in this paper treats each of the solution's components differently. These differences permit the selection of the parameter, α , in a manner oriented toward achieving different objectives at various points in a numerical integration.

Figures 2 and 3 graphically depict the amplification factor and phase error for various frequencies as a function of the choice of the parameter, α . The physical mode is damped for all frequencies which are computationally stable. The computational mode is severely damped. Low frequency waves have their computational mode damped by the factor α in just one time step. The computational mode associated with higher frequency oscillations is not damped as much.

In Asselin's paper, the use of several different values of the smoothing parameter during the calculation of a five day forecast was indicated. In terms of the parameter α , he used .14 for 36 hrs, .50 for the next 12 hrs, .80 from 48 to 60 hrs and $\alpha = 1.0$ for the rest. His calculation was made with the semi-implicit integration method which has different computational stability characteristics than the explicit scheme considered in this paper. However, it may be noted from figure 2 that the choice $\alpha = .14$ gives maximum damping of the high frequency physical modes. Continual use of such a value would however severely damp the low frequency oscillations. This consideration explains the reason for shifting to larger values of α later on. The values of $\alpha = .5$ and .8 provide significant damping of all the computational modes with modest damping of the physical modes. The shift to $\alpha = 1.0$ toward the end of the forecast period suggests that noise was no longer present in the fields.

The linear stability criterion may then be expressed, combining both temporal and spatial truncation error effects, in the form,

$$\Delta t < \frac{\Delta x}{\hat{c}} \quad (27)$$

With reference to the use of variable α in the forecast calculations, it should be noted that in an explicit integration the use of $\alpha = .14$ would require the use of a time step about 6/10 ths of that admissible when $\alpha = 1.0$. For $\alpha = .5$, one must use a time step about 3/4 ths that admissible with $\alpha = 1.0$.

We see then that Asselin's selection of α 's is efficient as against the use of an Euler-backward scheme, mainly because of his use of the semi-implicit rather than the explicit integration method.

4. Conclusions

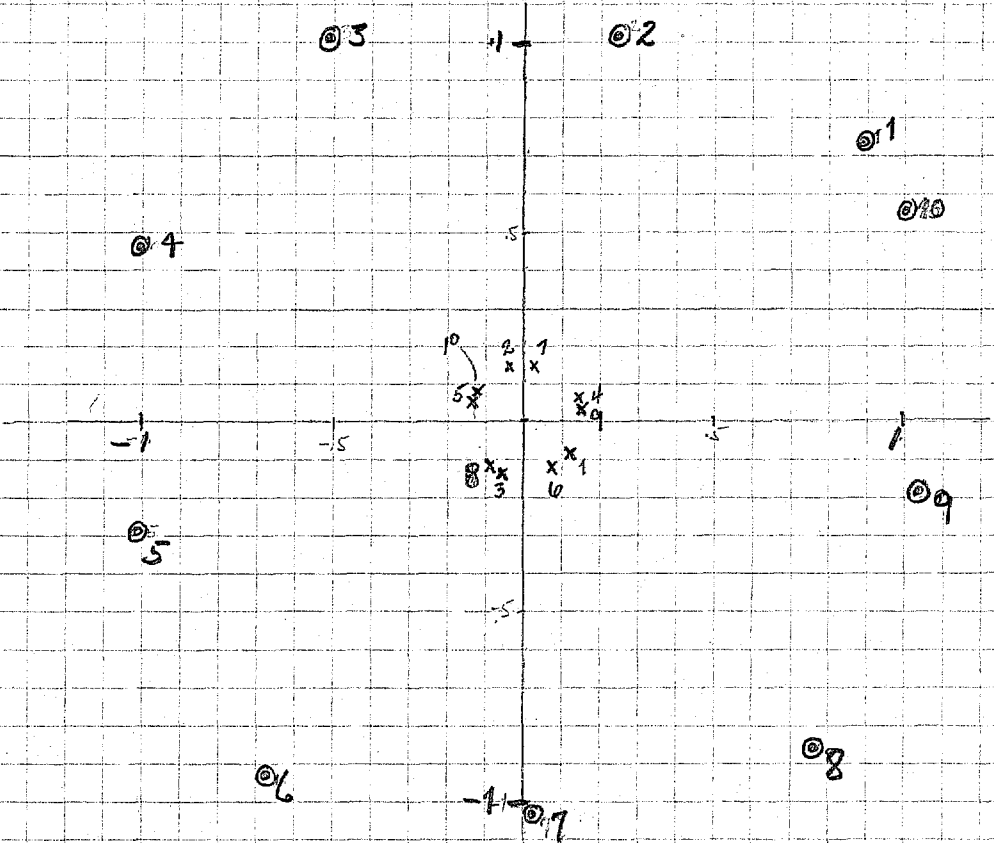
The analysis of Robert's time filter scheme for numerical integration of hyperbolic equations shows that selective damping of high frequency and computational modes may be achieved at relatively little cost in additional calculation. The testing of this method in complex numerical weather prediction models would appear to be desirable. Care must be exercised, however, to insure that the computational stability criterion is satisfied throughout the integration. This is particularly the case because the use of a variable smoothing index, α , appears to be profitable.

