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FURTHER PROPERTIES OF THE METHOD OF TIME AVERAGING AS APPLIED TO WAVE TYPE AND DAMPING TYPE EQUATIONS

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Introduction

In a previous office note [1], the properties of the "time averaging" method were explored by a numerical experiment. It has since been found possible to perform a closed form stability analysis. The present note is intended to document the results obtained.

If the scheme is indicated schematically by

$$u_*^{n+1} = u^{n-1} + 2\Delta t\, F(u,u_*) \tag{1a}$$

$$u^n = \alpha u_*^n + (1-\alpha)\tfrac{1}{2}(u^{n-1} + u_*^{n+1}) \tag{1b}$$

The results obtained fall into two classes.

First, if $F(u,u_*)$ is taken to represent a tendency of the wave type

$$F(u,u_*) = i\omega u \simeq i\omega u_*^n \tag{2}$$

as in the previous office note, then we find the linear stability criterion:

$$\omega\Delta t < \frac{1+\alpha}{2} \tag{3}$$

Secondly, if $F(u,u_*)$ is taken to represent a tendency of the damping type,

$$F(u,u_*) = -\gamma u \simeq -\gamma u^{n-1} \tag{4}$$

then we find the stability criterion,

$$\Delta t < \frac{1}{2\gamma} \tag{5}$$

under the constraint on α,

$$0. \leq \alpha \leq 1.0 \tag{6}$$

The remainder of this note presents the necessary analysis.

1. General Remarks

For a first order partial differential equation of the type used in numerical weather prediction,

$$\frac{\partial u}{\partial t} = F(u) \tag{7}$$

it has been proposed to use a modification of the leapfrog scheme, called the "time-averaging method."

We shall consider separately two types of equation 7, a wave type and a damping type. The wave type has the form,

$$\frac{\partial u}{\partial t} = i\omega u \tag{7a}$$

whereas the damping type has the form

$$\frac{\partial u}{\partial t} = -\gamma u \tag{7b}$$

In both cases, the parameters (ω and γ) are considered to be positive, real numbers.

2. The Wave Type

The time averaging approximation to equation 7a has the form,

$$u_*^{n+1} = u_*^{n-1} + 2\Delta t\ i\omega u_*^n \tag{8a}$$

$$u^n = \alpha u_*^n + (1-\alpha)\tfrac{1}{2}(u^{n-1} + u_*^{n+1}) \tag{8b}$$

Provided that

$$[\alpha + (1-\alpha)i\omega\Delta t] \neq 0 \tag{9}$$

which is generally satisfied, one may manipulate (8a) and (8b) to obtain the expression

$$u^{n+1} - 2[\frac{1-\alpha}{2} + i\omega\Delta t]u^n - [\alpha - i\omega\Delta t(1-\alpha)]u^{n-1} = 0 \tag{10}$$

From the form (10), it is readily seen that the method will be linearly stable only if the roots of the quadratic,

$$\zeta^2 - 2B\zeta - C = 0 \tag{11}$$

are both less than unity in magnitude.

One has

$$B \equiv \frac{1-\alpha}{2} + i\omega\Delta t \tag{11a}$$

$$C = \alpha - i\omega\Delta t(1-\alpha) \tag{11b}$$

The roots of (11) are,

$$\zeta_{\pm} = [\frac{1-\alpha}{2} + i\omega\Delta t] \pm [(\frac{1+\alpha}{2})^2 - (\omega\Delta t)^2]^{\frac{1}{2}} \tag{12}$$

Provided that

$$\omega\Delta t < \frac{1+\alpha}{2} \tag{13}$$

then one may derive without approximation,

$$|\zeta_{\pm}|^2 = \frac{1+\alpha^2}{2} \pm (1-\alpha)\ [(\frac{1+\alpha}{2})^2 - (\omega\Delta t)^2]^{\frac{1}{2}} \tag{14}$$

The only problem occurs with ζ_+

$$|\zeta_+|^2 = \frac{1+\alpha^2}{2} + (1-\alpha)\ [(\frac{1+\alpha}{2})^2 - (\omega\Delta t)^2]^{\frac{1}{2}} \tag{15}$$

Since, $1 > (1-\alpha) > 0$, we may use an upper bound on the right hand side second term by letting $\omega\Delta t \sim 0$, one then gets

$$|\zeta_+|^2 \le \frac{1+\alpha^2}{2} + \frac{(1-\alpha)(1+\alpha)}{2} = 1 \tag{16}$$

The conclusion is that the method is stable for wave type equations provided that

$$\omega\Delta t < \frac{1+\alpha}{2} \tag{17}$$

3. The Damping Type

We next consider the method applied to the damping problem (cf. eq. 4). The scheme has the form

$$u_*^{n+1} = u^{n-1} + 2\Delta t(-\gamma)u^{n-1} = (1-2\gamma\Delta t)u^{n-1} \tag{18a}$$

$$u^n = \alpha u_*^n + (1-\alpha)\tfrac{1}{2}[u^{n-1} + u_*^{n+1}] \tag{18b}$$

These equations may be manipulated to the form,

$$u^{n+1} - 2\left[\frac{(1-\alpha)(1-\gamma\Delta t)}{2}\right]u^n - \alpha(1-2\gamma\Delta t)u^{n-1} = 0 \qquad (19)$$

Stability of the scheme again requires that the roots of the quadratic formed from (19) have magnitude less than unity.

The roots are

$$\zeta_\pm = \frac{(1-\alpha)(1-\gamma\Delta t)}{2} \pm \left\{\left[\frac{(1-\alpha)(1-\gamma\Delta t)}{2}\right]^2 + \alpha(1-2\gamma\Delta t)\right\}^{\frac{1}{2}} \qquad (20)$$

If we assume that

$$0 < 2\gamma\Delta t < 1 \qquad (20)$$

then we may write

$$|\zeta_\pm| = \left|\frac{(1-\alpha)(1-\gamma\Delta t)}{2}\right| \left|1 \pm \left(1+K^2\right)^{\frac{1}{2}}\right| \qquad (22)$$

with

$$K^2 \equiv \frac{4\alpha(1-2\gamma\Delta t)}{(1-\alpha)^2(1-\gamma\Delta t)^2} \qquad (23)$$

The only difficulty will be associated with ζ_+. However, given (20) and assuming that

$$0 \le \alpha \le 1. \qquad (24)$$

one may readily show that $|\zeta_+| < 1$.

We conclude that for damping type systems, the method will be stable provided that

$$\Delta t < \frac{1}{2\gamma}$$

and

$$0. \le \alpha \le 1. \qquad (25)$$

4. <u>Concluding Remarks</u>

Additional analysis of the time averaging method is called for. It is desirable to clarify the method's treatment of the "computational mode" and damping as a function of frequency in wave-type equations.

The problems associated with mixed types of equations (in which wave and damped forms appear together) call for careful treatment. The "splitting method" of Marchuk offers a possible solution but numerical experimentation may be necessary.

5. <u>References</u>

Gerrity, J., and P. Polger, NMC Office NOte #51, "Comparative Analysis of a New Integration Method with Certain Standard Methods."