5. References

- Asselin, R., 1971: A Time Filter for Time Integrations, Manuscript submitted for publication in MWR.
- Robert, A. J., 1965: The Behaviour of Planetary Waves in an Atmosphere Model Based on Spherical Harmonics, Arctic Meteor. Res. Group, McGill Univ., Publ. in Meteor. No. 77, pp. 59-62.
- Vanderman, L., 1971: Personal Communication.

FIG 1 loci of Numerical Solutions

$\alpha = .999$ $R = 10$



$\alpha = .75$ $R = 10$

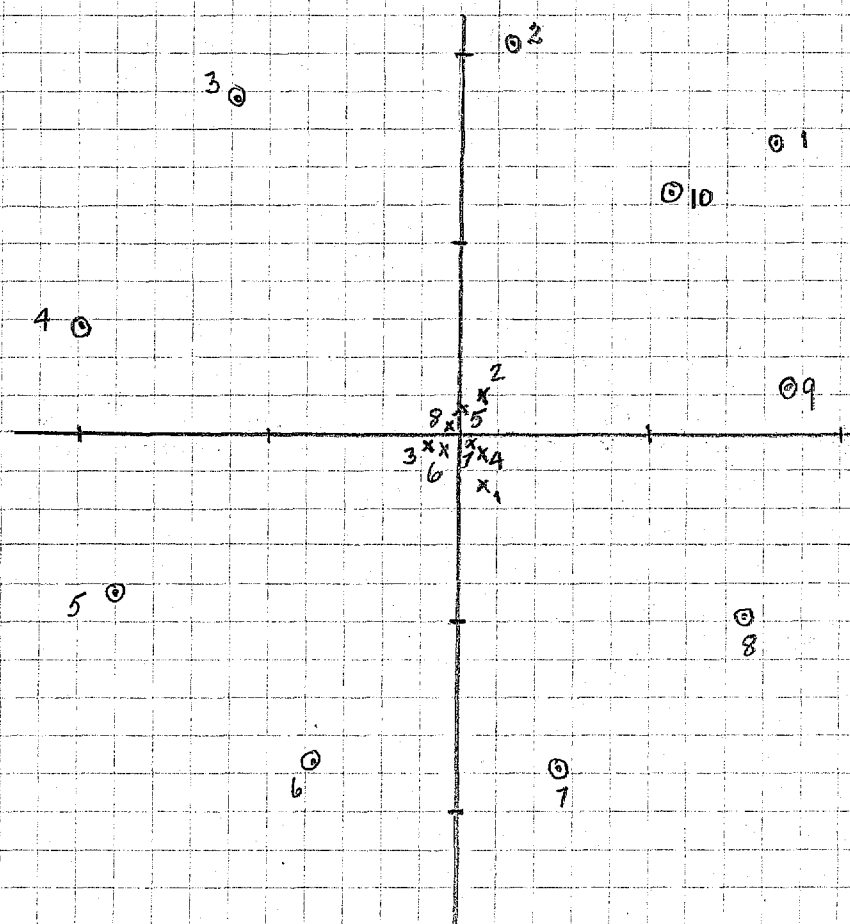


FIGURE 2. Amplification Factor

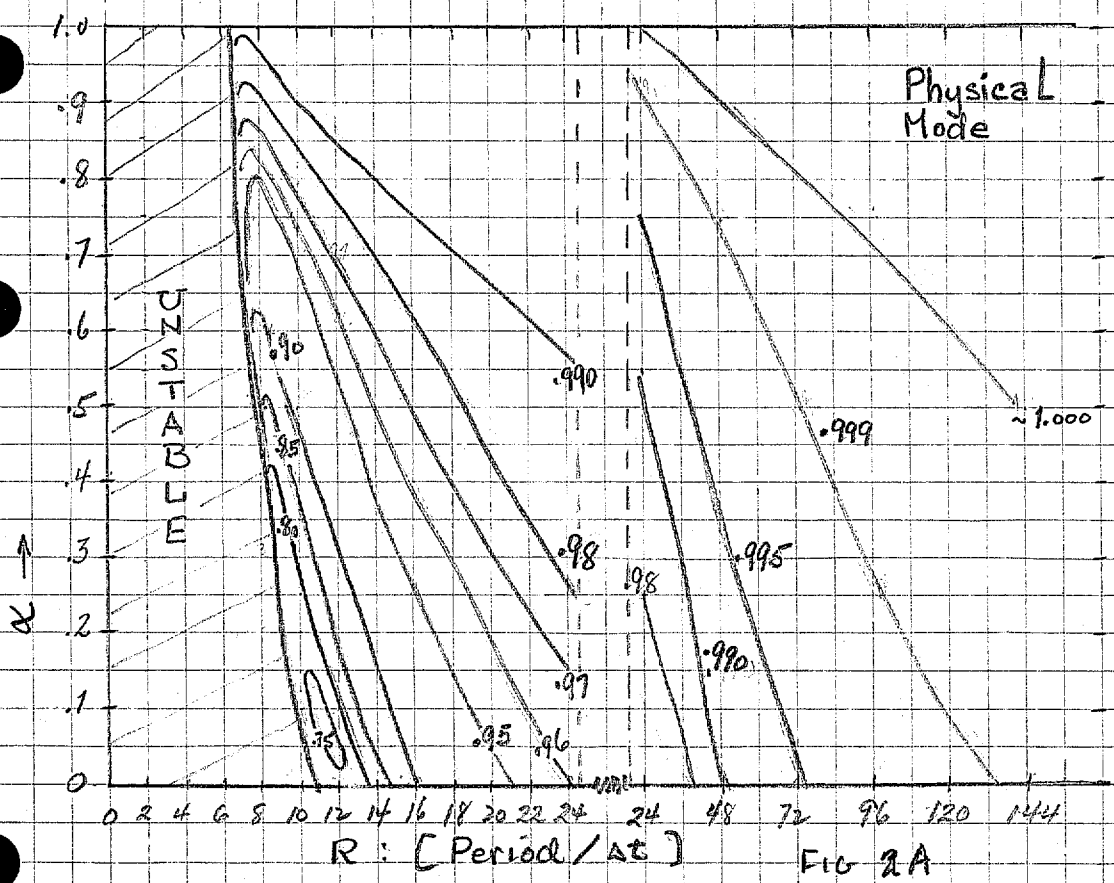


FIG 2A

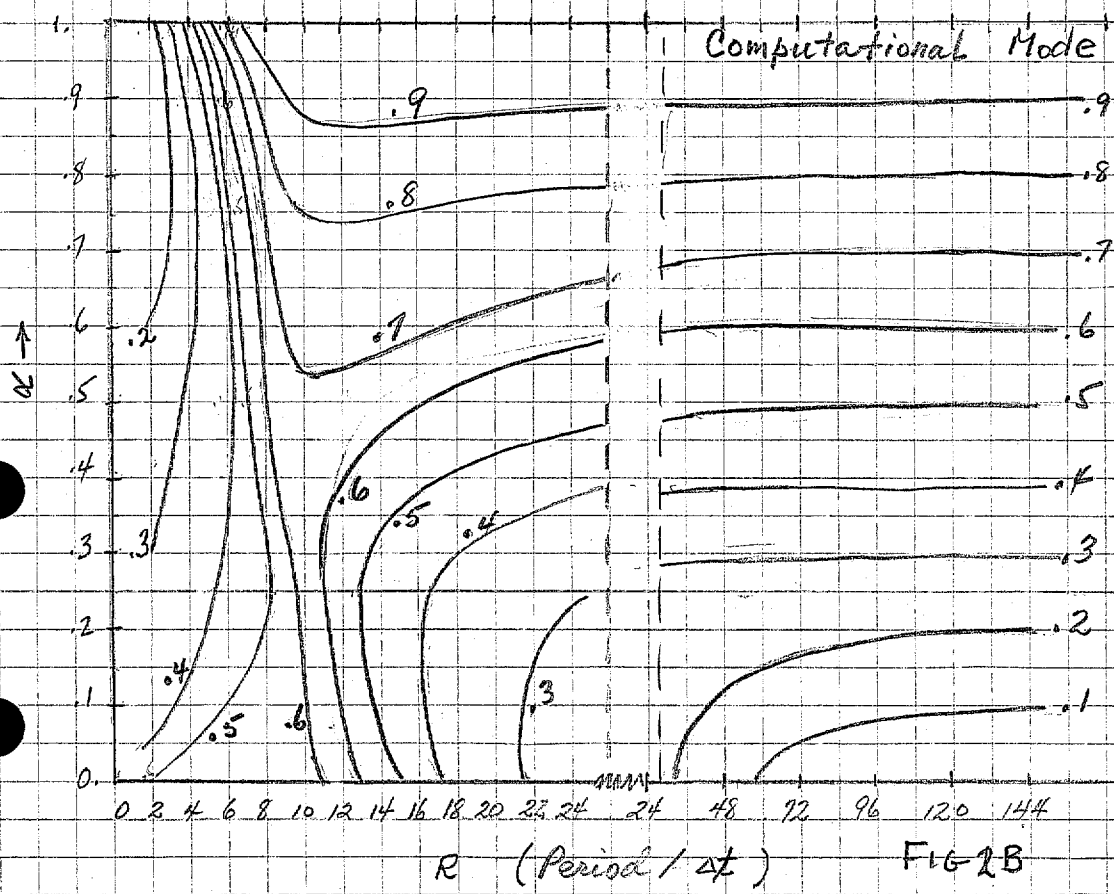


FIG 2B

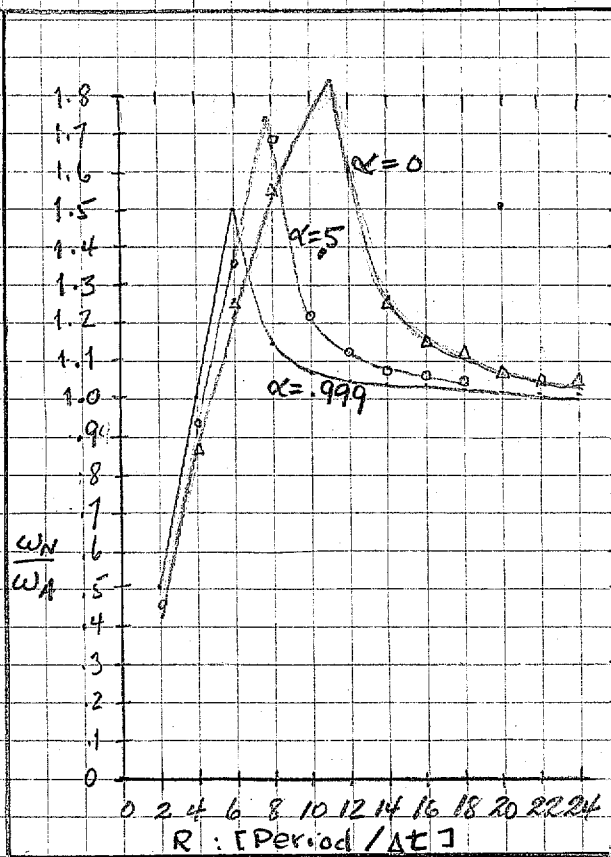


FIG 3a Ratio of numerical to analytic frequency for physical mode.

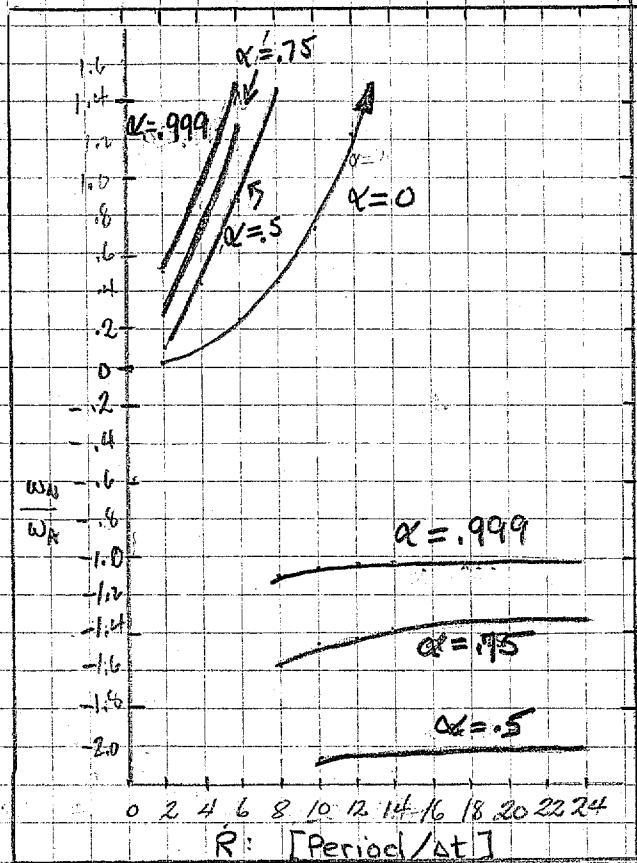


FIG 3b Ratio of numerical to analytic frequency for computational mode.